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THE PURE THEORY OF LARGE TWO CANDIDATE ELECTIONS

by

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INTRODUCTION

A long standing goal of political theorists has been the development of a coherent, consistent, and non-vacuous theory of elections, particularly of those using majority rule. If possible, such a theory is to be based on rational individual and group behavior. In spite of extensive effort, recent writings (see, for example, Ordeshook and Shepsle, 1982) reveal that many may now be prepared to give up this research program on the grounds that no such model exists. There appear to be two main stumbling blocks to a consistent theory based on the rational behavior of participants: (1) the theoretical proposition that, given any realistic assumption about the cost of voting, rational voters will not participate in elections, and (2) even if they do vote, majority-rule equilibria rarely exist. The first result is obviously contradicted by the facts, the second means that the theory as we know it is fundamentally flawed. Faced with these results, those who have not given up on political theory all together have gone in two other obvious directions. They have either given up on "rational" behavior (see for example Hinich et. al, 1972; Coughlin 1979) or they have given up on "equilibrium" models and have turned to "process" models (see for example Kramer, 1978.)

It is my belief that this retreat is premature. In particular, I intend to show in this paper that even under assumptions of extremely rational behavior, it is possible to combine voters, who may or may not vote depending on the benefits and costs, with candidates who game against each other, and end up with equilibria which not only exist but which also have a remarkable social welfare property. The approach is a straight-forward extension of the now standard spatial competition model of elections. Voters have preferences (utility functions) over issues, candidates choose a platform (a point in the issue space) and then voters vote for their most prefered candidate--in two candidate elections -- if and only if the expected benefits from so doing outweigh the costs. Given this voter behavior, candidates are assumed to maximize expected plurality (a very good approximation to the probability of winning). A full general equilibrium occurs when no voter or candidate wishes to alter their strategy.

	RATIONAL CHOICE	PROBABALISTIC VOTING
NO ABSTENTIONS	TRADITIONAL	COUGHLIN
ABSTENTIONS	THIS PAPER	HINICH, LEDYARD & ORDESHOOK

FIGURE 1

To show where the theory posed in this paper fits into the literature on the theory of majority rule elections, I refer the reader to figure 1 in which existing theories are divided into four "boxes" depending upon the assumptions concerning voting behavior. In the traditional theory it is assumed that all vote (no abstentions) and that choice behavior is rational (some form of utility maximization). It is this theory for which equilibria rarely exist. Hinich et al changed both of these behavioral hypotheses by allowing abstentions due to indifference, alienation, etc., and by modeling the decision to vote as probabalistic while leaving the choice of candidate to be based on utility. Although equilibria exist in this modification, voting behavior is somewhat ad hoc and certainly not rational choice based. Coughlin maintained the traditional assumption of no abstentions but removed the voter's choice of candidate from rational theory. Instead he adopted the decision theoretic framework of Luce (1959, 1977) by assuming that choice is probabalistic, where probabilities are proportional to utility. With this model of voter behavior, equilibria exist and have an interesting welfare property albeit different from that in this paper. It is not known what occurs in Coughlin's models if abstentions are allowed.[1]

The model in this paper departs from the traditional by allowing rational abstention behavior.[2] We begin by recognizing the obvious fact that when voters make the decision to vote they do not know how many others have voted, or plan to vote, or, especially, how these others have voted. They face a decision -- or better a game -- under uncertainty similar in spirit to a sealed-bid auction. In modeling this simultaneous decision problem for all voters we impose as much rationality as possible -- rational choice and rational expectations — and arrive at a model in which turnout is usually neither the 100% nor the 0% that have traditionally been implied by rational choice models. It is this model of the voters' behavior which constitutes the "new" component of the theory in this paper. Most of our rest of our model is standard, although the implications derived from this combination of new and old are not.

In section 1 we describe the behavior of a single voter in much the same way as that posed by Downs (1957), Tullock, and others. In section 2 we consider the simultaneous behavior of all voters and present the equilibrium concept first introduced in Ledyard (1981). In section 3 we define and describe both the behavior of candidates and the equilibrium which arises when all actors --candidates and voters -- are combined into a general equilibrium. In section 4 we explore the welfare properties of those equilibria, in section 5 we examine the existence of equilibrium, and some concluding remarks are added in section 6.

I. THE VOTER

The voter is assumed to choose whether to vote or abstain, as well as for whom to vote, consistent with the expected utility hypothesis. This model has already received much attention in the literature so I will not dwell on its rationale but will immediately turn to the notation and definitions. The interested reader can consult Ferejohn and Fiorina (1974), for a good survey.

For now we assume that there are only two candidates, A and B. Candidate A chooses a platform which we denote by A and candidate B chooses a platform denoted by B. We assume that the voter knows the candidates' choices and has a utility function over all possible platforms, R, given by $u(R,x)$ where x represents the appropriate utility parameters for this voter. We assume throughout that u is continuous in R. We sometimes call x the "type" of this voter. If this voter decides to go to the polls, he will cast his vote for A over B if and only if $u(A,x) > u(B,x)$. We assume the voter receives no consumption benefit from voting. Therefore, whether this voter will vote instead of abstaining depends on a simple benefit-cost calculation. The expected benefits from voting are equal to the probability of affecting the outcome times the gain from so doing. Letting P be the probability that this particular voter will alter the outcome, and assuming that $u(A,x) > u(B,x)$, the expected benefits are $(P)(u(A,x) - u(B,x))/2$. The utility difference is divided by 2 since a voter affects the outcome only if they create a tie or break one. Assuming that ties are broken by a fair coin toss, the gain from either event is the utility difference divided by 2. We assume that the voter faces a known cost of voting equal to $c>0$ and that this cost enters the utility calculation linearly. Therefore, if candidate A wins and the voter went to the poll, he receives $u(A,x) - c$ in utility.

In order to complete this model of rational voting behavior, we must provide a basis for the voter's beliefs about P_a and P_b, where P_j is the probability that candidate j either ties the other or loses by one vote. We assume, at this point, that the voter knows the probability that a voter, randomly selected from all other voters, will vote for A, vote for B, or abstain. (We will see in the next section how these can be estimated.) Using these probabilities, denoted respectively Q_a, Q_b, and Q_o, where $Q_a + Q_b + Q_o = 1$, it is a standard exercise to calculate the probability of a tie when there are n other voters. It is also easy to calculate the probability that A loses to B by one vote. Adding these together we find that $P_a = f(Q_a,Q_b)$ where $f(z,y) =$

$$\sum_{k=0}^{\left|\frac{n}{2}\right|} \frac{n!}{k!k!n-2k!} z^k y^k (1-z-y)^{n-2k} + \sum_{k=0}^{\left|\frac{n-1}{2}\right|} \frac{n!}{k+1!k!n-2k-1!} z^k y^{k+1} (1-z-y)^{n-2k-1}$$

A symmetric calculation yields $P_b=f(Q_b,Q_a)$. Gathering this all together we have described

THE VOTER

A voter with characteristics (x,c) who is faced with a choice between two candiates, A and B, and who thinks the probability that a randomly selected voter will vote for candidate j is Q_j will

(a) vote for A if $c < (P_a/2)(u(A,x) - u(B,x))$

(b) vote for B if $c < (P_b/2)(u(B,x) - u(A,x))$

(c) abstain otherwise,

where $P_a = f(Q_a,Q_b)$, $P_b = f(Q_b,Q_a)$ and f is defined above.

This model assumes rational behavior in the form of expected utility maximization, no income effects, no candidate specific preferences other than the platform choice, positive costs of voting, and knowledge by the voter of x,c,A,B, P_a and P_b.

At this point most writers reach an unsettling conclusion. "Since the expected benefit from voting is obviously small (if $Q_a = Q_b$ and $Q_o = 0$ then P_a and P_b are of order of magnitude 1/2n — see Chamberlain and Rothschild, 1980) and since the cost of voting is not small, no rational voter will ever vote in large elections. Therefore, something must be wrong with the theory." This is a not an unreasonable conclusion but the analysis is incomplete since it is based on a partial equilibrium view which is simply not appropriate. If this voter and others are embedded in a general equilibrium model, the apparent failure of rational choice to explain voting disappears. We turn to that task next.

II. RATIONAL VOTERS' EQUILIBRIUM

We now explore what happens when voters take into account the fact that other voters are also rational. The logic is simple and compelling and is contained in Ferejohn and Fiorina (1974). If eveyone is rational and carries out the partial equilibrium calculus in the previous section then, presumably, no one will vote. But then the probability of a tie is 1. If this is true and if these same rational, partial equilibrium non-voters redo their calculus most will find that it is now definitely in their interest to vote; they will be able to determine the outcome by themself. And so on. Somewhere between no one voting and everyone voting lies a situation in which some vote and in which the probability of a tie is consistent with those numbers and with the beliefs of all voters. It is this stable, rational, intermediate situation that we capture in the voters equilibrium defined below.

To close the partial equilibrium model in the previous section, it remains only to specify how a voter estimates Q_a and Q_b. We assume that it is common knowledge among all voters that each is rational and, therefore, that each follows the model of section I. What is not known to each voter, and never will be, are the values of the others' characteristics (x,c). We do, however, assume that the distribution of these characteristics is known, by all, to be described by the density functions h(c) and g(x). That is, c and x are independently distributed, where g(x) is the probability that a randomly selected voter will have characteristic x, and h(c) is the probability that a randomly selected voter will have a cost of voting equal to c.[3]

Given these densities, one can compute the probability that a randomly selected voter will vote for a candidate. We already know that the voter will vote for A if and only if her characteristic, (x,c), satifies

$$c < (P_a/2)(u(A,x) - u(B,x)).$$

Using the densities g and h we can compute that the probability of this is[4]

$$Q_a = \int_{X+(A,B)} H((P_a/2)(u(A,x) - u(B,x)))g(x)dx$$

where $X+(A,B) = \{x | u(A,x) > u(B,x)\}$,and $H(r) = \int_0^r h(c)dc$.
Writing this as $Q_a = t(P_a,A,B;g,h)$, it is easy to show that $Q_b = t(P_b,B,A;g,h)$ and $Q_o = 1 - Q_a - Q_b$.

We can thus compute Q_a and Q_b from P_a and P_b. In the previous section we computed P_a and P_b from Q_a and Q_b. A fully rational voter with fully rational expectations will require these calculations to be consistent with one another, and will be able to compute the values of Q and P for which consistency obtains.

RATIONAL VOTERS' EQUILIBRIUM

Given the densities on characteristics, h and g, and given the candidate platforms A and B, we call $\langle P_a, P_b, Q_a, Q_b \rangle$ a RATIONAL VOTERS' EQUILIBRIUM if and only if

$P_a = f(Q_a, Q_b)$ and $P_b = f(Q_b, Q_a)$ and

$Q_a = t(P_a, A, B; g, h)$ and $Q_b = t(P_b, B, A; g, h)$,

where f(,) is defined in section 1 and t(r,s,w;g,h) is defined above.

As an aside the reader should note that if we were to model the voters as playing a game of incomplete information, as is done in modeling auctions, the three pure strategies would be vote A, vote B, and abstain, and the Bayes equilibria of that game would be exactly the Rational Voters Equilibrium defined above. I chose the approach above for its expositional simplicity.

To complete this section, we consider several properties of the rational voters equilibrium.

PROPOSITION 1: (EXISTENCE) If H(c) C (that is, if H is continous), then a rational voters equilibrium exists.

PROOF: If A = B, then $Q_a = Q_b = 0$, $P_a = P_b = 1$ is an equilibrium. If $A \neq B$, then define the functions $P_a = N1(P_a, P_b) = f(t(P_a, A, B), t(P_b, B, A))$ and $P_b = N2(P_a, P_b) = f(t(P_b, B, A), t(P_a, A, B))$. It is easy to show that N1 and N2 are continuous in (P_a,P_b) since f is polynomial and therefore continuous, while t is continuous in P since H is by assumption. Further, N1 and N2 map $[0,1]x[0,1]$ into itself. Therefore Brouwer's fixed point theorem can be applied. There is at least one pair $P^* = (Pa^*,Pb^*)$ such that $P^* = N(P^*)$. Let $Q_a^* = t(P_a^*,A,B)$ and $Q_b^* = t(P_b^*,B,A)$. Then (P^*,Q^*) is a rational voters equilibrium.

QED

PROPOSITION 2: (SYMMETRY) (P_a,P_b,Q_a,Q_b) is a rational voters equilibrium given (A,B) if and only if (P_b,P_a,Q_b,Q_a) is a rational voters equilibrium given (B,A).

PROOF: Immediate.

QED

This is the first of several propositions concerning the symmetry of the model in this paper. The primary reason for symmetry is that we have assumed that voters care only about the platform candidates adopt and not the name of the candidate.

Another interesting property of equilibrium is uniqueness, or lack thereof. We have two propositions to present, both of which depend on the turnout probabilities.

DEFINITION: (Maximum Turnout Probability) Given the candidates' platforms, A and B, and given the distribution of voters' characteristics, we can compute an upper limit on turnout which is independent of the particular voter equilibrium which is arrived at. In particular, let

$$M(A,B,g,h) = H((1/2)|u(A,x) - u(B,x)|)g(x)dx.$$

We call M( ) the maximum turnout probability.

We have defined M( ) this way since M is the probability that a randomly selected voter will go to the polls if he thinks the probability of a tie is 1. To see this, remember that

$$Q_a + Q_b = \int_{X+(A,B)} H((P_a/2)(u(A,x) - u(B,x)))g(x)dx$$

$$+ \int_{X+(B,A)} H((P_b/2)(u(B,x) - u(A,x)))g(x)dx.$$

Let $P_a = P_b = 1$. The observation follows immediately since H is a distribution function, and $H' \geq 0$.

The next property is of interest for its implications about the voting probabilities in equilibrium.

PROPOSITION 3: In any rational voters equilibrium, $(P_a - P_b) = (Q_b - Q_a)F$ where $F >= 0$.

PROOF: $P_a - P_b = f(Q_a,Q_b) - f(Q_b,Q_a) =$

$$(Q_b - Q_a) \sum_{k=0}^{\left|\frac{n-1}{2}\right|} \frac{n!}{k!k+1!n-2k-1!} x^k y^k (1-x-y)^{n-2k-1}.$$

QED

It should be noted for completeness that F = 0 if and only if the number of voters is even and $Q_a + Q_b = 1$, (i.e.,turnout is 100%).

PROPOSITION 4: (Uniqueness 1) If M(A,B,g,h) = 0 then (1,1,0,0) is the unique rational voters equilibrium.

(M( ) = 0 if no rational voter will go to the polls even when the probability of influencing the election is 1. For an example of this, assume H(0) = 0 and let A = B.)

PROOF: Under the hypothesis, $Q_a = Q_b = 0$ for all values of P_a and P_b since $H(c) \geq H(c')$ whenever $c > c'$. But if Q_a and Q_b are 0 it follows that $P_a = P_b = 1$.

QED

It would be nice if we were also able to exhibit a proposition listing sufficient conditions for the uniqueness of the voter equilibrium when the maximum turnout probability is positive. Unfortunately I have not yet discovered such a result. It is, however, true that if the candidates' platforms are close enough then M is near 0 and the equilibrium will be both unique and continuous in (A,B).

PROPOSITION 5: (Uniqueness 2) Suppose $M(A,B,g,h) > 0$, $u(\ ,x)\varepsilon C1$ for all x, and $H(c)\varepsilon C1$ for all c. If $M(A,B,g,h)$ is near 0, (which is true, for example, if A is near B), the equilibrium (P_a,P_b,Q_a,Q_b) is unique and is C1 in A and B (for $A \neq B$).[5]

PROOF: Let $Q_a(P_a) = t(P_a,A,B)$ and $Q_b(P_b) = t(P_b,B,A)$. (P,Q) is an equilibrium if and only if P solves

$P_a - f(Q_a(P_a),Q_b(P_b)) = 0$ and

$P_b - f(Q_b(P_b),Q_a(P_a)) = 0$. The Jacobian of this system of equations is

$$J = \begin{vmatrix} 1 - f_1(Q_a,Q_b)Q_a' & -f_2(Q_a,Q_b)Q_b' \\ -f_2(Q_b,Q_a)Q_a' & 1 - f_1(Q_b,Q_a)Q_b' \end{vmatrix} .$$

$$Q_a' = \partial Q_a/\partial P_a = \partial \int_{X^+} H\big[(P_a/2)\big(u(A,x) - u(B,x)\big)\big]\, g(x)\, dx/\partial P_a$$

$$= \int_{X^+} h\big[(P_a/2)\big(U(A,x) - U(B,x)\big)\big]\, g(x)\, \big(U(A,x) - U(B,x)\big)\, \big(1/2\big)\, dx$$

Since this integral is taken over X^+ its value is positive. Similarly for Q_b'.

From Ledyard <18> we know that

$$f_1(x,y) = (y-x) \sum_{k=1}^{\left|\frac{n-1}{2}\right|} \frac{n!}{k+1!k-1!n-2k-1!} x^{k-1} y^{k} (1-x-y)^{n-2k-1}$$

$$- \sum_{k=1}^{\left|\frac{n-1}{2}\right|} \frac{n!}{k!k+1!n-2k!} x^{k} y^{k} (1-x-y)^{n-2k-1}$$

$$- n(1-x-y)^{n-1}$$

and

$$f_2(x,y) = (x-y) \sum_{k=1}^{\left|\frac{n}{2}\right|} \frac{n!}{k!k-1!n-2k!} x^{k-1} y^{k-1} (1-x-y)^{n-2k} .$$

From these it can be seen that the signs of f_i are:

	$f_1(Q_a,Q_b)$	$f_2(Q_a,Q_b)$	$f_1(Q_b,Q_a)$	$f_2(Q_b,Q_a)$
if $Q_a > Q_b$	−	+	?	−
if $Q_a = Q_b$	−	0	−	0
if $Q_a < Q_b$	?	−	−	+

Since f_1, f_2 are continuous if Q_a is near Q_b then $f_1(Q_a,Q_b) < 0$ and $f_1(Q_b,Q_a) < 0$. Therefore,

for $Q_a > Q_b$ $\quad J = \begin{vmatrix} + & - \\ + & + \end{vmatrix}$

for $Q_a = Q_b$ $\quad J = \begin{vmatrix} + & 0 \\ 0 & + \end{vmatrix}$

for $Q_a < Q_b$ $\quad J = \begin{vmatrix} + & + \\ - & + \end{vmatrix}$

Now we know that any solution must satisfy $0 \leq Q_a + Q_b \leq M(A,B)$, and $Q_a, Q_b \geq 0$. Therefore $|Q_a - Q_b| \leq M(A,B)$ and if $M(A,B)$ is small enough, Q_a is always near Q_b.

J is positive definite for all such Q_a, Q_b. Therefore, the equilibrium is unique. (Gale-Nakaido, 1965)

Continuity follows from the Implicit Function Theorem.

QED

To summarize, if the maximum turnout probability is small enough (or if the candidate's platforms are close enough) then the voter equilibrium is unique and C1 in the platforms. I do not know how close is "enough". If A is not near B, then it seems that multiple equilibria may be possible.

A final comment seems in order about the amount of turnout predicted by this model. We have seen that if the maximum turnout probability is 0 or if A = B then turnout is predicted to be 0. Since we have adopted a rational behavior hypothesis, one might suspect that, in fact, turnout is never positive. Such a suspicion would be false.

PROPOSITION 6: (Positive expected turnout) If the maximum turnout probability is positive, given A and B, then expected turnout is positive in any rational voters equilibrium.

PROOF: Remember that expected turnout is $(n + 1)(Q_a + Q_b)$. Suppose that $Q_a + Q_b = 0$. Then $Q_a = Q_b = 0$. But $f(0,0) = 1$. Therefore, $P_a = P_b =$

1. It follows that expected turnout is then $(n + 1)M(A,B) > 0$ which is a contradiction.

QED

<u>Corollary:</u> If there is a set of x, with positive measure, such that $u(A,x) - u(b,x) \neq 0$ and if $H(c) > 0$ when $c > 0$ (i.e., $h(c) > 0$ for all $c > = 0$) then expected turnout will be positive in equilbrium. M( ) gives an upper-bound on expected turnout.

Thus, contrary to naive expectations based on partial equilibrium analysis, a full rational general equilbrium consideration of voting behavior yields positive turnout in equilibrium unless each voter refuses to vote even when they know they are the only voter.

III. THE CANDIDATE AND THE ELECTION EQUILIBRIUM

In this section, we model how candidates determine their platforms and, therefore, the outcome of the election. We begin by considering what it is that motivates the candidates. Since this is a static model and since we have been assuming that platforms will be implemented and that the extent of implementation does not depend on the margin of victory, it seems reasonable to me to assume that these candidates care, ex post, only about winning. The appropriate outcome space then is simply the two point set $\{W,L\} = \{win,lose\}$. The simultaneous choice of platforms by the candidates determines a probability distribution (R_a,R_b) on this set and the rational, expected utility maximizing, candidate A will want to choose the platform to maximize $R_a*V(W) + R_b*V(L)$. Thus, this candidate will always want to maximize the probability of winning. We assume that these candidates know the model of the previous section, or at least act as if they know it. From that model they can determine an election outcome function, or more properly an outcome correspondence, which maps pairs of platforms (A,B) into sets of 4-tuples (P_a, P_b, Q_a, Q_b). It is possible for candidates to compute various implications of their choices such as the probability of winning.

THE PROBABILITY A WINS: Given A,B h(c), and g(x), and a rational voters equilibrium of a two-candidate election, the probability that A wins is:

$$R_a = \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{r=1}^{n-2k+1} \frac{n+1!}{k+r!\ k!\ n-2k-r+1!} (q_A)^{k+r}(q_B)^k(1-q_A-q_B)^{n-2k-r+1}$$

$$+ \frac{1}{2} \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \frac{n+1!}{k!\ k!\ n-zk+1} (q_A)^k(q_B)^k(1-q_A-q_B)^{n-2k+1}$$

where $Q_a(A,B)$ and $Q_b(A,B)$ are the appropriate parts of a rational voter equilibrium for A,B.

Although the analysis can be carried out using it, this is a remarkably unwieldy function. To simplify, let us use an approximation of Ra which is appropriate for large elections. (See Hinich, 1977.) If n is large, then $Q_a - Q_b$ is a good aproximation for a candidate to use in place of Ra. To see this, let Si = 1 if voter i votes for A, Si = 0 if i abstains, and Si = -1 if i votes for B. Then A wins if and only if $\sum_{i=1}^{n+1} Si > 0$. This is true if and only if $(1/(n+1)) \sum_{i=1}^{n+1} Si = \overline{S} > 0$. Since the Si are independently and identically distributed, it follows from a Law of Large Numbers that

$$\lim_{n \to \infty} \Pr \overline{S} > 0 = \begin{cases} 1 & \text{if } Q_a > Q_b \\ 1/2 & \text{if } Q_a = Q_b \\ 0 & \text{if } Q_a < Q_b. \end{cases}$$

Therefore maximizing $Q_a - Q_b$ maximizes (in the limit) the probability that A wins. Using this approximation, we posit the following model of the candidates.

THE CANDIDATES' OBJECTIVES

In a large, two-candidate election, each candidate will try to maximize expected plurality. In particular, the objective function of candidate A is

$$W(A,B) = Q_a(A,B) - Q_b(A,B)$$

and that of candidate B is

$$V(A,B) = Q_b(A,B) - Q_a(A,B)$$

where (P_a,P_b,Q_a,Q_b) is a Rational Voters Equilibrium for the platform choices A,B.[6] The observant reader will have already noticed a potential difficulty with this model---a rational voters equilibrium may not be unique and, therefore, the mapping W(A,B) may not be a function. We do have to confront this problem, but if W( ) and V( ) were unique, the above objective functions would point instantly to the appropriate behavior for the candidates in their choice of a platform, since this is a two-person zero-sum game for which game-theorists are in agreement about a solution concept. We adopt the concensus solution concept with modifications because of the non-uniqueness.

THE CANDIDATES' BEHAVIOR

In a two-candidate large election, candidates will choose platforms (A^*,B^*) which satisfy

$$\min W(A^*,B^*) \geq \max W(A,B^*) \qquad \text{for all } A$$

$$\min V(A^*,B^*) \geq \max V(A^*,B) \qquad \text{for all } B$$

where $W(A,B) = \{w \mid Q_a - Q_b = w \text{ for some rational voters equilibrium}\}$, and $V(A,B) = \{v \mid Q_b - Q_a = v \text{ for some rational voters equilibrium}\}$. We call (A^*,B^*) a (STRONG) RATIONAL ELECTION EQUILIBRIUM.

If W( ) and V( ) are single-valued this definition corresponds to the non-cooperative equilibrium (or maximin solution) of this game. Due to the modification, we have called this a strong equilibrium since if the candidates choose these strategies then even if candidate A could choose from the multiple set W( ) after changing her strategy, she could do no better than now. Weaker equilibria may also exist since a risk averse candidate might choose to play a strategy A^*, even though $\max W(A,B^*) > \min W(A^*,B^*)$ for some other A, in order to avoid a possible loss if $\min W(A,B^*) < \max W(A^*,B^*)$. I have chosen the stronger version since a more strategic candidate would notice that even if such a loss occured they could regain at least a payoff of 0 by choosing $A = B^*$. Thus, no outcome which yields less than 0 to some candidate should survive as an equilibrium. A strong equilibrium has the property that each player receives 0.

PROPOSITION 6: (VALUE) If (A^*,B^*) is a strong rational election equilibrium, then $W(A^*,B^*) = V(A^*,B^*) = 0$

PROOF: Given any B, since candidate A can always choose the same platform, B, it must be true that min $W(A^*,B^*) \geq 0$. Also, min $V(A^*,B^*) \geq 0$. But it is easy to see that if $w \epsilon W(A^*,B^*)$ then $-w \epsilon V(A^*,B^*)$. Therefore, min $W(A^*,B^*) \geq$ max $W(A^*,B^*) = 0$.

QED

This means that (A^*,B^*) is a strong equilibrium if and only if max $W(A,B^*) \geq 0$ for all A and max $V(A^*,B) \leq 0$ for all B. Even if there are "weaker" equilibria, one suspects that only strong equilibria are permanent. We, therefore, concentrate on them.

IV. EQUILIBRIUM AND OPTIMALITY

In this section we show that if utility functions are concave and have continuous derivatives in A, and costs are distributed from zero, then all equilibria can be characterized in a remarkably simple manner; the candidates choose the same platform, the chosen platform maximizes $\int u(A,x)g(x)dx$, and no one votes. Thus if an equilibrium exists there is a very simple maximization problem by which it can be computed. We give several examples at the end of this section.

To show these properties of equilibrium, we need to establish some intermediate results. The first of these occurs because of the symmetry of the model; candidates are essentially anonymous in all respects except their platforms.

PROPOSITION 7: (SYMMETRY) If (A,B) is a strong rational election equilibrium then so are (A,A), (B,A), and (B,B).

PROOF: Since (A,B) is an equilibrium, $W(A,B) = 0 = V(A,B)$ from proposition 4. For all D and w if $w \epsilon W(D,B)$ then $w \leq 0$. For all D and v if $v \epsilon V(A,D)$ then $v \leq 0$. Further, we know that $w \epsilon W(A,B)$ if and only if

$-w \varepsilon V(A,B)$ if and only if $w \varepsilon V(B,A)$. Now suppose that $w \varepsilon V(B,D)$ for some D. Then, $-w \varepsilon V(D,B)$ which implies that $w \varepsilon W(D,B)$ and therefore $w \leq 0$. Therefore, (B,B) is a strong rational election equilibrium. The rest follows in a similar manner.

QED

Now we take up a couple of lemmas which allow us to use calculus in the analysis of equilibrium.

LEMMA 1: (A^*,B^*) is a strong equilibrium if and only if $\int H((P_a/2)|D|)I(D)g(x)dx = 0$ for all A where $D = u(A,x) - u(B^*,x)$ and where $I(D) = 1$ if $D > 0$, $I(D) = 0$ if $D = 0$, and $I(D) = -1$ if $D < 0$.

PROOF: By definition (A^*,B^*) is a strong equilibrium if and only if

(1) $\int_{X+} H((P_a/2)(|D|))g(x)dx - \int_{X-} H((P_b/2)(|D|))g(x)dx \leq 0$ for all A. This is true if and only if

(2) $\int H((P_a/2)(|D|))I(D)g(x)dx + \int_{X-} \left[H((P_a/2)(|D|)) - H((P_b/2)(|D|))\right] g(x)dx \leq 0$ for all A. This in turn is true if and only if

(3) $\int H((P_a/2)(|D|))I(D)g(x)dx \leq 0$ for all A.

Statement (1) follows from the remark after Proposition 6 above. Statement (2) follows by adding and subtracting $\int_{X-} H((P_a/2)(|D|))g(x)dx$ to and from the left side of (1). To establish (3) takes a bit more work. I will prove that (2) implies (3) and leave the converse to the reader. Assume that $\int H((P_a/2)(|D|))g(x)dx > 0$ and that (2) is correct for some A. It must then be true that

$$\int (H((P_a/2)|D|)) - H((P_b/2)|D|)) g(x)dx < 0.$$

Therefore, $P_a < P_b$. Refering to Lemma 3 in section II we see that $Q_b <$

Q_a. But this implies that (1) is >0 since (1) is $Q_a - Q_b$. This in turn implies that (2) is > 0 which contradicts our initial assumption.

QED

LEMMA 2: Given (A^*,B^*,P_a) where P_a is a voters equilibrium for A^*,B^*. If A^* is "near" B^* and if $u\varepsilon C1$ and $H\varepsilon C1$ and if their derivatives are bounded then

$$d \int H((P_a/2)|D|I(D))g(x)dx \;/dA =$$
$$\int h((P_a/2)|D|) \left[(dP_a/dA)(D/2) + (P_a/2)(dD/dA)\right] g(x)dx.$$

PROOF: For any A and x such that $I(D) \neq 0$, we find that

$$d(H((P_a/2)|D|I(D))/dA =$$
$$h((Pa/2)|D|)\left[(D/2)(dPa/dA) + (Pa/2)(dD/dA)\right].$$

It follows that equality is also true if $I(D)=0$.[7] From Proposition 5 of the previous section dH/dA exists for all x since A^* is near B^*. The Lemma then follows from the Lebesque Dominated Convergence Theorem.

QED

Lemma 2 is valid even if A is n-dimensional where A is replaced by A_i for $i = 1,\ldots,n$.

We now have all the tools we need to establish the main proposition of this section.

THEOREM 1: Given the distribution of voters' types, g(x) and h(c), such that $u\varepsilon C1$, $H\varepsilon C1$, their derivatives are bounded, $h(0) > 0$, and u is concave in A for all x and strictly concave for some x. If (A^*,B^*) is a strong rational election equilibrium, then $A^* = B^*$, $P_a = P_b = 1$, $Q_a = Q_b = 0$, and A^* maximizes $\int u(A,x)g(x)dx$.

PROOF: From proposition 7 we know if (A^*,B^*) is an equilibrium then so is (A^*,A^*). We concentrate on the latter. Suppose that (A^*,A^*) is an equilibrium. We know that $\max W(A,A^*) \leq 0$ for all A. From Lemma 1 it must be true that $J = \int H((Pa/2)|D|)I(x)g(x)dx \leq 0$ for all A. From Lemma 2 and the first order conditions for maximization it must therefore be true that

$dJ/dA = 0$ at $A = A^*$. If $A = A^*$ then $D = 0$, $P_a = 1$, and

$$h(0) \int (du(A^*,x)/dA)g(x)dx = 0$$

Since $\int u(A,x)g(x)dx$ is a strictly concave function, A^* maximizes that function.

To finish the proof, we need to show that if (A^*,B^*) is an equilibrium then $A^* = B^*$. Suppose not. From propositon 7, both (A^*,A^*) and (B^*,B^*) are equilibria. Therefore both A^* and B^* are maximizers of $\int u(A,x)g(x)dx$. But u is strictly concave for some x which implies that there is a unique maximizer; that is, $A^* = B^*$.

QED

Theorem 1 fully characterizes the rational election equilibrium if it exists. In that equilibrium, even though no one votes — thus avoiding all the non-productive costs of voting -- candidates are led to select a platform which maximizes a social welfare function, the sum of voters' utilities. The existence of voters who are on the margin of voting, those with costs near 0, leads candidates to take the preferences of these voters into account. Because of the linearity of utility in the costs of voting, the change in the probability that a voter will vote, due to a change in a candidate's position, is "locally" proportional to the extra utility received by the voter if that candidate is elected. It is always in the interest of the candidates to change their position in the direction which maximizes the "aggregate marginal utility of the marginal voters". This leads them inexorably to a position which maximizes the aggregate utility of the voters whose costs are minimal.[8]

Because of the similarity of this theorem to the fundamental welfare theorem that competitive market equilibrium allocations are Pareto-Optimal, I am finding it difficult to refrain from phrases like "the invisible hand of the electorate". However, the fact that equilibrium platforms maximize a "social utility function" should not lead the reader to conclude that election equilibrium allocations are also Pareto-Optimal. The next few examples help illustrate this and other implications of the model.

EXAMPLE 1: Supppose there is a one dimension issue space and that the class of utility functions which any voter can have is given by $u(A,x) = -|A-x|$. x is usually interpreted to be voter x's ideal platform. For this type of example, traditional theory tells us that the election equilibrium will be the ideal platform of the median voter, $A = x^*$ where $\int^{x^*} g(x)dx = 1/2$. Let us calculate the rational election equilibrium. A^* will maximize $\int u(A,x)g(x)dx = \int -|A-x|g(x)dx$. It is easy to see that A^* will also be the median of the density $g(x)$. The two theories yield the same predicted equilibrium platform although turnout is predicted to be 100% by the traditional theory but 0% by this theory.

EXAMPLE 2: Let us now look at a well used example. We suppose that preferences over a one dimension issue space are given by the Type 1 utility functions $u(A,x) = -(A-x)^2$. In this case traditional theory still predicts the platforms will be the median voter's ideal platform.

The rational election equilibrium is, however, the mean voter's ideal platform. That is, A^* maximizes $\int -(A - x)^2 g(x)dx$. Differentiating, one gets $\int -2(A - x)g(x)dx = 0$. From this, we know that $\int Ag(x)dx = \int xg(x)dx$ or $A = \int xg(x)dx$, the mean of $g(x)$.

This simple example illustrates that there is absolutely nothing sacred about the median voter.[9] One might just as easily be concerned about the mean or, indeed, any other moment. For example, if $u = -(A-x)^n$ then the (n-1)st moment is the equilibrium platform. The predicted equilibrium platform depends on the composition of the class of utility functions. An important implication of this and the prior example is that functional forms are important. The functions $-|x - A|$ and $-(x - A)^2$ each represent the same ordinal risk-free preferences on the set of A. However they do represent different attitudes towards risk and different indifference surfaces between c and A. These differences are reflected in different equilibria. The intensity of preference for A, as opposed to c, as measured by the willingness to vote is what drives the result.

One other fact to note in this example. A multiple issue space will not eliminate this equilibrium. If A and x are, say, n-dimensional then the equilibrium is the mean of the multivariate distribution $g(x)$.

Example 3: Finally let us look at a simple application of this theory and consider what happens if the election is held to decide the allocation of a public good and the assignment of the taxes needed to pay for that good. Let $u(y,I,x)$ be the utility of voter x for the public good level, y, when that voter's income is I. We assume that x and I are not correlated and are distributed according to $r(x)s(I)$. Platforms will be of the form $(y,t(\ ))$ where the function, $t(I)$, indicates the tax to be paid if income is I. I am assuming that taxes cannot be placed directly on the unobservable x. If the cost of the public good is $C(y)$ we require that $\iint t(I)r(x)s(I)dIdx = C(y)$ for all platforms -- no deficit or surlplus financing is allowed. Given this model, we know that, in a rational election equilibrium, y and $t(\ )$ maximize $\iint u(y,I - t(I),x)r(x)s(I)dIdx$ subject to the above constraint. Letting L be the Lagrangian multiplier associated with the constraint, it follows from first order conditions that

$$d \iint u(y,I-t(I),x)r(x)s(I)dIdx)/dy - L\ dC(y)/dy = 0,$$

$$-\int (du(y,I-t(I),x)/dI)r(x)dxs(I) + L\int r(x)dx\ s(I) = 0 \text{ for all } I$$

and $C(y) = \iint t(I)r(x)s(I)dIdx$. Let I^* solve $\int (du(y,I^*,x)/dI)r(x)dx = L \int r(x)dx$. The second equation above implies that in equilibrium $I - t(I) = I^*$ for all I; that is, $t(I) = I - I^*$. This means, among other things, that everyone's after tax income will be identical-- income is redistributed towards the mean. Using the constraint we find that $I^* = RM - (1/N)C(y)$ where $N = \int s(I)dI$, $R = \int r(x)dx$ and $M = \int Is(I)dI$. Therefore, everyone's after tax income is $I^* = (RM - C(y))/N$. Turning to the first of the first order conditions, it can be shown that if $d(du/dI)/dx = 0$, that is if du/dI is constant over all x, then

$$N \int (du/dy)/(du/dI)\ r(x)dx = dC/dy.$$

This is simply the Samuelson-Lindahl condition for the Pareto-optimal allocation of the public good. Thus we can conclude that if the post-tax marginal utility of income is independent of the voter's type then large two-candidate elections allocate resources efficiently. There are no 'free riders' in this situation.[10]

Some examples of utility functions for which $d(du/dI)/dx = 0$ can be given:

$u = v(y,x) + I$

$u = v(y,x) + w(y,I)$, and, as a special case,

$u = x \ln y + \ln I$. I leave it to the interested reader to show that if income and type are not independent, then in general the efficiency disappears and redistribution will no longer require equal post-tax income. One can also show that if costs of voting and income are positively correlated, as is sometimes argued, then low income types will have a larger impact on the extent of redistribution.

These three examples are only a small indication of the powerful use one can make of the rational election equilibrium. I am sure the eager reader can provide many more.

To prove that all the above is not vaccuous we move next to the question of existence.

V. EQUILIBRIUM AND EXISTENCE

In the traditional theory of majority rule equilibrium with no abstentions, existence of equilibrium is an unusual occurance. One implication is that we cannot rely on theorems which assume existence. For example, local public goods theories using the median voter should be highly suspect; the results are likely to be vaccuous. The equilibrium described in this paper, on the other hand, exists in a large number of cases. These equuilibria can potentially provide the foundation for many models which make social choices by majority rule elections.

In the last section we proved that if A* was a rational election equilibrium then A* maximized aggregate utility. If we could prove the converse, that if A* maximizes aggregate utility then A* is a rational election equilibrium, we would be done since the appropriate compactness and continuity conditions which ensure the existence of a maximum (the Weierstrauss Theorem) are well known. Unfortunately the converse is not true without some additional conditions on the densities g and h.[11] It is our task to delineate as much as possible the set of distributions for which the following is true;

(S) if A* solves max $\int u(A,x)g(x)dx$ then A* is a rational election equilibrium.

If we knew for which (g,h) the function W(A,B) were concave in A and convex in B, with V(A,B) behaving symmetrically, we would be done since under these conditions the game-theoretic solution to the candidates' problem is known to exist. Unfortunately, one cannot take this approach. Remember that $W(A,B) = Q_a - Q_b$ where $Q_a = \int_{X+} H((P_a/2)(D))g(x)dx$ and $Q_b = \int_{X-} H((P_b)(-D))g(x)dx$, and where D is concave in A and convex in B from the concavity of u, (leaving aside the behavior of Pa and Pb for the moment). If H is a concave function of c then Q_a is concave, but we can't tell about Q_b which, in this instance, is a concave function of a convex function. If H is convex then we have a symmetric problem since $-Q_b$ is concave but we can say nothing about Q_a. Only if H is linear, both concave and convex, can we say something about the concavity properties of W. We capture this intuition in the next proposition.

PROPOSITION 8: If h( ) is the uniform density on [0,k], k > 0, then (S) is true.

Proof: Let $J = \int H((P_a/2)(|D|)I(D)g(x)dx = \int(1/k)(P_a/2)|D|I(D)g(x)dx$ $= \int(1/k)(P_a/2)\, Dg(x)dx$. At A*, the maximizer, letting $D = u(A,x) - u(A^*,x)$, we see that $J = 0$. At any other A, $J \leq 0$. Refering to lemma 1 we can now conclude that A* is a rational election equilibrium.

QED

Absolutely no conditions have been placed on g. That is, we need not worry about single-peakedness, symmetry, unimodality, or unidimensionality. Any old density over concave utility functions can be accomodated. The second thing to notice is that we have been pretty precise about h. An obvious question is whether (S) is true when h is not uniform. The answer is no if we require all g to be accomodated.

PROPOSITION 9: (S) is true for all g if and only if h is uniform.
Proof: The "if" is simply proposition 8. We prove the "only" if statement.

Suppose we have a non-uniform g, an A, and an A* such that A* solves $\max\int u(A,x)gdx$ and such that $\int H((P_a/2)|D|)I(D)gdx \leq 0$ when $D = u(A,x) - u(A^*,x)$. If there are no such g and A then we are done. If we can perturb g to g' such that $\int g'Ddx \leq 0$ and $\int H((P_a/2)|D|)I(D)g'dx > 0$ then we will prove that (S) is true. To see how the perturbation works, consult Figure 2.

Let S1 be the set of x for which $D > 0$ and $H(c) > c/k$, let S2 be those x for which $D > 0$ and $H(c) < c/k$, let S3 be the x for which $D < 0$ and $H(c) > c/k$, and let S4 be the set of x such that $D < 0$ and $H(c) < c/k$. We make g larger on S1 and S3 and smaller on S2 and S4 by letting $g'(x) = g(x) + ei$ when $x \in Si$ such that $\int e1\, dx + \int e2\, dx + \int e3\, dx + \int e4\, dx = 0$, such that $e1\int Ddx + e2\int Ddx + e3\int Ddx + e4\int Ddx \leq 0$ and such that $e1\int H((P_a/2)D)I(D)dx + e2\int HIdx + e3\int HIdx + e4\int HIdx > 0$. The careful reader can check to see that as long as H is not uniform this perturbation will be possible, since the sets Si will be non-empty.

QED

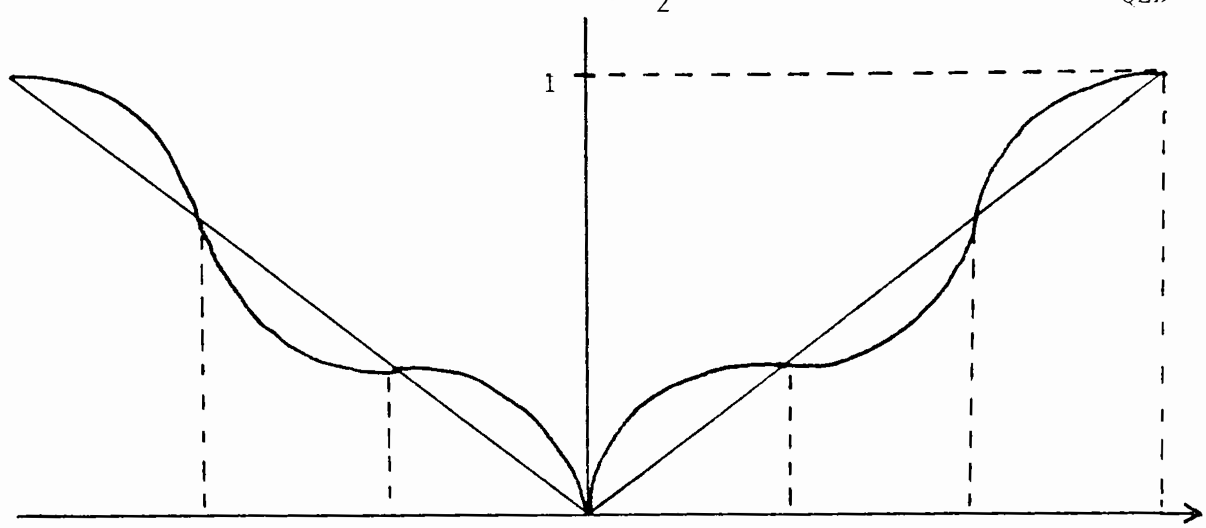

Figure 2

If we want a simple existence theorem and we want it to be applicable to all possible g, we must restrict our attention to only uniform distributions of costs. Suppose we instead wanted a theorem which would be applicable to all distibutions of costs? The answer is similar to that in proposition 9 —– statement (S) is true for all cost distributions if and only if we severely restrict the distribution of x. In order to see why, let us first define the derived distribution of utility differentials. Let $J(r) = \int_{X(r)} g(x)dx$ where $X(r) = \{x | u(A,x) - u(A^*,x) \leq r\}$, and let $j(r)dr = dJ(r)$. Finally let $l(r) = j(r) - j(-r)$ for all $r \geq 0$. A* maximizes $\int u(A,x)g(x)dx$ if and only if

(5.1) $\int_0^\infty r l(r)dr \leq 0$ for all A.

Also expected plurality $W(A,A^*)$ =

(5.2) $\int_0^{\infty}\{ H((P_a/2)r)/r\} r l(r)dr.$

Statement (S) is true when (5.1) implies that (5.2) is less than or equal to 0. Therefore in order for statement (S) to be true, the function $H((P_a/2)r)/r$ cannot weight r relatively more heavily when $l(r) > 0$ than when $l(r) < 0$. Notice that an equal relative weighting, of $(P_a/2)r$, occurs exactly when H is uniform. If we require that (S) be true for all possible h then we must not allow $l(r) > 0$, for otherwise there will be at least one H which weights $l(r)$ incorrectly. We capture all of this in the following:

PROPOSITION 10: (S) is true for all h if and only if $\int_z l(r)dr \leq 0$ for all $z \geq 0$, and all A at A*.

Proof: (if)Let (5.1) be true and let $z1 = \sup \{r | l(r) > 0 \}$. If $l(r) \leq 0$ for all r then we are done since $\int(H/r)rl(r)dr \leq 0$. If z1 = 00 then there is a z* such that $l(r)dr > 0$ which is impossible by assumption. Therefore, if z1 does not exist we are done. Let $z2 = \sup \{r | 0 \leq r <= z1,\ l(r) > 0\}$. Let I1 = [z1,00) and I2 = (z2,z1]. Since $\int_z l(r)dr \leq 0$, it follows that $\int_{I1} l(r)dr + \int_{I2} l(r)dr \leq 0$. Further, since H is a distributuion function, $H(kz) \geq H(kz1)$ if $z \geq z1$ and $H(kz) \leq H(kz1)$ if $z \leq z1$. Therefore, letting $p = P_a/2$,

$$\int H(pr)l(r)dr = \int_{\infty}^{z1} H(pr)l(r)dr + \int_{z1}^{\infty} H(pr)l(r)dr \leq H(p(z1))\int l(r)dr \leq 0.$$

One can iterate this proof for all zi until zi = 0.

(only if) Suppose that $\int_z l(r)dr > 0$ for some $z^* \geq 0$. Let $H''(z) = 1$ if $z \geq z^*$ and $= 0$ if $z < z^*$. Then $\int\{H''((P_a/2)r)/r\} l(r)dr = \int l(r)dr > 0$. But then A* cannot maximize $\int u(A,x)g(x)dx$.

QED

Restricting ouselves to H which are continuous is no problem since we can always find a continuous H which is near to H" and which also is an appropriate perturbation. Thus the above proof remains applicable with minor adjustments.[12]

In this section, we have proven results only about the extreme limits of the set of (g,h) for which rational elction equilibrium exists. That is, we have required existence to occur either for all g or for all h. If we are willing to consider only some g or h we should be able to do better. One can show that there is an open set of (g,h) for which existence obtains. In particular, if h is almost uniform or if g is almost 'symmetric' then equilibrium will exist. I suspect that there is a rather large set of such (g,h) but its precise characterization remains an open question.

Since the results of Coughlin, of Hinich, Ledyard and Ordeshook, of Hinich, and of this paper all point to the fact that multi-dimensional election equilibria exist more often than suspected and that they rarely involve the median voter, one might speculate whether it is my assumption of rationality or the role of uncertainty which drives these results.[13] I suspect uncertainty is the key to existence and that some form of rationality is the key to "optimality". This remains a future research issue.

VI. VARIATIONS ON A THEME

As I have presented this paper in many places, a number of issues have been raised which seem to be easily handled within the framework of the above model. Let me address these variations one at a time.

1) INCOME EFFECTS

In the analysis of the rational voter I assumed that the cost of voting entered the voter's utility function linearly. This assumption is not necessary and can be eliminated. In particular, let u(A,0,x) be the utility received by the voter if A wins and this voter did not vote. Let u(A,c,x) be the utility if A wins and this voter voted where x and c are as in the original model. Assume that du/dc exists and is less than zero (that is, an increase in the cost of voting lowers x's utility, ceteris paribus). Although the analysis is messier than above, one can derive similar results. For example, at an equilibrium

$$h(0)\int u_A\big((A,0,x)/-u_c(A,0,x)\big)g(x)dx = 0.$$

This is identical to the earlier result if we "normalize" marginal utility by the marginal utility of voting costs at 0. That is if an equilibrium exists $A^* = B^*$ and A^* maximizes

$$\int\big[U(A,Q,x)/U_A(A^*,0,x)\big]\ g(x)\ dx.$$

I do not yet know how other results translate. For example, establishing existence appears to be more difficult.

2) NEGATIVE VOTING COSTS

I find myself suspicious of any one who claims to vote no matter what the issues or how close the election. In almost every election held there are some frictions, or other phenomena ignored by this model, which cause there to be some difference in the candidates, and which might lead low cost voters to vote. As far as I can tell there is still no common agreement on the facts about voter behavior. In spite of my scepticism it is important for completeness of the theory to explore what will happen to the equilibrium if there are indeed voters who derive some utility from the act of voting itself. It is easiest to model these as voters whose cost, c, is negative. In the model of the calculus of voting any voter with $c < 0$ will always vote for their most prefered candidate. With this in mind consider now the equilibrium in which those voters with $c < 0$ always vote and those voters with $c > 0$ behave as described before. If $A = B$ then only the voters with $c < 0$ will vote and, therefore, if $A = B$ in equilibrium it must be true that A is the ideal platform of the median voter, the median of those who always vote, if one exists. (We know from standard theory that existence can be problematical.) If c and x are uncorrelated and if that median platform also maximizes $\int u(A,x)g(x)dx$, then A will be the equilibrium. However, if the median either does not exist or does not equal the maximizer of aggregate utility then we must look elesewhere. It is an open question as to whetheran equilibrium even exists in this situation and, if so, what it is. I am not even sure whether candidates will choose the same platform in equilibrium. All I can conclude so far is that "irrational" voters who derive utility from the act of voting create an externality which interferes with the selection by the election of a socially desirable outcome. Perhaps we should educate voters not to be "citizens" but to be selfish?

3) MINIMAX REGRET VOTERS

If some of the voters in the electorate use the minimax regret criteria of Savage (made popular by Ferejohn and Fiorina 1974, 1975). Then the analysis remains pretty much the same but the conclusions are slightly altered. As I showed in < >, if we replace Pa by 1/2 in the model of rational voting behavior we will have modeled the behavior of a minimax regret voter. If one then follows the model to its conclusion one will see that, in equilibrium, $A^* = B^*$ and A^* will maximize $\int u(A,x)\left(g(x) + (1/2)g^*(x)\right)dx$ where g( ) is the density of the expected utility maximizing voters and g*( ) is the density of the minimax voters. It appears that because minimax voters don't care about closeness they end up being weighted at half that of utility maximizers in their effect on the outcome. At the margin, when A is near B, they react more slowly to changes in platforms and, thus, lose their effectiveness.

4) VOTE MAXIMIZING CANDIDATES

It is sometimes argued that candidates care about other things than just winning. This is another of those areas of disagreement in political theory. There is no agreement on the factors which motivate candidates. Although it is obviously of little use to a candidate to have a large vote if that candidate does not win, some argue that candidates should want to maximize votes, not the probability of winning. Several of our conclusions change if that is the case. First of all, candidates will not choose the same platform. If they did, one of them could increase their votes (from 0) by simply moving away from the other candidate. (Of course this could lead to an election loss.) In equilibrium, if one exists, turnout will occur.

5) TURNOUT

A major issue raised by many who see this model for the first time is the lack of turnout in equilibrium. While I see this as a good (the deadweight loss of voting costs is avoided), many see this as a prediction of the model which is clearly contradicted by the facts. It must be remembered that, because of the many possible frictions, actual elections will rarely match this theory. Among other things, most elections are held to decide several contests simultaneously and political activitists, ignored in my model, operate to interfere with the natural forces. It is true that single issue elections with few activists and with little at stake do have very little turnout. Examples abound but the normal school tax election is the obvious one. In a New Hampshire town an election was held to fill the school board. Only one slate was on the ballot. No one voted. I am not sure why the judges did not write in their own names but the moral is clear; when there is no choice it pays not to go to the polls.

I am not sure what an appropriate example is for the model in this paper but the following does provide an ease of computation. Let $u = -(A-x)^2$, H( ) = 1-exp(-ac), and let g( ) = (R)(b) exp(-bx). Going through the appropriate manipulations one can, somewhat tediously, discover that given the platforms A and B, the maximum turnout M( ) is

$$1 + (b/(aD-b))((\exp -2aDS) - ((2aD/(aD+b))\exp -bS)).$$

Here, D = (A-B)/2 and S = (A+B)/2.

It can be easily shown that M is near 1 if D,S,a,b are large. M is near 0 if D,S,a,b are near 0. I have no idea what "reasonable" values of these parameters are. Does anyone want to make a guess?

If one wishes to estimate equilibrium turnout, given A and B, one must solve the following equation; let M(a,b,D,S) by the equation above, then solve N = M(a/N,b,D,S) for N. N/R will then be an estimate of the % turnout since 1/N estimates P_a.

These are but a few of the possibilities for refinement of the model. Others which I think are as important, but of which I know little, follow.

a) Three candidate elections (and multiple candidates)

b) Political activists and parties

c) candidate choice and the role of primaries

d) intertemporal considerations

e) representative democracy and the responsiveness of the system.

f) multiple, simultaneous elections

g) empirical estimation.

FOOTNOTES

[1] Stop press: I have recently seen Coughlin (1983b) in which abstentions are allowed but only on an aggregate basis. Individual behavior remains unspecified.

[2] In many respects, I am finally getting around to answering the complaints of Slutsky (1975) about the ad hoc and unrealistic nature of our earlier paper (Hinich et al, 1972). In spite of my efforts he still remains unconvinced of the "reality" of the model.

[3] The assumption of independence is made only for espositional convenience. The eager reader can easily show that correlation between c and x in a density function like g(x,c) can be accomodated without destroying any of the results which are detailed below.

[4] Note that if A = B, then $Q_a = 0$ since U(A,x) = U(V,x), H(0) = 0 and to modify this, see section VI, (2).

[5] Q_a and Q_b may have discontinuous derivatives at A = B. I thank Peter Coughlin for noting this in an earlier version.

[6] The Rational Voters Equilibrium is defined in Section II.

[7] For arbitrary functions f(x), if $f'(x) \to a$ as $x \to 0$ for all sequences of x, then $f'(0) = a$. Let $A \to B$ so that $D \to 0$.
Then $dH/dA \to h(0)(1/2)(dU(A,x)/dx)$ for all such sequences.

[8]If types and costs are correlated, that is if the density is G(x,c) instead of g(x)h(c), then candidates will choose the platform A* which maximizes $\int u(a,x)G(x,0)dx$.

[9]Hinich(1977), Coughlin and Nitzan (1981) and Coughlin (1983a) also find the median to be unimportant when uncertainty is included in the voting model.

[10]A side issue: Since this case covers utility functions without income effects, it covers all situations covered by the Demand Revealing Mechanisms. Therefore, it dominates that method for social choice.

[11]Economists will notice that this phenomenon also arises when considering the welfare theorems about competitive equilibria.

[12]As a side note, the cost distribution, H", used in this proof, which assumes equal costs of voting which are known to all, is the same distribution used in Ledyard (1981). The fact that this distribution causes the most difficulties for existence partly explains the rather weak theorem in that paper.

[13]These remarks are motivated by an insightful remark of Howard Rosenthal.

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such that $e1\int Ddx + e2\int Ddx + e3\int Ddx + e4\int Ddx \leq 0$ and such that $e1\int H((P_a/2)D)I(D)dx + e2\int HIdx + e3\int HIdx + e4\int HIdx > 0$. The careful reader can check to see that as long as H is not uniform this perturbation will be possible, since the sets Si will be non-empty.

QED

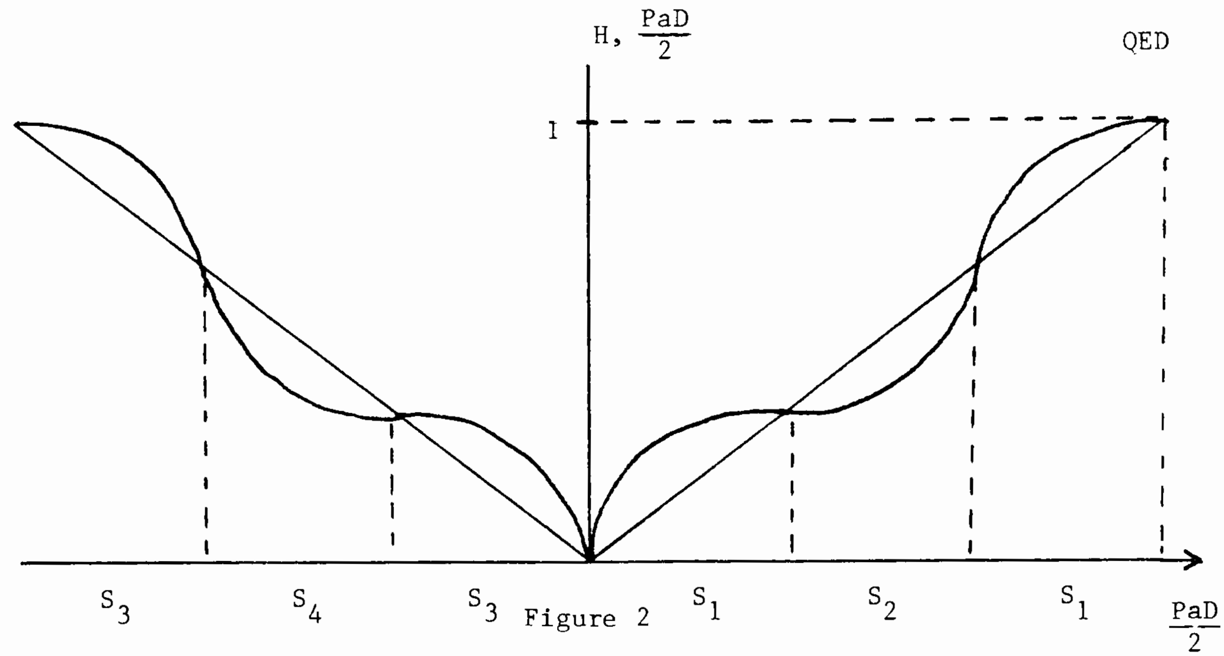

Figure 2

If we want a simple existence theorem and we want it to be applicable to all possible g, we must restrict our attention to only uniform distributions of costs. Suppose we instead wanted a theorem which would be applicable to all distibutions of costs? The answer is similar to that in proposition 9 —– statement (S) is true for all cost distributions if and only if we severely restrict the distribution of x. In order to see why, let us first define the derived distribution of utility differentials. Let $J(r) = \int_{X(r)} g(x)dx$ where $X(r) = \{x | u(A,x) - u(A^*,x) \leq r\}$, and let $j(r)dr = dJ(r)$. Finally let $l(r) = j(r) - j(-r)$ for all $r \geq 0$. A* maximizes $\int u(A,x)g(x)dx$ if and only if

(5.1) $\int_0^\infty r l(r)dr \leq 0$ for all A.

Also expected plurality W(A,A*) =