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# THE $Q$-PROPERTY OF A MULTIPLICATIVE TRANSFORMATION IN SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS*

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**Abstract.** The $Q$-property of a multiplicative transformation $AXA^T$ in semidefinite linear complementarity problems is characterized when $A$ is normal.

**Key words.** Multiplicative transformations, $Q$-property, Complementarity.

**AMS subject classifications.** 90C33, 17C55.

**1. Introduction.** Let $S^n$ be the space of all real symmetric matrices of order $n$. Suppose that $L : S^n \to S^n$ is a linear transformation and $Q \in S^n$. We write $X \succeq 0$, if $X$ is symmetric and positive semidefinite. The semidefinite linear complementarity problem, SDLCP$(L, Q)$ is to find a matrix $X$ such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0, \quad \text{and} \quad XY = 0.$$

SDLCP has various applications in control theory, semidefinite programming and other optimization related problems. We refer to [2] for details. SDLCP can be considered as a generalization of the standard linear complementarity problem [1]. However many results in the linear complementarity problem cannot be generalized to SDLCP, as the semidefinite cone is nonpolyhedral and the matrix multiplication is noncommutative.

We say that a linear transformation $L$ defined on $S^n$ has the $Q$-property if SDLCP$(L, Q)$ has a solution for all $Q \in S^n$. Let $A \in R^{n\times n}$. Then the double sided multiplicative linear transformation $M_A : S^n \to S^n$ is defined by $M_A(X) := AXA^T$. One of the problems in SDLCP is to characterize the $Q$-property of a multiplicative linear transformation. When $A$ is a symmetric matrix, Sampangi Raman [6] proved that $M_A$ has the $Q$-property if and only if $A$ is either positive definite or negative definite and conjectured that the result holds when $A$ is normal. In this paper, we prove this conjecture.

The transformation $M_A$ has the following property: $X \succeq 0 \Rightarrow M_A(X) \succeq 0$. In other words, the multiplicative transformation leaves the positive semidefinite cone invariant. Using this interesting property, Gowda et al. [3] derived some specialized results for the multiplicative transformation. However, the problem of characterizing the $Q$-property of $M_A$ remains open.

We recall a theorem due to Karamardian [5].

THEOREM 1.1. *Let $L$ be a linear transformation on $S^n$. If* SDLCP$(L, 0)$ *and* SDLCP$(L, I)$ *have unique solutions then $L$ has the $Q$-property.*

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The following theorem is well known, see for example [3].

THEOREM 1.2. *Let $A$ be a $n \times n$ matrix. Then the following are equivalent:*

1. *$A$ is positive definite or negative definite.*
2. *$\mathrm{SDLCP}(M_A, Q)$ has a unique solution for all $Q \in S^n$.*

We mention a few notations. If $k$ is a positive integer, let $I_k$ be the $k \times k$ identity matrix. Let $\mathrm{SOL}(M_A, Q)$ be the set of all solutions to $\mathrm{SDLCP}(M_A, Q)$. Suppose that $F$ is a $n \times n$ matrix. Then $f_{ij}$ will denote the $(i, j)$-entry of $F$. Given a vector $x \in R^n$, we let $\mathrm{diag}(x)$ to denote the diagonal matrix with the vector $x$ along its diagonal.

**2. Main Result.** We introduce the following definitions.

DEFINITION 2.1. Let $A$ be a $k \times k$ matrix. We say that $A$ is of *type*$(*)$, if $A = I + B$ where $B$ is a $k \times k$ skew-symmetric matrix.

EXAMPLE 2.2. Let $A := \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}$. Then $A$ is a *type*$(*)$ matrix.

DEFINITION 2.3. Let $A \in R^{n \times n}$. We say that $A$ is of $form(n_1, n_2)$, if there exist *type*$(*)$ matrices $S$ and $T$ of order $n_1$ and $n_2$ respectively such that $n_1 + n_2 = n$ and

$$A = \begin{pmatrix} S & 0 \\ 0 & -T \end{pmatrix}.$$

DEFINITION 2.4. Let $m > 2$ and $A \in R^{m \times m}$. We say that $A$ is of $form(*)$, if there exists a skew-symmetric matrix $W$ of order $k \geq 2$ such that

$$A = \begin{pmatrix} W & 0 \\ 0 & \widehat{A} \end{pmatrix},$$

where $\widehat{A} \in R^{(m-k)\times(m-k)}$.

DEFINITION 2.5. We say that an $n \times n$ symmetric matrix $D = (d_{ij})$ is a *corner* matrix if its rank is one, $d_{11}$, $d_{1n}$, $d_{n1}$ and $d_{nn}$ are nonzero real numbers and all the remaining entries are zeros.

DEFINITION 2.6. We say that an $n \times n$ symmetric matrix $Q$ is of *type*$(n_1, n_2)$, if $Q$ is not positive semidefinite and there exist integers $n_1$ and $n_2$ and a rank one matrix $Q_1 \in R^{n_1 \times n_2}$ such that $n_1 + n_2 = n$ and $Q = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & I_{n_2} \end{pmatrix}$.

Define

$$\widetilde{Q} := \begin{pmatrix} I_{n-1} & q \\ q^T & 1 \end{pmatrix},$$

where $q := (2, 0, \cdots, 0)^T$.

By the well-known formula of Schur, $\det \widetilde{Q} = -3$. Therefore $\widetilde{Q}$ is not positive semidefinite. It is clear that if $n_1$ and $n_2$ are any two positive integers such that $n_1 + n_2 = n$, then $\widetilde{Q}$ can be written as a *type*$(n_1, n_2)$ matrix. Throughout the paper, we use $\widetilde{Q}$ to denote this matrix.

We will make use of the following proposition. The proof is a direct verification.

PROPOSITION 2.7. *Let $A \in R^{n \times n}$. Then the following statements are true.*

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1. *If* $0 \in \mathrm{SOL}(M_A, Q)$, *then* $Q \succeq 0$.
2. *Suppose that* $P$ *is a nonsingular matrix. Then*

$$X \in \mathrm{SOL}(M_A, Q) \Leftrightarrow P^{-1}XP^{-T} \in \mathrm{SOL}(M_{P^TAP}, P^TQP).$$

*Thus* $M_A$ *has the* $Q$*-property iff* $M_{PAP^T}$ *has the* $Q$*-property.*

3. *If* $M_A$ *has the* $Q$*-property, then* $A$ *must be nonsingular.*

We will use the following property of positive semidefinite matrices.

THEOREM 2.8. *Suppose that* $X := \begin{pmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{pmatrix} \succeq 0$. *If* $X_1 = 0$ *or* $Z_1 = 0$, *then* $Y_1 = 0$.

We begin with the following lemma.

LEMMA 2.9. *Suppose that* $U_1$ *and* $U_2$ *are orthogonal matrices of order* $n_1$ *and* $n_2$ *respectively where* $n_1 + n_2 = n$. *Let* $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$. *If* $A \in R^{n\times n}$ *is of* $form(n_1, n_2)$, *then* $UAU^T$ *is of* $form(n_1, n_2)$.

*Proof.* Let $B := UAU^T$. Then there exist $type(*)$ matrices $S$ and $T$ of order $n_1$ and $n_2$ respectively such that

$$A = \begin{pmatrix} S & 0 \\ 0 & -T \end{pmatrix}.$$

It is easy to see that

$$B = \begin{pmatrix} S_1 & 0 \\ 0 & -T_1 \end{pmatrix}$$

where $S_1 = U_1SU_1^T$ and $T_1 = U_2TU_2^T$.

Let $S = I_{n_1} + W$, where $W$ is a skew-symmetric matrix. Then $W_1 := U_1WU_1^T$ will be skew-symmetric. Therefore $S_1 = I_{n_1} + W_1$. So $S_1$ is of $type(*)$. Similarly, $T_1$ is of $type(*)$. Thus $B$ is of $form(n_1, n_2)$. ∎

LEMMA 2.10. *Let* $A \in R^{n\times n}$. *Suppose* $X$ *is a solution to* $\mathrm{SDLCP}(M_A, P\widetilde{Q}P^T)$, *where* $P$ *is a permutation matrix. Then rank of* $X$ *must be one.*

*Proof.* Let $\widehat{Q} := P\widetilde{Q}P^T$ and $Y := AXA^T + \widehat{Q}$. Let $K$ be the leading principal $(n-1)\times(n-1)$ submatrix of $Y$. Then it can be easily verified that $K$ is positive definite. Therefore the rank of $Y$ must be at least $n-1$.

Since $X \in \mathrm{SOL}(M_A, \widehat{Q})$, $XY = 0$. Suppose that $U$ is a orthogonal matrix which diagonalize $X$ and $Y$ simultaneously. Let $D = UXU^T$ and $E = UYU^T$, where $D$ and $E$ are diagonal. Then $DE = 0$. The rank of $E$ is at least $n-1$. Therefore the rank of $D$ can be at most one. If $D = 0$, then $X = 0$. This implies that $\widehat{Q} \succeq 0$ (Proposition 2.7) which is a contradiction. This means that the rank of $X$ is exactly one. ∎

LEMMA 2.11. *Let* $A \in R^{n\times n}$. *Suppose that* $A$ *is of* $form(n_1, n_2)$. *If* $X \in \mathrm{SOL}(M_A, \widetilde{Q})$ *then there exists a* $form(n_1, n_2)$ *matrix* $B$ *and a* $type(n_1, n_2)$ *matrix* $\widehat{Q}$ *such that* $\mathrm{SDLCP}(M_B, \widehat{Q})$ *has a corner solution.*

*Proof.* Write

$$X = \begin{pmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{pmatrix},$$

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where $X_1 \in S^{n_1 \times n_1}$ and $Z_1 \in S^{n_2 \times n_2}$. The above lemma implies that rank of $X$ is one. Therefore rank of $X_1$ can be at most one. We now claim that rank of $X_1$ is exactly one. Let $Y := AXA^T + \widetilde{Q}$.

Since $A$ is of $form(n_1, n_2)$, there exist $type(*)$ matrices $S_1$ and $S_2$ of order $n_1$ and $n_2$ respectively such that

$$A = \begin{pmatrix} S_1 & 0 \\ 0 & -S_2 \end{pmatrix}.$$

Now

$$\widetilde{Q} = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & I_{n_2} \end{pmatrix},$$

where $Q_1$ is of rank one. Suppose $X_1 = 0$. Then Theorem 2.8 implies that $Y_1 = 0$. Thus,

$$AXA^T = \begin{pmatrix} 0 & 0 \\ 0 & S_2 Z_1 S_2^T \end{pmatrix}$$

and hence

$$Y = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & S_2 Z_1 S_2^T + I_{n_2} \end{pmatrix}.$$

From the condition $XY = 0$, we see that

$$Z_1(S_2 Z_1 S_2^T + I_{n_2}) = 0.$$

This implies that $Z_1 = 0$; so $X = 0$. Therefore $\widetilde{Q} \succeq 0$ (Proposition 2.7) which is a contradiction. Thus, $X_1$ is of rank one. Similarly we can prove that $Z_1$ and $Y_1$ are of rank one.

Since $X_1$ is a rank one matrix, we can find an orthogonal matrix $U_1$ such that

$$D := U_1 X_1 U_1^T = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $d > 0$.

Let $U_2$ be an orthogonal matrix such that

$$R := U_2 Z_1 U_2^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & r \end{pmatrix},$$

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where $r > 0$. Let $G = U_1 Y_1 U_2^T$. Then rank of $G$ must be one as rank of $Y_1$ is one. Define

$$U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}.$$

Then $U$ is orthogonal. Let $Z := UXU^T$. Now

$$Z = \begin{pmatrix} D & G \\ G^T & R \end{pmatrix}.$$

Since $Z \succeq 0$, by Theorem 2.8,

$$Z = \begin{pmatrix} d & 0 & ... & 0 & e \\ 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e & 0 & ... & 0 & r \end{pmatrix}.$$

As $G$ is of rank one, $e$ is nonzero. Thus $Z$ is a *corner* matrix.

Let $B := UAU^T$. Then by Proposition 2.7, $Z$ is a solution to $\mathrm{SDLCP}(M_B, \widehat{Q})$, where $\widetilde{Q} := U\widetilde{Q}U^T$. By Lemma 2.9, $B$ must be of $form(n_1, n_2)$. It is direct to verify that $\widehat{Q}$ is of $type(n_1, n_2)$. This completes the proof. ∎

Lemma 2.12. *Let $Q$ be a $m \times n$ matrix defined as follows:*

$$Q = \begin{pmatrix} 0 & 0 & ... & 0 & \pm 1 \\ q_{21} & q_{22} & ... & q_{2n-1} & 0 \\ ... & ... & ... & ... & ... \\ q_{m1} & q_{m2} & ... & q_{mn-1} & 0 \end{pmatrix}.$$

*Suppose the rank of $Q$ is one. Then the submatrix of $Q$ obtained by deleting the first row and the last column is a zero matrix.*

*Proof.* We claim that $q_{21} = 0$. Consider the $2 \times 2$ submatrix

$$\begin{pmatrix} 0 & \pm 1 \\ q_{21} & 0 \end{pmatrix}.$$

Since $Q$ is of rank one, $q_{21} = 0$. By repeating a similar argument for the remaining entries we get the result. ∎

Lemma 2.13. *Suppose that $\widehat{B}$ is of $form(n_1, n_2)$. Let $\widehat{Q}$ be a $type(n_1, n_2)$ matrix. Then a corner matrix cannot be a solution to* $\mathrm{SDLCP}(M_{\widehat{B}}, \widehat{Q})$.

*Proof.* Since $\widehat{B}$ is of $form(n_1, n_2)$, there exist $type(*)$ matrices $B$ and $C$ of order $n_1$ and $n_2$ respectively such that

$$\widehat{B} = \begin{pmatrix} B & 0 \\ 0 & -C \end{pmatrix}.$$

Let $B = (b_{ij})$ and $C = (c_{ij})$. Then $b_{ii} = c_{ii} = 1$. Every off-diagonal entry of $B$ and $C$ will now satisfy $b_{ij} + b_{ji} = 0$ and $c_{ij} + c_{ji} = 0$.

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Suppose that $X$ is a *corner* matrix and solves SDLCP$(M_{\widehat{B}}, \widehat{Q})$. Let

$$X = \begin{pmatrix} d & 0 & ... & 0 & e \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ e & 0 & ... & 0 & r \end{pmatrix}.$$

Let $\widehat{Q} = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & I_{n_2} \end{pmatrix}$ where

$$Q_1 = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n_2} \\ q_{21} & q_{22} & \cdots & q_{2n_2} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n_1 1} & q_{n_1 2} & \cdots & q_{n_1 n_2} \end{pmatrix}.$$

Suppose that $Y := \widehat{B} X \widehat{B}^T + \widehat{Q}$. Then

$$Y = \begin{pmatrix} d+1 & * & ... & * & q_{1n_2} - e \\ -b_{12}d & * & ... & * & b_{12}e + q_{2n_2} \\ ... & ... & ... & * & \\ -b_{1n_1}d & * & ... & * & b_{1n_1}e + q_{n_1 n_2} \\ c_{1n_2}e + q_{11} & * & ... & * & -c_{1n_2}r \\ c_{2n_2}e + q_{12} & * & ... & * & -c_{2n_2}r \\ ... & ... & ... & * & \\ -e + q_{1n_2} & * & ... & * & r+1 \end{pmatrix}.$$

Suppose that $y_1, y_2, ..., y_n$ are the columns of $Y$ and $x_1, x_2, ..., x_n$ are the columns of $X$. Since $X$ is a solution to SDLCP$(M_{\widehat{B}}, \widehat{Q})$, $XY = 0$. Therefore for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., n\}$, we must have $y_i^T x_j = 0$.

From the equations $y_1^T x_1 = 0$ and $y_n^T x_n = 0$, we have

$$d(d+1) + e(q_{1n_2} - e) = 0, \tag{2.1}$$

$$r(r+1) + e(q_{1n_2} - e) = 0. \tag{2.2}$$

Equations (2.1) and (2.2) imply that

$$d(d+1) = r(r+1).$$

As $d$ and $r$ are positive, $d = r$. Since $X$ is a *corner* matrix, rank of $X$ must be one and hence

$$d = r = \pm e.$$

Now $d^2 = e^2$, and therefore from (2.1) we have

$$q_{1n_2} = \pm 1.$$

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Let $i \in \{2, \cdots, n_1\}$. Then $y_i^T x_1 = 0$ gives

$$-b_{1i}d^2 + b_{1i}e^2 + q_{in_2}e = 0.$$

As $d^2 = e^2$ and $e$ is nonzero,

$$q_{in_2} = 0.$$

Thus the last column of $Q_1$ is $(\pm 1, 0, \cdots, 0)^T$.

Let $i \in \{1, \cdots, n_2 - 1\}$. Then

$$c_{in_2}ed + q_{1i}d - c_{in_2}re = 0.$$

Using $r = d$, we have

$$q_{1i} = 0.$$

Thus the first row of $Q_1$ is $(0, \cdots, 0, \pm 1)$.

Now $\widehat{Q}$ is a *type*$(n_1, n_2)$ matrix and hence $Q_1$ is of rank one. Thus $Q_1$ satisfies the conditions of Lemma 2.12 and therefore the submatrix obtained by deleting the first row and last column of $Q_1$ is a zero matrix. Thus

$$\widehat{Q} = \begin{pmatrix} I_{n-1} & e \\ e^T & 1 \end{pmatrix},$$

where $e$ is the $n-1$ vector $(\pm 1, 0, \cdots, 0)^T$.

If $x \in R^n$, then

$$x^T \widehat{Q} x = (x_1 \pm x_n)^2 + \sum_{i=2}^{n-1} x_i^2 \geq 0.$$

Hence $\widehat{Q} \succeq 0$. This contradicts that $\widehat{Q}$ is a *type*$(n_1, n_2)$ matrix. This completes the proof. ☐

Lemmas 2.11 and 2.13 now implies the following result.

Lemma 2.14. *Let $A$ be a $form(n_1, n_2)$ matrix. Then $M_A$ cannot have the $Q$-property.*

We now claim that a skew-symmetric matrix cannot have $Q$-property.

Lemma 2.15. *If $A$ is a $n \times n$ skew-symmetric matrix, then* SDLCP$(M_A, \widetilde{Q})$ *has no solution.*

*Proof.* Suppose that $X$ is a solution. Then the rank of $X$ must be one. Therefore $X = xx^T$ for some vector $x \in R^n$. By the skew-symmetry of $A$, $x^T Ax = 0$; hence $XAX = 0$. Now $X(AXA^T + \widetilde{Q}) = 0$. So $X\widetilde{Q} = 0$. Since $\widetilde{Q}$ is nonsingular, $X = 0$. This implies that $\widetilde{Q} \succeq 0$ (Proposition 2.7) which is a contradiction. ☐

Lemma 2.16. *Let $A \in R^{n\times n}$. If $A$ is a $form(*)$ matrix, then $M_A$ cannot have the $Q$-property.*

*Proof.* Suppose that $M_A$ has the $Q$-property. Since $A$ is of $form(*)$,

$$A = \begin{pmatrix} W & 0 \\ 0 & B \end{pmatrix},$$

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where $W$ is skew-symmetric of order $k \geq 2$ and $B$ is of order $l$.

Define a $k \times k$ matrix by

$$Q_{11} = \begin{pmatrix} I_{k-1} & p \\ p^T & 1 \end{pmatrix},$$

where $p := (2, 0, ..., 0)^T$.

Now define

$$Q' = \begin{pmatrix} Q_{11} & 0 \\ 0 & I_l \end{pmatrix}.$$

Note that there exists a permutation matrix $P$ such that $P\widetilde{Q}P^T = Q'$. Suppose that $X$ is a solution to SDLCP$(M_A, Q')$. Write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{pmatrix},$$

where $X_1$ is of order $k$.

Suppose that $X_3 = 0$. Then, as $X \succeq 0$, $X_2 = 0$.

Now

$$AXA^T + Q' = \begin{pmatrix} WX_1W^T + Q_{11} & 0 \\ 0 & I_l \end{pmatrix}.$$

It is now easy to verify that $X_1$ is a solution to SDLCP$(M_W, Q_{11})$. However by applying the previous lemma, we see that SDLCP$(M_W, Q_{11})$ has no solution. Thus, we have a contradiction. Therefore $X_3$ cannot be zero.

In view of Lemma 2.10, rank of $X$ must be one. Hence the rank of $X_1$ can be at most one and the rank of $X_3$ is exactly one.

Let $U_1$ be a orthogonal matrix such that

$$U_1X_1U_1^T = \begin{pmatrix} d & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix},$$

and $U_2$ be a orthogonal matrix such that

$$U_2X_3U_2^T = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & r \end{pmatrix}.$$

Define an orthogonal matrix $U$ by

$$U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}.$$

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Suppose that $Z := UXU^T$. Then by Theorem 2.8

$$Z = \begin{pmatrix} d & 0 & ... & e \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ e & 0 & ... & r \end{pmatrix}.$$

Note that $r > 0$. Now $Z$ is a solution to SDLCP$(M_{UAU^T}, UQ'U^T)$. Suppose that $Y := M_{UAU^T} + UQ'U^T$.

Now

$$UQ'U^T = \begin{pmatrix} U_1Q_{11}U_1^T & 0 \\ 0 & I_l \end{pmatrix} \text{ and } UAU^T = \begin{pmatrix} U_1WU_1^T & 0 \\ 0 & U_2BU_2^T \end{pmatrix}.$$

Let $\alpha$ be the $(n, n)$-entry of $UBU^T$. Clearly, $U_1WU_1^T$ is skew-symmetric. Let the last row of $Y$ be the vector $\mathbf{y} := (y_1, ..., y_n)^T$. Then by a direct verification, $y_1 = 0$ and $y_n = \alpha^2 r + 1$. By the complementarity condition, $\mathbf{y}$ is orthogonal to $(e, 0, ..., 0, r)^T$. Thus, $r(\alpha^2 r + 1) = 0$, which is a contradiction. This completes the proof. ☐

The next result is apparent from Theorem 2.5.8 in Horn and Johnson [4]; hence we omit the proof.

LEMMA 2.17. *Suppose that $A \in R^{n\times n}$ is a nonsingular normal matrix. If $A$ is neither positive definite nor negative definite, then one of the following statements must be true:*

1. *There exists a nonsingular matrix $Q$ and positive integers $n_1$ and $n_2$ such that $QAQ^T$ is of $form(n_1, n_2)$.*
2. *There exists a nonsingular matrix $Q$ such that $QAQ^T$ is a $form(*)$ matrix.*
3. *$A$ is skew-symmetric.*

Now the following theorem which is our main result follows from item (2) of Proposition 2.7 and the above results.

THEOREM 2.18. *Let $A \in R^{n\times n}$ be normal. Then the following are equivalent:*

*(i)* *$\pm A$ is positive definite.*
*(ii)* SDLCP$(M_A, Q)$ *has a unique solution for all $Q \in S^n$.*
*(ii)* *$M_A$ has the $Q$-property.*

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