E C O N O M I C S   B U L L E T I N

The pure Nash equilibrium property and the quasi−acyclic condition

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Abstract

This paper presents a sufficient condition for the quasi−acyclic condition. A game is quasi−acyclic if from any strategy profile, there exists a finite sequence of strict best replies that ends in a pure strategy Nash equilibrium. The best−reply dynamics must converge to a pure strategy Nash equilibrium in any quasi−acyclic game. A game has the pure Nash equilibrium property (PNEP) if there is a pure strategy Nash equilibrium in any game constructed by restricting the set of strategies to a subset of the set of strategies in the original game. Any finite, ordinal potential game and any finite, supermodular game have the PNEP. We show that any finite, two−player game with the PNEP is quasi−acyclic.

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1. Introduction

Consider the following best-reply dynamics: at each period, only one player gets the opportunity to revise his strategy. The probability of getting the revision opportunity is equal among all players. If the strategy of a player with the revision opportunity is one of his best replies to the current strategy profile, then he continues to take the current strategy. Otherwise, he switches to any of his best replies to the current strategy profile with equal probability.

Friedman and Mezzetti (2001) introduce the concept of *quasi-acyclic* games. A game is quasi-acyclic if from any strategy profile, there exists a finite sequence of strict best replies that ends in a pure strategy Nash equilibrium. The best-reply dynamics must converge to a pure strategy Nash equilibrium in any quasi-acyclic game. In this paper, we present a sufficient condition for the quasi-acyclic condition.

Friedman and Mezzetti (2001) also introduce the concept of the *weak finite improvement property* (*week FIP*) and show that the better-reply dynamics must converge to a pure strategy Nash equilibrium in any game with the weak FIP. The class of games with the weak FIP includes the classes of quasi-acyclic games, finite, supermodular games, and generic, continuous, two-player, quasi-concave games. The class of quasi-acyclic games includes the class of games with the *finite improvement property* (*FIP*) defined in Monderer and Shapley (1996), which, in turn, includes the classes of finite, dominance solvable games, and finite, ordinal potential games.

2. Notation and Definitions

Let $g = \langle N, A, \pi \rangle$ be a game, where N is the set of players, A_i is the finite set of all strategies of player i, $A = \prod_{i \in N} A_i$ is the set of all strategy profiles, π_i is player i's payoff function over A, and $\pi = (\pi_i)_{i \in N}$. For any strategy profile $a = (a_i)_{i \in N}$, $a_{-i} = (a_j)_{j \neq i}$ denotes a strategy profile of all players other than player i. For any nonempty subset B_i of A_i for each player i and $B = \prod_{i \in N} B_i$, let $g|_B = \langle N, B, \pi|_B \rangle$ be a *subgame* of g, where $\pi|_B$ is the restriction of π on B.[1]

Player i's strategy a'_i is said to be one of *player i's strict best replies* to a if a'_i is one of player i's best replies to a_{-i} and $\pi_i(a'_i, a_{-i}) > \pi_i(a)$.[2] a' is said to be a (single-player) *strict best reply* to a if there exists a player i such that $a'_{-i} = a_{-i}$ and a'_i is one of player i's strict best replies to a. A strategy

[1] This paper uses the term "subgame" in the sense of Shapley (1964). This usage is different from the usual interpretation in extensive form games.

[2] Some authors use the term "player i's strict best reply" for player i's unique best reply. Note that their usage is different from ours.

profile a^* is a pure strategy Nash equilibrium if and only if there is no strict best reply to a^*. A game g is *quasi-acyclic* if, for any strategy profile a, there exists a finite sequence of strict best replies that starts in a and ends in a pure strategy Nash equilibrium of g.

A strategy profile a' is said to be a *single-player improvement* over a if there exists a player i such that $a'_{-i} = a_{-i}$ and $\pi_i(a') > \pi_i(a)$. A game g has the *weak FIP* if, for any strategy profile $a \in A$, there exists a finite sequence of single-player improvements that starts in a and ends in a pure strategy Nash equilibrium of g. Any quasi-acyclic game has the weak FIP.

Definition 1 A game $g = \langle N, A, \pi \rangle$ has the *pure Nash equilibrium property* (*PNEP*) if $g|_B$ has a pure strategy Nash equilibrium for any product subset B of A.

Any finite, ordinal potential game has the PNEP since it has a pure strategy Nash equilibrium and its subgame is also a finite, ordinal potential game. Any finite, supermodular game has the PNEP since it has a pure strategy Nash equilibrium and its subgame is also a finite, supermodular game.[3] Shapley (1964) shows that any finite, two-player, zero-sum game has the PNEP if and only if any of its 2×2 subgames has a pure strategy Nash equilibrium.

3. Results

Proposition 1 *Any finite, two-player game with the PNEP is quasi-acyclic.*

Proof. Let $g = \langle \{1, 2\}, A, \pi \rangle$ be a finite, two-player game with the PNEP. We will prove this by mathematical induction on the cardinality of A.

If A is a singleton $\{a\}$, then a is a sequence that ends in a Nash equilibrium.

For any $k \geqslant 2$, suppose that Lemma 2 holds for $|A| < k$. Consider a game with $|A| = k$. Without loss of generality, $|A_1| \geqslant 2$. Also, since g has the PNEP, g has a Nash equilibrium a^*. Let $B_1 := A_1 \setminus \{a_1^*\}$ and $B := B_1 \times A_2$. We have $|B| < k$. Since g has the PNEP, $g|_B$ also has the PNEP. Hence, by the induction hypothesis, $g|_B$ has a sequence of strict best replies from any strategy profile that ends in a Nash equilibrium.

[3] As in Friedman and Mezzetti (2001), when we consider a supermodular game, we assume that A_i is a totally ordered set. If g is a supermodular game where A_i is a lattice, then its subgame $g|_B$ is not necessarily a supermodular game unless B_i is a sublattice of A_i.

Given any strategy profile $a^0 \in A$, we will construct a sequence of strict best replies in g from a^0 that ends in a Nash equilibrium.

Case 1. Suppose that $a^0 \in B$. Then $g|_B$ has a sequence $(a^0, a^1, \dots, a^t)$ of strict best replies, where a^t is a Nash equilibrium of $g|_B$. It suffices to consider the case where the sequence is not a sequence of strict best replies in the original game g that ends in a Nash equilibrium of g. Then there exists t^* such that (i) $0 \leqslant t^* \leqslant t$, (ii) for any s such that $0 \leqslant s < t^*$, there exists player i such that i has a strict best reply to a^s, a_i^{s+1} is one of player i's strict best replies to a^s in g, and $a_{-i}^{s+1} = a_{-i}^s$, (iii) a_1^* is one of player 1's strict best replies to a^{t^*} in g.

Let $b^0 = (a_1^*, a_2^{t^*})$. It follows from Conditions (ii) and (iii) that if b^0 is a Nash equilibrium of g, then $(a^0, a^1, \dots, a^{t^*}, b^0)$ is a sequence of strict best replies in g that ends in a Nash equilibrium of g and if b^0 is not a Nash equilibrium of g, then $(a^0, a^1, \dots, a^{t^*}, b^0, a^*)$ is a sequence of strict best replies in g that ends in a Nash equilibrium of g.

Case 2. Suppose that $a^0 \notin B$. If a^0 is a Nash equilibrium, then a^0 is a sequence that ends in a Nash equilibrium of g. If a^0 is not a Nash equilibrium, then (a^0, a^*) is a sequence of strict best replies in g that ends in a Nash equilibrium of g. $\blacksquare$

Friedman and Mezzetti (2001) claim in Lemma 2 that any finite, two-player, supermodular game has the weak FIP. In that paper, they extend this result to any finite, more-than-two-player, supermodular game. Proposition 1 in this paper is another extension of this result, but in a different direction. Note that a finite, two-player game with the PNEP is not necessarily quasi-supermodular.[4] Consider the game $g = \langle \{1, 2\}, A, \pi \rangle$, where $A = \{t, m, b\} \times \{l, c, r\}$ and π is given by

	l	c	r
t	$1,1$	$2,2$	$0,0$
m	$2,2$	$0,0$	$2,1$
b	$0,0$	$1,2$	$1,1$

.

By easy but tedious calculation, we can see that g has the PNEP but is not quasi-supermodular under any pair of orders on $\{t, m, b\}$ and $\{l, c, r\}$.[5]

[4]A game $g = \langle N, A, \pi \rangle$ is said to be a quasi-supermodular game if for any player i, A_i is a totally ordered set and

$$\pi_i(a_i, a'_{-i}) \geqslant \pi_i(a') \Rightarrow \pi_i(a) \geqslant \pi_i(a'_i, a_{-i}), \quad \pi_i(a_i, a'_{-i}) > \pi_i(a') \Rightarrow \pi_i(a) > \pi_i(a'_i, a_{-i})$$

for any strategy profiles a and a' such that $a_j \geqslant a'_j$ for any player j. Any supermodular game is a quasi-supermodular game but the converse is not true.

[5]To see that g is not quasi-supermodular in any pair of orders, suppose that g is

Any quasi-acyclic game has the weak FIP, but the converse is not true. Friedman and Mezzetti (2001) give an example of a two-player game that has the weak FIP, but is not quasi-acyclic. In Proposition 2, however, we can see that the weak FIP of *all* subgames is equivalent to the quasi-acyclicity of *all* subgames.

Proposition 2 *Let g be a finite, two-player game. Then the following are equivalent:*

(i) *g has the PNEP;*

(ii) *$g|_B$ is quasi-acyclic for any product subset B of A;*

(iii) *$g|_B$ has the weak FIP for any product subset B of A.*

Proof. (i) $\Rightarrow$ (ii): Take any product subset B of A. Since g has the PNEP, $g|_B$ also has the PNEP. By Proposition 1, $g|_B$ is quasi-acyclic.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are immediate because any quasi-acyclic game has the weak FIP and any game with the weak FIP has a pure Nash equilibrium. ∎

4. A Counterexample

A finite, more-than-two-player game with the PNEP is not necessarily quasi-acyclic. Consider the game $g = \langle\{1,2,3\}, A, \pi\rangle$, where $A = \{t,m,b\} \times \{l,c,r\} \times \{x,y,z\}$ and π is given by

	l	c	r	l	c	r	l	c	r
t	$0,0,0$	$2,2,1$	$2,1,2$	$1,2,2$	$1,0,0$	$2,1,1$	$2,2,1$	$2,1,2$	$2,0,0$
m	$2,1,2$	$0,0,1$	$1,2,2$	$0,1,0$	$2,2,2$	$0,0,0$	$1,2,1$	$0,0,0$	$1,1,1$
b	$1,2,2$	$1,1,2$	$0,0,2$	$2,2,1$	$0,0,0$	$1,1,1$	$0,2,0$	$1,1,1$	$0,0,0$
		x			y			z	

.

g has the PNEP but is not quasi-acyclic because the only sequence of strict best replies from (b,l,x) is cyclic as follows:

$$\begin{aligned}(b,l,x) &\to (m,l,x) \to (m,r,x) \to (t,r,x) \to (t,c,x) \\ &\to (t,c,z) \to (t,l,z) \to (t,l,y) \to (b,l,y) \to (b,l,x) \to \cdots .\end{aligned}$$

quasi-supermodular under some pair of orders. Without loss of generality, we assume $t < m$. Then we have $c < r$ because player 1's payoff function π_1 is quasi-supermodular on $\{t,m\} \times \{c,r\}$. Similarly, by the quasi-supermodularity of π_1, $c < r$ implies $t < b < m$. Moreover, $t < b$ implies $l < r$ by the quasi-supermodularity of π_1, but the orders of $b < m$ and $l < r$ contradict the quasi-supermodularity of π_2 on $\{m,b\} \times \{l,r\}$.

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