

The Prolongation for Bayesian estimation of Scale – parameter of Double exponential distribution

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Abstract : In this paper, based on the traditional Bayesian risk, aiming at the empirical Bayesian estimation of the double exponential distribution's scale parameter, through an improved inequality about the LINEX loss function, we reduce the order of the convergence speed of the parameter θ in Bayesian estimation, which is from the prior distribution \tilde{F} .

Introduction

The INEX loss function put forward by Varian in 1975 is as follows (Varian,1975):

$$L(\hat{\theta} - \theta) = L(\Delta) = be^{a\Delta} - c\Delta - b \quad (1)$$

$\Delta = \hat{\theta} - \theta$ represent estimate error of $\hat{\theta}$, $a, c \neq 0, b > 0$, a and b respectively represent dimension parameters and shape parameters. According to the definition of LINEX loss function, the minimum value of the loss function $L(\Delta)$ exist when $\Delta = 0$ if only the parameters a , b and c satisfy the condition of $ab = c$. Obviously, when $|\Delta| = |\hat{\theta} - \theta|$ is very small, the image of the loss function is almost symmetrical, and nearly close to the square loss. In addition, when the parameter $a > 0$, if $\Delta = \hat{\theta} - \theta < 0$ then the loss function is in linear growth; If $\Delta = \hat{\theta} - \theta > 0$, the loss function is in exponential growth.

In recent years, there is a lot paper on LINEX, under LINEX loss function, in the condition of Lomax distribution scale parameter is known. You and Zhouling (2015) studied the empirical Bayesian estimation of the position parameter of double exponential distribution under LINEX loss function..Xie Xiaoyi, Cao Hong etc (2016) discussed the problem of multilayer Bayesian estimation and the comparison of the prior distribution of the location parameters of the normal distribution model based on LINEX loss function. In this paper, we expand the Bayesian estimation under LINEX loss function and its promotion.

Considering the following density function:

$$f(x; \theta, \sigma) = \frac{1}{2\sigma} e^{-\frac{x-\theta}{\sigma}} I(x > 0) \quad (2)$$

Here, θ represent the location parameters of the double exponential distribution, σ represent the scale parameter, and satisfied the condition: $-\infty < \theta < +\infty, \sigma > 0$. For arbitrary given parameter σ , let $\theta = 0$, the marginal distribution density of random variable X is:

$$f(x | \sigma) = \frac{1}{2\sigma} e^{-\frac{x}{\sigma}} I(x > 0) \quad (3)$$

$G(\sigma)$ is prior distribution, $g(\sigma)$ is probability density, then the edge probability density of the random variable X is:

$$f(x) = \int_{\theta} \frac{1}{2\sigma} e^{-\frac{x}{\sigma}} dG(\sigma) \quad (4)$$

The LINEX loss function is shown as follows:

$$L(\sigma, a) = e^{c(a-\sigma)} - c(a-\sigma) - 1 \quad (5)$$

In representation (5), a is estimation of σ , c is the scale parameter of the LINEX loss function $L(\sigma, a)$. $c \in R, c \neq 0$. Under the loss function $L(\sigma, a)$, if $G(\sigma)$ is known, we can calculated the Bayesian estimation based on random variable X . $\delta = \delta(x)$ is decision function, $B(\sigma, \delta)$ is the risk function of σ , $B(\delta)$ is the corresponding risk function of δ . Then we have:

$$B(\delta) = \int_0^{+\infty} f(x) \left[\int_0^{+\infty} L(a, \delta) g(\sigma | x) \right] dx \quad (6)$$

In representation (6), $g(\sigma | x)$ is the Conditional probability density of σ when given $X = x$. $f(x)$ is the marginal distribution density function of random variable X .

When we apply the Bayesian method, if $\delta_G = \delta_G(x)$ is the estimation of parameter σ , then we have $B(\delta_G) = \inf_{\delta} B(G)$. And when $B(G)$ reach the minimum value, $B(\delta) = \int_0^{+\infty} L(a, \delta) g(\sigma | x)$ also reach the minimum. Through calculating, we obtain the Bayesian Estimation of the distribution groups under LINEX loss function.

$$\delta_G(x) = \frac{1}{c} \frac{f(x+c)}{f(x)} \quad (7)$$

In this paper, $E_{(x, \sigma)}$ represent the mathematical expectation of the joint distribution of (x, σ) . Then we can calculate the Bayesian risk of $\delta_G(x)$:

$$\begin{aligned} R(G, \delta_G) &= E_{(x, \delta)} [L(\sigma, \delta_G)] = \int_0^{+\infty} [e^{c\delta_G} E(e^{-c\delta} | x) - c\delta_G - cE(e | x) - 1] f(x) dx \\ &= \int_0^{+\infty} [-c\delta_G + cE(e | x)] f(x) dx \end{aligned} \quad (8)$$

In summary, if $G(\sigma)$ is known, then the Bayesian risk of parameter σ could be determined precisely. However, if $G(\sigma)$ is unknown, then the empirical Bayesian estimation methods will need to be introduced.

Lemma

Suppose $k(x)$ is Borel Measurable function, $k(x)$ is zero when x is outside the interval $(0, 1)$, and satisfies the following conditions:

$$\int_0^1 y^t k(y) dy = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0, t = 1, 2, \dots, s-1 \end{cases} \quad (9)$$

Now define the kernel estimation of the probability density as follows:

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^n k\left(\frac{x-x_i}{h_n}\right) \quad (10)$$

When $n \rightarrow \infty$ and $h_n \rightarrow 0$, $k(x)$ is the kernel function which satisfying the condition.

Lemma 1 $\forall a, b, a, b \geq 2 \ln \frac{1+\sqrt{5}}{2}$, if $\frac{1}{2} \leq k < 1$, then the inequality exists as follows:

$$e^{a-b} - (a-b) - 1 \leq (e^{ka} - e^{kb})^2$$

Proof: (i) We prove the inequality when $k = \frac{1}{2}$. Suppose $x \in [0, b)$, we introduce function :

$$F(x) = e^{x-b} - (x-b) - 1 - (e^{x/2} - e^{b/2})^2$$

then

$$F'(x) = e^{x-b} - 1 - e^{x/2} (e^{x/2} - e^{b/2}) = e^{-b} (e^{x/2} - e^{b/2}) [(e^{x/2} + e^{b/2}) - e^{b+x/2}] \geq 0$$

In the same way, when $x > b$, we have $F'(x) < 0$.

(ii) when $k = 1$, let $F(x) = e^{x-b} - (x-b) - 1 - (e^x - e^b)^2$

$$\text{Then } F'(x) = e^{x-b} - 1 - 2(e^x - e^b)e^x = \frac{1}{e^b} [e^x - e^b - 2(e^x - e^b)e^x]$$

$$= \frac{1}{e^b} (e^x - e^b) (1 - 2e^x e^b) \geq 0, x \in (a, b)$$

In the same way, when $x > b$, we have $F'(x) < 0$.

(iii) when $\frac{1}{2} \leq k \leq 1$, we introduce the function:

$$F(x, k) = e^{x-b} - (x-b) - 1 - (e^{kx} - e^{kb})^2, \frac{\partial F}{\partial k} = 2xe^{kx}(e^{kx} - e^{kb}) \leq 0, (0 \leq x \leq b)$$

From the above (i) and (ii), we find

$$F(x, 1) \leq 0, F(x, \frac{1}{2}) \leq 0$$

Therefore, the function $F(x, k)$ which is about variable k on interval $[1/2, 1]$ is monotonically non-increasing function. So we have $F(x, k) \leq 0, (\frac{1}{2} \leq k < 1)$.

Theorem

Suppose that X_1, X_2, \dots, X_n independent and identically distributed sample. In order to build Bayesian estimation function, at first we should construct estimator of $\tau_G(x)$. Suppose the probability density function is estimated at (14), the kernel estimation, definite the estimator of $\tau_G(x)$ as follows:

$$\tau_n(x) = \begin{cases} \frac{f_n(x)}{f_n(x+c)} & 1 < \frac{f_n(x)}{f_n(x+c)} < n^v \\ 1 & \text{other} \end{cases} \quad (12)$$

Here, $0 < v < 1$ and v is indeterminate. Definite the empirical Bayesian estimation of σ as follows:

$$\delta_n(x) = c^{-1} \ln \tau_n(x) \quad (13)$$

Suppose $R(G, \delta_n | X_n)$ is the conditional Bayesian risk of EB estimation $\delta_n(x)$ when $X_n = (X_1, \dots, X_n)$ is given. $R(G, \delta_n) = E[R(G, \delta_n | X_n)]$ represent the full Bayesian risk of $\delta_n(x)$.

If for arbitrary prior distribution G under prior distribution family \tilde{F} , when $n \rightarrow \infty$, we have $R(G, \delta_n) \rightarrow R(G, \delta_G)$. Then we call $\{\delta_n(x)\}$ as the asymptotically optimal empirical Bayesian estimation of θ under the prior distribution group \tilde{F} (a.0. EB estimation).

If $R(G, \delta_n) - R(G, \delta_G) = o(n^{-q})$, then we call q as the order of the convergence speed of the parameter θ in Bayesian estimation, which is from the prior distribution \tilde{F} .

Theorem 1 (Guan Ming, 2010) For arbitrary given natural number $s > 1, 0 < \lambda < 1$, if the following conditions are satisfied:

- (i) $\int_0^1 y^t k(y) dy = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0, t = 1, 2, \dots, s-1 \end{cases}$;
- (ii) s order derivation of $f(x)$ exists, and $\sup_x f(x) < +\infty, \sup_x |f^s(x)| < +\infty$;
- (iii) $\int_0^\infty f^{1-\lambda}(x) dx, \int_0^\infty f^{-\lambda}(x+c)f(x) dx$ and $\int_0^\infty \tau_G^l(x)f(x) dx$ exist.

Then when $\delta_n(x)$ is defined by formula (13) and h_n satisfies $h_n = n^{-\frac{1}{2s+1}}$:

$$R(G, \delta_n) - R(G, \delta_G) = o(n^{-q}), \quad q = \frac{\lambda s(l-2)}{(2s+1)l}$$

Theorem 2 For arbitrary given natural number $s > 1, 0 < \lambda < 1$, and $a, b \geq 2 \ln \frac{1+\sqrt{5}}{2}$, if conditions of

theorem 1 are satisfied. Then when $\delta_n(x)$ is defined by formula (13) and h_n satisfies $h_n = n^{-\frac{1}{2s+1}}$:

$$R(G, \delta_n) - R(G, \delta_G) = o(n^{-q}), \quad q = \frac{\lambda s(l-2k)}{(2s+1)l} \quad (14)$$

Proof: In the light of (9), we have

$$0 < R(G, \delta_n) - R(G, \delta_G) = \int_0^{+\infty} E_n(e^{c(\delta_n - \delta_G)} - c(\delta_n - \delta_G) - 1) f(x) dx \quad (15)$$

Denote $A_n = \{x | x \in (0, +\infty), 1 < \tau_G(x) < n^v\}, B_n = (0, +\infty) - A_n$.

According to the Lemma 1, it is easy to obtain

$$\begin{aligned}
& E[\tau_n^k(x) - \tau_G^k(x)]^2 \leq (2n^{kv})^{2-\lambda} E_n[|\tau_n(x) - \tau_G(x)|_{2n^v}^\lambda] \\
& = (2n^{kv})^{2-\lambda} E_n \left[\left| \frac{f_n^k(x)}{f_n^k(x+c)} - \frac{f^k(x)}{f^k(x+c)} \right| \right]^\lambda
\end{aligned} \tag{16}$$

According to the Lemma 3 and Lemma 4(Wei Laisheng, 2002), we have

$$\begin{aligned}
& E[\tau_n^k(x) - \tau_G^k(x)]^2 \leq (2n^{kv})^{2-\lambda} \cdot 2|f(x+c)|^{-\lambda} \left[\left(\left| \frac{f^k(x)}{f^k(x+c)} \right| + 2n^v \right) E|f_n^k(x+c) - f^k(x)|^\lambda \right] \\
& \leq Mn^{2kv} \cdot [f^{-k\lambda}(x) E|f_n^k(x) - f^k(x)|^\lambda] + Mn^{2kv} \cdot [f^{-k\lambda}(x+c) E|f_n^k(x+c) - f^k(x)|^\lambda]
\end{aligned}$$

$$\begin{aligned}
\text{Then } & \int_{A_n} E[\tau_n^k(x) - \tau_G^k(x)]^2 f(x) dx \\
& \leq M \int_{A_n} n^{2kv} \cdot [f^{k-\lambda k}(x) E|f_n^k(x) - f^k(x)|^\lambda + f^{-k\lambda}(x+c) E|f_n^k(x+c) - f^k(x+c)|^\lambda] f(x) dx \\
& \leq Mn^{\frac{2kv-\lambda s}{2(s+2)}} \left[\int_0^{+\infty} f^{k-k\lambda}(x) dx + \int_0^{+\infty} f^{-k\lambda}(x+c) dx \right]
\end{aligned}$$

According to the condition (i) and Hölder inequality

$$\int_{A_n} E[\tau_n^k(x) - \tau_G^k(x)]^2 f(x) dx \leq Mn^{\frac{2kv-\lambda s}{2(s+2)}} \tag{17}$$

Let E_* represents the mathematics expectations of the joint distribution of $(X_1, \dots, X_n, (X, \theta))$.

When $x \in B_n$, $1 < \tau_n(x) < n^v < \tau_G(x)$, by using Hölder inequality and Markov inequality, we get

$$\begin{aligned}
& \int_{B_n} E[\tau_n^k(x) - \tau_G^k(x)]^2 f(x) dx \\
& \leq 4 \int_{B_n} \tau_G^{2k}(x) f(x) dx = 4E_*(\tau_G^{2k}(x) I_{B_n}(x)) \leq Mn^{-v(l-2k)}
\end{aligned} \tag{18}$$

$$\text{Let } -v(l-2k) = 2kv - \frac{\lambda s(l-2)}{(2s+l)l}, \text{ we have } v = \frac{\lambda s}{2(s+2)l}$$

$$\text{Therefore, } R(G, \delta_n) - R(G, \delta_G) = O(n^{-q}), \quad q = \frac{\lambda s(l-2k)}{(2s+l)l}.$$

Conclusions

In this article, we use LINEX loss function to estimate the scale parameter of based on empirical Bayesian method. At first, we use inequality conversion to transform the LINEX loss function, then use the optimized inequality to estimate the rate of convergence of Empirical Bayesian estimation. And use lemma 1, Markov inequality and Hölder inequality to calculate the optimized convergence rate.

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