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The pulsations of ZZ Ceti stars – III. The driving mechanism

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SUMMARY

The outer layers of the variable white dwarfs are in a state of partial ionization. During the pulsation cycle the base of the ionization zone is strongly heated by the radiative layers below, in phase with the pressure perturbation. If this excess heat is not quickly lost at the surface, then the driving effect is strong. The surface flux perturbation tends to be small and delayed in phase because the surface flux is remarkably insensitive to temperature changes in the deeper layers of the ionization zone. This insensitivity is closely associated with the well known inward divergence of the solutions for the equilibrium thermal structure in the convective layers. The mechanism which excites the oscillations could be called 'convective driving'.

1 INTRODUCTION

The mechanism which drives the pulsations of the variable white dwarfs has not been identified because of difficulties associated with convection. In the linear non-adiabatic calculations of Dolez & Vauclair (1981), Winget *et al.* (1982), Starrfield *et al.* (1982), Koester *et al.* (1985), Cox *et al.* (1987) and Bradley *et al.* (1988) convection was included in the unperturbed models but the perturbation to the convective flux was neglected. This approximation has been successfully used in the study of variable stars where convection carries a small fraction of the total flux, but in the white dwarf variables practically all the flux is carried by convection. I argued in Paper I (Brickhill 1983) that, in this case, the calculations would almost inevitably indicate strong driving if the convective flux perturbation was neglected. More recently a careful examination of the solutions to the linear non-adiabatic equations by Pesnell (1987) has shown that the instabilities which have been found are associated with the assumption that the convective flux does not vary. Pesnell called this effect 'convective blocking'.

A second problem with these calculations is the neglect of the momentum exchange (turbulent viscosity) associated with the convective motions. The results indicate marked changes in phase and amplitude of the horizontal displacement through the convective layers. It is easy to show that this is inconsistent with the neglect of the turbulent viscosity (Brickhill 1990, Paper II).

In view of these problems I introduced an alternative approach in Paper I. This approach is based on the assumptions (a) that the horizontal motions (and hence also the relative pressure perturbation) change little with depth through the ionization zone, (b) that the convection adjusts instantly to the changing conditions and (c) that the flux changes in the layers below the ionization zone are quasi-adiabatic. Assumption (a) was discussed in Paper II. It was shown there that the turbulent viscosity associated with the convection is easily sufficient to ensure that the horizontal displacement does not change significantly with depth. Assumption (b) is justified by the very short time-scale of the convective motions. Assumption (c) will be discussed below. With these simplifications there is little difficulty in following the flux and temperature changes in the ionization zone numerically, and thus obtaining the surface flux perturbation and the driving rate in a way that includes a consistent treatment of convection.

The constancy of the relative pressure perturbation in the driving zone also greatly simplifies the discussion of the driving mechanism. We are able to make use of expressions for the driving rate in the ionization zone and for the damping rate in the layers below which were given by Eddington (1941). These expressions show that if the local luminosity perturbation at the lower boundary of the ionization zone is in phase with the pressure perturbation (which is the case if the flux changes there are quasi-adiabatic) and if the luminosity perturbation at the surface is relatively small or substantially delayed in phase, then the driving rate is greater than the damping rate in the radiative layers below. In order to explain the instability one need only explain why the surface flux perturbation is small or delayed in phase.

The explanation given here is based on the assumption that the outer part of the ionization zone is close to thermal equilibrium. Therefore, one can use the equilibrium models to show that the surface flux perturbation is very small unless the temperature changes in the deeper parts of the ionization zone are large – much larger than the temperature changes associated with purely adiabatic effects. It follows that the surface flux perturbation tends to be small and delayed in phase unless the period is long in comparison with the thermal time-scale of the ionization zone.

The problem of including convection in the work integral is discussed in Section 2. It is argued there that the uncertainties associated with the convection are not significantly reduced by separating the energy and pressure associated with the turbulence from the thermal terms. In this case the work integral retains its usual form, except that an additional term which describes the turbulent viscous dissipation must be included. In Section 3 we use linear, quasi-adiabatic calculations to show that Eddington's expression for the damping rate in the radiative layers below the ionization zone is reasonably accurate. The explanation of the driving mechanism given above is expanded in Section 4 and in Sections 5 and 6 we describe and discuss the detailed model calculations. The results confirm the semi-quantitative discussion in Section 4. The assumption that the flux changes at the base of the ionization zone are quasi-adiabatic is discussed in Section 7. It is argued that the maximum of ~ 1000 s in the periods of ZZ Ceti stars is associated with the failure of this approximation in modes of high radial order. The effect of neglecting the perturbation to the convective flux is also examined in Section 7. The result is surprising. Driving is generally much stronger when this perturbation is included. It was this that suggested the name 'convective driving'.

2 TURBULENT CONVECTION AND THE WORK INTEGRAL

For a fluid element of mass m, internal energy U and volume V the equation of conservation of thermal energy can be written

$$dU/dt + V \operatorname{div} \boldsymbol{F} = m\varepsilon - P\,dV/dt, \tag{1}$$

where P is the pressure, ε the rate of viscous dissipation of kinetic energy into heat and $\boldsymbol{F}$ the radiative flux. (Nuclear energy generation has been neglected.) This expression gives the rate of conversion of mechanical energy into heat. For an element in the (non-turbulent) radiative layers it gives the rate of dissipation of oscillation energy within the element but the situation is greatly complicated in the convective layers by the rapid exchanges between the internal energy and the turbulence (e.g. Unno *et al.* 1979). It is desirable to avoid the complications if possible, not only to retain clarity, but also because existing theories of convection are based on a simplified view of the gas dynamics and it is not clear that there is much to be gained.

In the convective layers we regard the gas as consisting of elements enclosed by surfaces which move with the velocity of the mean motion, with horizontal dimensions much larger than the scale of the convective motions but much smaller than the horizontal wavelength and vertical dimensions smaller than both the scaleheight and the vertical wavelength (see Paper II). Then the equation describing the conservation of thermal and turbulent kinetic energy can be written in a form identical to equation (1) except that now U represents the total (internal plus turbulent) energy, $V \operatorname{div} \boldsymbol{F}$ represents the total energy flux across the enclosing surface, $m\varepsilon$ represents the rate of dissipation of kinetic energy of the mean motion into turbulence and heat due to turbulent viscosity and P represents the sum of the mean thermodynamic pressure and the turbulent pressure (see Paper II, equation 9).

We are interested in the work term $P\,dV/dt$ in equation (1). In the absence of turbulence the volume change is given by the equation of state:

$$dV/dt = -(dP/dt)\,V/(\Gamma_1 P) + (m\varepsilon - V \operatorname{div} \boldsymbol{F})(\Gamma_3 - 1)/(\Gamma_1 P) \tag{2}$$

(Cox & Guili 1968). Substitution from equation (2) for dV/dt and integration over a complete cycle gives the required expression for the work integral, but is it reasonable to use equation (2) when the turbulent energy and pressure are included in U and P? In fact, if the turbulence is isotropic and the gammas are close to 5/3 then the exchange of energy between the turbulence and the internal energy at fixed total energy does not affect the total pressure and equation (2) is accurate ($P = [2/3]\,U/V$). The situation is more complicated if the turbulence is not isotropic but in any case the turbulent energy and pressure are very small in the white dwarf envelopes, where the convective velocities are an order of magnitude smaller than the sound speed. It seems unlikely that the uncertainties associated with the approximate models for the convection are significantly increased by assuming that the turbulent energy and pressure are included with the thermal terms. This approach is consistent with the usual practice in calculating non-pulsating DA envelopes and there is clearly an advantage in retaining the simplicity of the standard mixing length theory at this stage. In what follows we use P and $\boldsymbol{F}$ to denote the total pressure and energy flux in the presence of convection. Moreover we neglect the effect of ordinary viscosity on the oscillation and use ε to denote the rate of turbulent viscous dissipation of oscillation energy.

Integrating the right-hand side of equation (1) over a complete cycle, assuming a steady state where the fluid element returns to the same volume at the end of the cycle, so that $P\,dV/dt$ can be replaced by $\delta P\,dV/dt$, where δP represents the Lagrangian perturbation about the mean, and using equation (2) for dV/dt, we obtain

$$d = (1/\mathscr{P}) \int_0^{\mathscr{P}} [m\varepsilon + \nabla_{\rm ad}(\delta P/P)(V \operatorname{div} \boldsymbol{F} - m\varepsilon)]\,dt, \tag{3}$$

where d is the average rate of dissipation of oscillation energy into turbulent and thermal energy, $\mathscr{P}$ is the period and $\nabla_{\rm ad}$ is the adiabatic temperature gradient, $(\Gamma_3 - 1)/\Gamma_1$. The first term in the integrand represents the viscous dissipation. The term in $V \operatorname{div} \boldsymbol{F}$ is just the usual expression (e.g. Unno *et al.* 1979, equation 19.5) except that the relative temperature perturbation, $\delta T/T$, is replaced by its adiabatic part $\nabla_{\rm ad}\,\delta P/P$. The non-adiabatic part contributes nothing on average because it is out of phase with $V \operatorname{div} \boldsymbol{F}$. The term in $\delta P\varepsilon$ is negligible. ε is already second order in the perturbations and the time varying part is in phase with the velocity, out of phase δP. (The term in dP/dt has vanished because it is out of phase with δP.)

From now on we will use P and $\boldsymbol{F}$ for example, to denote the mean, unperturbed quantities and, avoiding the usual representation in terms of complex quantities because the second order terms are given by the product of either the real or the imaginary parts, we assume that the perturbations can be expressed in the form

$$\delta f(r)\, Y_\ell^m \,\mathrm{e}^{-im\phi} \cos[\sigma t + m\phi + \psi_{\mathrm{f}}(r)]$$

where Y_ℓ^m is the spherical harmonic describing the surface distortion, normalized so that

$$\iint Y_\ell^m Y_\ell^{m*} \sin\theta\, d\theta\, d\phi = 4\pi.$$

Furthermore, we will neglect horizontal heat exchange (because the horizontal wavelength is orders of magnitude greater than the vertical in the DA variables) and evaluate equation (3) correct to second order in the perturbations. Integrating over all the elements between the radii r and $r+dr$, we obtain

$$d(r, dr) = d_{\mathrm{t}}(r, dr) + (dr/\mathscr{P}) \int_0^{\mathscr{P}} [\nabla_{\mathrm{ad}} p \cos(\sigma t + m\phi + \psi_{\mathrm{p}})$$

$$\times\, d\delta L \cos(\sigma t + m\phi + \psi_{\mathrm{L}})/dr]\, dt, \qquad (4)$$

where d_{t} is the turbulent viscous dissipation associated with the first term in equation (3), p is the amplitude of the relative pressure perturbation, $\delta P(r)/P$, δL is the amplitude of the local luminosity perturbation, $4\pi r^2 \delta F(r) + 2L\xi_{\mathrm{r}}/r$, and ξ_{r} is the vertical displacement. The term in $\delta\nabla_{\mathrm{ad}}\delta P dL/dr$ has been neglected because $\delta\nabla_{\mathrm{ad}}$ is negligible except in the ionization zone where dL/dr is negligible.

3 THE DAMPING IN THE RADIATIVE LAYERS

The Lagrangian perturbation to the radiative luminosity can be written

$$\delta L/L = (4-\kappa_{\mathrm{T}})\,\delta T/T - (1+\kappa_\rho)\,\delta\rho/\rho + 2\xi_{\mathrm{r}}/r$$

$$+\,\delta(d \ln T/dr)/(d \ln T/dr), \qquad (5)$$

where κ_{T} and κ_ρ represent the relative rates of change of opacity with $\ln T$ and $\ln\rho$ at constant density and temperature respectively. In the radiative layers the term in $\delta T/T$ dominates, because $(4-\kappa_{\mathrm{T}}) \sim 8$. The third term is negligible in the variable white dwarfs. For the present we neglect the last term as well and assume that the temperature changes are approximately adiabatic, writing

$$\delta L/L = [(4-\kappa_{\mathrm{T}})\nabla_{\mathrm{ad}} - (1+\kappa_\rho)/\Gamma_1]\, p. \qquad (6)$$

As $\nabla_{\mathrm{ad}} \sim 0.4$, $\kappa_\rho \sim 1$ and $\Gamma_1 \sim 5/3$ we have $\delta L/L \sim 2p$.

The viscous dissipation given by the first term in equation (4) can be neglected in the radiative layers. We will assume that the luminosity perturbation is given by the quasi-adiabatic relation (6). As the presssure and luminosity perturbations are in phase, the second term in equation (4) becomes $(\nabla_{\mathrm{ad}} p\, d\delta L/dr)\, dr/2$. Substituting from equation (6) for δL and integrating from the centre out to the base of the ionization zone, assuming that ∇_{ad} and the terms in square brackets in equation (6) are constant in the layers where the amplitude is large, we obtain

$$d_{\mathrm{R}} = \nabla_{\mathrm{ad}} p\, \delta L_l/4, \qquad (7)$$

where ∇_{ad}, p and δL_l refer to the values at the lower boundary of the ionization zone. This is the expression for the damping rate given by Eddington.

Table 1. Quasi-adiabatic damping rates of the *g*-modes in a ZZ Ceti model.

k	$\ell = 1$ $\mathscr{P}$ (s)	$d_{\mathrm{R}}/\nabla_{\mathrm{ad}} p \delta L$	$\ell = 3$ $\mathscr{P}$ (s)	$d_{\mathrm{R}}/\nabla_{\mathrm{ad}} p \delta L$
1	269	0.32	111	0.33
2	318	0.31	131	0.31
3	372	0.30	158	0.31
4	414	0.29	192	0.28
5	478	0.27	216	0.27
6	541	0.27	235	0.27
7	592	0.27	263	0.27
8	653	0.26	292	0.26
9	725	0.26	313	0.26
10	777	0.26	333	0.26
11	824	0.26	351	0.26
12	879	0.26	379	0.26
13	936	0.26	401	0.26

Notes: The model is for a 0.6 $M_\odot$ star, $T_{\mathrm{e}} = 10\,750$ K. The thermal time-scale of the convective zone is typical of models in the instability strip, $\tau_{\mathrm{t}} \sim 100$ s.

The damping rate has been calculated in the quasi-adiabatic approximation for a ZZ Ceti model using the expressions (22.12), (22.13) given by Unno *et al.* (1979). The model and the adiabatic eigenfunctions were obtained using methods described previously (Brickhill 1975). Apart from the neglect of the viscous dissipation and the quasi-adiabatic approximation, the expressions do not include any of the assumptions used to obtain equation (7). The calculations show that the rate of radiative dissipation given by equation (7) is accurate to within about 30 per cent (Table 1). The relatively large difference between the value given by equation (7) and the calculated value for the low order modes is due to a slow change in the expression in square brackets $[(4-\kappa_{\mathrm{T}})\nabla_{\mathrm{ad}} - (1+\kappa_\rho)/\Gamma_1]$ in the layers below the ionization zone. In the modes of high radial order, where the amplitude decreases rapidly below the ionization zone, this expression *is* effectively constant in the region where the damping is marked.

The horizontal heat losses and the term in ξ_{r}/r are entirely negligible as expected. The term in the temperature gradient does not have a significant effect either, but this conclusion is not expected to persist for modes of radial order much higher than those included in the calculations. Comparison with the calculations shows that equation (7) underestimates the damping in the deeper layers of the envelope where the solutions for p and δL oscillate. It is accurate for the modes considered because the amplitude in these layers is very small and the dissipation is completely dominated by the layers above the outermost node in p. However, as the radial order increases the outermost node in p approaches the base of the ionization zone. For modes of roughly twice the period of the modes of radial order 13 in Table 1 the damping rate is expected to be much greater than the value given by equation (7) (see Paper II, section 3).

A complete investigation of the role of the damping in the radiative layers should be based on the non-adiabatic calculations. Our main intention in this section has been to show that Eddington's result provides an excellent starting point. Nonetheless it may be worth noting that the results obtained here appear to contradict the results of Winget *et al.* (1982) and Bradley *et al.* (1988). These authors found that the blue edge of the instability strip is sensitive to the mass of the hydrogen layer, M_H. For models where $M_H > 10^{-8}\ M_*$ the blue edge was found to shift to a much lower temperature, *apparently* because of increased damping in the deeper layers. In the model used here $M_H \sim 10^{-7}\ M_*$ but the contribution of layers below $\sim 10^{-8}\ M_*$ to the damping is very small. The situation would be clarified if the damping rates in the radiative layers were calculated in units of $\nabla_{ad} p\, \delta L_l$. It would then be obvious whether the change in composition causes increased damping. The sensitivity of the blue edge to M_H was not confirmed by Cox *et al.* (1987).

4 THE DRIVING MECHANISM

We refer to the entire region where the opacity is significantly affected by hydrogen ionization as the ionization zone. It extends from the photosphere to a point roughly one scale-height below the base of the convectively unstable zone. This whole region is expected to be in turbulent motion because of convective overshooting and the horizontal motions and the relative pressure changes are therefore assumed to be effectively independent of depth. The turbulent viscous dissipation given by the first term in equation (4) may not be negligible in the ionization zone but this term has been discussed in Paper II. We will not consider it again here.

The second term in equation (4) gives the rate of damping due to heat losses. As the phase of the pressure perturbation does not change significantly in these layers this term can be simplified by writing $\psi_L = \psi_p + \theta$ and using the usual identity for $\cos(x+\theta)$. The second term becomes $\nabla_{ad} p(d\delta L \cos\theta/dr)\,dr/2$. In the outer part of the ionization zone, where the heat capacity is small, $d\delta L \cos\theta/dr \approx 0$ so these layers do not contribute to the damping or driving. We will refer to the remainder of the ionization zone as the driving zone. Integrating equation (4) from the base of the ionization zone out to the surface, neglecting the first term on the right-hand side, we obtain an expression for the damping rate in the ionization zone

$$d_I = \tfrac{1}{2}\int_r \nabla_{ad} p(d\delta L \cos\theta/dr)\,dr$$

$$= \tfrac{1}{2}\langle\nabla_{ad}\rangle p(\delta L_u \cos\theta_u - \delta L_l \cos\theta_l), \tag{8}$$

where the angular brackets denote the weighted mean, θ_u and θ_l the phase of the luminosity perturbation relative to the pressure perturbation at the upper and lower boundaries of the driving zone and δL_u and δL_l denote the amplitudes.

The region where ∇_{ad} is substantially depressed by ionization is largely confined to the outer part of the ionization zone, above the driving zone, so $\langle\nabla_{ad}\rangle$ is approximately equal to the value of ∇_{ad} in the radiative layers (0.4). For the present we will assume that the luminosity perturbation at the lower boundary of the ionization zone is in phase with the pressure perturbation so $\cos\theta_l \approx 1$. This is the case if the luminosity changes there are quasi-adiabatic (equation 6). The base of the ionization zone is heated from below in phase with the pressure perturbation. Provided that this heat is not lost rapidly at the upper surface (i.e. if $\delta L_u \cos\theta_u \ll \delta L_l$) equation (8) gives

$$d_I \approx -\tfrac{1}{2}\nabla_{ad} p\, \delta L_l. \tag{9}$$

The driving in the ionization zone $(-d_I)$ given by equation

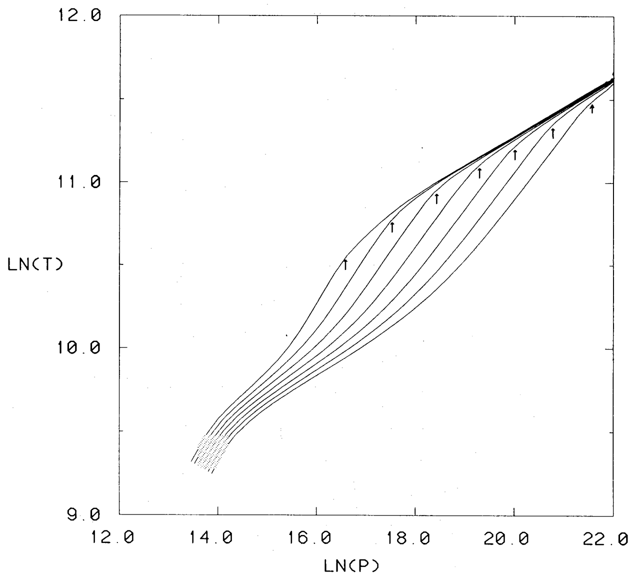

Figure 1. The temperature structure of ZZ Ceti models in thermal equilibrium; in cgs units $\log g = 8$ and T_e decreases from 11 125 K to 10 375 K in steps of 125 K (from the top). The construction of the models will be discussed in Paper IV. The curves start at an optical depth of 2/3. The top of the convective zone lies above this point. The arrows indicate the lower boundary. The natural logarithm has been used because we require the relative temperature change, $\Delta T/T = \Delta \ln T$, from one model to the next (see text).

(9) is twice the damping rate in the radiative layers below (equation 7). Therefore, if we can explain why the luminosity perturbation at the top of the driving zone is small or substantially delayed in phase, then we will have explained why the pulsations tend to grow to a finite amplitude.

To see why the luminosity perturbation at the top of the driving zone tends to be small we make use of the definition of this point, namely that δL is independent of depth in the layers above. Thus these layers are effectively in thermal equilibrium. The outer layers of the pulsating white dwarfs are also very close to hydrostatic equilibrium and the perturbation to the gravitational acceleration and to the radius are negligibly small (see Paper II, equation 13). The instantaneous temperature–pressure relationship must therefore be similar to that of an equilibrium model of luminosity $L + \delta L_u$.

Fig. 1 shows the relationship between temperature and pressure in seven ZZ Ceti model envelopes in thermal equilibrium which differ in luminosity by 5 per cent from one model to the next. The region which is effectively in thermal equilibrium corresponds to the outer part of the convective zone, where the solutions in Fig. 1 converge towards the surface. It is evident that the δL_u will be small unless the temperature changes on a given isobar near the top of the driving zone are large. If ΔT represents such a temperature change then it can be seen from the figure that

$$\delta L_u/L \sim 0.25\,\Delta T/T \tag{10}$$

The temperature change ΔT, measured by a sensor which moves vertically relative to the Lagrangian elements so as to remain at constant pressure during the pulsation cycle, is given by

$$\Delta T/T=(\nabla_{\rm ad}-\nabla)\,\delta P/P+\delta T_{\rm n}/T, \tag{11}$$

where $\nabla_{\rm ad}\,\delta P/P$ is the adiabatic part of the Lagrangian temperature perturbation and $\delta T_{\rm n}/T$ is the non-adiabatic part. The additional term $\nabla\,\delta P/P$, where ∇ is the temperature gradient ($d\ln P/d\ln T$) in the star, is due to the vertical displacement of the sensor relative to the Lagrangian elements. The superadiabatic temperature gradient tends to be very small in the ionization zone, $\nabla_{\rm ad}-\nabla\sim-10^{-1}$, so if the non-adiabatic part of the temperature perturbation was negligible then the temperature change ΔT would be very small. This adiabatic temperature change would cause a luminosity change in the outer layers of only $\sim 10^{-2}\ \delta P/P$ (equations 10 and 11). Moreover, $\Delta T/T$ would have the opposite sign to $\delta P/P$: the luminosity would *decrease* slightly during the phase of compression. On the other hand, if $\delta L_{\rm u}$ is comparable with the luminosity perturbation at the base of the ionization zone, then the non-adiabatic part of the temperature perturbation must be at least an order of magnitude greater than the adiabatic part.

Provided the period of the pulsation is short in comparison with the thermal time-scale of the driving zone, the non-adiabatic part of the temperature perturbation will be small and delayed in phase relative to the pressure perturbation because it takes time to heat the ionization zone. Thus we have explained why the pulsations tend to grow, but we can take this discussion one stage further without detailed calculations by defining what we mean by 'short in comparison with the thermal time-scale'. The thermal time-scale of the layers above the interior mass point M can be defined as

$$\tau_{\rm t}=(1/L)\int_M^{M_*}(C_{\rm p}T)\,dM, \tag{12}$$

where M_* is the mass of the star and $C_{\rm p}$ is the specific heat at constant pressure. But this gives the time for the luminosity to drain the thermal and potential energy of the outer layers. For our purpose the thermal time-scale D introduced in Paper I is more relevant.

$$D=(1/\Delta L)\int_M^{M_*}(C_{\rm p}\Delta T)\,dM, \tag{13}$$

where ΔT is the difference in temperature (at fixed M) between two model envelopes which differ in luminosity by a small amount ΔL. The integral in equation (13) is the amount of energy which must be provided before the thermal structure of the layers above M can approach that of a model of luminosity $L+\Delta L$ and D is a measure of the time taken before a sudden change in the luminosity at M results in a comparable change in the surface luminosity. In this respect it is a much better measure than $\tau_{\rm t}$. In the convective zone $D\sim 4\tau_{\rm t}$ because of the large temperature differences illustrated in Fig. 1. In the radiative layers where a small difference in temperature corresponds to a large difference in luminosity, $D\sim\tau_{\rm t}/10$.

During the pulsation the luminosity at the base of the ionization zone changes in phase with the pressure but the change in the thermal structure due to purely adiabatic effects is very small (equation 11). If the period is much smaller than D the non-adiabatic heating from below does not change the thermal structure significantly and the amplitude of the surface flux change is very small. If $\mathscr{P}$ is much greater than D then the surface luminosity changes follow the luminosity changes at the base of the ionization zone closely and there is no driving. The case of greatest interest arises when $D\sim\frac{1}{2}\mathscr{P}$. Then there is just enough time to heat the ionization zone substantially before the phase changes. The surface luminosity perturbation is expected to be large (but not as large as δL_l) and substantially delayed in phase so the driving remains strong.

5 THE MODEL

Consider a representative volume of gas reaching from the base of the ionization zone to the surface. The sides of the volume are vertical and the horizontal dimensions are much smaller than the horizontal wavelength of the pulsations but much larger than the scale of the convective motions. Suppose that the volume is divided into ~ 20 'elements' by horizontal surfaces and that the surfaces enclosing the volume and separating the elements move with the velocity of the mean motion. It follows from the definition of the mean motion that the mass of the elements is constant. The problem is to calculate the driving rate and the surface flux perturbation. Our calculations are based on the following assumptions.

(i) The horizontal velocity is independent of depth in the ionization zone (so the sides of the volume remain vertical). This assumption seems very secure. It is easy to show that if the time and length scales of the convective motions are anything like the values indicated by the mixing length theory then the momentum exchange associated with these motions is much larger than the total horizontal force unless the horizontal velocity is effectively independent of depth (Paper II).

(ii) The horizontal motions remain very much larger than the vertical motions when turbulent momentum exchange is included. This assumption was shown to be self consistent in Paper II. It follows that the pressure perturbation is caused by the horizontal motions; it is inversely related to the change in the horizontal area of the elements ($\delta P/P=-\delta A/A$). The vertical acceleration and viscous force have a negligible effect (so the vertical stratification is always very close to hydrostatic equilibrium) and the terms due to changes in the gravitational acceleration and the radius are also negligible (Paper II, equation 13).

(iii) The pressure and the horizontal area vary sinusoidally in time. The period is supposed to correspond to one of the *g*-mode periods but for the purpose of the calculation the period and the amplitude are free parameters.

(iv) The equation of state relating the mean pressure, density, temperature and internal energy in the volume elements is given by the usual formulae, including the effects of ionization and radiation but not the turbulence (see Section 2).

(v) Horizontal heat transfer can be neglected. Convection adjusts instantaneously to the changing conditions and the flux can be calculated using the standard mixing length formalism. This is justified by the fact that the convective time-scale is two orders of magnitude smaller than the pulsation time-scale. The vertical component of the radiative flux is given by the diffusion approximation.

(vi) Near the surface the instantaneous thermal structure is similar to that of a DA model in thermal and hydrostatic equilibrium. This assumption follows directly from the condition of hydrostatic equilibrium [assumption (ii)], from the fact that δL is constant in the outermost layers and from the short time-scale of the convective motions. It is thus possible to establish a relationship between the surface luminosity perturbation and the temperature perturbation at the base of the photosphere, where the diffusion approximation of radiative transfer is appropriate. This procedure avoids over-simplifying the radiative transfer problem in the photosphere (e.g. Gabriel 1989).

(vii) Below the base of the ionization zone the relative pressure perturbation does not change substantially in one scaleheight and the temperature perturbation is approximately adiabatic. In this case the flux perturbation at the base of the ionization zone is given by equation (19) of Paper I. Equation (6) above gives almost the same boundary condition. This boundary condition is expected to fail when the radial order is high. This failure will be discussed in Section 7.

The change in the pressure and the horizontal area are constrained by assumptions (i)–(iii). In view of assumptions (iv)–(vii), there is no difficulty in calculating the rate of energy exchange across the surfaces enclosing the volume elements and rate of mechanical work, and so obtaining the thermal structure for the next time-step. The average rate of work in the model is calculated by summing PdV/dt over a complete cycle. Provided the amplitude is small the driving rate is proportional to the square of the amplitude and the rate of work in the entire ionization zone follows immediately. Further details of the methods used are given in Paper I. The calculations presented here are similar except that the equation of state includes the effects of ionization and the upper boundary of the model is situated immediately below the photosphere in the version used here.

6 THE RESULTS

The calculations are based on six 'models' corresponding to different values of the surface gravity, the effective temperature and the mixing length, with variations which differ in the number of elements or in the position of the upper boundary. After an initial crude guess at the thermal structure, each model was allowed to evolve towards thermal equilibrium keeping the pressure and the flux at the base of the model constant. The resulting thermal structure was compared with the equilibrium models, which will be discussed in Paper IV, and found to be in close agreement (provided the number of elements was not too small). The flux at the base of the model could also be changed discontinuously by a small amount and then kept constant while the model relaxed. The results confirm that the time delay D given by equation (13) provides a reliable measure of the time taken for a thermal disturbance to propagate through the ionization zone. The rapid propagation through the radiative layers and the marked delay in the convective zone are illustrated in Paper I, fig. 4.

The pressure in each element, the horizontal area and the flux at the base of the model were then varied sinusoidally as described above. In the calculations used here $p \leq 0.01$ and

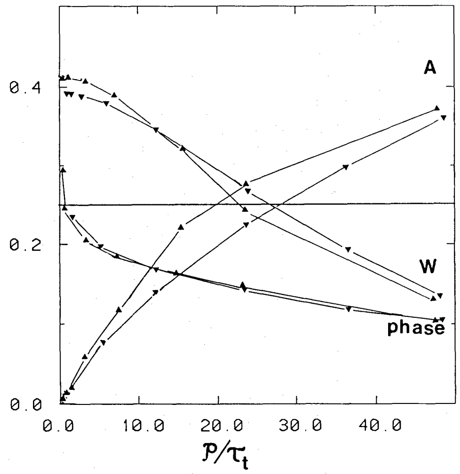

Figure 2. The fractional change in the surface flux when the amplitude of the relative pressure perturbation is 0.1, $A=(F_{\max}-F_{\min})/(10pF)$, the driving rate $W=-d_l/(\nabla_{\rm ad}p\,\delta L_l)$, and the phase lag, phase $=\theta_u/2\pi$. The horizontal line gives the approximate rate of damping in the radiative layers, $d_R/(\nabla_{\rm ad}p\,\delta L_l)$ (see Section 3). For the models shown the mixing length (ML) is equal to the pressure scaleheight (H) and $\log(g)=8$. The upward pointing arrows indicate the results obtained for the $T_e=10\,500$ K model, the downward pointing arrows correspond to the $T_e=11\,000$ K model. The results obtained for the four models corresponding to $\log(g)=7.5$, 8.5 and $ML=H$, $2H$ are similar. The differences are of the same size as the difference between the two models shown.

the results are independent of amplitude. The validity of assumption (vi) (the thermal structure in the surface layers is similar to that in the equilibrium models) was checked by changing the position of the upper boundary. The results were not affected provided that the upper boundary was chosen more than ~2 scaleheights above the base of the convective zone (see Fig. 1). This result also justifies the use of equilibrium models to explain the driving mechanism in Section 4.

The results are summarized in Fig. 2. The driving rate, the amplitude of the surface flux changes and the phase difference between the surface flux perturbation and the pressure perturbation are shown as a function of $\mathcal{P}/\tau_t$ where τ_t is the thermal time-scale of the convective zone (equation 12). The results obtained with the earlier (Paper I) version of the program were similar to those presented here except that the driving rate was slightly greater. The reason is that $\nabla_{\rm ad}$ had the value 0.4 throughout the driving zone in the older calculations. In the more realistic recent version $\nabla_{\rm ad}$ *is* significantly depressed and the driving rate is reduced (see equations 8 and 9).

The results are consistent with the discussion in Section 4. The delay time D is $\sim 4\tau_t$. When the period is much smaller than D the amplitude of the surface flux change is very small and the driving rate is close to the value given by equation (9). When $\mathcal{P}\sim 2D$, the changes in surface flux are marked but smaller than the perturbation at the base of the ionization zone and the driving is still strong. When $\mathcal{P}>6D$, surface flux change is large but the driving in the ionization zone is not sufficient to compensate for the damping in the layers below.

The most obvious weakness of the non-adiabatic calculations which have been used up till now to investigate the

ZZ Ceti instability is the neglect of the perturbation to the convective flux. To see how this affects the results we write the perturbation to the luminosity in the form

$$\delta L/L = \delta L_c/L + (\delta L_R/L_R)(L_R/L), \tag{14}$$

where the symbols have their usual meanings. The factor L_R/L is $\sim 10^{-2}$ in the convective zone so it appears that δL_u must inevitably be small if δL_c is neglected. Equation (8) shows that this would lead to strong driving and Pesnell (1987) has shown that the driving which has been found in the calculations is associated with the rapid outward decrease in L_R/L at the base of the convective zone.

In order to obtain some idea of how changes in the convective flux affect the driving, Cox *et al.* (1987) assumed that $\delta L_c/L_c = \delta L_R/L_R$. In the model which they tested all the modes which had been found to be unstable previously were stabilized. This result suggested that 'convective blocking' must operate, at least to some extent.

The effect of the neglect of the convective flux perturbation can also be tested by setting $\delta L_c = 0$ in the calculations described above. The result is surprising. The amplitude of the surface flux changes *increased* and the driving rate *decreased* relative to the models where the convective flux was allowed to vary. The period where the driving rate first exceeds the damping rate in the radiative layers changed from $\sim 25\tau_t$ to $\sim 4\tau_t$. The reason is clear from the detailed output. In the lower part of the convective zone the fraction of the flux carried by convection decreases rapidly as the temperature and the total flux increase: δL_c has the opposite sign to δL_R and it is nearly equal in magnitude so δL is much smaller than δL_R! Convective blocking is *more* effective when the perturbation to the convective flux is included.

7 DISCUSSION

The weakest aspect of our model is the assumption that the flux changes at the base of the ionization zone are quasi-adiabatic (vii). This approximation is satisfactory if the relative pressure perturbation does not change rapidly with depth in the layers below the ionization zone because the luminosity perturbation is also fairly constant in this case and the non-adiabatic heating is small (see equation 6). Therefore one expects the approximation to be satisfactory for modes of low radial order. The calculations described in Section 3 show that the quasi-adiabatic approximation is self consistent out to the base of the ionization zone for the modes considered there. However, this does not ensure that our assumption is reliable because large non-adiabatic changes in the ionization zone may cause rapid changes in the pressure perturbation in the layers immediately below, where the turbulent viscosity vanishes.

On the basis of the discussion in Paper II, section 6, we estimate that the fractional change in the relative pressure perturbation at the base of the ionization zone is of the order $\mathscr{L}^2/\sigma^2$ where $\mathscr{L}$ is the value of Lambs acoustic cut-off frequency at the base of the ionization zone and σ is the oscillation frequency. When $\mathscr{L}^2/\sigma^2 \gtrsim 1$, the phase of the flux perturbation at the base of the ionization zone is expected to be ahead of the phase of the pressure perturbation and driving may be drastically reduced (equation 8). The maximum of ~ 1000 s in the periods of ZZ Ceti stars indicates that modes where $\mathscr{L}^2/\sigma^2 > 1$ are not observed. I argued in Paper I that turbulent viscous damping is responsible but it appears here that the driving mechanism itself may fail. Linear non-adiabatic calculations which include a consistent treatment of convection (e.g. Xiong 1989) are probably needed to investigate this point but we note that a strong tendency towards stability is evident in the results of Dolez & Vauclair (1981) for modes where $\mathscr{L}^2/\sigma^2 > 1$.

8 CONCLUSION

The simple model described here is intended as a complement to the linear non-adiabatic (LNA) approach. In order to compare results it will be necessary to determine the driving rate in the ionization zone and the damping rate in the layers below from the LNA solutions. There are advantages in discussing the driving mechanism (for example) in terms of the driving rate. The LNA results are nearly always expressed in terms of the growth rate which is given by the difference between the driving and damping rates, divided by the energy of oscillation (e.g. Unno *et al.* 1979, equation 19.2). These three quantities are, to a large extent, associated with different parts of the star and the underlying physics *must* be clarified if they are considered separately. Our results indicate that changes in the driving and damping mechanisms are immediately evident if the driving and damping rates are expressed in units of $\nabla_{ad} p\, \delta L_l$. These rates would appear meaningless if they were given in erg s^{-1} for example.

Our conclusion that the driving is strong and that the surface flux perturbation is small, when $\mathscr{P}/\tau_t < 1$ is not expected to change when non-adiabatic effects in the layers below the ionization zone are fully taken into account because the non-adiabatic temperature changes in and below the ionization zone are small in this case. The conclusion that the modes are stable when $\mathscr{P}/\tau_t \gtrsim 25$ is also expected to be correct. The ionization zone is not able to 'trap' excess heat for a significant fraction of a cycle in this case and it is very difficult to see how the failure of the quasi-adiabatic approximation in the layers below the ionization zone could substantially *increase* the driving. On the other hand the conclusion that modes where $\mathscr{P}/\tau_t \sim 10$ are unstable is sensitive to the assumption that the flux changes at the base of the ionization zone are quasi-adiabatic. (These modes are of the greatest interest because they cause marked brightness changes at the surface and detection is likely.) A substantial phase shift between the pressure perturbation and the flux perturbation would stabilize these modes. We have suggested that the cut-off at ~ 1000 s in the periods of ZZ Ceti stars is caused by such a phase shift, but LNA calculations which take account of convection are needed to investigate this point.

The amplitude of the relative Lagrangian pressure perturbation in the ionization zone is a natural measure of amplitude in ZZ Ceti stars. If p were to approach one the pressure would approach zero at minimum pressure and the amplitude of the displacement would be comparable with the wavelength: the non-linear damping would be huge. In real stars p must be much smaller than one. Therefore Fig. 2 shows that there is a limited interval in frequency for each star where large amplitude brightness changes can be observed. If $P/\tau_t \lesssim 1$, the driving is strong but the amplitude of the surface flux changes is so small that detection is unlikely. If $P/\tau_t \gtrsim 25$ the driving rate is smaller than the radiative damping rate and the mode is stable. As the pulsation periods of ZZ Ceti stars lie between 100 and 1000 s this result provides the basis for a determination of the position of both the red and blue edges of the theoretical instability strip from the DA models (Paper IV).

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