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THE PROPAGATION OF WAVE MODES IN ULTRARELATIVISTIC MAGNETOACTIVE PLASMA—I

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SUMMARY

Some results on the propagation of undamped wave modes in fully relativistic, magnetoactive plasma are presented. We introduce a specific isotropic particle distribution function that is uniform in phase space up to a limiting particle energy and zero for higher energies. If only the electrons contribute significantly to the dispersion, it is shown that undamped 'quasi-ordinary' and 'quasi-extraordinary' modes can propagate when the wave frequency is less than the relativistic gyro frequency of the most energetic electrons. When this inequality is reversed, there are limited ranges of field strength and propagation direction for which one quasi-ordinary and two quasi-extraordinary modes can propagate. There are additional propagating modes if the protons contribute significantly to the dispersion.

I. INTRODUCTION

In this paper we discuss the propagation of electromagnetic waves in fully relativistic plasmas—ionized gases in which the majority of the particles of at least one species (usually the electrons) move with relativistic speeds. Radio and infrared observations of the Crab Nebula, pulsars and active galactic nuclei suggest that relativistic plasmas are not uncommon in astrophysical environments, and it is possible that cosmic rays of at least low and intermediate energy are accelerated under these conditions.

The calculations presented below are based on the linearized Boltzmann equation and are therefore specifically applicable to waves that only perturb the particle trajectories. This is in direct contrast to the cold (or nearly cold) theory of Akhiezer & Polovin (1956) that has been interestingly developed in recent years (see Arons & Max 1975 and references therein) in which the wave induced motions are taken to be dominant. The present investigation is complementary to these discussions and should be relevant when typical particle energies significantly exceed $eEc\omega^{-1}$ where E is the electric field strength of the wave and ω the angular frequency. However, as the field strength increases, non-linear effects can be expected to become increasingly important. Also we do not assume the presence of a dominant sub-relativistic plasma which exists in the interstellar medium.

In Section 2 we review wave propagation in unmagnetized plasmas. In Section 3 we consider the magnetoactive case, introducing a specific unperturbed distribution function. The investigation is confined to modes that are undamped by this distribution function, for reasons that are outlined in Paper II, where damping is discussed.

2. WAVE PROPAGATION IN AN UNMAGNETIZED ULTRARELATIVISTIC PLASMA

First, we review wave propagation in an unmagnetized ultrarelativistic plasma in the presence of a uniform neutralizing positive ion background which does not contribute to the conductivity tensor. This problem appears to have been first discussed by Silin (1960) and is considered further in Prentice (1967, 1968), Arons & Max (1974) and references therein.

The procedure followed is to linearize the collisionless, relativistic Boltzmann equation (e.g. Clemmow & Dougherty 1969) in the normal fashion and to search for plane wave modes with space and time variation $\propto \exp[i(\omega t - \mathbf{k}.\mathbf{r})]$. As a simplification, we introduce the ultrarelativistic approximation

$$\gamma = (1+\mathbf{u}.\mathbf{u})^{1/2} \sim u \gg 1,$$

with u the proper velocity. This can be shown to be adequate as long as the refractive index, $\mu = k/\omega \lesssim 1 - 1/2\gamma^2$ (setting $c = 1$). When this inequality is violated, the fact that the particle speeds are not quite c becomes important. When $\mu \gtrsim 1 + 1/2\gamma^2$, the waves are Landau damped.

If the particle distribution function is isotropic two modes can propagate in this approximation, a longitudinal, purely electrostatic mode and a transverse electromagnetic mode. The relevant dispersion relations are

$$X\phi_1(\mu) = \tfrac{3}{2}; \quad [\ln\langle\gamma^{-1}\rangle^{-1}]^{-1} \lesssim X \leqslant \tfrac{3}{2} \tag{1}$$

for the longitudinal mode, and

$$X\phi_2(\mu) = \tfrac{3}{2}; \quad 0 \leqslant X \leqslant \tfrac{3}{2} \tag{2}$$

for the transverse mode, with

$$X = \frac{Ne^2\langle\gamma^{-1}\rangle}{m\epsilon_0\omega^2}, \quad \langle\gamma^{-1}\rangle \ll 1$$

$$\phi_1(\mu) = \frac{3}{2\,\mu^3}\left[\ln\left(\frac{1+\mu}{1-\mu}\right) - 2\,\mu\right]$$

$$\phi_2(\mu) = \frac{3}{4\,\mu^3(1-\mu^2)}\left[2\,\mu - (1-\mu^2)\log_e\left(\frac{1+\mu}{1-\mu}\right)\right]. \tag{3}$$

(Silin 1960)

(Note that when $X = \frac{3}{2}$, $\mu = 0$ for both modes, consistent with the mode being oscillatory rather than wave-like.)

More complete dispersion relations can be obtained if we stipulate a specific unperturbed distribution function (usually a relativistic Maxwellian). (For further details, see the above references.) In Blandford (1973), the treatment in the ultrarelativistic approximation is discussed further and extended to include distribution functions that are axisymmetric with respect to the wave propagation direction. The condition that the ions (assumed to be protons) have an insignificant effect on the dispersion relations is

$$1 \ll \langle\gamma_e^{-1}\rangle^{-1} \ll 1840\langle\gamma_p^{-1}\rangle^{-1},$$

where the subscripts e, p refer to electrons and protons. If the second inequality is not satisfied and $\langle\gamma_p^{-1}\rangle \ll 1$, the influence of the protons can be included by replacing X by $X_e + X_p$. If neither of these inequalities is satisfied, the protons are non-relativistic and can be treated as a cold component of the plasma.

3. WAVE PROPAGATION IN A MAGNETOACTIVE, UNRELATIVISTIC PLASMA

We again use as a starting point, the linearized relativistic Boltzmann equation, this time including a uniform static magnetic field. The initial development has been given by several authors, particularly Montgomery & Tidman (1964) to which reference should be made for further details and discussions of technical difficulties involving the initial value problem, that are ignored in the following. We specialize immediately to the particular isotropic unperturbed electron distribution function,

$$f_0(u) = FH(U-u), \quad u \geqslant 0 \tag{4}$$

where $H(x)$ is the step function. We consider initially only the gyrations of electrons in a uniform stationary positive background. The extension to include ion motions is discussed later. The generalization to anisotropic distribution functions of the form $f_0(\mathbf{u}) = f_0(u_\perp, u_\parallel)$ where the subscripts $_\parallel$ and $_\perp$ refer to components resolved parallel and perpendicular to the magnetic field, is in principle straightforward (Skilling 1971), but leads to fairly complex expressions.

There are two advantages in using distribution function (4). First it is an adequate approximation to a relativistic Maxwellian distribution function, which might be expected under equilibrium conditions, and is considerably easier to manipulate. Secondly, under non-equilibrium but possibly stationary conditions, it is apparent, at least observationally, that a high energy power-law spectrum of particle energies develops. It is improbable that such a high energy development could be stable without the presence of low energy particles obeying an expression similar to (4). It turns out that under a wide range of conditions, we can have a situation where the low energy particles effectively determine the real part of the refractive index and the imaginary part, that is associated with damping, results from the high energy particles. If we make the low damping assumption, we can self-consistently decouple the two parts of the problem and then derive criteria for the applicability of the results. In this paper we therefore, use (4) to investigate dispersion and postpone a discussion of damping to Paper II.

The linearized Boltzmann equation is written

$$\frac{\partial f_1}{\partial t}+\mathbf{v}.\frac{\partial f_1}{\partial \mathbf{x}}-(\mathbf{v}\times\mathbf{\Omega}).\frac{\partial f_1}{\partial \mathbf{u}} = \frac{e\mathbf{E}}{m}.\frac{\partial f_0}{\partial \mathbf{u}}. \tag{5}$$

(Note that an additional term arises if f_0 is anisotropic.) $\mathbf{\Omega}$, the formal gyro frequency, like e is taken as a positive quantity. We define Cartesian axes x, y, z so that

$$\begin{aligned}
\mathbf{\Omega} &= (0, 0, \Omega) = (0, 0, eB/m)\\
\mathbf{k} &= (k_\perp, 0, k_\parallel) = \mu\omega(\sin\eta, 0, \cos\eta)\\
\mathbf{u} &= (u_\perp\cos\phi, u_\perp\sin\phi, u_\parallel) = u(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\\
\mathbf{v} &= \gamma^{-1}\mathbf{u},\ \Gamma = (1+U^2)^{1/2},\ V = U\Gamma^{-1}.
\end{aligned}$$

Without loss of generality, we restrict η to the range $0 \leqslant \eta \leqslant \pi/2$.

Again, we look for solutions of the form f_1, $E \propto \exp[i(\omega t-\mathbf{k}.\mathbf{x})]$ with ω, $\mathbf{r}$ assumed real. We obtain

$$\frac{\partial f_1}{\partial \phi}+Pf_1 = Q$$

where

$$P = i\gamma\Omega^{-1}(\omega - \mathbf{k}.\mathbf{v})$$

$$Q = \frac{e\gamma}{m}\mathbf{E}.\frac{\partial f_0}{\partial \mathbf{u}} = -\frac{e\gamma F}{m\Omega}\frac{(\mathbf{E}.\mathbf{u})}{u}\delta(u-U).$$

Hence

$$f_1 = -\int_\phi^\infty Q(\phi')\exp\left[\int_\phi^{\phi'} P(\phi'')\,d\phi''\right]d\phi',$$

where the limits of the ϕ' integration have been chosen to satisfy causality. Thus

$$f_1 = -\int_0^\infty Q(\phi+\alpha)\exp\left[\int_\phi^{\phi+\alpha} P(\phi')\,d\phi'\right]d\alpha$$

Now,

$$\mathbf{j} = -e\int f_1\mathbf{v}\,d^3u,$$

and using the Bessel function identities quoted in Montgomery & Tidman (1964) and Skilling (1971), we eventually obtain an expression for the conductivity tensor σ_{ij}.

$$\sigma_{ij} = i\pi\epsilon_0 X\zeta\omega\int_0^\pi d\theta\sin\theta\,\mathrm{cosec}\,\pi\xi\, M_{ij}$$

where

$$\begin{aligned}
X &= \frac{3}{2}\frac{N_e e^2}{\Gamma m_e\omega^2\epsilon_0} = \frac{\omega_p{}^2}{\omega^2}\langle\gamma^{-1}\rangle\\
\zeta &= \Gamma\omega\Omega^{-1} \quad (>0),\\
\xi &= \zeta(1-\mu V\cos\theta\cos\eta),\\
M_{11} &= \tfrac{1}{4}\sin^2\theta\,[J_{\xi-1}J_{-\xi+1}+J_{\xi+1}J_{-\xi-1}+2J_{\xi+1}J_{-\xi+1}]\\
M_{12} &= M_{21}{}^* = i/4\sin^2\theta[J_{\xi-1}J_{-\xi+1}-J_{\xi+1}J_{-\xi-1}]\\
M_{13} &= M_{31}{}^* = \tfrac{1}{2}\sin\theta\cos\theta[J_\xi J_{-\xi+1}-J_{\xi+1}J_{-\xi}]\\
M_{22} &= \tfrac{1}{4}\sin^2\theta[J_{\xi-1}J_{-\xi+1}+J_{\xi+1}J_{-\xi-1}-2J_{\xi+1}J_{-\xi+1}]\\
M_{23} &= M_{32}{}^* = \tfrac{1}{2}\sin\theta\cos\theta[J_\xi J_{-\xi+1}-J_{\xi+1}J_{-\xi}]\\
M_{33} &= -\cos^2\theta J_\xi J_{-\xi}
\end{aligned} \tag{6}$$

and the argument of the Bessel functions is

$$z = \Gamma k_\perp V_\perp\Omega^{-1} = \zeta\mu V\sin\theta\sin\eta.$$

Note that the ratio,

$$\frac{\text{Particle energy density}}{\text{Magnetic energy density}}$$

given by $X\zeta^2$.

We again make the ultrarelativistic approximation, $V = 1$ which will be adequate as long as

$$\left|\frac{1-n\zeta^{-1}}{\mu \cos \eta}\right| > 1, \quad n \text{ integral}, \tag{7}$$

i.e. as long as we are not too close to a resonance for any of the particles with energy Γm. A weaker condition is sometimes appropriate and this is discussed further in Paper II.

Now,

$$|(1-\mu^2)\,\delta_{ij} + \mu^2 \hat{k}_i \hat{k}_j - i\sigma_{ij}/\epsilon_0 \omega| = 0, \tag{8}$$

with $\hat{\mathbf{k}}$ a unit vector in the direction of $\mathbf{k}$. Thus the dispersion relations for undamped modes can be obtained by setting the determinant of the Hermitian matrix, D_{ij}, equal to zero, where

$$D_{xx} \equiv d_1 = 1-\mu^2 \cos^2 \eta$$
$$+\frac{1}{4}\int_0^\pi d\theta \sin^3 \theta \operatorname{cosec} \pi\xi (J_{\xi-1}J_{-\xi+1} + J_{\xi+1}J_{-\xi-1} + 2J_{\xi+1}J_{-\xi+1})$$

$$D_{xy} \equiv id_2 = \frac{i}{4}\pi X\zeta \int_0^\pi d\theta \sin^3 \theta \operatorname{cosec} \pi\xi (J_{\xi-1}J_{-\xi+1} - J_{\xi+1}J_{-\xi-1})$$

$$D_{xz} \equiv d_3 = \mu^2 \cos \eta \sin \eta$$
$$+\frac{1}{2}\pi X\zeta \int_0^\pi d\theta \sin^2 \theta \cos \theta \operatorname{cosec} \pi\xi (J_\xi J_{-\xi+1} - J_{\xi+1}J_{-\xi})$$

$$D_{yy} \equiv d_4 = 1-\mu^2$$
$$+\frac{1}{4}\pi X\zeta \int_0^\pi d\theta \sin^3 \theta \operatorname{cosec} \pi\xi (J_{\xi-1}J_{-\xi+1} + J_{\xi+1}J_{-\xi-1} - 2J_{\xi+1}J_{-\xi+1})$$

$$D_{yz} \equiv id_5 = \frac{-i\pi X\zeta}{2}\int_0^\pi d\theta \sin^2 \theta \cos \theta \operatorname{cosec} \pi\xi (J_\xi J_{-\xi+1} + J_{\xi+1}J_{-\xi})$$

$$D_{zz} \equiv d_6 = 1-\mu^2 \sin^2 \eta - \pi X\zeta \int_0^\pi d\theta \sin \theta \cos^2 \theta \operatorname{cosec} \pi\xi J_\xi J_{-\xi}$$

$$D_{ij} = D_{ji}^*.$$

The dispersion relation is thus given formally by

$$d_1 d_4 d_6 - 2d_2 d_3 d_5 - d_1 d_5^2 - d_4 d_3^2 - d_6 d_2^2 = 0. \tag{9}$$

This can be regarded as a cubic polynomial in X, in which the coefficients are functions of η, ζ, μ. Thus the solution of the dispersion relation is simply a matter of finding the positive real for roots physically meaningful values of the coefficients. The polarization of the corresponding modes can then be represented by the vector

$$\mathbf{E} = E(e_1, ie_2, e_3) \tag{10}$$

with

$$e^2 + e_2^2 + e_3^2 = 1$$

where

$$e_1 : e_2 : e_3 = (d_2 d_5 + d_3 d_4) : (d_1 d_5 + d_2 d_3) : (d_2^2 - d_1 d_4).$$

The dispersion relations and associated polarizations have been determined by solving these equations numerically and the results are presented below. However, we first discuss some limiting forms which are more amenable to direct calculation and provide a check on the accuracy of the numerical evaluation.

For large values of the static magnetic field strength, ζ will be small and so we can expand D_{ij} in powers of ζ, up to and including ζ^2. We find

$$\begin{aligned}
d_1 &= 1-a^2+\tfrac{2}{3}X\zeta^2[1-b^2\phi_3(a)] \\
d_2 &= \tfrac{2}{3}X\zeta \\
d_3 &= ab \\
d_4 &= 1-a^2-b^2+\tfrac{2}{3}X\zeta^2[1-b^2\phi_3(a)] \\
d_5 &= -\tfrac{2}{3}X\zeta ab\phi_4(a) \\
d_6 &= 1-b^2-\tfrac{2}{3}X[\phi_1(a)-\zeta^2b^2\phi_4(a)]
\end{aligned}$$

where

$$a = \mu\cos\eta, \quad b = \mu\sin\eta$$

$$\phi_3(a) = \frac{3}{8}\int_0^\pi \frac{d\theta\sin^5\theta}{(1-a\cos\theta)} = \frac{3}{8a^5}\left[\frac{10a^3}{3}-2a+(1-a^2)^2\ln\left(\frac{1+a}{1-a}\right)\right]$$

$$\phi_4(a) = \frac{3}{4}\int_0^\pi \frac{d\theta\sin^3\theta\cos^2\theta}{(1-a\cos\theta)} = \frac{3}{4a^5}\left[-\frac{4}{3}a^3+2a-(1-a^2)\ln\left(\frac{1+a}{1-a}\right)\right]$$

and $\phi_1(a)$ is given in (3). (Note that we are still employing the ultrarelativistic approximation.) For a general angle, η, the dispersion relation still requires numerical solution. However, a simple forms results when $\eta = 0, \pi/2$. There are of course corresponding simplifications in the general expression involving (8), but these do not yield dispersion relations sufficiently compact for analytical use.

With $\eta = 0$ (parallel propagation), the dispersion relation factorizes to give

$$\begin{aligned}
X\phi_1(\mu) &= 0 \\
1-\mu^2+\tfrac{2}{3}X\zeta^2 &= \pm\tfrac{2}{3}X\zeta.
\end{aligned}$$

The first relation describes a purely electrostatic mode propagating along $\mathbf{\Omega}$ in which the particle motions are totally uninfluenced by magnetic forces. It is in fact valid for all ζ subject to (7) and corresponds exactly to (1). The solutions of the second relation, $1-\mu^2 = \pm\frac{2}{3}X\zeta+o(\zeta^2)$, correspond to circularly polarized purely transverse waves. In the mode with phase velocity exceeding c (i.e. $\mu<1$), the sense of the circular polarization is the opposite of that of the gyration of the relativistic electrons in the magnetic field. The mode with phase velocity less than c can in fact propagate as condition (7) need only be satisfied for $n = \pm 1$ when $\eta = 0$.

With $\eta = \pi/2$ (perpendicular propagation),

$$X = \frac{3}{2}\frac{(1-\mu^2)}{(1-\zeta^2\mu^2/5)}$$

$$(1-\mu^2)+\tfrac{4}{3}X\zeta^2(1-\tfrac{7}{10}\mu^2) = \tfrac{4}{9}X^2\zeta^2.$$

The first relation describes an ordinary mode in which the electric field oscillates along $\mathbf{\Omega}$. As X rises from 0 to $\frac{3}{2}$, μ falls from 1 to 0. In much denser plasma, a mode with $\mu > \sqrt{5}\zeta^{-1}$ can in principle propagate. The second relation describes an extraordinary mode in which the electric field oscillates in the x–y plane. For $X \ll 1$, we see that the refractive index rises from unity with X, so that μ reaches a maximum value $\sim 1+9\zeta^2/200$ when $X \sim \frac{9}{20}$ and thereafter falls to 1 when $X \sim \frac{9}{10}$ and to zero when $X \sim 3/(2\zeta)$. Note that a mode can still propagate when $X \gg 1$.

Alternative limiting forms can be obtained by setting $\mu = 0$. These correspond to strict plasma oscillations and so the dispersion relations are independent of η. After performing elementary integrals we find

$$\begin{aligned} d_1 &= d_4 = 1+2X\zeta^2/\{3(1-\zeta^2)\} \\ d_2 &= 2X\zeta/\{3(1-\zeta^2)\} \\ d_3 &= d_5 = 0 \\ d_6 &= 1-2X/3. \end{aligned}$$

We thus isolate three normal modes as follows:

(i) A longitudinal oscillation parallel to the magnetic field with $X = \frac{3}{2}$.

(ii) A circularly polarized oscillation with **E** rotating in a plane perpendicular to $\mathbf{\Omega}$ in the opposite sense to the electrons with $X = 3(\zeta+1)/(2\zeta)$, $\zeta > 0$.

(iii) A similarly circularly polarized oscillation with **E** rotating in the same sense as the electrons with $X = 3(\zeta-1)/(2\zeta)$, $\zeta > 1$.

The complete numerical solutions of the dispersion relation (9) for values of $\zeta < 1$ are displayed in Figs 1 and 2 in the form of refractive index surfaces. These are essentially polar plots of $\mu(\eta)$. Each quadrant represents sections of a family of biaxial ellipsoids parametrized by different values of X for a specific value of ζ. The values of the refractive index for which damping by particles contained in (4) sets in is indicated. The associated polarizations can be seen from (10). **E** describes an ellipse, one axis of which lies along the y direction and the other in the x–z plane. The plane of the ellipse is termed the polarization plane. **H** is given by $\mu_0\omega^{-1}(\mathbf{k}\times\mathbf{E})$ and therefore describes an ellipse in a plane normal to **k**. The Poynting vector, represents the electromagnetic energy flux and is instantaneously directed along the projection of **k** perpendicular to **E**. However, in the presence of spatial dispersion, there is an additional energy flux carried by the particles. The net result (in a linearized theory) is that the total mean energy flux is directed along the group velocity, $\partial\omega/\partial\mathbf{k}$ (Bekefi 1966). In the present notation this is along a vector normal to refractive index surface and therefore at an angle

$$\eta - \tan^{-1}\left(\frac{\partial \log \mu}{\partial \eta}\right)_{X,\zeta} \quad \text{to } \mathbf{\Omega}.$$

The polarization can be described by two parameters; the axial ratio, $r = e_2(e_1^2+e_3^2)^{-1/2}$ and the inclination angle $\chi = \tan^{-1}(e_1/e_3)$. Thus $r = \pm 1$ corresponds to a circularly polarized wave, the helicity of which can be determined by inspection, $r = 0$ when the wave is linearly polarized with **E** lying in the x–z plane, and r becomes infinite when the wave is linearly polarized with **E** in the y direction, normal to **k**, $\pi/2-\chi$ is the angle between the normal to the polarization plane and the magnetic field. Therefore when $\chi \sim 0$ (or π) and $r \ll 1$, the electric field oscillates more or less parallel to $\mathbf{\Omega}$ and when $r \sim 1$, $\chi \sim \pi/2$, the wave is approximately circularly polarized with **E** rotating in a plane normal to $\mathbf{\Omega}$. It turns out that when $\zeta < 1$ there exists under most circumstances up to two propagating

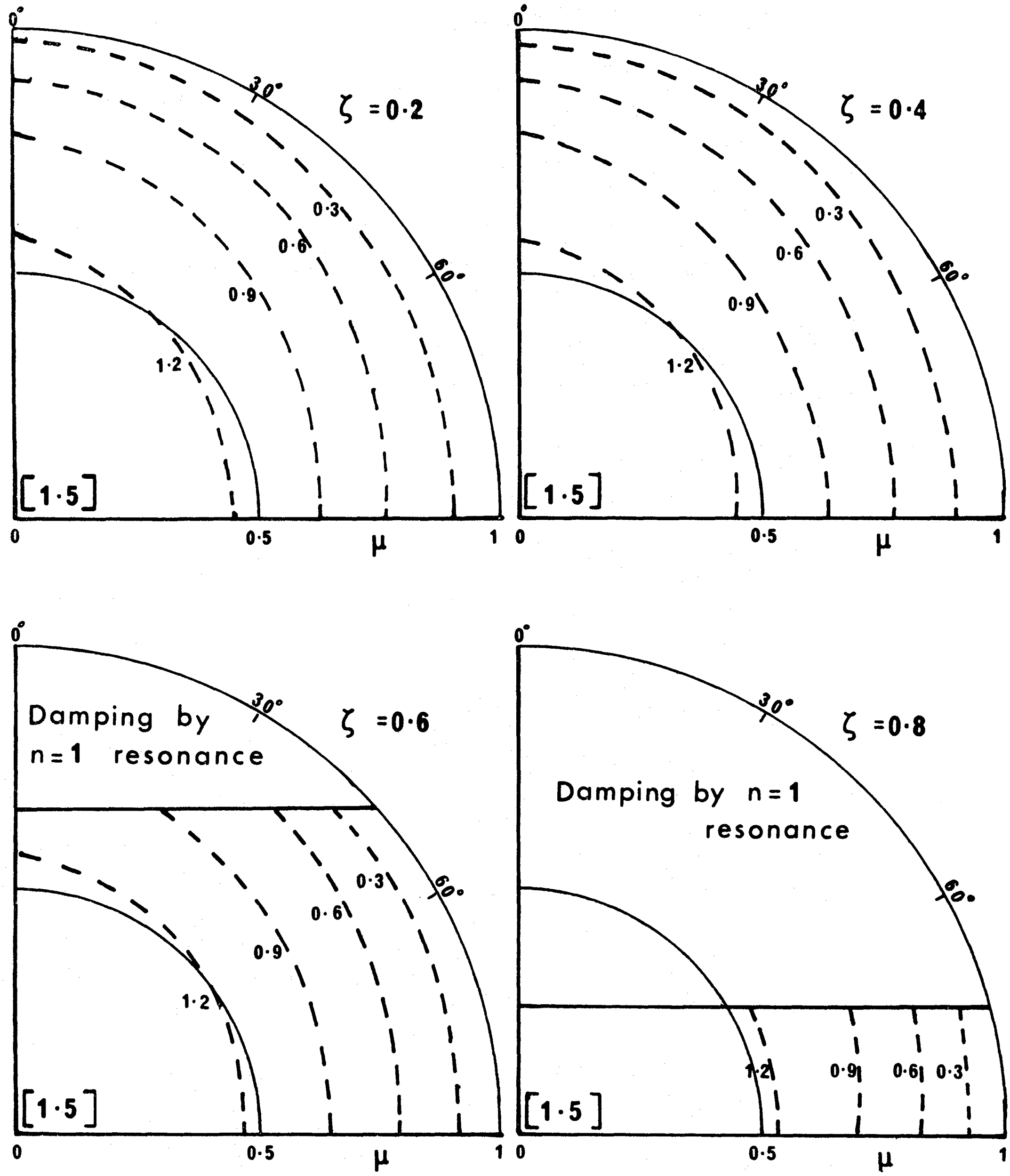

FIG. 1. *Sections of refractive index surfaces, $\mu(\eta)$, for the quasi-ordinary mode. Families of curves parametrized by X are plotted for different values of $\zeta < 1$. The value of X quoted in square brackets is the limiting value when $\mu = 0$. The group velocity is directed normal to the surface. Regions in which undamped modes cannot propagate are indicated.*

undamped modes having these two characteristic polarizations. They are termed the quasi-ordinary and quasi-extraordinary mode respectively.

From Fig. 1 and the limiting form when $\mu = 0$, it can be seen, that the quasi-ordinary wave propagates when $0 \leqslant X \leqslant \frac{3}{2}$. As X increases through this range, the refractive index decreases monotonically to zero for all angles η. We also find that

$$\left(\frac{\partial\mu}{\partial\eta}\right)_{X,\zeta} < 0$$

$$\left(\frac{\partial\mu}{\partial\zeta}\right)_{X,\eta} > 0, \quad 0 < X < \tfrac{3}{2}, \quad 0 < \eta < \pi/2.$$

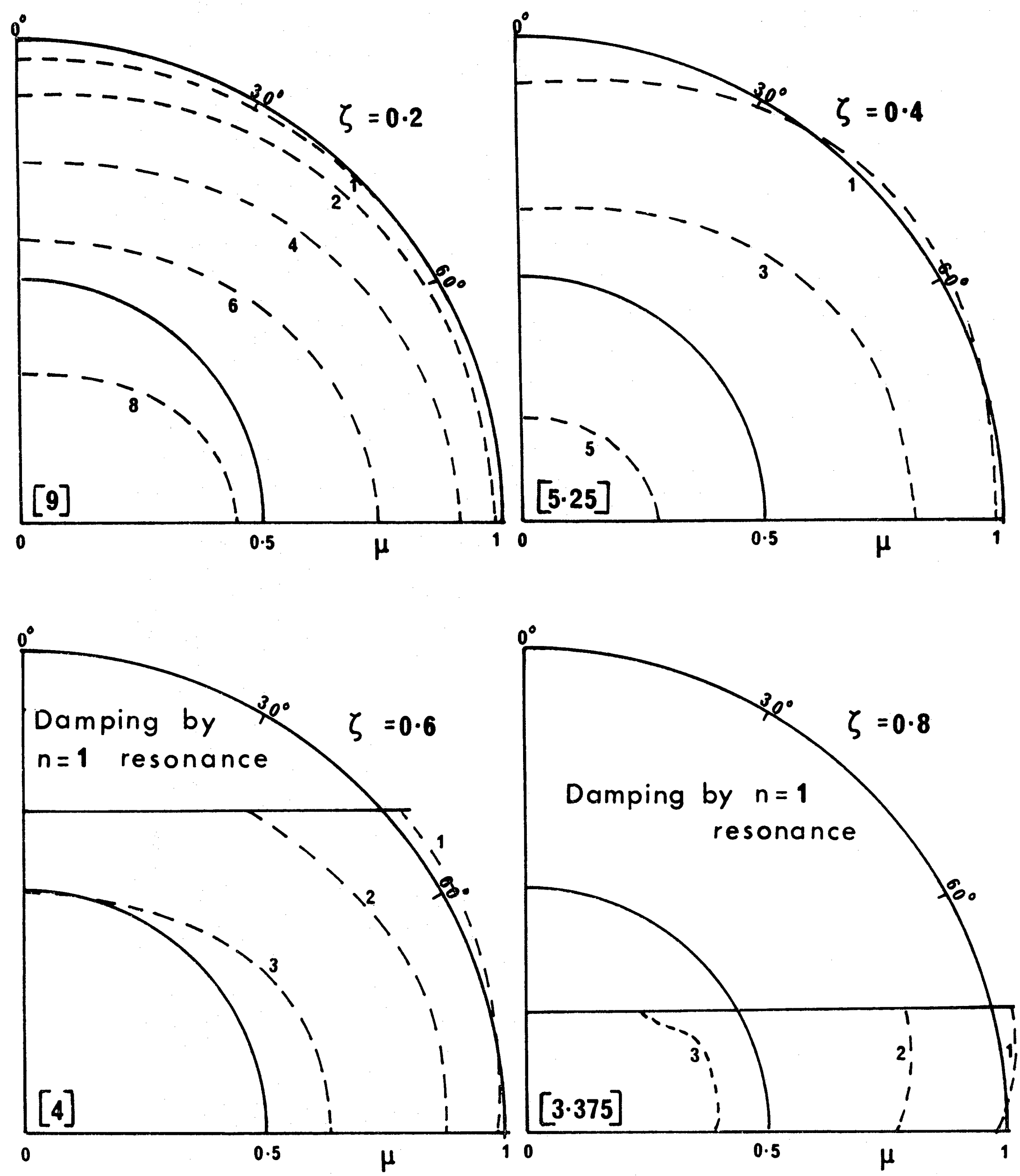

FIG. 2. *Sections of refractive index surfaces, $\mu(\eta)$, for the quasi-extraordinary mode.* (*See caption to Fig.* 1.)

The polarization also exhibits a smooth variation. When $\mu \ll 1$, the wave consists almost entirely of electric field oscillations along $\mathbf{\Omega}$ and as μ increases (X decreases) the polarization plane rotates about the y axis until it is normal to the $\mathbf{k}$ vector. The amount of elliptical polarization also increases, and there is an appreciable degree of circular polarization (measured by $2r/(1+r^2)$) when $1 > \mu \gtrsim 0{\cdot}8$. These two variations are plotted for specific intermediate values of ζ, η in Figs 3 and 4.

The behaviour of the quasi-extraordinary mode is shown in Fig. 2. As X increases from zero, μ rises from unity to reach a maximum only slightly greater than unity and then falls monotonically through 1, when $X \lesssim 1$, to zero when $X = 3(\zeta+1)/(2\zeta)$. Thus when $\frac{3}{2} \leqslant X \leqslant 3(\zeta+1)/(2\zeta)$ only the quasi-extraordinary mode can propagate. As long as $X \gtrsim \frac{3}{2}$, we find,

$$\left(\frac{\partial \mu}{\partial \eta}\right)_{X,\zeta} > 0$$

$$\left(\frac{\partial \mu}{\partial \zeta}\right)_{X,\eta} < 0, \quad \tfrac{3}{2} \lesssim X < 3(\zeta+1)/(2\zeta), \quad 0 < \eta < /\pi 2.$$

When $X < \frac{3}{2}$, the behaviour is more complex.

The basic polarization pattern when $0 < \mu \ll 1$, is for the electric vector to describe a circle (with the opposite helicity to the gyrating electrons) in a plane normal to $\mathbf{\Omega}$. As X decreases, the polarization plane rotates to become perpendicular to $\mathbf{k}$. The degree of circular polarization correspondingly falls. This is also displayed in Figs 3 and 4, for specific intermediate values of ζ, η.

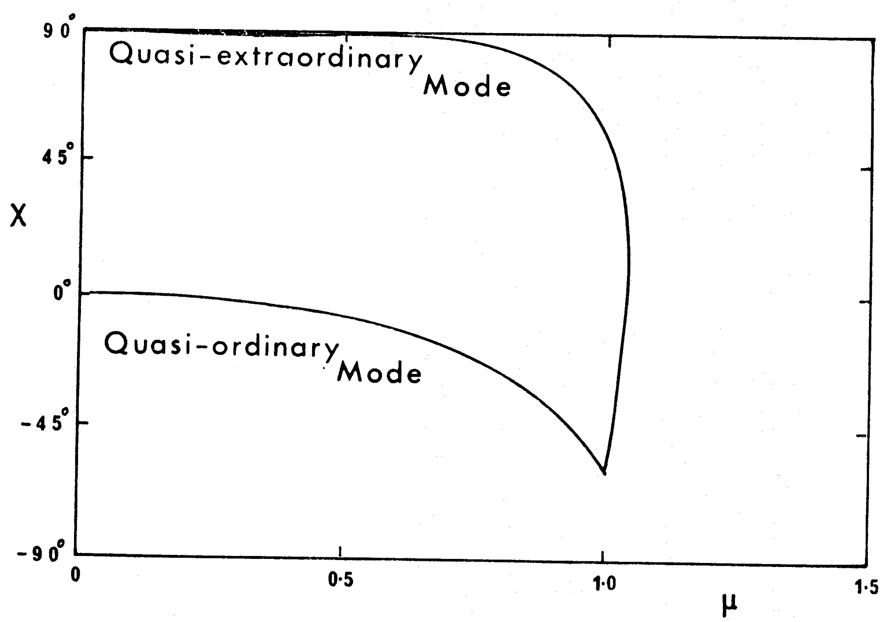

FIG. 3. *Polarization angle,* $\chi = \tan^{-1}(e_1/e_3)$, *for undamped modes when* $\zeta = 0{\cdot}4$, $\eta = 30°$.

The above discussion is restricted to undamped modes when $\zeta < 1$. When ζ is integral, a resonance is always possible (with $\theta = \pi/2$), but there are discrete ranges of ζ within which undamped propagation becomes possible for limited values of η. In particular from (7), it is apparent that if m is the greatest (positive) integer less than $\zeta + \frac{1}{2}$, we require $\mu \cos \eta < |1 - m/\zeta|$ for an undamped mode. If ζ lies between n and $n+1$, the maximum range of $\mu \cos \eta$ occurs when $\zeta = n + \frac{1}{2}$ and is then given by $(2n+1)^{-1}$. Thus as ζ increases, i.e. the static magnetic field decreases, propagating modes are restricted to smaller indices and directions almost perpendicular to the magnetic field with $\zeta > 1$. However, there are now three possible modes, a quasi-ordinary (O) and a quasi-extraordinary mode (E1) corresponding to the modes that propagate when $\zeta < 1$ and in addition there is a second circularly polarized, quasi-extraordinary (E2) mode consistent with the results in the limiting case when $\mu = 0$. The modes when $\zeta = 1{\cdot}5$ are displayed in Fig. 5 as a plot of $\mu(X)$ rather than as a refractive index surface. Note that X need not be a monotonic function of μ, and that there can be a critical value of X above or below which the undamped mode disappears for a given value of 7.

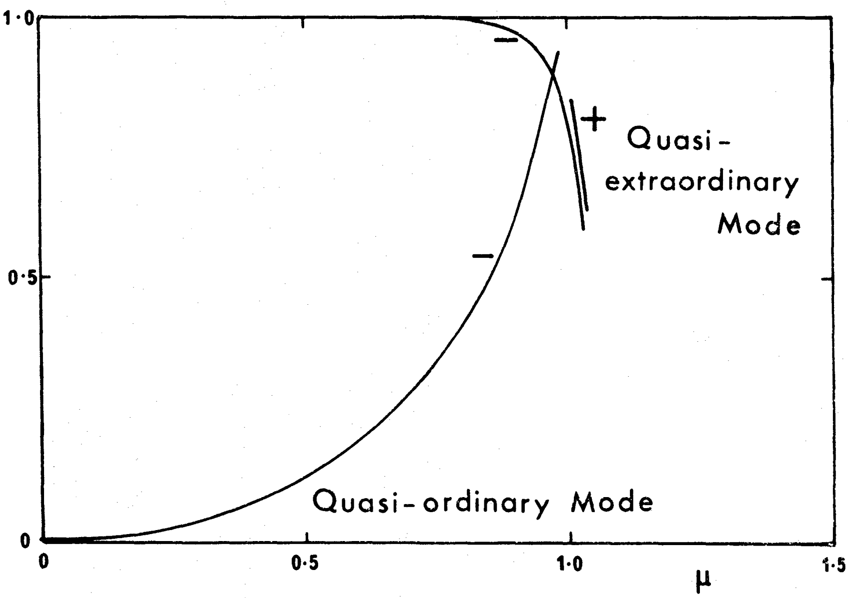

FIG. 4. *Degree of circular polarization* $|C| = 2e_2(e_1^2+e_3^2)^{-1/2}$ *for undamped modes when* $\zeta = 0{\cdot}4$, $\eta = 30°$. *The* $\pm$ *indicate the helicity of the mode relative to that of the gyrating electrons.* $|C|$ *for the extraordinary mode changes rapidly when* $\mu \sim 1$.

The analysis can be extended to include the motions of the protons. Formally this requires that we add a second conductivity tensor analogous to that describing the electron motions. If we assume that a distribution function similar in form to (4) is also applicable to the protons and that the ultrarelativistic approximation is

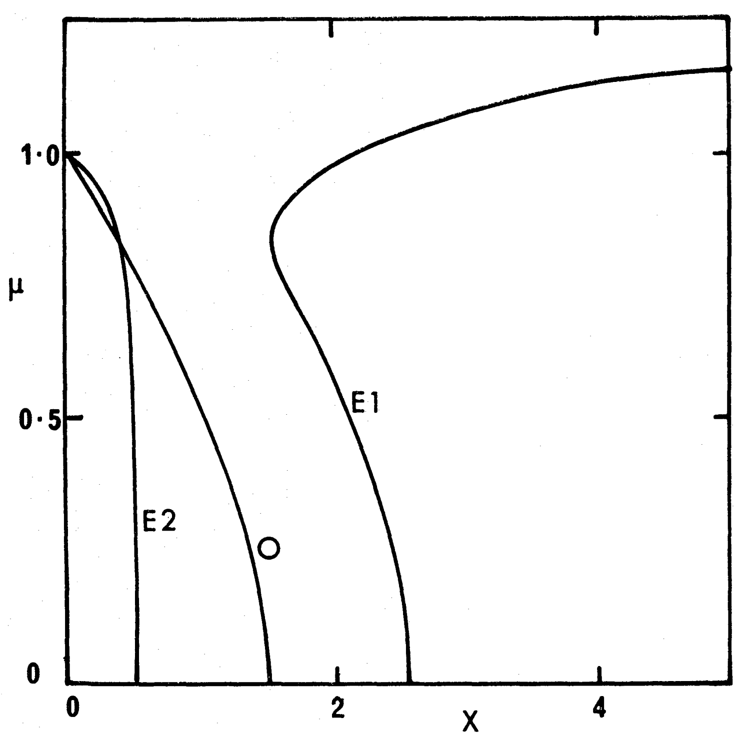

FIG. 5. *Dispersion relation for* $\zeta = 1{\cdot}5$. *The refractive indices are not very sensitive to* η *except that damping sets in when* $\mu \cos \eta > \frac{1}{3}$. *The three curves shown describe the quasi-ordinary* (O), *first quasi-extraordinary* (*E*1) *and second quasi-extraordinary* (*E*2) *modes.*

still valid, then the proton conductivity tensor is derived from the electron conductivity tensor by means of the substitutions:

$$X_e \to X_p = X_e \Gamma_e m_e / (\Gamma_p m_p)$$
$$\zeta_e \to -\zeta_p = -\zeta_e \Gamma_p m_p / (\Gamma_e m_e).$$

Inevitably, this greatly increases the number and complexity of the propagating modes. However, the preceding analysis should be approximately applicable if the protons are non-relativistic and ζ is not too low. As the previously described modes all have $\mu \lesssim 1$, we can treat the protons as a cold plasma and ignore spatial dispersion. As we show in Paper II, modes with μ significantly greater than unity will be damped by the electrons under realistic conditions. We thus have additional contributions, d_i^p, to the vector d_i in (9).

$$d_1^p = d_3^p = 2XR\zeta^2/3\{(1-R^2\zeta^2)\}$$
$$d_2^p = -2X\zeta/\{3(1-R^2\zeta^2)\}$$
$$d_4^p = d_5^p = 0$$
$$d_6^p = -2X/(3R)$$

where

$$R = (\Gamma_p m_p)/(\Gamma_e m_e).$$

Comparison with the contributions arising from the electrons shows that if $R \gg 1$ ($1 \ll \Gamma_e \ll 1840$) and $\zeta^2 \gg R^{-1}$, the proton contribution is certainly ignorable to first order. (When $0 < \mu \ll 1$, the weaker condition $1 \gg \zeta \gg R^{-1}$ is adequate.) When $\zeta \sim R^{-1}$, there is the possibility of a proton cyclotron resonance and the proton thermal motions must be included.

A further tractable case arises if there is equipartition between the electrons and the protons, so that $R = 1$, $\Gamma_p \gg 1$. In this case,

$$d_1 = 1-\mu^2\cos^2\eta + \frac{1}{2}\pi X\zeta \int_0^\pi d\theta \sin^3\theta \operatorname{cosec} \pi\xi (J_{\xi-1}J_{-\xi+1} + J_{\xi+1}J_{-\xi-1} + 2J_{\xi+1}J_{-\xi+1})$$

$$d_2 = d_5 = 0$$

$$d_3 = \mu^2 \cos\eta \sin\eta + \pi X\zeta \int_0^\pi d\theta \sin^2\theta \cos\theta \operatorname{cosec} \pi\xi (J_\xi J_{-\xi+1} - J_{\xi+1}J_{-\xi})$$

$$d_4 = 1-\mu^2 + \frac{1}{2}\pi X\zeta \int_0^\pi d\theta \sin^3\theta \operatorname{cosec} \pi\xi (J_{\xi-1}J_{-\xi+1} + J_{\xi+1}J_{-\xi-1} - 2J_{\xi+1}J_{-\xi+1})$$

$$d_6 = 1-\mu^2\sin^2\eta - 2\pi X\zeta \int_0^\pi d\theta \sin\theta \cos^2\theta \operatorname{cosec} \pi\xi J_\xi J_{-\xi}.$$

The quasi-ordinary mode remains qualitatively the same, except that the cut-off occurs when $X = \frac{3}{4}$ rather than $\frac{3}{2}$. However, the quasi-extraordinary modes are substantially altered and in particular are no longer circularly polarized. When $\zeta < 1$, $\mu \ll 1$ no extraordinary modes can propagate. When $(R\zeta) \lesssim 1$ low frequency Alfvén and magnetosonic modes can propagate. These have been considered by Barnes & Scargle (1973). For different values of R, the vector d_i can still be written in the form $\alpha_i + \beta_i X$ and the dispersion relation solved as above.

4. SUMMARY

The aim of this paper has been to elucidate the principal qualitative features of some of the propagating undamped modes in a fully relativistic plasma. In order to perform the calculations, it has been necessary to make several simplifying assumptions such as using the distribution function (4). The precise forms of the dispersion relations and the associated polarizations are therefore unimportant.

We have introduced the three parameters,

$$X = \frac{N_e e^2 \langle \gamma_e^{-1} \rangle}{m_e \epsilon_0 \omega^2}, \quad \zeta = \Gamma\omega|\Omega|^{-1}, \quad \eta = \cos^{-1}\left(\frac{\mathbf{k}.\boldsymbol{\Omega}}{k\Omega}\right)$$

of which ζ, η are irrelevant to an unmagnetized plasma. The refractive index, μ, can thus be regarded as being a many valued function of these parameters, each value corresponding to an individual mode.

In an isotropic unmagnetized plasma there are two principle propagating modes.

(i) *Longitudinal plasma oscillation*. In an isotropic, single component plasma, μ falls monotonically with X from ~ 1 for small X to zero at $X = \frac{3}{2}$. When μ is close to unity, the ultrarelativistic approximation breaks down and there may be significant damping. The point of onset of this damping depends critically on the high energy shape of the particle distribution function. When $X > \frac{3}{2}$ the mode becomes evanescent and decays rapidly.

(ii) *Transverse electromagnetic mode*. In this mode in an isotropic, single component plasma, $\mu(X)$ also falls monotonically from $\mu(0) = 1$ to $\mu(0) = \frac{3}{2}$ with evanescent solutions for larger values of X. However, in this case, there should be no damping when $\mu \sim 1$.

For both modes the inclusion of proton motions has the qualitative effect of reducing the values of X at which cut-offs occur. The effect of small anisotropies in the distribution function is complex but examinable when they are axisymmetric with respect to the wave direction. Non-axisymmetric distribution functions will break the degeneracy of the two transverse modes.

In a magnetized electron plasma for which the distribution function satisfies (4), there are three basic undamped modes.

(i) *Quasi-ordinary mode*. In this mode $\mu(X)$ falls monotonically from $\mu(0)$ to $\mu(\frac{3}{2}) = 0$ for all values of η and $\zeta < \frac{1}{2}$ and for decreasing ranges of $\mu \cos \eta$ locally maximized when $\zeta \sim n+\frac{1}{2}$ with n an integer $\geqslant 1$. When $\mu \ll 1$, the mode consists predominantly of an electric field oscillating parallel to the static magnetic field analogous to the longitudinal oscillation in an unmagnetized plasma. However, as X decreases the polarization changes continuously to that of a transverse circularly polarized electromagnetic wave. The inclusion of proton motions does not seriously affect the propagation of this mode except perhaps when $\mu \sim 1$.

(ii) *First quasi-extraordinary mode*. In this mode $\mu(X)$ falls from $\mu(0) = 1$ to $\mu[3(\zeta+1)/(2\zeta)] = 0$, $\zeta < 1$ although it is only monotonic when $X \gtrsim 1$. When $\mu \ll 1$, the wave is circularly polarized, the electric vector rotating perpendicular to the static field with sense opposite to that of the gyrating electrons. Again as X decreases, the polarization changes to that of a transverse mode. For $\zeta < 1$, this mode propagates over a significantly larger range of X than the ordinary mode. When $\zeta > 1$, there is again undamped propagation for decreasing ranges of $\mu \cos \eta$ when ζ lies within ranges centred on $n+\frac{1}{2}$. This mode is not seriously affected by the inclusion of non-relativistic protons provided that $1 \ll \Gamma_e \ll 1840$ and $\zeta^2 \gg \Gamma_e/1840$. (For $\mu \ll 1$, the weaker condition $\zeta > \Gamma_e/1840$ is sufficient.) However, when the protons are relativistic with $\Gamma_e m_e \sim \Gamma_p m_p$ this mode and the second extraordinary mode will merge to form a linearly polarized mode.

(iii) *Second quasi-extraordinary mode*. This mode only exists when $\zeta > 1$ and is similar in polarization to the first quasi-extraordinary mode except that the sense is opposite (i.e. the same as the gyrating electrons). $\mu(X)$ falls monotonically from $\mu(0) = 1$ to $\mu[3(\zeta-1)/(2\zeta)] = 0$. Similar remarks to those appropriate for the first quasi-extraordinary mode apply concerning damping and the influence of the protons.

One general trend displayed by all three modes in a magnetized plasma is that in general modes becomes less likely to be dampled as the angle η between $\mathbf{k}$ and $\mathbf{\Omega}$ increases for $\zeta \geqslant \frac{1}{2}$. From Figs 2 and 3 it is seen that the refractive index surfaces are approximately spherical and so the direction of the total energy flux makes a small angle to $\mathbf{k}$. Thus if $\zeta \gtrsim \frac{1}{2}$, wave energy transport is easier perpendicular rather than parallel to the magnetic field direction. This may counterbalance the opposite effect displayed by the particles.

In this paper we have confined our attention to strictly undamped modes. In Paper II we discuss the mechanism of damping both by particles contained within the distribution function (4) and by an additional high energy development to the distribution function.

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