The Propagation of Phase Boundaries in Elastic Bars

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Introduction

When the stress-deformation relation governing static, one dimensional elasticity is not monotone, certain static solutions may contain stationary shock waves, or *phase boundaries*. These static solutions have been studied in [1]. One would expect that the corresponding dynamic theory would admit solutions containing propagating phase boundaries. In this paper I investigate the existence and properties of such phase boundaries in dynamic, one dimensional elasticity. A phase boundary is conceived as kinematically like a shock wave, but the non-monotonicity of the stress-deformation relation permits the phase boundary to come to rest.

The motivation for this study arises from experiments on phase transitions in solid bars, though the familiar van der Waals equation of state in the context of one dimensional gas dynamics yields essentially the same theory. A host of materials, including various polymers [2], natural rubbers [3, chapter 7], and metals [4] undergo first order phase transitions in which sharply defined phase boundaries travel through the body. In a uniaxial tensile experiment on a bar in a dead loading device the specimen appears to deform more or less homogeneously until a certain value of the applied load is reached. Then, typically, a phase boundary appears at the end of the bar, or a pair of phase boundaries separate at some interior point of the bar. These phase boundaries travel through the specimen at much lower speeds than the shock or sound speeds of either of the homogeneous phases. Studies of static elastic bar theory have indicated that the main features of these transitions are predicted by use of a non-convex stored energy, or, equivalently, a non-monotone stress-deformation relation.

Here, I establish the existence of a family of solutions in which a moving phase boundary separates two classical solutions (Theorem 1). Solutions can be found whereby the moving phase boundary comes to rest and a static solution results; or, a stationary phase boundary in a static solution may begin to move. I investigate in Section 2 the possibility that a global solution containing a phase boundary may exist. The interaction of a sound wave and a phase boundary is explored in Section 4, where two kinds of problems are formulated. In the first a sound wave moves down the bar and strikes a moving phase boundary; generally, a reflected and a transmitted wave will emerge, but it is found that either one or the other, but not both, can be suppressed. In the second problem I ask whether a phase boundary can spontaneously emit sound waves. This turns out to be possible only if the motion of the phase boundary itself experiences a weak singularity. Two kinds of solutions to the Riemann problem are found in Section 5. The first is a one parameter family of solutions each containing a single phase boundary. The second is a two parameter family containing two phase boundaries which emerge from constant initial data. The admissibility of these solutions according to the static theory, a criterion of viscosity, and the entropy rate criterion are investigated.

1. Dynamic Elastic Bar Theory

An elastic bar is described by a single material co-ordinate

$$X \in [-L, L]. \tag{1.1}$$

A **transplacement** of the bar is a function

$$y = y(X, t), \quad X \in [-L, L], \quad t \in [0, T], \tag{1.2}$$

which assigns the position $y(X, t)$ to the point X at time t. A prime and superimposed dot will represent derivatives with respect to X and t, respectively. Whenever y is continuously differentiable, we call $u = y'(X, t)$ the **deformation** and $v = \dot{y}(X, t)$ the **velocity** at point X and time t.

The constitutive function $\sigma(u)$ delivers the **stress** corresponding to the deformation u. The function $\sigma(\cdot)$ will be defined on a closed interval $[\alpha, \beta]$. Let α^1 and β^1 be assigned constants consistent with the requirements

$$0<\alpha<\alpha^1<\beta^1<\beta<\infty. \tag{1.3}$$

We assume that

$$\begin{aligned} &\text{(i)}\ \sigma(\cdot)\in C^2_{[\alpha,\beta]},\\ &\text{(ii)}\ \sigma_u>0 \quad \text{on } [\alpha,\alpha^1)\cup(\beta^1,\beta],\\ &\text{(iii)}\ \sigma_u<0 \quad \text{on } (\alpha^1,\beta^1). \end{aligned} \tag{1.4}$$

The constitutive function restricted to the interval $[\alpha, \alpha^1]$ will be called the α-branch; the restriction of the constitutive function to the interval $[\beta^1, \beta]$ will be called the β-branch. If at some pair (X, t) the value of $y'(X, t)$ lies in the interval $[\alpha, \alpha^1]$, (or in the interval $[\beta^1, \beta]$), then we shall say that the point X is in the **α-phase (β-phase)** at time t. The function $(\sigma_u)^{\frac{1}{2}}$, for u in the domain of the α- and β-branch, will be denoted by $c(u)$.

The classical equations of motion for dynamic elastic bar theory are written

$$\ddot{y}=(\sigma(y'))'=\sigma_u(y')\,y''. \tag{1.5}$$

A transplacement will be a *solution* if it satisfies the **balance of momentum** on each subinterval of the bar. That is, $y(X,t)$ is a solution if its distributional derivatives exist and are measurable, and if *

$$\begin{gathered}\int_{X_1}^{X_2} \dot{y}(X,t_2)-\dot{y}(X,t_1)\,dX=\int_{t_1}^{t_2}\sigma(y'(X_2,t))-\sigma(y'(X_1,t))\,dt\\ \forall\ -L\leqq X_1<X_2\leqq L,\ 0\leqq t_1<t_2\leqq T.\end{gathered} \tag{1.6}$$

If $y(X,t)$ is a solution which is twice continuously differentiable except on a finite number of smooth curves S_i: $X=\chi_i(t)$, $i=1,2,\ldots,N$, then it follows from (1.6) that $y(X,t)$ is a classical solution of (1.5) on $\{(-L,L)\times(0,T)\}-\bigcup_{i=1}^{N} S_i$, and $y(X,t)$ satisfies the Rankine-Hugoniot conditions on each curve:

$$\begin{aligned}\dot{\chi}_i(y'_{i+}(t)-y'_{i-}(t))+(\dot{y}_{i+}(t)-\dot{y}_{i-}(t))&=0,\\ \sigma(y'_{i+}(t))-\sigma(y'_{i-}(t))+\dot{\chi}_i(\dot{y}_{i+}(t)-\dot{y}_{i-}(t))&=0,\end{aligned} \tag{1.7}$$

In (I.7)

$$\begin{gathered}y'_{i+}(t)\equiv y'(\chi_i(t)+0,t),\\ y'_{i-}(t)\equiv y'(\chi_i(t)-0,t),\\ \text{etc.}\end{gathered} \tag{1.8}$$

Conversely, if $y(X,t)$ satisfies (1.5) in the classical sense, except on a finite number of smooth curves, and the Rankine-Hugoniot conditions are satisfied on those curves, then $y(X,t)$ is a solution.

* It is known that if y satisfies (1.6), then necessarily $\dot{y}$ and y' are of bounded variation, and for such functions (1.6) is equivalent to the standard definition of a weak solution of (1.5). Implicit in the definition (1.6) is the requirement that $\alpha\leqq y'\leqq\beta$ a.e.

A curve S: $X=\chi(t)$ will be called a **phase boundary** if the Rankine-Hugoniot conditions hold on S, if the limiting values y'_+, y'_-, $\dot{y}_+$, $\dot{y}_-$ exist, and if y'_+ is contained in the domain of the α-branch (β-branch) and y'_- is contained in the domain of the β-branch (α-branch).

Equation (1.5) can be decomposed into a system of two first order equations in several ways. One formulation, which is useful for calculations involving the Rankine-Hugoniot conditions, follows by changing to the variables deformation and velocity,

$$u(X,t)=y'(X,t), \qquad v(X,t)=\dot{y}(X,t). \tag{1.9}$$

A statement equivalent to (1.5) is

$$\begin{aligned} \dot{u}&=v', \\ \dot{v}&=\sigma_u(u)\,u', \end{aligned} \tag{1.10}$$

in which u and v belong to $C^1_{[-L,L]\times[0,T]}$. The system (1.10) is well defined whenever $\alpha\leqq u\leqq\beta$, that is, whenever u belongs to the domain of the constitutive function. When dealing with characteristics, it is more convenient to diagonalize the operator on the right hand side of (1.10) by defining the Riemann invariants,

$$\begin{aligned} r&=R(u,v)\equiv\int^u c(w)\,dw+v, \\ s&=S(u,v)\equiv-\int^u c(w)\,dw+v. \end{aligned} \tag{1.11}$$

Whenever u is restricted to an open subset of either the α-branch or the β-branch, the mapping $(u,v)\to(r,s)$ is a diffeomeorphism. We denote the inverse of this map by $u=U(r,s)$, $v=V(r,s)$. We invoke (1.11) only when $c(u)$ is well defined. It is evident from (1.11) that u can be expressed as a strictly monotone function of $(r-s)$, and v can be expressed as a strictly monotone function of $(r+s)$, *viz.*,

$$\begin{aligned} u&=\tilde{u}(r-s), \\ v&=\tilde{v}(r+s). \end{aligned} \tag{1.12}$$

Let

$$\tilde{c}(r-s)\equiv c(\tilde{u}(r-s)). \tag{1.13}$$

When u belongs to the domain of the α- and β-branch, the Riemann invariants satisfy a system of equations equivalent to the system (1.10), or equivalent to the single equation (1.5). From the definition (1.11), this system may be written

$$\begin{aligned} \dot{r}&=\tilde{c}(r-s)\,r', \\ \dot{s}&=-\tilde{c}(r-s)\,s'. \end{aligned} \tag{1.14}$$

Let $X=\chi(t)$ be a phase boundary contained in the domain of a solution $u(X,t)$, $v(X,t)$. It will be necessary in the following treatment to impose the restriction that *the speed of the phase boundary be less than the acoustic speed on at least one side of the phase boundary.* It will be shown in Section 4 that the acoustic velocities are given by

$$\pm c(u_+(t)) \tag{1.15}$$

on the (+) side of the phase boundary, and by

$$\pm c(u_-(t)) \tag{1.16}$$

on the (−) side. Here, as before,

$$\begin{aligned} u_+(t) &= u(\chi(t)+0, t), \\ u_-(t) &= u(\chi(t)-0, t). \end{aligned} \tag{1.17}$$

By assuming $v_+(t) \neq v_-(t)$, and then eliminating $v_+(t) - v_-(t)$ from the Rankine-Hugoniot conditions (1.7), we obtain the squared velocity of the phase boundary,

$$\dot{\chi}^2 = \frac{\sigma(u_+(t)) - \sigma(u_-(t))}{u_+(t) - u_-(t)}. \tag{1.18}$$

Suppose $u(t)$ equals either $u_+(t)$ or $u_-(t)$. The condition we seek is

$$|\dot{\chi}| < c(u(t)), \tag{1.19}$$

which, according to (1.19) and the definition of $c(u)$, can be written

$$\frac{\sigma(u_+(t)) - \sigma(u_-(t))}{u_+(t) - u_-(t)} < \sigma_u(u(t)). \tag{1.20}$$

Viewed in this way, the condition (1.19) expresses the fact that in the graph of σ vs. u, the slope of the line connecting $(u_+, \sigma(u_+))$ to $(u_-, \sigma(u_-))$ is less than the slope of the tangent at u_+ or u_-.

2. Existence of Solutions with Phase Boundaries

In this section an existence theorem will be proved for a solution which contains a single phase boundary. The path of the phase boundary, $X = \chi(t)$, $t \in (-\infty, \infty)$, will be prescribed, and a solution $y(X, t)$ will be found in the neighborhood

$$\{(X, t) \in \mathbb{R}^2 \mid \chi(t) - \varepsilon \leqq X \leqq \chi(t) + \varepsilon\}, \tag{2.1}$$

for some $\varepsilon > 0$. This solution will be a classical solution of (1.5) except on the phase boundary, where the Rankine-Hugoniot conditions (1.7) will hold. Therefore, $y(X, t)$ will satisfy the balance of momentum (*cf.* equation (1.6)).

We begin with the simplest case. A static solution is prescribed on the (−) side of a given phase boundary, and the Rankine-Hugoniot conditions yield Cauchy data for the (+) side. We construct a smooth solution having this Cauchy data which covers the region $\{(X, t) \mid \chi(t) \leqq X \leqq \chi(t) + \varepsilon\}$.

Let $u=\alpha_-$ be a constant deformation contained in the domain of the α-branch, and let $X=\chi(t)$ be an assigned curve in (X,t) space, which fulfills the requirements

(i) $\chi(\cdot)\in C^2_{(-\infty,\infty)}$. $\ddot{\chi}(\cdot)$ is Lipschitz continuous and bounded:

$$|\ddot{\chi}(t_2)-\ddot{\chi}(t_1)|\leqq\lambda|t_2-t_1|,\qquad |\ddot{\chi}(t)|\leqq m. \tag{2.2}$$

(ii) For some sufficiently small $\tau>0$,

$$\frac{\sigma(\beta^1+\tau)-\sigma(\alpha_-)}{(\beta^1+\tau)-\alpha_-}<\dot{\chi}^2<\frac{\sigma(\beta-\tau)-\sigma(\alpha_-)}{(\beta-\tau)-\alpha_-}. \tag{2.3}$$

(iii) For the same constant τ, and some positive constant k,

$$c(\beta_+)-|\dot{\chi}|>k, \qquad \text{whenever } \beta^1+\tau\leqq\beta_+\leqq\beta-\tau. \tag{2.4}$$

The assumption of smoothness $\chi(\cdot)\in C^2_{(-\infty,\infty)}$ is equivalent to the assertion that shock and acceleration waves do not impinge on the phase boundary, as will be made evident in Section 4. The Lipschitz condition implies that the amplitudes of third order waves are bounded (*cf.* Section 4), the bounding constant being dependent only upon λ and the constitutive equation. Conditions (2.3) and (2.4) insure that $\dot{\chi}$ takes on values in the range of all possible velocities, and that the acoustic speed evaluated on the $(+)$ side of the phase boundary is uniformly greater than the speed of the phase boundary.

The inequalities (2.3) and the assumption that $\sigma(u)$ is strictly increasing on the β-branch imply that

$$\frac{\sigma(u_+)-\sigma(\alpha_-)}{u_+-\alpha_-}=\dot{\chi}(t)^2 \tag{2.5}$$

has a unique solution $u_+(t)\in C^1_{(-\infty,\infty)}$, which lies in the domain of the β-branch. Let

$$v_+(t)\equiv-\dot{\chi}(t)\left(u_+(t)-\alpha_-\right). \tag{2.6}$$

Then the Rankine-Hugoniot conditions are satisfied for $\dot{\chi}(t)$, $u_+(t)$, $v_+(t)$, $u_-(t)=\alpha_-$, $v_-(t)=0$.

We assign the static solution

$$u(X,t)=\alpha_-, \qquad v(X,t)=0, \tag{2.7}$$

for $X<\chi(t)$. The functions $u_+(t)$, $v_+(t)$ represent Cauchy data for a solution extending on the $(+)$ side of the phase boundary.

The corresponding Cauchy data for the Riemann invariants is calculated through the mapping (1.11):

$$r_+(t)\equiv R\left(u_+(t),v_+(t)\right),\qquad s_+(t)\equiv S\left(u_+(t),v_+(t)\right). \tag{2.8}$$

Lemma 1. *Suppose $\chi(t)$ satisfies (2.2), (2.3) and (2.4). Then $|\dot{r}_+(t)|$ and $|\dot{s}_+(t)|$ are bounded and uniformly Lipschitz continuous on $(-\infty,\infty)$.*

Proof. By differentiation of (2.5),

$$\dot{u}_+\left\{\frac{\frac{d\sigma}{du}(u_+)}{u_+-\alpha_-}-\frac{\sigma(u_+)-\sigma(\alpha_-)}{(u_+-\alpha_-)^2}\right\}=2\dot{\chi}\ddot{\chi}, \tag{2.9}$$

which immediately implies that $|\dot{u}_+(t)|$ is bounded. Equations (2.6) and (2.3) show that $|\dot{v}_+(t)|$ is bounded. By differentiating (2.8) and using the definition (1.12), it follows that $|\dot{r}_+|$ and $|\dot{s}_+|$ are bounded on $(-\infty, \infty)$.

Equations (2.9) and (2.2) show that $\dot{u}_+$ is Lipschitz constant, say, λ_u. It follows from (2.6) that $\dot{v}$ is Lipschitz continuous with constant, say, λ_v. It is easy to derive from (1.12) that $\dot{r}_+$ and $\dot{s}_+$ are Lipschitz continuous. □

Let the bounds of $|\dot{r}_+|$ and $|\dot{s}_+|$ be δ_{r+} and δ_{s+}, and let $\tilde{\lambda}$ be a Lipschitz constant for both $\dot{r}_+$ and $\dot{s}_+$.

Theorem 1. *Suppose $X=\chi(t)$ is assigned consistent with* (2.2), (2.3), *and* (2.4). *Let $u_+(t)$ and $v_+(t)$ be the unique solutions of* (2.5) *and* (2.6). *Then, there is a continuously differentiable solution $u(X,t)$, $v(X,t)$ defined on*

$$\mathscr{D}_\varepsilon:\ -\infty<t<\infty, \qquad \chi(t)\leqq X\leqq\chi(t)+\varepsilon, \tag{2.10}$$

which satisfies the Cauchy data,

$$u\big(\chi(t),t\big)=u_+(t) \qquad v\big(\chi(t),t\big)=v_+(t). \tag{2.11}$$

The combined solution consisting of u, v defined on $\mathscr{D}_\varepsilon$ and the static solution (2.7) *defined for $X<\chi(t)$ satisfies the balance of momentum* (1.6). *For $X>\chi(t)$ the bar is in the β-phase; for $X<\chi(t)$ the bar is in the α-phase.*

Proof. We shall use the Riemann invariants as dependent variables, instead of deformation-velocity variables. Let $r_+(t)$ and $s_+(t)$ be defined by (2.8). The appropriate equations are

$$\begin{aligned}\dot{r}&=\tilde{c}(r-s)\,r', & r\big(\chi(t),t\big)&=r_+(t),\\ \dot{s}&=-\tilde{c}(r-s)\,s', & s\big(\chi(t),t\big)&=s_+(t).\end{aligned} \tag{2.12}$$

To ease the analysis, it is profitable to straighten the path of the phase boundary by changing variables according to the prescription,

$$\begin{aligned}\hat{X}&=X-\chi(t),\\ \hat{t}&=t.\end{aligned} \tag{2.13}$$

Upon substitution of (2.13) into (2.12), (2.12) becomes

$$\begin{aligned}\dot{r}&=\big(\tilde{c}(r-s)+\dot{\chi}\big)\,r', & r(0,t)&=r_+(t),\\ \dot{s}&=\big(-\tilde{c}(r-s)+\dot{\chi}\big)\,s', & s(0,t)&=s_+(t).\end{aligned} \tag{2.14}$$

In (2.14) and until the end of the proof, dot and prime denote derivatives with respect to $\hat{t}$ and $\hat{X}$, respectively; by (2.13) the region $\mathscr{D}_\varepsilon$ of (2.10) is mapped onto

$$\hat{\mathscr{D}}_\varepsilon\colon\ -\infty<t<\infty, \qquad 0\leqq X\leqq\varepsilon \tag{2.16}$$

For the remainder of the proof the superimposed hats will be omitted from t and X.

As a preface to the quasilinear problem (2.14), I shall treat the corresponding linear problem,

$$\begin{aligned} \dot r &= \big(\hat c(X,t)+\dot\chi(t)\big)\, r', & r(0,t)&=r_+(t),\\ \dot s &= \big(-\hat c(X,t)+\dot\chi(t)\big)\, s', & s(0,t)&=s_+(t), \end{aligned} \tag{2.17}$$

in which $\hat c(X,t)-|\dot\chi|>k$ on $\hat{\mathscr{D}}_\varepsilon$. We assume $\hat c(X,t)$ is continuously differentiable on $\mathscr{D}_\varepsilon$. The problem (2.17) can be solved by the method of characteristics. Here, X is the natural choice of parameter along the characteristic. Let $t=\tau_r(X,\gamma)$ be the solution of

$$\frac{dt}{dX}=\frac{-1}{\hat c(X,t)+\dot\chi(t)}, \qquad t(0)=\gamma, \tag{2.18}$$

and let $t=\tau_s(X,\gamma)$ be the solution of

$$\frac{dt}{dX}=\frac{1}{\hat c(X,t)-\dot\chi(t)}, \qquad t(0)=\gamma. \tag{2.19}$$

At fixed γ, $t=\tau_r(X,\gamma)$ is the equation of the *r-characteristic*, and $t=\tau_s(X,\gamma)$ is the equation of the *s-characteristic*. The r- and s-characteristics can be used as coordinate curves on $\hat{\mathscr{D}}_\varepsilon$. That is, from the theory of ordinary differential equations [5, Theorem 3.1], τ_r and τ_s belong to class $C^1_{\hat{\mathscr{D}}_\varepsilon}$. Moreover, the assumption $\hat c-|\dot\chi|>k>0$ implies that $\tau_r(X,\cdot)$ and $\tau_s(X,\cdot)$ are strictly monotone. Hence, there are continuously differentiable inverses,

$$\begin{aligned} \gamma&=\Gamma_r(X,t),\\ \gamma&=\Gamma_s(X,t), \end{aligned} \qquad (X,t)\in\hat{\mathscr{D}}_\varepsilon. \tag{2.20}$$

A straightforward calculation shows that a solution of the system (2.17) is

$$\begin{aligned} r&=r_+\big(\Gamma_r(X,t)\big),\\ s&=s_+\big(\Gamma_s(X,t)\big). \end{aligned} \tag{2.21}$$

According to the classical theory of such equations, this solution is unique within the class of C^1 solutions of (2.17). It follows directly from (2.21) that r is constant along r-characteristics, and s is constant along s-characteristics.

We shall find the solution of the nonlinear problem to be the fixed point of a certain mapping Ξ. To construct Ξ, we let $\{r(X,t),\ s(X,t)\}$ be a pair of continuously differentiable functions defined on $\hat{\mathscr{D}}_\varepsilon$, which satisfy the data

$$\begin{aligned} r(0,t)&=r_+(t),\\ s(0,t)&=s_+(t). \end{aligned} \tag{2.22}$$

We also assume that r, $\dot r$, s, $\dot s$ are sufficiently close (uniformly) to r_+, $\dot r_+$, s_+, $\dot s_+$, respectively. We define

$$\hat{c}(X,t)=\tilde{c}\big(r(X,t)-s(X,t)\big), \tag{2.23}$$

and solve the linear problem (2.17) to obtain functions

$$\rho(X,t), \qquad \sigma(X,t). \tag{2.24}$$

Let Ξ be the mapping $(r,s)\to(\rho,\sigma)$ obtained in this manner. We regard Ξ as defined on a metric space of pairs of continuously differentiable functions (r,s) endowed with the supremum norm on r, s, $\dot{r}$, $\dot{s}$, r', s'. It is a straightforward but routine calculation to show that Ξ is a contraction mapping for ε sufficiently small. The estimate makes use of Lemma 1, equations (2.2) and (2.4). The fixed point of Ξ is a solution of (2.14). □

3. Extensions of Theorem 1

A. Data prescribed on both sides of the phase boundary

The restriction that the solution on one side of the phase boundary be a static solution was unnecessary. The proof of Theorem 1 can be carried out for the Cauchy problem on each side of the phase boundary as long as the Cauchy data $\{u_+(t),\ v_+(t),\ u_-(t),\ v_-(t)\}$ are consistent with Lemma 1. This consistency can be insured in many ways.

One way to do so is to prescribe a phase boundary which satisfies (2.2), (2.3) and (2.4), and to assign Cauchy data $u_-(t)$, $v_-(t)$ on one side of the phase boundary that have bounded, uniformly Lipschitz continuous derivatives, and for which $u_-(t)$ belongs to the domain of the α-branch. The condition (2.4) that the acoustic speed on the β-branch be greater than the speed of the phase boundary permits the Rankine-Hugoniot conditions to be solved for $u_+(t)$, $v_+(t)$. It can easily be shown that the data $u_+(t)$, $v_+(t)$ obtained in this way satisfy Lemma 1.

B. Global solutions

Theorem 1 established the existence of solutions on a region $\mathscr{D}_\varepsilon$: $\chi(t)\leqq X \leqq\chi(t)+\varepsilon$, $-\infty<t<\infty$, for some $\varepsilon>0$. It is well known [6] that solutions of the equations of dynamic elastic bar theory are not globally smooth, in general. Except for a certain class of Cauchy data, the values of the first derivatives of u and v blow up after a finite distance along a characteristic. This result does not contradict Theorem 1, because, there, each point in $\mathscr{D}_\varepsilon$ could be connected to the phase boundary by a short characteristic. Here, "short" refers to the comparison of ε with the other constants introduced in the proof of Theorem 1. However, under some special conditions on $r_+(t)$ and $s_+(t)$, globally smooth solutions exist.

To investigate this probability within the context of Theorem 1, we imagine that a solution has been proved to exist on $\mathscr{D}_\varepsilon$, for some $\varepsilon>0$. This solution provides Cauchy data $r\big(\chi(t)+\varepsilon,t\big)=\bar{r}_+(t)$, $s\big(\chi(t)+\varepsilon,t\big)=\bar{s}_+(t)$, on the curve X

$=\chi(t)+\varepsilon$. In the following theorem, we prove that under special conditions on the original Cauchy data $r_+(t)$, $s_+(t)$ the procedure of Theorem 1 delivers a global solution by continuation.

Theorem 2. *Suppose that, in addition to the hypotheses of Theorem* 1, $r_+(t)$ *and* $s_+(t)$ *belong to the class* $C^2_{(-\infty,\infty)}$. *Assume that either*

$$\dot{r}_+ \geqq 0, \quad \dot{s}_+ \leqq 0 \quad \text{and} \quad \sigma_{uu}(u_+(t)) \geqq 0 \tag{3.1}$$

or

$$\dot{r}_+ \leqq 0, \quad \dot{s}_+ \geqq 0 \quad \text{and} \quad \sigma_{uu}(u_+(t)) \leqq 0. \tag{3.2}$$

Then there is a smooth solution $r(X,t)$, $s(X,t)$ *on* $\mathscr{D}_\infty$: $-\infty<t<\infty$, $X \geqq \chi(t)$.

Proof. For this proof it is convenient not to make the change of variables (2.13). Let $r(X,t)$, $s(X,t)$ be the solutions of the system

$$\begin{aligned} \dot{r} &= \tilde{c}(r-s)\, r', & r(\chi(t),t) &= r_+(t), \\ \dot{s} &= -\tilde{c}(r-s)\, s', & s(\chi(t),t) &= s_+(t), \end{aligned} \tag{3.3}$$

which have been proved to exist in Theorem 1. Suppose r, s are defined on $\mathscr{D}_\varepsilon$. To carry out the proof along standard lines, first we must show that $r, s \in C^2_{\mathscr{D}_\varepsilon}$. The equations for the characteristics of (3.3) are

$$\begin{aligned} \frac{d\tau_r}{dX} &= \frac{-1}{\tilde{c}(r-s)}, & \tau_r(\chi(\gamma),\gamma) &= \gamma, \\ \frac{d\tau_s}{dX} &= \frac{1}{\tilde{c}(r-s)}, & \tau_s(\chi(\gamma),\gamma) &= \gamma. \end{aligned} \tag{3.4}$$

By the theory of ordinary differential equations [5], τ_r and τ_s belong to $C^1_{\mathscr{D}_\varepsilon}$, and the partial derivatives

$$\frac{\partial^2 \tau_r}{\partial\gamma\,\partial X}, \frac{\partial^2 \tau_r}{\partial X^2}, \frac{\partial^2 \tau_s}{\partial\gamma\,\partial X}, \frac{\partial^2 \tau_s}{\partial X^2} \tag{3.5}$$

exist and are continuous. We shall show that the assumption $r_+, s_+ \in C^2_{(-\infty,\infty)}$ implies that $\frac{\partial\tau_r}{\partial\gamma}, \frac{\partial\tau_s}{\partial\gamma} \in C^1_{\mathscr{D}_\varepsilon}$. This result will in turn imply that $\tau_r, \tau_s \in C^2_{\mathscr{D}_\varepsilon}$ and that $r, s \in C^2_{\mathscr{D}_\varepsilon}$. The function $\partial\tau_r/\partial\gamma$ satisfies the linear equation which follows from $(3.4)_1$ by differentiation,

$$\frac{d}{dX}\left(\frac{\partial\tau_r}{\partial\gamma}\right) = \frac{1}{\tilde{c}^2}\left[\frac{d\tilde{c}}{d(r-s)}(\dot{r}-\dot{s})\right]\frac{\partial\tau_r}{\partial\gamma}, \tag{3.6}$$

and the side condition

$$\frac{\partial\tau_r}{\partial\gamma}(\chi(\gamma),\gamma) = 1 + \frac{\dot{\chi}(\gamma)}{\tilde{c}(r_+(\gamma)-s_+(\gamma))}. \tag{3.7}$$

Since

$$\frac{d}{dX}\left\{\frac{1}{2}\log\tilde{c}(r(X,\tau_r(X,\gamma))-s(X,\tau_r(X,\gamma)))\right\}=\frac{1}{\tilde{c}^2}\frac{d\tilde{c}}{d(r-s)}\dot{s}, \tag{3.8}$$

then (3.6) may be written

$$\frac{d}{dX}\left(\frac{\partial\tau_r}{\partial\gamma}\right)=\frac{1}{\tilde{c}^2}\frac{d\tilde{c}}{d(r-s)}\frac{dr_+}{d\gamma}-(\tfrac{1}{2}\log\tilde{c})_X\frac{\partial\tau_r}{\partial\gamma}. \tag{3.9}$$

To derive (3.9), the relation $\dot{r}\frac{\partial\tau_r}{\partial\gamma}=\frac{dr_+}{d\gamma}$ has been used. The solution of (3.9) with the side condition (3.7) is

$$\begin{aligned}\frac{\partial\tau_r}{\partial\gamma}(X,\gamma)=&\left[\frac{\tilde{c}(r_+(\gamma)-s_+(\gamma))}{\tilde{c}(r(X,\tau_r(X,\gamma))-s(X,\tau_r(X,\gamma)))}\right]^{\frac{1}{2}}\\ &\cdot\left\{1+\frac{\dot{\chi}(\gamma)}{c(r_+(\gamma)-s_+(\gamma))}+\int_{\chi(\gamma)}^{X}\frac{1}{\tilde{c}(r_+(\gamma)-s_+(\gamma))^{\frac{1}{2}}}\frac{1}{\tilde{c}^{\frac{3}{2}}}\frac{d\tilde{c}}{d(r-s)}\frac{dr_+}{d\gamma}\bigg|_{(z,\gamma)}dz\right\}.\end{aligned} \tag{3.10}$$

Hence $\frac{\partial\tau_r}{\partial\gamma}\in C^1_{\mathcal{D}_\varepsilon}$. It follows that $\tau_r\in C^2_{\mathcal{D}_\varepsilon}$ and that $\Gamma_r\in C^2_{\mathcal{D}_\varepsilon}$. Since

$$r(X,t)=r_+(\Gamma_r(X,t)), \tag{3.11}$$

and by assumption $r_+\in C^2_{(-\infty,\infty)}$, then $r\in C^2_{\mathcal{D}_\varepsilon}$. By similar reasoning as above, carried out for the s-characteristic, $s\in C^2_{\mathcal{D}_\varepsilon}$.

The assumptions (3.1) and (3.2) now permit the application of a standard theorem [6] on the growth of the derivatives of r and s. That theorem implies that

$$\begin{aligned}&\frac{d}{dt}r(\chi(t)+\varepsilon,t)=\dot{r}\left(\frac{\dot{\chi}+c}{c}\right)\bigg|_{(\chi(t)+\varepsilon,t)},\\ &\frac{d}{dt}s(\chi(t)+\varepsilon,t)=\dot{s}\left(\frac{\dot{\chi}+c}{c}\right)\bigg|_{(\chi(t)+\varepsilon,t)}\end{aligned} \tag{3.12}$$

are bounded by constants which only depend upon the constitutive function, $\chi(t)$, $r_+(t)$, $s_+(t)$, but not upon ε. Also, the curve $X=\chi(t)+\varepsilon$ satisfies the hypotheses (2.2), (2.3), and (2.4), with the same constants $\tilde{\lambda}$, m, τ, and k as with $X=\chi(t)$. Therefore the solution can be extended to $\mathcal{D}_{\varepsilon+\varepsilon_1}$, for some $\varepsilon_1>0$. At $\chi(t)+\varepsilon$ the solution is C^2. At $(\chi(t)+\varepsilon+\varepsilon_1,t)$, $\frac{d}{dt}r(\chi(t)+\varepsilon+\varepsilon_1,t)$, *etc.* can be bounded by the same bounds given in this proof, so the solution can be extended to $\mathcal{D}_{\varepsilon+2\varepsilon_1}$. Continuing in this fashion the solution can be extended to $\mathcal{D}_\infty$. □

Clearly, it is possible to have globally smooth solutions on *both sides* of the phase boundary. This can be accomplished if, in addition to (3.1) and (3.2), the data (r_-,s_-) satisfy the analogues of (3.1) and (3.2), *i.e.*

$$r_-\geqq 0,\quad s_-\leqq 0\quad\text{and}\quad \sigma_{uu}(u_-(t))\leqq 0, \tag{3.13}$$

$$r_-\leqq 0,\quad s_-\geqq 0\quad\text{and}\quad \sigma_{uu}(u_-(t))\geqq 0. \tag{3.14}$$

C. Invariance groups

The equations (1.10) or (1.14) generally admit four invariance groups aside from the trivial change of origin. If we take proper account of the change of the domain induced by the transformation, the balance of momentum (1.6) is also invariant under these transformations.

1. Quasilinearity.
   Deformation-velocity variables: $t \to \nu t\ X \to \nu X,\ \nu = \text{const.}$
   Riemann invariants: $t \to \nu t\ X \to \nu X.$
2. Time reversal.
   Deformation-velocity variables: $t \to -t,\ v \to -v.$
   Riemann invariants: $t \to -t,\ r \to -s,\ s \to -r.$
3. Space reversal
   Deformation-velocity variables: $X \to -X,\ u \to u,$
   $v \to -v.$
   Riemann invariants: $X \to -X,\ r \to -s,$
   $s \to -r.$
4. Galilean transformation.
   Deformation-velocity variables: $u \to u,\ v \to v + k,\ k = \text{const.}$
   Riemann invariants: $r \to r + k,\ s \to s + k.$

These transformations can be used to produce a class of solutions from any single solution delivered by Theorem 1. Specific use will be made of the first group in Section 5.

4. Interaction of Sound Waves with a Phase Boundary

Within the framework of elastic bar theory, a *sound wave* is a smooth curve in the domain of a solution across which $\dot{r}$ or $\dot{s}$, or higher derivatives of r and s, experience a finite jump, while r and s themselves remain continuous. So that the jump is well defined, it is also required that the limiting values of $\dot{r}$ or $\dot{s}$, or the higher derivatives, exist as the curve is approached from either side. If $\dot{r}$ or $\dot{s}$ experience jumps, which, of course, entails that r' and s' experience jumps, the wave is called an *acceleration wave*. If some of the derivatives of $(n-1)^{\text{st}}$ order of r or s jump, but the derivatives of $(n-2)^{\text{nd}}$ order remain continuous, the sound wave is said to be of n^{th} *order*. In elasticity theory it is known that the behavior of acceleration waves is analogous to the behavior of higher order waves; that is, the waves lie on curves determined by the same equations, and the jumps satisfy analogous compability conditions.

Solutions obtained from Theorem 1 cannot propagate acceleration waves since $\dot{r}$, $\dot{s}$, r', s' are continuous in those solutions. We shall therefore be concerned with the propagation of third and higher order waves.

I shall formulate two kinds of problems. Imagine an elastic bar containing a single, generally moving, phase boundary. In the first problem, I assume that one end of the bar has been hit, so a sound wave moves toward the phase boundary. I investigate the possibility that no wave is reflected from, or that no wave is transmitted through the phase boundary, or that neither reflected nor transmitted waves emerge. That is, I find conditions under which a bar containing a phase boundary can be used as a damping device for sound waves. In the second problem, I consider the possibility that a phase boundary spontaneously emit sound waves.

To begin the analysis, we recall the theory of third order waves. Let $X=\xi(t)$ be a curve in the domain $\mathscr{D}_\varepsilon$ of a solution taken from Theorem 1. Suppose that across $X=\xi(t)$, $\ddot{r}(X,t)$ experiences a jump, but the limiting values $\ddot{r}(\xi(t)+0,t)$ and $\ddot{r}(\xi(t)-0,t)$ exist. I shall use the notation

$$[\ddot{r}]\equiv\ddot{r}(\xi(t)+0,t)-\ddot{r}(\xi(t)-0,t) \tag{4.1}$$

for jumps across third order waves. Let

$$[\dot{r}']=a \tag{4.2}$$

and assume $a\neq 0$. Since $\dot{r}$ is continuous across $X=\xi(t)$,

$$[\ddot{r}]=-a\frac{d\xi}{dt}.$$

By differentiating the equation $r=c(r-s)\,r'$ and subtracting across $X=\xi(t)$, we arrive at the relation

$$\frac{d\xi}{dt}=-\tilde{c}(r-s). \tag{4.4}$$

By comparison with (3.4), (4.4) implies that *third order waves in which* $[\dot{r}']\neq 0$ *travel along r-characteristics.* It follows by a similar argument that *third order waves in which* $[\dot{s}']\neq 0$ *travel along s-characteristics.* Furthermore, only second and higher derivatives of r can experience a jump across an r-characteristic which is a third order wave, and only second and higher derivatives of s can experience a jump across an s-characteristic which is a third order wave. That is, the equations for the characteristics (3.4) show that the r-and s-characteristics meet non-tangentially. Also, s is constant along an s-characteristic and r is constant along an r-characteristic. *Therefore,* $[\ddot{s}]=[\dot{s}']=0$ *across an r-characteristic which is a third order wave, and* $[\ddot{r}]=[\dot{r}']=0$ *across an s-characteristic which is a third order wave.*

Let $r(X,t)$, $s(X,t)$ be a piecewise C^2 solution taken from Theorem 1. Suppose a third order sound wave travels down an r-characteristic $X=\zeta_r(t)$ and impinges at $t=0$ on the phase boundary $X=\chi(t)$. Let the Cauchy data for the solution on the $(+)$ side of the phase boundary be $r_+(t)$, $s_+(t)$. Then

$$\begin{aligned}\frac{d^2 r_+}{dt^2}&=\frac{d^2}{dt^2}\,r(\chi(t),t)\\ &=r''\,\dot{\chi}^2+r'\,\ddot{\chi}+2\,\dot{r}'\,\dot{\chi}+\ddot{r}\\ &=\frac{\dot{\chi}^2}{c}\left(\dot{r}'-\frac{d\tilde{c}}{d(r-s)}(r'-s')\,r'\right)+r'\,\ddot{\chi}+2\,\dot{r}'\,\dot{\chi}\\ &\quad+\left(\tilde{c}\,\dot{r}'+\frac{d\tilde{c}}{d(r-s)}(\dot{r}-\dot{s})\,r'\right).\end{aligned} \tag{4.5}$$

Hence

$$[\ddot{r}_+] = [\dot{r}'] \left(\frac{\dot{\chi}^2}{c} + 2\dot{\chi} + c \right) = [\dot{r}'] \left(\frac{1}{c} \right) (\dot{\chi} + c)^2. \tag{4.6}$$

But (2.4) shows that $\frac{1}{c}(\dot{\chi}+c) > 0$. By assumption $[\dot{r}'] \neq 0$. Therefore *a third order sound wave on an r-characteristic impinges on the phase boundary at* $t=0$ *if and only if* $[\ddot{r}_+](0) \neq 0$. *By similar reasoning, a third order sound wave on an s-characteristic impinges on the phase boundary at* $t=0$ *if and only if* $[\ddot{s}_+](0) \neq 0$.

The results just stated presume the existence of a piecewise twice continuously differentiable solution. Suppose Cauchy data $r_+(t)$, $s_+(t)$ is prescribed consistent with the hypotheses of Theorem 1. Suppose $r_+(t)$, $s_+(t)$ are twice continuously differentiable except at $t=0$, where the limits $\ddot{r}_+(0+)$ and $\ddot{r}_+(0-)$ exist and are unequal. Let $r(X,t)$, $s(X,t)$ be the corresponding solution. Then the argument leading to (3.11) shows that the $r(X,t)$ is twice differentiable on its domain of existence, except for points lying on the r-characteristic $X = \xi_r(t)$ issuing from $(\chi(0), 0)$. Moreover, the limiting values $\ddot{r}(\xi_r(t)+0, t)$ and $\ddot{r}(\xi_r(t)-0, t)$ exist. Therefore, *if* $r_+(t)$ *is assigned in the class* $C^2_{(-\infty,0)\cup(0,\infty)} \cap C^1_{(-\infty,\infty)}$, *and the limits* $\ddot{r}_+(0+)$ *and* $\ddot{r}_+(0-)$ *exist, and* $s_+(t)$ *is assigned in the class* $C^2_{(-\infty,\infty)}$, *then there is a solution* $r, s \in C^2_{\mathscr{D}_\varepsilon - R} \cap C^1_{\mathscr{D}_\varepsilon}$ *containing a third order sound wave on the r-characteristic R, which impinges on the phase boundary at* $(\chi(0), 0)$. A similar statement holds for the s-characteristic.

In the first problem we imagine that we strike the end of the bar which is near the $(+)$ side of the phase boundary. The equations (3.4) for the characteristics show that the wave must travel down an r-characteristic, in order that it move toward the phase boundary as time increases. Hence, we assume $[\ddot{r}_+] \neq 0$ at, say, $t=0$. A wave will be reflected if $[\ddot{s}_+]$ $(0) \neq 0$, and a wave will be transmitted if $[\ddot{r}_-]$ $(0) \neq 0$. We stipulate that we hit *only one* end of the bar; therefore, $[\ddot{s}_-]$ $(0) = 0$. That is, the only wave which can travel backward in time as it recedes from $(\chi(0), 0)$ lies on the r-characteristic on the $(+)$ side of the phase boundary. In summary, for

$$\textit{Problem 1:} \quad [\ddot{r}_+](0) \neq 0, \qquad [\ddot{s}_-](0) = 0. \tag{4.7}$$

In the second problem we investigate the conditions which imply that a phase boundary spontaneously generate sound waves. These conditions expressed in terms of the Cauchy data are, for

$$\textit{Problem 2:} \quad [\ddot{r}_+](0) = 0, \qquad [\ddot{s}_-](0) = 0. \tag{4.8}$$

We now analyze the jump conditions on the phase boundary. The Rankine-Hugoniot conditions,

$$\begin{aligned} F(t) &\equiv \dot{\chi}(t)\big(u_+(t) - u_-(t)\big) + \big(v_+(t) - v_-(t)\big) = 0, \\ G(t) &\equiv \sigma\big(u_+(t)\big) - \sigma\big(u_-(t)\big) + \big(v_+(t) - v_-(t)\big)\,\dot{\chi}(t) = 0, \end{aligned} \tag{4.9}$$

must be satisfied across the phase boundary. Let the Cauchy data r_+, s_+ belong to the class $C^2_{(-\infty,)\cup(0,\infty)} \cap C^1_{(-\infty,\infty)}$ and have limiting values at $t=0$ from the right and left. By the assumptions of smoothness on $\sigma(\cdot)$, u_+ and v_+ also belong to the same class, and have limiting values. Let (4.9) be differentiated twice and the limiting values subtracted across $t=0$. The first differentiation shows that $[\ddot{\chi}](0)=0$. The second may be written

$$\begin{gathered}[\dddot{\chi}](u_+ - u_-) + \dot{\chi}[\ddot{u}_+ - \ddot{u}_-] + [\ddot{v}_+ - \ddot{v}_-] = 0, \\ \sigma_u(u_+)[\ddot{u}_+] - \sigma_u(u_-)[\ddot{u}_-] + [\ddot{v}_+ - \ddot{v}_-]\dot{\chi} + (v_+ - v_-)[\dddot{\chi}] = 0.\end{gathered} \tag{4.10}$$

The following theorem reduces the study of the interaction of sound waves with the phase boundary to a study of the conditions (4.10).

Theorem 3. *Let $r(X,t)$, $s(X,t)$ be a solution containing a phase boundary $X=\chi(t)$, defined in a neighborhood of the point $(\chi(0),0)$. Suppose the data*

$$\begin{aligned} r_+(t) &= r(\chi(t)+0,t), \qquad r_-(t) = r(\chi(t)-0,t), \\ s_+(t) &= s(\chi(t)+0,t), \qquad s_-(t) = s(\chi(t)-0,t) \end{aligned} \tag{4.11}$$

belongs to the class $C^2_{(-\delta,0)\cup(0,\delta)} \cap C^1_{(-\delta,\delta)}$ for some $\delta>0$ and has limiting values as $t=0$ is approached from either side. Then the jump conditions (4.10) *are satisfied.*

Conversely, suppose finite values are assigned for

$$\dot{\chi}(0),\ u_+(0),\ u_-(0),\ v_+(0),\ v_-(0) \tag{4.12}$$

such that the Rankine-Hugoniot conditions (4.9) *and the conditions*

$$\begin{aligned} c(u_+(0)) - |\dot{\chi}(0)| &> 0, \\ c(u_-(0)) - |\dot{\chi}(0)| &> 0 \end{aligned} \tag{4.13}$$

are satisfied. Suppose finite values are assigned for

$$[\dddot{\chi}],\ [\ddot{u}_+],\ [\ddot{u}_-],\ [\ddot{v}_+],\ [\ddot{v}_-] \tag{4.14}$$

so that the jump conditions (4.10) *hold for the set* (4.12) *and* (4.14). *Then, for some $\delta>0$, there is a phase boundary*

$$X = \chi(t) \in C^3_{(-\delta,0)\cup(0,\delta)} \cap C^2_{(-\delta,\delta)}$$

and Cauchy data

$$u_+(t),\ u_-(t),\ v_+(t),\ v_-(t) \in C^2_{(-\delta,0)\cup(0,\delta)} \cap C^1_{(-\delta,\delta)},$$

both consistent with (4.9) *and* (4.10). *Furthermore, there is a solution $u(X,t)$, $v(X,t)$ defined in a neighborhood of $(\chi(0),0)$ having this Cauchy data on the phase boundary $X=\chi(t)$.*

Proof. The first part of the theorem has been proved by the derivation of (4.10). To prove the converse, we begin with the once differentiated form of the jump conditions, evaluated at $t=0$,

$$\begin{aligned} \dot F(0) &= \dot\chi(u_+ - u_-) + \chi(\dot u_+ - \dot u_-) + (\dot v_+ - \dot v_-) = 0, \\ \dot G(0) &= \sigma_u(u_+)\,\dot u_+ - \sigma_u(u_-)\,\dot u_- + \dot\chi(\dot v_+ - \dot v_-) + \ddot\chi(v_+ - v_-) = 0. \end{aligned} \tag{4.15}$$

Let $\dot u_+(0)$, $\dot u_-(0)$, $\dot v_+(0)$, $\dot v_-(0)$, $\ddot\chi(0)$ be prescribed so that $\dot F(0) = \dot G(0) = 0$. This is always possible because (4.15) can be solved explicitly for $\dot u_+$, $\dot u_-$. The twice differentiated form of the jump conditions are

$$\begin{aligned} \ddot F(t) &= \dddot\chi(u_+ - u_-) + 2\dot\chi(\dot u_+ - \dot u_-) + \chi(\ddot u_+ - \ddot u_-) + (\ddot v_+ - \ddot v_-) = 0, \\ \ddot G(t) &= \sigma_u(u_+)\,\ddot u_+ - \sigma_u(u_-)\,\ddot u_- + \sigma_{uu}(u_+)\,\dot u_+^2 - \sigma_{uu}(u_-)\,\dot u_-^2 \\ &\quad + \dot\chi(\ddot v_+ - \ddot v_-) + 2\ddot\chi(\dot v_+ - \dot v_-) + (v_+ - v_-)\,\dddot\chi = 0 \end{aligned} \tag{4.16}$$

The condition (4.13) permits (4.16) to be solved for $\ddot u_+(t)$ and $\ddot u_-(t)$. Let $\ddot u_+(0+)$, $\ddot u_-(0+)$, $\ddot v_+(0+)$, $\ddot v_-(0+)$, $\dddot\chi(0+)$ be assigned so that $\ddot F(0+) = \ddot G(0+) = 0$. Since the jump conditions (4.10) are fulfilled for the set (4.14), $F(0-) = G(0-) = 0$. Let $v_+(t)$, $v_-(t) \in C^2_{(-\delta,0)\cup(0,\delta)} \cap C^1_{(-\delta,\delta)}$ and $\chi(t) \in C^3_{(-\delta,0)\cup(0,\delta)} \cap C^2_{(-\delta,\delta)}$ be prescribed consistent with the assignments of $v_+(0)$, $v_-(0)$, $\dot v_+(0)$, $\dot v_-(0)$, $\ddot\chi(0)$, $\dot\chi(0)$, made above. If (4.16) is formally solved for $\ddot u_+(t)$, $\ddot u_-(t)$, there results a second order ordinary differential system for $u_+(t)$, $u_-(t)$, with initial values $u_+(0)$, $u_-(0)$, $\dot u_+(0)$, $\dot u_-(0)$. This equation is solved for functions $u_+(t)$, $u_-(t)$, $t \geqq 0$, and for functions $u_+(t)$, $u_-(t)$, $t \leqq 0$. All of the functions involved having been determined on a sufficiently small neighborhood of $t=0$, it is clear that by the construction

$$\begin{aligned} &\ddot F(t) = \ddot G(t) = 0, \qquad t \in (-\delta, 0) \cup (0, \delta), \\ &\dot F(0) = \dot G(0) = 0, \\ &F(0) = G(0) = 0. \end{aligned} \tag{4.17}$$

Also, $u_+(t)$, $u_-(t) \in C^2_{(-\delta,0)\cup(0,\delta)} \cap C^1_{(-\delta,\delta)}$. Therefore, $F(t) = G(t) = 0$, $t \in (-\delta, \delta)$. With this collection of Cauchy data and the hypothesis (4.13), a solution $r(X,t)$, $s(X,t)$ can be found by the method of Theorem 1 in a neighborhood of $(\chi(0), 0)$. □

It is evident from the theorem that the resulting Cauchy data are not unique, since $v_+(t)$, $v_-(t)$ and $\chi(t)$ were chosen arbitrarily, except at $t=0$.

The two problems set forth in (4.7) and (4.8) are now laid open for investigation. Those problems have been formulated in terms of the Riemann invariants r, s because the geometry of the characteristics is immediately plain for that choice of dependent variables. However, the Rankine-Hugoniot conditions are formulated naturally for the variables u and v. The relation between the two sets of variables is the equation (1.11). We shall need the implied relations between the jumps of $\ddot r_+$, $\ddot s_+$, $\ddot r_-$, $\ddot s_-$, and the jumps of $\ddot u_+$, $\ddot v_+$, $\ddot u_-$, $\ddot v_-$. By differentiation of (1.11)

$$\begin{aligned} [\ddot r_+] &= c(u_+)\,[\ddot u_+] + [\ddot v_+], \\ [\ddot s_+] &= -c(u_+)\,[\ddot u_+] + [\ddot v_+]. \end{aligned} \tag{4.18}$$

The analogous equations hold for $(-)$.

Problem 1. In this problem it is necessary to solve (4.10) under the restriction (4.7). All functions below will be evaluated at $t=0$. We assume that the basic inequalities (4.13) hold. Let

$$\rho=[\ddot{r}_+]. \tag{4.19}$$

By assumption, $\rho \neq 0$. In terms of the deformation-velocity variables, (4.7) becomes

$$\begin{aligned}\rho &= c(u_+)[\ddot{u}_+]+[\ddot{v}_+],\\ c(u)[\ddot{u}_-]&-[\ddot{v}_-]=0.\end{aligned} \tag{4.20}$$

As in Theorem 3, we assume that $\dot{\chi}$, u_+, u_-, v_+, v_- have been assigned consistent with the Rankine-Hugoniot conditions. Equations (4.20) are introduced into the equation (4.10); by elimination of $[\ddot{v}_+]$ and $[\ddot{v}_-]$, and by the use of (4.9) evaluated at $t=0$, the result may be written

$$\begin{aligned}&[\ddot{u}_+](\dot{\chi}-c(u_+))+[\ddot{u}_-](-c(u_-)-\dot{\chi})+[\ddot{\chi}](u_+-u_-)+\rho=0,\\ &[\ddot{u}_+](c(u_+)^2-\dot{\chi}\,c(u_+))+[\ddot{u}_-](-\dot{\chi}\,c(u_-)-c(u)^2)\\ &\quad+[\ddot{\chi}](u_+-u_-)\,\dot{\chi}+\rho\,\dot{\chi}=0.\end{aligned} \tag{4.21}$$

These are the basic equations for problem 1. Viewed as a system of equations for the vector $([\ddot{u}_+], [\ddot{u}_-])$, (4.21) has determinant

$$(c(u_+)+c(u_-))(c(u_+)-\dot{\chi})(c(u_-)+\dot{\chi}) \tag{4.22}$$

which can never vanish. Therefore, *let arbitrary values be assigned to* $[\ddot{\chi}]$ *and* ρ. *Then there is a solution* $[\ddot{u}_+]$, $[\ddot{u}_-]$ *so that* (4.21) *is satisfied.* Combining this result with Theorem 3 and (4.6), we deduce that for arbitrary assignment of $[\ddot{\chi}]$, and for arbitrary amplitude of the incoming wave, there is a solution. Generally a wave is reflected and a wave is transmitted.

Suppose, in addition, we assume that no wave is transmitted. Then

$$[\ddot{r}_-]=0 \Rightarrow [\ddot{u}_-]=[\ddot{v}_-]=0. \tag{4.23}$$

In this case (4.21) becomes,

$$\begin{aligned}&[\ddot{u}_+](\dot{\chi}-c(u_+))+[\ddot{\chi}](u_+-u_-)+\rho=0,\\ &[\ddot{u}_+](c(u_+)^2-\dot{\chi}\,c(u_+))+[\ddot{\chi}](u_+-u_-)\,\dot{\chi}+\rho\,\dot{\chi}=0.\end{aligned} \tag{4.24}$$

Viewed as a system of equations for the vector $([\ddot{u}_+], [\ddot{\chi}])$, this system of equations has determinant equal to

$$(u_+-u_-)(\dot{\chi}-c(u_+)(\dot{\chi}+c(u_+))\neq 0. \tag{4.25}$$

Therefore, if ρ *is assigned an arbitrary value, there are unique values of* $[\ddot{u}_+]$ *and* $[\ddot{\chi}]$ *so that no wave is transmitted across the phase boundary at* $(\chi(0), 0)$. It might be interesting for some applications to determine if (4.24) can be solved without inducing a weak singularity in the motion of the phase boundary, that is, if (4.24) can be solved when $[\ddot{\chi}]=0$. It is easy to show that it cannot. Therefore, *if* $[\ddot{\chi}]=0$, *a third order wave must be transmitted.*

Now assume that no wave is reflected at $(\chi(0), 0)$, but that a wave may be transmitted. Then

$$\begin{aligned}[\ddot{s}_+]=0 \Rightarrow [\ddot{v}_+]=c(u_+)[\ddot{u}_+] \Rightarrow \\ \rho = 2c(u_+)[\ddot{u}_+].\end{aligned} \tag{4.26}$$

After substitution of (4.26), (4.21) becomes,

$$\begin{aligned}\left\{\frac{\dot{\chi}+c(u_+)}{2c(u_+)}\right\}\rho+[\ddot{u}_-](-c(u_-)-\dot{\chi})+[\ddot{\chi}](u_+-u_-)=0,\\ \left\{\frac{c(u_+)+\dot{\chi}}{2}\right\}\rho+[\ddot{u}_-](-\dot{\chi}c(u_-)-c(u_-)^2)+[\ddot{\chi}](u_+-u_-)\dot{\chi}=0.\end{aligned} \tag{4.27}$$

Viewed as an equation for the vector $([\ddot{u}_-], [\ddot{\chi}])$, the system (4.27) has determinant

$$(u_+-u_-)(\dot{\chi}+c(u_+))(\dot{\chi}-c(u_-)) \neq 0. \tag{4.28}$$

Therefore, *if* ρ *is assigned an arbitrary value, there are unique values of* $[\ddot{u}_-]$ *and* $[\ddot{\chi}]$ *so that no wave is reflected at* $(\chi(0), 0)$. However, *if* $[\ddot{\chi}]=0$ *there is a solution* $[\ddot{u}_-]$ *of* (4.27) *for every value of* ρ *if and only if* $c(u_+)=c(u_-)$. When this latter condition holds, there is said to be an *impedance match* across the phase boundary.

Finally, assume that third order waves are neither transmitted nor reflected. In that case, the jump conditions (4.21) become

$$\begin{aligned}\left\{\frac{\dot{\chi}+c(u_+)}{2c(u_+)}\right\}\rho+[\ddot{\chi}](u_+-u_-)=0,\\ \left\{\frac{c(u_+)+\dot{\chi}}{2}\right\}\rho+[\ddot{\chi}](u_+-u_-)\dot{\chi}=0.\end{aligned} \tag{4.29}$$

This system has no solutions $[\ddot{\chi}]$ under the basic restriction (4.13). Therefore, *every incident third order sound wave must be either reflected or transmitted, or both reflected and transmitted.*

Problem 2. Problem 2 can be analyzed in the same way as problem 1. The defining equations,

$$[\ddot{r}_+]=[\ddot{s}_-]=0, \tag{4.30}$$

are equivalent to the conditions

$$c(u_+)[\ddot{u}_+]=-[\ddot{v}_+], \qquad c(u_-)[\ddot{u}_-]=[\ddot{v}_-]. \tag{4.31}$$

Therefore, we begin with (4.21) and put $\rho=0$. The result follows immediately. *If* $[\ddot{\chi}]=0$, *no third order waves can be spontaneously emitted from the phase boundary. If* $[\ddot{\chi}]\neq 0$, *there is always a spontaneous emission of at least one third order sound wave.* The word spontaneous here means (4.30) holds, *i.e.* there are no incident waves.

This completes the analysis of problems 1 and 2. For waves of higher order, the results are analogous, at every stage, to the results derived here.

5. The Riemann Problem and Admissibility

A. The Riemann problem. Single phase boundary

The problem of admissibility of solutions for dynamic elastic bar theory has received widespread attention only for the initial value problem. Particular emphasis has been expended on the initial value problem known as the *Riemann problem*, in which the data are

$$(u, v)(X, 0) = \begin{cases} (u_+, v_+)\, X \geqq 0 \\ (u_-, v_-)\, X < 0, \end{cases} \tag{5.1}$$

u_+, u_-, v_+, and v_- being constants. In order to apply the various concepts of admissibility that have been formulated to the propagation of a phase boundary, it is necessary first to solve the Riemann problem when the initial data contain a phase boundary. This can be accomplished by assigning u_+ on the β-branch and u_- on the α-branch, for example.

I shall present a kind of local solution to the Riemann problem, based upon the original treatment of LAX [7]. Initial values u_+, u_-, v_+, v_- will be assigned consistent with the Rankine-Hugoniot conditions. Under certain assumptions on the constitutive relation and initial data, which amount to the requirements that the initial speed of the phase boundary be less than the acoustic speeds at u_+ and u_-, and that the constitutive function be genuinely nonlinear at u_+ and u_-, there will exist a one parameter family of solutions to the Riemann problem.

Recalling the basic definition (1.4), we let u_+ and u_- be assigned constants satisfying

$$\begin{aligned} &\alpha < u_- < a^1, \\ &\beta^1 < u_+ < \beta, \end{aligned} \qquad \sigma(u_+) \geqq \sigma(u_-). \tag{5.2}$$

Let the velocity of the phase boundary ϑ_0 be defined by

$$\vartheta_0 = \pm \sqrt{\frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-}}, \tag{5.3}$$

and let constants v_+ and v_- be chosen so that

$$(v_+ - v_-) = -\vartheta_0 (u_+ - u_-). \tag{5.4}$$

Then the Rankine-Hugoniot conditions are satisfied for u_+, u_-, v_+, v_- and ϑ_0. Evidently, one of the values v_+ or v_-, *and the sign of* ϑ_0, can be arbitrarily prescribed.

We shall assume that

$$\text{(i)}\ \ c(u_+) - |\vartheta_0| > 0, \qquad c(u_-) - |\vartheta_0| > 0, \tag{5.5}$$

$$\text{(ii)}\ \ \sigma_{uu}(u_+) \neq 0, \qquad \sigma_{uu}(u_-) \neq 0. \tag{5.6}$$

The first condition is familiar by now; the second requires genuine nonlinearity at u_+ and u_-. We shall also assume that LAX's criterion for admissible shocks [7] holds for shock waves connecting values of u on the α-branch, or connecting values of u on the β-branch, that is, the usual kinds of shock waves. To express this condition, let $X=\gamma(t)$ be a shock wave, and let

$$\begin{aligned} u_r &\equiv u(\gamma(t)+0, t), \\ u_l &\equiv u(\gamma(t)-0, t). \end{aligned} \tag{5.7}$$

Then we assume, according to LAX, that

$$\begin{aligned} &\text{(iii) either } c(u_r)<\dot{\gamma}(t)<c(u_l), \\ &\qquad\text{or } -c(u_r)<\dot{\gamma}(t)<-c(u_l). \end{aligned} \tag{5.8}$$

We shall piece together the solution from two other solutions, defined on overlapping intervals. The first kind of solution will be either a simple wave or a shock solution connecting (u_-, v_-) to a one parameter family of values $(u_-(\varepsilon), v_-(\varepsilon))$, $(u_-(0), v_-(0))=(u_-, v_-)$, $-\mu\leqq\varepsilon\leqq\mu$, for some constant $\mu>0$. In the same way, the constant solution (u_+, v_+) will be connected to a one parameter family of values $(u_+(\delta), v_+(\delta))$, $-\mu\leqq\delta\leqq\mu$ by shock or simple wave solutions. The two families $(u_-(\varepsilon), v_-(\varepsilon))$ and $(u_+(\delta), v_+(\delta))$ will then be connected at $\frac{X}{t}=\mathfrak{s}$ by a phase boundary.

We suppose, without loss of generality, that $\sigma_{uu}(u_-)<0$. The genuine non-linearity condition then implies that the equation

$$\frac{\sigma(w)-\sigma(u_-)}{w-u_-}=\left(-c(u_-)+\frac{\varepsilon}{2}\right)^2 \tag{5.9}$$

has a solution $w=u_-(\varepsilon)$, $u_-(0)=u_-$, defined in a neighborhood of $\varepsilon=0$. In order that LAX's criterion for admissible shocks (5.8) be satisfied, $\varepsilon\leqq 0$. Let

$$v_-(\varepsilon)\equiv v_- -\left(-c(u_-)+\frac{\varepsilon}{2}\right)(u_-(\varepsilon)-u_-), \tag{5.10}$$

so that $v_-(0)=v_-$. Equations (5.9) and (5.10) imply that the Rankine-Hugoniot conditions are satisfied for a shock connecting (u_-, v_-) to $(u_-(\varepsilon), v_-(\varepsilon))$. Hence, there is a one parameter family of values

$$\begin{aligned} &(u_-(\varepsilon), v_-(\varepsilon)), \qquad \varepsilon\leqq 0, \\ &(u_-(0), v_-(0))=(u_-, v_-), \end{aligned} \tag{5.11}$$

that can be connected to (u_-, v_-) by shocks. The corresponding shock speeds are $(-c(u_-)+\varepsilon/2)$.

For $\varepsilon\geqq 0$, (u_-, v_-) can be connected to a one parameter family of values $(u_-(\varepsilon), v_-(\varepsilon))$ by simple waves. A simple wave is a solution of the form

$$\begin{aligned} u(X,t)&=\hat{u}(X/t), \\ v(X,t)&=\hat{v}(X/t). \end{aligned} \tag{5.12}$$

The choice (5.12) is a natural choice for the Riemann problem, in view of the fact that the equation (*cf.* Section 3C.) and initial data are invariant under the transformation $t \to at$, $X \to aX$, $a = \text{const}$. By introducing (5.12) into the basic equations (1.10), we derive that either

$$\begin{aligned} \hat{u}(\xi) &= \hat{u} = \text{const.} \\ \hat{v}(\xi) &= \hat{v} = \text{const.}, \end{aligned} \tag{5.13}$$

or

$$\sigma_u(\hat{u}(\xi)) = \xi^2, \quad \frac{d\hat{v}}{d\xi} = -\xi \frac{d\hat{u}}{d\xi}. \tag{5.14}$$

Let $\hat{u}(\xi)$, $\hat{v}(\xi)$ be the unique local solutions of (5.14) having data $\hat{u}(-c(u_-)) = u_-$, $\hat{v}(-c(u_-)) = v_-$. The functions $\hat{u}(\xi)$, $\hat{v}(\xi)$ will be considered only for $\xi \geqq -c(u_-)$ and ξ sufficient close to $-c(u_-)$. From $(5.14)_1$,

$$\sigma_{uu}(\hat{u}(\xi)) \frac{d\hat{u}}{d\xi} = 2\xi \Rightarrow \frac{d\hat{u}}{d\xi} > 0. \tag{5.15}$$

Let

$$\begin{aligned} u_-(\varepsilon) &= \hat{u}(-c(u_-) + \varepsilon), \\ v_-(\varepsilon) &= \hat{v}(-c(u_-) + \varepsilon), \end{aligned} \tag{5.16}$$

so that $(u_-(\varepsilon), v_-(\varepsilon))$ are defined for sufficiently small $\varepsilon \geqq 0$. Let $(u_-(\varepsilon), v_-(\varepsilon))$, $-\mu \leqq \varepsilon \leqq \mu$ be the composite of (5.16) and (5.11).

It is not difficult to prove that $(u_-(\varepsilon), v_-(\varepsilon))$ $-\mu \leqq \varepsilon \leqq \mu$ is continuously differentiable. The proof for $u_-(\varepsilon)$ follows immediately by differentiating (5.9) twice, taking the limit at $\varepsilon = 0-$, and comparing the result with (5.15) evaluated at $-c(u_-)$. The proof for $v_-(\varepsilon)$ follows by comparison of $(5.14)_2$ with the derivative of (5.10) at $\varepsilon = 0$.

If $\sigma_{uu}(u_-) > 0$, a similar procedure can be used to connect (u_-, v_-) to a one parameter family of values $(u_-(\varepsilon), v_-(\varepsilon))$, $-\mu \leqq \varepsilon \leqq \mu$. In that case the admissible shock solutions are defined for $\varepsilon \geqq 0$, and the simple wave solutions are defined for $\varepsilon \leqq 0$.

The domain of the solutions so far constructed is the neighborhood

$$-c(u_-) - f(\varepsilon) \leqq \frac{X}{t} \leqq -c(u_-) + f(\varepsilon) \tag{5.17}$$

for some continuous function $f(\varepsilon) > 0$, $\varepsilon \neq 0$, $f(0) = 0$.

The same procedure yields a continuously differentiable one parameter family of values $(u_+(\delta), v_+(\delta))$, $-\mu \leqq \delta \leqq \mu$, $(u_+(0), v_+(0)) = (u_+, v_+)$, that can be connected to (u_+, v_+) by shock or simple wave solutions. These solutions are defined in a neighborhood,

$$\begin{aligned} &c(u_+) - g(\delta) \leqq \frac{X}{t} \leqq c(u_+) + g(\delta), \\ &g(0) = 0, \\ &g(\delta) > 0, \quad \delta \neq 0, \\ &g \in C^1. \end{aligned} \tag{5.18}$$

The functions $(u_+(\delta), v_+(\delta))$ and $(u_-(\varepsilon), v_-(\varepsilon))$ are now inserted into the Rankine-Hugoniot conditions for the phase boundary,

$$\begin{aligned} \mathscr{s}(u_+(\delta)-u_-(\varepsilon))+(v_+(\delta)-v_-(\varepsilon))&=0,\\ \sigma(u_+(\delta))-\sigma(u_-(\varepsilon))+\mathscr{s}(v_+(\delta)-v_-(\varepsilon))&=0. \end{aligned} \tag{5.19}$$

$\mathscr{s}$ being the velocity of the phase boundary. The equations (5.19) are satisfied when $\delta=\varepsilon=0$, and $\mathscr{s}=\mathscr{s}_0$. We view them as conditions on (δ, ε). The determinant of (5.19), evaluated at $\delta=\varepsilon=0$, $\mathscr{s}=\mathscr{s}_0$, is

$$\frac{du_+}{d\delta}(0)\frac{du_-}{d\varepsilon}(0)\left(c(u_+)+c(u_-)\right)\left(c(u_+)-\mathscr{s}_0\right)\left(c(u_-)+\mathscr{s}_0\right), \tag{5.20}$$

which, according to (5.5), (5.6), and (5.15), is unequal to zero. Therefore, there is a solution

$$\begin{cases}\delta(\mathscr{s}),\\ \varepsilon(\mathscr{s}),\end{cases} \qquad \delta(\mathscr{s}_0)=\varepsilon(\mathscr{s}_0)=0, \tag{5.21}$$

defined for $\mathscr{s}$ near $\mathscr{s}_0$.

We construct the solution to the Riemann problem in the following way. The half plane $t\geqq 0$, $-\infty<X<\infty$ is partitioned into two parts by the phase boundary, which occurs on the line $\frac{X}{t}=\mathscr{s}$. Each part is further subdivided by the two solutions which connect u_- to $u_-(\varepsilon(\mathscr{s}))$ and u_+ to $u_+(\delta(\mathscr{s}))$, respectively. When $\mathscr{s}$ is sufficiently close to $\mathscr{s}_0$, the two solutions and the phase boundary lie on mutually exclusive domains. To complete the construction of the solution, constant solutions are prescribed consistent with the Rankine-Hugoniot conditions on the included regions.

This solution differs drastically from the corresponding solution for a monotone, genuinely nonlinear constitutive relation. The latter is composed of *three* constant solutions separated from one another by shock or simple wave solutions, and it is locally unique under the restrictions imposed by LAX's criterion for admissible shocks. The solutions exhibited here are composed of *four* constant solutions separated by conventional shock or simple waves and a phase boundary, and there is a one parameter family of such solutions.

Recall that the sign in (5.3) was arbitrarily prescribed. Hence, for given values of u_+ and u_- there are actually two one parameter families of solutions corresponding to the choices $\pm\mathscr{s}_0$. (They coincide when $\mathscr{s}_0=0$.) In part C of this section we explore the consequences of dynamic conditions which may serve to single out one solution, or perhaps a subfamily, from these two families.

To conclude this section, we record some results which may be helpful for the interpretation of these solutions. Let $u_+(\mathscr{s})$ and $u_-(\mathscr{s})$ denote the functions $u_+(\delta(\mathscr{s}))$, $u_-(\varepsilon(\mathscr{s}))$. By differentiation of the system (5.19) with respect to $\mathscr{s}$, we deduce the simple relations,

$$\begin{aligned} \frac{du_+}{d\mathscr{s}}(\mathscr{s}_0)&=\frac{(u_+-u_-)}{\det}\left(c(u_-)+\mathscr{s}_0\right)^2=\frac{(u_+-u_-)\left(c(u_-)+\mathscr{s}_0\right)}{\left(c(u_+)+c(u_-)\right)\left(c(u_+)-\mathscr{s}_0\right)},\\ \frac{du_-}{d\mathscr{s}}(\mathscr{s}_0)&=\frac{(u_+-u_-)}{\det}\left(c(u_+)-\mathscr{s}_0\right)^2=\frac{(u_+-u_-)\left(c(u_+)-\mathscr{s}_0\right)}{\left(c(u_+)+c(u_-)\right)\left(c(u_-)+\mathscr{s}_0\right)}. \end{aligned} \tag{5.22}$$

Here

$$\det = \big(c(u_+) + c(u_-)\big)\big(c(u_+) - \dot{s}_0\big)\big(c(u_-) + \dot{s}_0\big). \tag{5.23}$$

Therefore, regardless of the sign of σ_{uu}, $u_+(\dot{s})$ and $u_-(\dot{s})$ are both either increasing or decreasing functions of $\dot{s}$.

B. The Riemann problem. Double phase boundary

A similar method to the one used in the preceding section may be used to establish the existence of a two parameter family of double phase boundary solutions. These solutions are more naturally associated with experiments involving "necking". All of the solutions will contain two phase boundaries which separate at some point in the bar and travel toward the ends. At the point where the phase boundaries separate in the bar a pair of ordinary shocks or rarefaction waves will emerge and will propagate out ahead of the phase boundaries.

We shall begin with a "basic solution" and seek a *two parameter* family of solutions close to the basic solution. Following the procedure of the preceding section, we do not want the basic solution to contain any rarefaction waves or ordinary shock waves. Also, the initial data shall be assigned as a homogeneous, static solution:

$$(u, v)(X, 0) = (u_0, 0), \qquad u_0 = \text{const.} \in (\alpha, \alpha^1). \tag{5.24}$$

By accounting for Galilean invariance (*cf.* Section 3C), we have chosen the initial velocity to be zero, without loss of generality. We shall also assume that $\sigma(u_0)$ is in the range of the interior of the β-branch, *e.g.* that there is a value $\hat{u} \in (\beta^1, \beta)$ such that $\sigma(\hat{u}) = \sigma(u_0)$.

Suppose that the basic solution consists of a phase boundary moving forward with velocity $\dot{s}_+$ and a phase boundary moving backward with velocity $\dot{s}_-$, the included region being a constant solution $(\hat{u}_0, \hat{v}_0)$. The region between each phase boundary and the X-axis is assigned as the constant solution $(u_0, 0)$. Necessary and sufficient conditions that the Rankine-Hugoniot conditions hold across the phase boundaries are that

$$\hat{v}_0 = 0, \qquad \dot{s}_+ = 0, \qquad \dot{s}_- = 0, \qquad \sigma(\hat{u}_0) = \sigma(u_0). \tag{5.25}$$

Equations (5.25) define the *basic solution.* It is actually just the static solution $(u_0, 0)$ because $\dot{s}_+ = \dot{s}_-$, so that the region included by the phase boundaries is null.

In the manner of the preceding section, the value $(u_0, 0)$ can be connected for $X < 0$ to a one parameter family of values $(u_-(\varepsilon), v_-(\varepsilon))$, $(u(0), v(0)) = (u_0, 0)$, on the right by a one parameter family of solutions. This one parameter family of solutions consists either of shock waves or simple wave solutions bounded by acoustic waves, and both of these waves move nearly at the acoustic speed $c(u_0)$. For $X > 0$ the value $(u_0, 0)$ can be connected to a one parameter family of values $(u_+(\delta), v_+(\delta))$, $(u_+(0), v_+(0)) = (u_0, 0)$ on the left by shock or simple wave solutions. Here, as always, left and right refer to the $X - t$ plane with the t-axis drawn vertical. These sets of solutions are defined for ε and δ near zero. We wish to connect the constant solution $u_+(\delta)$, $v_+(\delta)$ to a constant solution $(\hat{u}, \hat{v})$ by a phase boundary moving forward with velocity $\mathcal{V}_+$. Similarly, we wish to connect $u_-(\varepsilon)$, $v_-(\varepsilon)$ to the constant solution $(\hat{u}, \hat{v})$ by a phase boundary moving with velocity $\mathcal{V}_-$ backward. In order that this construction deliver a solution, we must check that the Rankine-Hugoniot conditions be satisfied across the two phase boundaries:

$$\begin{aligned} \mathcal{V}_+\big(u_+(\delta)-\hat{u}\big)+\big(v_+(\delta)-\hat{v}\big)&=0, \\ \sigma\big(u_+(\delta)\big)-\sigma(\hat{u})+\mathcal{V}_+\big(v_+(\delta)-\hat{v}\big)&=0, \\ \mathcal{V}_-\big(\hat{u}-u_-(\varepsilon)\big)+\big(\hat{v}-v_-(\varepsilon)\big)&=0, \\ \sigma(\hat{u})-\sigma\big(u_-(\varepsilon)\big)+\mathcal{V}_-\big(\hat{v}-v_-(\varepsilon)\big)&=0. \end{aligned} \tag{5.26}$$

These equations are satisfied at the basic solution (5.25). We view them as restrictions on $\hat{u}$, $\hat{v}$, δ, ε, with $\mathcal{V}_+$ and $\mathcal{V}_-$ acting as parameters. The determinant of the gradient of (5.26) with respect to $\hat{u}$, $\hat{v}$, δ, ε, evaluated at the basic solution, is

$$\frac{du_-}{d\varepsilon}(0)\,\frac{du_+}{d\delta}(0)\;c(u_0)^3\,c(\hat{u}_0)^3 \tag{5.27}$$

which, according to (1.4), (5.15), (5.16) and the definition of $c(u)$, is never zero. Hence, there is a two parameter family of solutions

$$\hat{u}(\mathcal{V}_-,\mathcal{V}_+),\;\hat{v}(\mathcal{V}_-,\mathcal{V}_+),\;\delta(\mathcal{V}_-,\mathcal{V}_+),\;\varepsilon(\mathcal{V}_-,\mathcal{V}_+). \tag{5.28}$$

These solutions reduce to the basic solution when $\mathcal{V}_-=\mathcal{V}_+=0$, and, otherwise, in order that the solutions be meaningful, $\mathcal{V}_-\leqq 0\leqq\mathcal{V}_+$.

In order to describe these solutions, imagine an infinite bar as the abscissa, and the time axis as the vertical ordinate, of a rectangular co-ordinate system. For $t<0$ the bar is homogeneously deformed in the α-phase. At $t=0$ an ordinary shock wave moves forward and an ordinary shock wave moves backward, or a rarefaction wave moves foreward and another backward. Simultaneous to the emergence of these fast moving waves, two slow moving phase boundaries emerge from the origin, one moving foreward and the other moving backward. The region included between the phase boundaries is homogeneously deformed in the β-phase.

It has been assumed merely for definiteness that the bar was initially in the α-phase. It could have been assumed that the bar was in the β-phase; then, the α-phase would have emerged between the phase boundaries.

C. *Admissibility*

The problem of admissibility of solutions is a deep and difficult one, especially when the constitutive equation permits the co-existence of phases.

The need for an admissibility criterion seems to have arisen from two sources. One school of thought, systematically and critically presented by RAYLEIGH [8], demands that the solutions of the equations of elastic bar theory be consistent with thermoelastic, viscoelastic, or thermoviscoelastic bar theory.

"Consistency" means that elastic bar theory is embedded as a special case of the broader theory, so the broader theory places additional restrictions on solutions of the special one. Typically, the mechanical equations are obtained from the thermoviscoelastic ones by setting the temperature equal to a constant, by setting the heat flux equal to zero, or by allowing the viscosity to approach zero. RAYLEIGH compares the earlier work by RANKINE [9], who develops an elegant and general thermodynamic theory, to that of HUGONIOT [10]; both of these earlier authors considered the embedding into thermodynamic theories (without viscosity), HUGONIOT employing the adiabatic law of LAPLACE-POISSON throughout, and RANKINE developing a thermodynamic theory for transition layers including the conduction of heat. RAYLEIGH adjoins to these thermodynamic theories a theory of viscosity, and seems to be the first to show concretely that some of the solutions may not be dissipative. He regards* it "a question of great interest to inquire what is the influence of viscosity and especially whether alone, or in co-operation with heat-conduction, it allows a wave of condensation to acquire a permanent regime"; since here the nature of an underlying thermodynamic theory is obscure, the use of a viscoelastic theory alone appears most fruitful.

Ever since LAPLACE [11] predicted to correct speed of sound for forbidding the conduction of heat, there has been a recurring belief that for most materials the disturbances which propagate at the speed of sound, or nearly the speed of sound, undergo adiabatic processes. Though weak shock waves fall into this category, the slow moving phase boundaries probably do not. The solutions constructed in parts A and B of this section may simultaneously contain a phase boundary which moves as slowly as desired, and a shock wave which moves as close to the acoustic speed as desired. This situation may be realized in part A by assigning the data u_+, u_- so that $\sigma(u_+)=\sigma(u_-)$; all of the solutions constructed in part B have this property. It is not inconceivable that shock waves ought to be regarded as adiabatic and phase boundaries as isothermal. Not all contemporary workers agree; DUNWOODY presents evidence based upon the theory of infinitesimal deformation superimposed on finite deformation that the temperature is continuous across weak shock waves [12]. DAFERMOS [13] has put forth a general scheme which is consistent with the isothermal hypothesis.

The second line of thought which has promoted the need for an admissibility criterion has been the desire to prove uniqueness for the initial value problem on the infinite bar. This view was promoted by LAX [7] in his influential paper on the subject, and mathematicians developing the theory of hyperbolic conservation laws have adopted it without question since then. Casual observation casts doubt on this prejudice, especially for constitutive relations allowing a change of phase. When a long, thin polyethylene bar of uniform cross section is loaded by a sufficiently large weight, a new phase forms; two distinct phase boundaries separate at some point in the bar, converting regions of low stretch into regions of high stretch. It is observed that for very long, very thin bars of uniform cross section the point of initiation of the new phase is quite variable. Experiments of this kind are not generally reproducible. The caution should be

* RAYLEIGH [8, p. 269].

extended to shock wave problems for ordinary stress-deformation relations as well.

Another touchstone for the dynamic theory which seems not to have been brought to bear on the problem of admissibility is static stability theory. The reason is plain. Static stability theory is trivial for strictly monotone stress-deformation functions, all solutions being absolutely stable according to the energy criterion for stability. On the other hand, for a non-invertible stress-deformation function some solutions are unstable, others metastable, others absolutely stable, still others neutrally stable. I refer to [1] for the complete story. Let us suppose that the two theories yield consistent predictions. The Riemann problem provides a basis of comparison for the two theories. That is, if the initial data for the Riemann problem is assigned so that $\sigma(u_+)=\sigma(u_-)$, u_+ lying on the β-branch and u_- lying on the α-branch, then *the initial data is a static solution.* For the double phase boundary solutions the initial data is always a static solution. If that static solution is stable according to the static theory, then the admissibility criterion for the dynamic theory should imply that the solution persists. If the static solution is unstable, it should not persist. If it is metastable or neutrally stable the question is moot. Observations suggest that when a new phase appears in a supercooled liquid or gas, it will pass quickly through the body, so that a metastable static solution containing two phases will not be sustained. For transitions between solid phases, however, metastable solutions containing several phases seem likely to persist.

In spite of these difficulties some progress can be made toward classifying solutions if we return to the suggestions of RAYLEIGH. The most commonly quoted criteria of admissibility, LAX's entropy* criterion [7] and OLEINIK's E-condition [15] and their generalizations are inapplicable to non-monotone stress-deformation relations because they presume monotonicity in their derivations. We fail to understand, however, why the "entropy" of the entropy criterion is required to be convex by definition. The equations used here admit an additional conservation law for the total energy $(\frac{1}{2}v^2+\int^u \sigma(u)\,du)$, but it is not convex. On the other hand, DAFERMOS' criterion [16], which allows solutions only if they maximize the rate of decrease of entropy, is meaningful in this context, and the evaluation of it for the equations of elastic bar theory [16, equn. (4.5)] is valid for a non-convex total energy. In the succeeding paragraphs we shall explore the entropy rate criterion and others which can be applied to the present situation. For the lack of a clear conceptual framework for a thermodynamic argument of admissibility, I shall omit it.

1. Consistency with static stability theory. We shall first focus attention on the solutions of the Riemann problem containing a single phase boundary.

As mentioned above, the initial data for those solutions is a static solution containing a stationary phase boundary if $\sigma(u_+)=\sigma(u_-)$, wherein $u_-\in(\alpha,\alpha^1)$ and $u_+\in(\beta^1,\beta)$. The Rankine-Hugoniot conditions (5.3) and (5.4) then imply that $\dot{s}_0=0$ and $v_+=v_-$, which, by accounting for Galilean invariance (*cf.* section 3C), can be replaced by $v_+=v_-=0$. The theory of such solutions is presented in [1],

* I adhere to the perhaps misleading choice of the word "entropy" in this context.

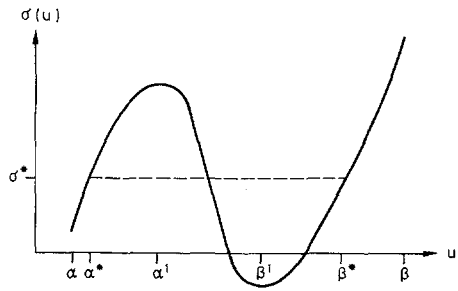

Fig. 1. The Maxwell line (dashed).

so we only recall the requisite facts here. To establish a full correspondence between the results there and the solution obtained in part A of this section, the bar must be placed in a certain loading device and must have finite length. It is possible to restrict the solutions of the Riemann problem to the interval $[-L, L] \times [0, T]$, in which T is chosen smaller than $L/(\max\{c(u_+), c(u_-)\})$. In that case, when ϑ is sufficiently close to ϑ_0, the shock and acoustic waves do not interact with the ends of the bar. Then an appropriate loading device for the Riemann problem is the dead loading device; that is, a neighboorhood of each end of the bar remains static with a constant stress $\sigma(u_+)$. Therefore, the dynamic solutions of the Riemann problem, restricted to the interval $[0, T]$, can be conceived as solutions for a finite bar $[-L, L]$ loaded by a dead loading device. The particular solution defined by $\vartheta = \vartheta_0 = 0$ is a static solution for the dead loading device.

In [1] the stability of such solutions has been investigated. Suppose

$$y'(X) = \begin{cases} u_-, & X \in [-L, 0], \\ u_+, & X \in (0, L], \end{cases} \qquad (5.29)$$

$$\sigma(u_+) = \sigma(u_-).$$

In this relatively simple situation, the stability of $y(X)$ according to the energy criterion for stability is determined completely by the **Weierstrass excess function**, which is defined as

$$\mathscr{E}(v, u) = \int_u^v \sigma(s)\, ds - (v - u)\, \sigma(u). \qquad (5.30)$$

Static stability theory implies that y is **absolutely stable** if $\mathscr{E}(v, y'(X)) > 0$ for almost all $X \in [-L, L]$ and all $v \neq y'(X)$ in the domain of σ; y is **neutrally stable** if $\mathscr{E}(v, y'(X)) \geqq 0$, $X \in [-L, L]$, for all v in the domain of σ, and if $\mathscr{E}(v(X), y'(X)) = 0$, $v(X) \neq y'(X)$ holds for $X \in \mathscr{S}$, $\mathscr{S}$ being a set of positive measure in $[-L, L]$; y is **metastable** if for some $\varepsilon > 0$, $\mathscr{E}(v, y'(X)) \geqq 0$ a.e. whenever $|v - y'(X)| < \varepsilon$. Finally, y is **unstable** if it is not metastable.

An analysis of the Weierstrass excess function for the constitutive class (1.4) yields the **Maxwell line** pictured in Figure 1. The stress σ^* is called the *Maxwell* *stress* and α^* and β^* are defined by,

$$\begin{aligned} \sigma(\alpha^*)&=\sigma^*, \quad \alpha^*\in[\alpha,\alpha^1), \\ \sigma(\beta^*)&=\sigma^*, \quad \beta^*\in(\beta^1,\beta]. \end{aligned} \tag{5.31}$$

The deformations α^* and β^* are the unique points on the α- and β-branches, respectively, that satisfy $\mathscr{E}(\beta^*,\alpha^*)=\mathscr{E}(\alpha^*,\beta^*)=0$, which is to say that the signed area between the curve $\sigma(u)$ and the Maxwell line is zero.

Briefly summarized, the static theory implies that if

α. $u_+=u_-\in[\alpha,\alpha^*)\cup(\beta^*,\beta]$, then y is absolutely stable;
β. $u_+=\alpha^*$ or β^*, then y is neutrally stable;
γ. u_+ and u_- belong to $[\alpha,\alpha^1)\cup(\beta^1,\beta]$, then y is metastable;
δ. u_+ or u_- belong to $[\alpha^1,\beta^1]$, then y is unstable.

Clearly, if y is absolutely stable, then $u_+=u_-$, and no phase boundaries are present. Let us assume that we may impose compatibility of the dynamic theory with the static theory, when the initial data is a static solution. If u_+ or u_- belong to $[a^1,\beta^1]$, the solution corresponding to $\sigma_0=0$ is an unstable static solution, so we would expect never to observe it. For this reason, and others which will become clear in the next section, u_+ and u_- have been chosen to lie on the β- and α-branches, respectively, in the analysis of the Riemann problem. No definite prediction is delivered in the cases of metastability and neutral stability, and the absolutely stable solutions do not contain any phase boundaries. Therefore, *if u_- is assigned on the α-branch and u_+ is assigned on the β-branch, the existence of a one parameter family of solutions is compatible with the static theory.*

Finally, we note that in [1] the results found for the dead loaded homogeneous bar were rather untypical. Slight inhomogeneities or body forces in the static theory tend to allow absolutely stable solutions to occur which contain one or more phase boundaries. The connection between these possibilities and the dynamic theory are as yet unclear.

We shall now focus attention on the solutions obtained in part B of this section. The initial data for those solutions is always the static solution $(u_0, 0)$, and as before the bar may be regarded as loaded by a dead loading device, for sufficiently short times. The static theory implies that if

α. $u_0\in[\alpha,\alpha^*)\cup(\beta^*,\beta]$, then the initial data is an absolutely stable static solution;
β. $u_0=\alpha^*$ or β^*, the initial data is neutrally stable;
γ. $u_0\in[\alpha,\alpha^1)\cup(\beta^1,\beta]$, the initial data is metastable;
δ. $u_0\in[\alpha^1,\beta^1]$, the initial data is unstable.

Contrary to the predictions of the static theory regarding the solutions with a single phase boundary, here the static theory is remarkably exclusive. That is, *if $u_0\in[\alpha,\alpha^*)\cup(\beta^*,\beta]$ only the particular solution $u=u_0$, $v=0$ is admissible from among the two parameter family* (5.28). *If $u_0\in[\alpha^*,\alpha^1)\cup(\beta^1,\beta^*)$, then the whole two parameter family is compatible with the static theory.*

In the construction of solutions containing two phase boundaries, the assignment $u_0\in[\alpha^1,\beta^1]$ has been avoided. The reason for this exclusion is now evident; if such had not been the case, the infinite bar, after any finite time had passed, would have contained subintervals on which were defined unstable static solutions.

2. *Consistency with viscoelastic bar theory.* Viscoelastic bar theory provides an example of a theory from which elastic bar theory can be easily obtained. Furthermore, while viscoelasticity embodies a concept of dissipation,* it does not require a decision on the nature of the head flux in the bar. The simplest viscoelastic bar theory follows by generalizing the constitutive function for the stress to

$$\hat{\sigma}(u, \dot{u}) = \sigma(u) + \varepsilon \dot{u}, \tag{5.32}$$

which yields the equations,

$$\begin{aligned} \dot{u} &= v' \\ \dot{v} &= \sigma_u(u)\, u' + \varepsilon \dot{u}'. \end{aligned} \tag{5.33}$$

Here the positive constant ε is the viscosity. Let $u_\varepsilon(X, t)$, $v_\varepsilon(X, t)$ be a solution of (5.33). We shall seek necessary conditions that a solution $u(X, t)$, $v(X, t)$ of the equations of elastic bar theory is the limit, as $\varepsilon \to 0$, of solutions of the viscoelastic equation. From the theorems of GREENBERG, MACCAMY & MIZEL [17], and DAFERMOS [18], we expect u_ε, v_ε to be at least twice continuously differentiable**. Also, studies of the viscosity method ([19], for example) suggest that $(u_\varepsilon, v_\varepsilon)$ will contain a subsequence $(u_{\varepsilon_k}, v_{\varepsilon_k})_{k=1}^{\infty}$, $\varepsilon_k \to 0$ as $k \to \infty$, which tends almost everywhere to a pair of functions (u, v) of bounded variation. From these two facts follows the result that (u, v) is a weak solution of the equations of dynamic elastic bar theory. I shall assume the validity of these statements. Let

$$E(u, v) = \tfrac{1}{2} v^2 + \int^{u} \sigma(u)\, du \tag{5.34}$$

* The positiveness of the viscosity is a result of thermodynamic origins, which may be derived from an entropy inequality or a "second law". Otherwise the theory is purely mechanical in nature.

** However, neither of these papers applies without alternation to the present situation; the first forbids the possibility that the constitutive function be non-monotone, and the second excludes the end conditions (5.1), (5.2). However, aspects of DAFERMOS' paper [18], in particular his theorem 3.1 of uniqueness, do apply to the situation at hand, but their relation to the solutions of the Riemann problem given in parts A and B of this section is hazy. For example, regarding the solutions obtained in part B, a solution of the corresponding viscoelastic problem is $(u_\varepsilon, v_\varepsilon) = (u_0, 0)$, and by DAFERMOS' result this solution is unique. The limit of these solutions, as $\varepsilon \to 0$, is simply $(u_0, 0)$. However, a two-parameter family of solutions has been produced in part B, and they all may be regarded as solving the same initial-boundary value problem for sufficiently short times. Ostensibly, then, all those solutions except the static one are inadmissible as limits of viscosity solutions. Our feeling about the matter runs contrary to this conclusion. We think that in some cases, at least, some non-static members of the two-parameter family are observable solutions, but that their counterparts for the viscoelastic problem are not quite classical solutions of the initial-boundary value problem. Evidence for this prejudice comes from the viscoelastic problem with ε replaced by εt (*cf.* equation (5.33)), which admits solutions as functions of the single variable $\frac{X}{t}$. Those equations appear to have solutions corresponding to non-static members of the two-parameter family (5.28).

denote the total energy, and let

$$Q(u,v) = -v\,\sigma(u) \tag{5.35}$$

denote the energy flux. Observe that E is not convex. A subscript ε will be attached to E and Q when they are calculated for a solution of (5.33). Because the viscosity is positive it follows from (5.33) that

$$\dot{E}_\varepsilon + Q'_\varepsilon \leqq \tfrac{1}{2}\varepsilon(v_\varepsilon^2)''. \tag{5.36}$$

Thus, for the limiting solution (u, v), the inequality

$$\dot{E} + Q' \leqq 0 \tag{5.37}$$

is satisfied in a weak sense. (That is, the inequality which results when the following operations are performed on (5.37) is satisfied: multiplication by a smooth non-negative test function with compact support in $(-L, L) \times (0, T)$, integration over $[-L, L] \times [0, T]$, integration by parts.) If (X, t) is a point of differentiability of (u, v), then (5.37) is satisfied there as an equality. If $X = \chi(t)$ is a shock wave or phase boundary in the domain of (u, v), then the inequality

$$\dot{\chi}(E_+ - E_-) \geqq Q_+ - Q_- \tag{5.38}$$

holds across it. By eliminating v from (5.38) and using the Rankine-Hugoniot conditions (1.7), we deduce that

$$\dot{\chi}\left\{W(u_+) - W(u_-) - (u_+ - u_-)\left(\frac{\sigma(u_+) + \sigma(u_-)}{2}\right)\right\} \geqq 0 \tag{5.39}$$

in which

$$\begin{aligned} W(u) &= \int^u \sigma(s)\,ds, \\ u_+ &= u(\chi(t) + 0, t), \\ u_- &= u(\chi(t) - 0, t). \end{aligned} \tag{5.40}$$

Let

$$\mathscr{A} = W(u_+) - W(u_-) - (u_+ - u_-)\left(\frac{\sigma(u_+) + \sigma(u_-)}{2}\right). \tag{5.41}$$

We shall refer to (5.39) as the **viscoelastic criterion of admissibility**. On genuinely nonlinear portions of the constitutive function it is equivalent to LAX's entropy admissibility criterion, but it has other implications as well. If $\sigma(u_+)$ is sufficiently close to $\sigma(u_-)$ and the Weierstrass condition,

$$\mathscr{E}(u_+, u_-) > 0 \quad (<0), \tag{5.42}$$

is satisfied with strict inequality, then $\mathscr{A} > 0$ (<0). Unlike the Weierstrass condition, the viscoelastic criterion cannot be analyzed explicitly for the constitutive class described here because its implications depend on the details of $\sigma_{uu}(u)$*. However, the solutions of the Riemann problem presented in parts A

* A useful formula for the analysis of explicit constitutive relations is

$$\frac{\partial^2 \mathscr{A}(v,u)}{\partial v^2} = -\frac{(v-u)}{2}\sigma_{uu}(v).$$

Regarding this equation as a differential equation for $\mathscr{A}(v, u)$ at fixed u, if the initial condition v_0 for v is assigned so that $\sigma(u) = \sigma(v_0)$, then $\mathscr{A}$ becomes the Weierstrass function initially, and

$$\frac{\partial \mathscr{A}}{\partial v}(v_0, u) = \left(\frac{u - v_0}{2}\right)\sigma_u(v_0).$$

and B of this section do submit to analysis by the viscoelastic criterion of admissibility.

We shall first concentrate on the admissibility of solutions to the Riemann problem containing a single phase boundary (part A). We focus attention on the phase boundaries in those solutions, since the shock waves are covered by LAX's criterion for admissible shocks. Let u_+ and u_- be the values of the initial data (5.2). When $\dot{s}$ is sufficiently close to $\dot{s}_0$, $\dot{s}_0$ being determined by (5.3), the analysis of part A provides a solution to the Riemann problem. Recall from (5.3) that $\dot{s}_0$ was determined up to a sign, and that either choice $\pm$ led to a one parameter family of solutions. Suppose $\mathscr{A}(u_+, u_-) > 0$. Then, by virtue of (5.39) the plus sign in (5.3) is preferred. It then follows that the viscoelastic criterion of admissibility holds for $\dot{s}$ near $\dot{s}_0$. *Therefore, of the two possible one-parameter families of solutions to the Riemann problem, the one with the phase boundary moving foreward is selected by the viscoelastic criterion of admissibility when* $\mathscr{A}(u_+, u_-) > 0$. *Alternatively, when* $\mathscr{A}(u_+, u_-) < 0$ *the phase boundary must move backward.* Of course, the solution corresponding to $\dot{s}_0 = 0$ is never excluded. If $\mathscr{A}(u_+, u_-) = 0$, it may be true* that $\frac{d\mathscr{A}}{ds}(u_+, u_-) \neq 0$; in that case, one half of each of the one parameter families is preferred according to the viscoelastic criterion.

In the special case $\sigma(u_+) = \sigma(u_-)$, it is possible to compare the predictions of both static stability theory and the viscoelastic criterion of admissibility. Recall that when $\sigma(u_+) = \sigma(u_-)$, $\mathscr{A}(u_+, u_-) = \mathscr{E}(u_+, u_-)$. Therefore, by comparing the viscoelastic criterion with the results of the previous section on the Weierstrass function, we deduce that, according to the viscoelastic criterion, if $\sigma(u_+) = \sigma(u_-)$ and

1. if $u_+ > u_-$ and $u_- \in [\alpha, \alpha^*)$, then $\dot{s} \geqq 0$;
2. if $u_+ > u_-$ and $u_- \in (\alpha^*, \alpha^1)$, then $\dot{s} \leqq 0$.

A physical description of this result is perhaps clearest. Suppose the initial data is a static solution containing a stationary boundary:

$$\begin{gathered} \sigma(u_+) = \sigma(u_-) = \sigma_0 = \text{const.} \\ \dot{s}_0 = 0. \end{gathered} \tag{5.43}$$

If $\sigma_0 > \sigma^*$, then the static theory predicts that this static solution is metastable, but that there is an absolutely stable solution for the same loading device in which the bar is homogeneously deformed in the β-phase (*i.e.* $u = \text{const.}$ on $[0, T] \times [-L, L]$ and $u \in (\beta^*, \beta]$). Under the same conditions the viscoelastic criterion implies that the Riemann problem must have solutions in which the phase boundary propagates with a non-positive velocity; that is, the viscoelastic criterion implies that either the solution remains static or that the phase

* By using (5.22), it is not difficult to show that

$$\begin{aligned} \frac{d\mathscr{A}}{d\dot{s}}(u_+, u_-) = \frac{-(u_+ - u_-)}{2 \det} \{ & (c_+^2 - c_-^2 - (u_+ - u_-) c_+^2)(c_- + \dot{s}_0)^2 \\ & + (c_+^2 - c_-^2 - (u_+ - u_-) c_-^2)(c_+ - \dot{s}_0)^2 \}. \end{aligned}$$

boundary moves backward, so as to increase the amount of β-phase. Alternatively, if $\sigma_0 < \sigma^*$, the static theory predicts that a bar homogeneously deformed in the α-phase is absolutely stable, the phase mixture corresponding to the initial data for the Riemann problem being only metastable. The viscoelastic criterion forces the phase boundary either to remain stationary or to convert β-phase into α-phase.

We turn now to the admissibility of the solutions containing two phase boundaries which were presented in part B. Those solutions were always uniformly close to the basic solution $(u_0, \hat{u}_0)$. Since the basic solution was a static solution, $\mathscr{A}(\hat{u}, u_0) = \mathscr{E}(\hat{u}, u_0)$ and $\mathscr{A}(u_0, \hat{u}) = \mathscr{E}(u_0, \hat{u})$. We suppose, without loss of generality, that u_0 lies on the α-branch and $\hat{u}$ on the β-branch. Suppose $u_0 \in [\alpha, \alpha^*)$ and consider the two parameter family of solutions (5.28). Since $\mathscr{A}(u_0, \hat{u}) = \mathscr{E}(u_0, \hat{u}) < 0$, it follows that $\mathscr{A}(u_+(\delta(\sigma_-, \sigma_+)), \hat{u}(\sigma_-, \sigma_+)) < 0$ for σ_- and σ_+ sufficiently close to zero.

Hence, the viscoelastic criterion is violated across the phase boundary which moves forward. Also, it is violated across the phase boundary which moves backward. By similar reasoning carried out for the β-branch, we arrive at the following conclusion. *If $u_0 \in [\alpha, \alpha^*) \cup (\beta^*, \beta]$, that is to say, if the initial data is a constant, absolutely stable static solution, then the only solution among the (local) two parameter family which is admissible according to the viscoelastic criterion is the static solution* $(u_0, 0)$. On the other hand, if $u_0 \in (\alpha^*, \alpha] \cup [\beta, \beta^*)$ it is easy to show that the viscoelastic criterion is satisfied across both phase boundaries present, if σ_+ and σ_- are sufficiently close to zero. *Therefore, the whole (local) two-parameter family of solutions that begin from a constant, metastable static solution which is neither neutrally stable nor absolutely stable is admissible by the viscoelastic criterion.*

One other line of thought may prove useful in the investigation of these solutions. The viscoelastic criterion was merely a necessary condition that the purely elastic solution be possible as the limit of viscoelastic solutions. It may not be sufficient. For example, although the full two parameter family of solutions with two phase boundaries was found to be admissible by the viscosity criterion, perhaps only one of that family, or some preferred subfamily, can be actually realized as limits of viscoelastic solutions as the viscosity vanishes. One way to investigate this possibility would be to replace the viscosity ε by a viscosity of the form εt, $\varepsilon = \text{const.}$, according to the suggestion of DAFERMOS* [19]. In that case the viscoelastic equations admit the invariance group $X \to \nu X$, $t \to \nu t$, $\nu = \text{const.}$, so solutions depending upon the single variable $\frac{X}{t}$ can be found. We shall not continue this line of thought here.

3. Consistency with the maximal rate of decay of entropy. In response to difficulties encountered in the application of the entropy criterion to constitutive

* His equations do not quite fit into the form (5.33) with ε replaced by εt. He has also added the term $\varepsilon t\, u''$ to $(5.33)_1$. While these alterations yield a neat existence theorem, we are reluctant to adopt the entire procedure for the purposes of admissibility, since $(5.33)_1$ loses physical meaning. On the other hand, merely the replacement $\varepsilon \to \varepsilon t$ seems not too objectionable, especially for bounded intervals of time, and we would be inclined to trust the predictions.

relations which are not genuinely nonlinear, DAFERMOS postulates the "*entropy rate admissibility criterion*" in [16]. He justifies it on the grounds that it represents a natural extension of the notion of decrease of entropy in the descension from a broader theory, and that it only makes use of a single entropy.

As it stands, DAFERMOS' criterion does not apply to the problem at hand because the entropy (mechanical energy) is required to be strictly convex. We can think of no convincing reason for this assumption. However, when the requirement of strict convexity is excluded from the criterion, the criterion remains useful and interesting, though perhaps a bit strong in its selectiveness.

The details of the application of DAFERMOS' criterion to elastic bar theory are contained in his paper [16, part 4]. I shall only briefly compare those results to the ones given here in the preceding paragraphs. DAFERMOS shows that the *rate of decay of entropy* $D_+ H_{(u,v)}(\tau)$ calculated at time τ for piecewise smooth solutions is given by

$$D_+ H_{(u,v)}(\tau) = -\sum_{\text{shocks}} \dot{\chi}(\tau)\, \mathscr{A}(u_+, u_-), \tag{5.44}$$

$\mathscr{A}(u_+, u_-)$ being given by (5.41). DAFERMOS' criterion states that the rate of decay of entropy be not greater for the admissible solution than for any other solution of the same initial value problem. We shall refer to this criterion, exclusive of the part which requires the energy to be strictly convex, as the *condition for the maximal rate of decay of entropy.*

The solutions of the Riemann problem that have been produced in parts A and B of this section are local in character and are certainly not exhaustive; we could combine several of them together on adjacent domains in the $X-t$ plane to build up another solution. Therefore, we cannot use the condition for the maximal rate of decay of entropy to select solutions, but only to exclude them. The analysis of (5.44) proceeds along the lines already established in the preceding subsection, so we only record the rather exclusive result here. *According to the condition for the maximal rate of decay of entropy, a piecewise smooth, metastable, static solution, which is neither absolutely stable nor neutrally stable, cannot persist.*

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