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PROPAGATION OF SINGULARITIES ALONG GLIDING RAYS

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Exposé N^{o} I

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Let $\Omega = \{x \in R^n; f(x) > 0\}$ be a domain with C^∞ boundary and let $P = P(x,D)$ be a second order differential operator of real principal type with coefficients in $C^\infty(\bar{\Omega})$. The subject of this lecture is to discuss the singularities of distributions u satisfying

$$Pu \in C^\infty(\bar{\Omega}), \quad u|_{\partial\Omega} \in C^\infty(\partial\Omega).$$

If $\wp$ denotes the set of zeros of the principal symbol $p(x,\xi)$, $\mathfrak{F} = \partial\Omega \times \dot{R}^n$ and

(1) $$\{p,f\} \neq 0 \text{ in } \wp \cap \mathfrak{F} \quad \text{(microlocally)},$$

then this problem has been treated by several authors (see [2], [8]). We shall consider a case where (1) is violated. If the boundary $\partial\Omega$ is non-characteristic, then

(2) $$\{p,f\}(z) = 0 \Rightarrow \{f,\{f,p\}\}(z) \neq 0 \ , \ z \in \wp \cap \mathfrak{F} \ .$$

We shall also assume that Ω is pseudo-convex with respect to P, i.e.

(3) $$\{p,f\}(z) = 0 \Rightarrow \{p,\{p,f\}\} < 0 \ , \ z \in \wp \cap \mathfrak{F} \ .$$

The opposite case, when Ω is pseudo-concave, has been treated in [4], [6] and [9].

When (2) and (3) are satisfied, it is possible to define so called boundary bicharacteristics for P as follows. Consider the restriction $\tilde{p}$ of p to the symplectic manifold $\Delta_{\mathfrak{F}} = \{z \in \mathfrak{F} ; \{p,f\}(z) = 0\}$. The projection, onto $T^*(\partial\Omega)$ of the null-bicharacteristics for $\tilde{p}$ are called boundary bicharacteristics for P. It is easy to check that the null-bicharacteristics for the restriction of f to $\Delta_{\wp} = \{z \in \wp ; \{p,f\}(z) = 0\}$ give the same curves as the null-bicharacteristics for $\tilde{p}$. In fact, the tangent of such a curve lies in the plane spanned by H_p and H_f and is orthogonal to the gradient of $\{p,f\}$.

In order to localize the concept of "regularity up to the boundary", we assume that Ω is given by $x_n > 0$ and denote points on the boundary $\partial\Omega$ by $x' = (x',0)$. If $(\bar{x}',\bar{\xi}') \in \dot{T}^*(\partial\Omega)$, we say that $u \in H_s(\bar{\Omega})$ at $(\bar{x}',\bar{\xi}')$ if there is a homogeneous symbol $\psi(x',\xi')$ such that $\psi(x',\xi') = 1$ in a neighborhood of $(\bar{x}',\bar{\xi}')$ and $\psi(x',D')u(x',x_n)$ belongs to $H_s(\bar{\Omega})$ close to $\bar{x}'$. Here it is implicitly assumed that $x_n \mapsto u(.,x_n)$ is well-defined. In the same manner we say that $(\bar{x}',\bar{\xi}') \notin WF(u;\bar{\Omega})$ if, for some ψ as above, $\psi(x',D')u \in C^\infty(\bar{\Omega})$ close to $\bar{x}'$. This definition of microlocal regularity up to the boundary has also been suggested by Chazarain [3].

<u>Theorem</u> : Suppose that Ω is pseudo-convex with respect to P and let γ be a boundary bicharacteristic for P. If $u \in \mathcal{D}'(\bar{\Omega})$, $Pu \in C^\infty(\bar{\Omega})$ and $\gamma \cap WF(u_{|\partial\Omega}) = \emptyset$, then either $\gamma \subset WF(u;\bar{\Omega})$ or $\gamma \cap WF(u;\bar{\Omega}) = \emptyset$.

<u>Sketch of proof</u> : In order to avoid technical complications, we only consider the case when P is hyperbolic. We also assume that Ω is given by $x_n > 0$. Let $(\bar{x}',\bar{\xi})$ be a point in $\mathcal{P} \cap \mathcal{F}$ where $\{p,f\} = 0$ and denote by $\gamma = \gamma(\bar{x}',\bar{\xi}')$ the boundary bicharacteristic through the projection $(\bar{x}',\bar{\xi}')$ of $(\bar{x}',\bar{\xi})$ onto $T^*(\partial\Omega)$. The main step in the proof is to construct a suitable symbol $a(x,\xi)$ of order zero such that

$$\{p,a\}(x,\xi) = 0 \tag{4}$$

$$a(x',\xi) = r(x',\xi)p(x',\xi) + q(x',\xi') \tag{5}$$

and $a(x,\xi) = 1$ on a conic neighborhood of $(\bar{x}',\bar{\xi})$. Here the condition (5) is required to be satisfied for <u>some</u> choice of r and q.

If (x',ξ') is close to $(\bar{x}',\bar{\xi}')$ then there is either no root or else two roots $\xi_n^{\pm}$ of the equation $p(x',\xi',\xi_n) = 0$. These roots coincide when $\{p,f\} = 0$. Since q is independent of ξ_n it follows from (5) that one must have

$$a(x',\xi',\xi_n^+) = a(x',\xi',\xi_n^-). \tag{5'}$$

Now (4) means that a is constant along the bicharacteristics for P so (5') implies that a has to be constant along the successively reflected bicharacteristics. In particular it follows that a must be constant along the lifting $\tilde{\gamma}$ to $\rho \cap \mathcal{F}$ of the boundary bicharacteristic $\gamma = \gamma(\bar{x}', \bar{\xi}')$. If we denote by $\Gamma(\bar{x}', \bar{\xi}')$ the flow-out along H_p from $\tilde{\gamma}$, the results of [7] imply that a can be chosen to satisfy (4) and (5) and to have support in an arbitrary small conic neighborhood of $\Gamma(\bar{x}', \bar{\xi}')$. Since $x_n = 0$ is non-characteristic, we can in particular assume that a has the transmission property (see [1]).

Let r be a homogeneous symbol of order -2 which has the transmission property and satisfies (5) and denote by R the corresponding operator. Let furthermore Q and A be operators with symbols q and $a + a_{-1}$, where

$$a_{-1} = p_1 r + i\left[\frac{\partial}{\partial \xi_n} \frac{\partial}{\partial x_n} (pr - a) - \sum_{k=1}^{n} \frac{\partial p}{\partial \xi_k} \frac{\partial r}{\partial x_k}\right] \tag{6}$$

Here p_1 denotes the symbol of order one of P.

Assume now that $u \in H_s(\bar{\Omega})$ along γ and put

$$v = Au^o + R((Pu)^o - Pu^o),$$

where u^o denotes the extension of u which vanishes outside Ω. Since $\gamma \cap WF(u_{|\partial\Omega}) = \emptyset$, it follows from (4) - (6) that

$$Pv = A((Pu)^o) \quad \text{mod } H_s(\bar{\Omega}). \tag{7}$$

Moreover (5) gives that

$$v_{|\partial\Omega} = (R((Pu)^o) + Qu)|_{\partial\Omega} \text{ mod } H_{s+1/2}(\partial\Omega). \tag{8}$$

Because A and R have the transmission property, $Pu \in C^\infty(\bar{\Omega})$ and $\gamma \cap WF(u_{|\partial\Omega}) = \emptyset$, we get from (7) and (8) that

$$Pv \in H_s(\bar{\Omega}), \; v_{|\partial\Omega} \in H_{s+1/2}(\partial\Omega).$$

If now some point on $\gamma(\bar{x}', \bar{\xi}')$ is outside $WF(u;\bar{\Omega})$ and $a(x,\xi)$ has support sufficiently close to $\Gamma(\bar{x}', \bar{\xi}')$, it follows that we can assume that v has initialdata in C^∞. Note that the pseudo-convexity means that the bicharacteristics in $\Gamma(\bar{x}',\bar{\xi}')$ leave Ω. Well-known regularity theorems for the mixed problem now implies that $v \in H_{s+1/2}(\bar{\Omega})$.

Since $a(x,\xi) = q(x',\xi') = 1$ in a neighborhood of the zeros of $p(x,\xi)$, when (x',ξ') is close to $\gamma(\bar{x}',\bar{\xi}')$ and x_n is small, we can assume that $r(x,\xi)$ satisfies

(9) $$a(x,\xi) = r(x,\xi)p(x,\xi) + q(x',\xi')$$

there. Remember that (5) is only required to be satisfied when $x_n = 0$. From (9) it follows that

$$v = R((Pu)^o) + Qu \bmod H_{s+1}(\bar{\Omega}),$$

close to γ. Thus

$$Qu = H_{s+1/2}(\bar{\Omega}),$$

close to γ, and the proof is finished.

Remark : Just before this lecture we received a manuscript from G. Eskin [5] where a parametrix is constructed for certain mixed hyperbolic problems with gliding rays. For second order Dirichlet problems such a parametrix has also been constructed by one of the authors (Melrose). This construction as well as a fuller account of the argument described above will be published elsewhere.

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