

# The quadratically cubic Burgers equation: an exactly solvable nonlinear model for shocks, pulses and periodic waves

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**Abstract** A modified equation of Burgers type with a quadratically cubic (QC) nonlinear term was recently pointed out as a new exactly solvable model of mathematical physics. However, its derivation, analytical solution, computer modeling, as well as its physical applications and analysis of corresponding nonlinear wave phenomena have not been published up to now. The physical meaning and generality of this QC nonlinearity are illustrated here by several examples and experimental results. The QC equation can be linearized and it describes the experimentally observed phenomena. Some of its exact solutions are given. It is shown that in a QC medium not only shocks of

compression can be stable, but shocks of rarefaction as well. The formation of stationary waves with finite width of shock front resulting from the competition between nonlinearity and dissipation is traced. Single-pulse propagation is studied by computer modeling. The nonlinear evolutions of N- and S-waves in a dissipative QC medium are described, and the transformation of a harmonic wave to a sawtooth-shaped wave with periodically recurring trapezoidal teeth is analyzed.

**Keywords** Strongly nonlinear systems · Nonlinear partial differential equation · Exact analytical solutions · Quadratically cubic equation · Shock fronts · Nonlinear acoustics and turbulence · Exact linearizations

**Mathematics Subject Classification** 35C07 · 35G20 · 35K55 · 74J30 · 74J40 · 76E30

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## 1 Introduction

The Burgers equation was until recently the only known nonlinear partial differential equation of the second order which simultaneously has the two important properties that: (i) it can be exactly linearized by a simple transformation (using the Hopf–Cole substitution), and (ii) it has a significant physical meaning. Suggested initially as a model to describe turbulent spectra [1], it became the basic mathematical model

of nonlinear waves in systems where the propagation velocity does not depend on frequency [2]. The Burgers equation adequately describes the physical phenomena of high-intensity wave propagation in dissipative non-dispersive media with quadratic nonlinearity. Its predictive efficiency has been verified many times in comparisons between experiments and numerical simulations [3–5].

A second nonlinear partial differential equation which can be linearized by a simple substitution was indicated recently by the authors [6–8]. This is a *quadratically cubic* (QC) Burgers-type equation. Its derivation and physical applications and the analysis of the corresponding nonlinear wave phenomena have not been published prior to this current paper. Like the usual Burgers equation, the QC equation also has a significant physical meaning, but its manifestations are completely different.

A version of Burgers' equation is written here for a nonlinear acoustic wave as [2]:

$$\frac{\partial p'}{\partial z} = \frac{\epsilon}{c^3 \rho} p' \frac{\partial p'}{\partial \tau} + \frac{b}{2c^3 \rho} \frac{\partial^2 p'}{\partial \tau^2}. \quad (1)$$

Here  $p'$  is the disturbance of pressure,  $\rho$  is the equilibrium density,  $\epsilon$  is the nonlinear coefficient,  $\tau = t - z/c$  is the time measured in a coordinate system accompanying the wave along the  $z$ -axis with sound velocity  $c$ , and  $b$  is the effective dissipation which depends on the shear and bulk viscosities and the thermal conductivity [2].

For the mathematical analysis, it is convenient to rewrite the Burgers equation in a dimensionless form:

$$\frac{\partial V}{\partial Z} = V \frac{\partial V}{\partial \theta} + \Gamma \frac{\partial^2 V}{\partial \theta^2}. \quad (2)$$

The following normalized variables are used in formula (2):

$$\begin{aligned} Z &= \frac{z}{z_{\text{SH}}}, & \theta &= \omega \tau, & V &= \frac{p'}{p'_0}, \\ \Gamma &= \frac{z_{\text{SH}}}{z_{\text{DISS}}} = \frac{b\omega}{2\epsilon p'_0}. \end{aligned} \quad (3)$$

Here  $\omega$  and  $p'_0$  are the typical frequency and amplitude of the initial wave, and  $z_{\text{SH}}$  and  $z_{\text{DISS}}$  are typical shock formation and dissipation lengths defined by

$$z_{\text{SH}} = \frac{c^3 \rho}{\epsilon \omega p'_0}, \quad z_{\text{DISS}} = \frac{2c^3 \rho}{b\omega^2} \quad (4)$$

The dimensionless parameter combination  $\Gamma$ —known as the acoustical Reynolds number or Goldberg's number—is the only similarity criterion for one-dimensional nonlinear waves in dissipative media. At  $\Gamma \gg 1$  dissipation dominates over nonlinearity, and at  $\Gamma \ll 1$  nonlinearity is stronger.

The QC Burgers-type equation can be written, by using the same variables as in (3), in the following dimensionless form [6–8]:

$$\frac{\partial V}{\partial Z} = \frac{1}{2} \frac{\partial}{\partial \theta} (|V|V) + \Gamma \frac{\partial^2 V}{\partial \theta^2}. \quad (5)$$

The difference between Eqs. (5) and (2) is that a QC nonlinearity  $|V|V$  is present in (5), instead of the quadratic nonlinearity  $V^2$  of the Burgers equation. The term  $|V|V$  is functionally rather similar to  $V^3$  in its symmetry. Therefore one can say that the usual cubic nonlinearity  $V^3$  [9, 10] is modeled here by the piecewise quadratic relation  $|V|V$ . This function is continuous, as is its first derivative, while the second derivative has a singularity at  $V = 0$ . Equation (5) is possible to linearize by using the substitution [8]

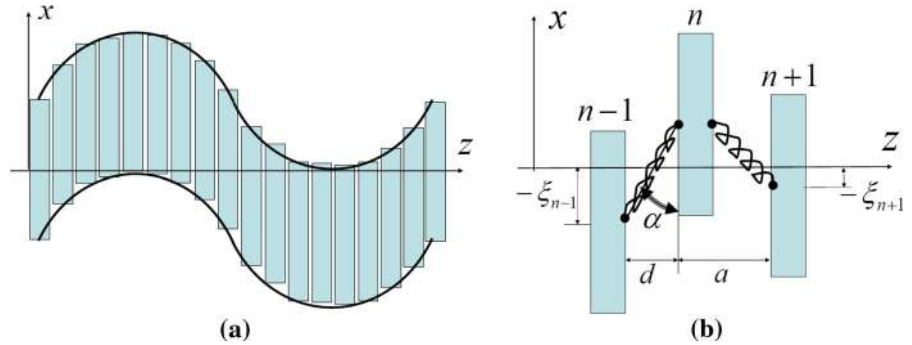
$$\begin{aligned} |V| &= 2\Gamma \frac{\partial}{\partial \theta} \ln U \Rightarrow \\ \frac{\partial U}{\partial Z} &= \Gamma \frac{\partial^2 U}{\partial \theta^2} + CU. \end{aligned} \quad (6)$$

Therefore Eq. (5) can be solved much more easily than the Burgers-type equation with the common cubic nonlinearity  $V^3$ . But the formal replacement  $V^3 \rightarrow V|V|$  is useful not only for the qualitative analysis of cubic nonlinear phenomena. It is more crucial to know that real systems with the nonlinearity  $V|V|$  exist. Examples of such will be provided in the next section.

## 2 Examples of physical systems with quadratically cubic nonlinearity

The first example is related to a strong shear wave propagating in the structure shown in Fig. 1. It consists of rigid plates located periodically along the  $z$ -axis. The plates are only allowed to move up and down in the  $x$ -direction (Fig. 1a). At equilibrium, the center of each plate lies on the  $z$ -axis. They are connected by linear elastic forces (Hookean springs) as shown in Fig. 1b.

**Fig. 1 a** Shear wave in the structure of *plane-parallel plates* interconnected by elastic forces, **b** image patch of three neighboring plates and springs is given for the derivation of equation of motion



The displacement of the center of the  $n$ -th plate up from  $z$ -axis is  $\xi_n(t)$ . An initial displacement of one or more of the plates from equilibrium leads to a shear wave propagating along the  $z$ -axis. In deriving an equation of motion, we consider Fig. 1b where  $a$  is the spatial period of the structure, and  $d$  is the thickness of the deformable layer where the elastic force acts. Let us now consider three neighboring plates, each having the same mass  $M$ . The equation of motion for the plate with number  $n$  is:

$$M \frac{d^2 \xi_n}{dt^2} = F_{n-1} + F_{n+1}. \tag{7}$$

The force acting from plate number  $n - 1$  is:

$$F_{n-1} = -k \left[ \sqrt{(\xi_n - \xi_{n-1})^2 + d^2} - d \right] \cos \alpha. \tag{8}$$

Here  $k$  is the stiffness coefficient of the linear spring. As we are interested in the projection of the restoring force on the  $x$ -axis, the factor  $\cos \alpha$  appears:

$$\cos \alpha = \frac{\xi_n - \xi_{n-1}}{\sqrt{(\xi_n - \xi_{n-1})^2 + d^2}}. \tag{9}$$

The displacements are considered to be small in comparison with the thickness  $d$  of the elastic layer. In this approximation the force (8) is:

$$F_{n-1} = -k \frac{(\xi_n - \xi_{n-1})^3}{2d^2}. \tag{10}$$

The force acting from plate number  $n + 1$  is calculated analogously, and Eq. (7) takes the form:

$$M \frac{d^2 \xi_n}{dt^2} = -k \frac{(\xi_n - \xi_{n-1})^3}{2d^2} - k \frac{(\xi_n - \xi_{n+1})^3}{2d^2}. \tag{11}$$

If all plates except the plate with number  $n$  are fixed, this equation becomes

$$\frac{d^2 \xi_n}{dt^2} + \frac{k}{M d^2} \xi_n^3 = 0. \tag{12}$$

It is similar to the well-known Duffing equation but is missing the linear term  $k \xi_n$ . This ordinary differential equation, which lacks a transition to linear vibration at infinitesimally small amplitude, was used by W. Heisenberg in his nonlinear quantum field theory [11].

We will pass now from the discrete chain of Eq. (11) to the continuum limit. Let the wavelength be much longer than the period  $a$  of the structure in Fig. 1. Assuming that in (11)

$$\xi_n = \xi(z), \quad \xi_{n+1} = \xi(z + a), \quad \xi_{n-1} = \xi(z - a), \tag{13}$$

and expanding the displacements (13) in powers of  $a$ :

$$\xi_{n\pm 1} = \xi(z \pm a) \approx \xi(z) \pm a \frac{\partial \xi}{\partial z} + a^2 \frac{\partial^2 \xi}{\partial z^2} \tag{14}$$

we derive the nonlinear partial differential equation

$$\frac{\partial^2 \xi}{\partial t^2} = 3\beta \left( \frac{\partial \xi}{\partial z} \right)^2 \frac{\partial^2 \xi}{\partial z^2}, \quad \beta = \frac{k a^4}{M d^2}. \tag{15}$$

It is now convenient to pass from Eq. (15) written for the displacement  $\xi$  to an equation for the dimensionless variable  $\zeta = \partial \xi / \partial z$ , which describes the deformation of the structure:

$$\frac{\partial^2 \zeta}{\partial t^2} = \beta \frac{\partial^2 \zeta^3}{\partial z^2}. \tag{16}$$

The linear term is missing in Eq. (16). Consequently, there is *no limiting transition* to a linear wave equation even for very weak disturbances. Therefore, in accordance with the classification suggested in Ref. [7], Eq. (16) describes a strongly nonlinear wave of the third type.

A simpler equation of the first order corresponding to the second-order Eq. (16) is:

$$\frac{\partial \xi}{\partial t} = \sqrt{3\beta} |\xi| \frac{\partial \xi}{\partial z}. \quad (17)$$

This can be proven by differentiating both sides of (17) with respect to  $t$ , and then switching the  $t$ -derivative to the  $z$ -derivative in the right-hand side using Eq. (17) once more. Thus, the existence of QC nonlinearity (17) for nonlinear shear waves is shown. This type of nonlinearity is essential for shear waves in soft biological tissues where a quadratic nonlinearity does not exist because of symmetry reasons, and the stress–strain relationship has no linear region [12].

Let us now briefly discuss other physical systems with QC nonlinearities. The nonlinear acoustic properties of an orifice drilled in a plate were studied in Ref. [13]. It was shown experimentally how the relation between pressure and velocity approaches a quadratic law at large disturbances:  $p' \sim u^2$ . Because the velocity reverses its sign during oscillation, this relation must be rewritten as  $p' \sim u|u|$  [13]. In the Cauchy–Lagrange integral for the case of a potential oscillating flow, an equivalent term with an absolute value would appear. More examples of general relations for obstacles in oscillating flows are known in engineering hydraulics [14]. In general, the pressure disturbance is the sum of two terms. The first term  $p_{AC}$  is caused by the compressibility of the fluid, and the second term  $p_{HYD}$  is connected with the oscillating flow around the obstacle:

$$p' = p_{AC} + p_{HYD} = c^2 \rho' + \gamma \rho u |u|. \quad (18)$$

Here  $\gamma$  is the coefficient of hydraulic resistance which depends on the shape of the body placed in the flow [14], and  $\rho'$  is the density disturbance caused by the acoustic wave. The QC nonlinearity also describes the nonlinear loss in the throat of Helmholtz resonator operating as a high-intensity sound absorber [15], and some models of dry friction are based on QC nonlinearity [16].

Interesting manifestations of QC nonlinearity were experimentally observed in solids. In grainy media the measured amplitude of the third harmonic depends on the squared amplitude of fundamental frequency wave ( $\sim p_0^2$ ) and linearly on the distance traversed in the medium ( $\sim z$ ). This may be compared to a normal quadratic nonlinear medium where the third har-

monic governed by Burgers equation is proportional to the cube of the amplitude of the fundamental harmonic ( $\sim p_0^3$ ) and grows with distance as  $z^2$ . The unusual behavior of polycrystalline aluminum alloy was explained by nonlinear friction at the grain boundaries [17]. A theoretical explanation of such dependencies ( $\sim p_0^2$  and  $\sim z$ ) is not difficult.

The series expansion of the exact solution of QC Eq. (5) at  $\Gamma = 0$  was calculated in Ref. [6]:

$$V = \sum_{n=1}^{\infty} [1 - (-1)^n] \frac{2}{nZ} \left[ \left( \frac{2}{n\pi} - E_n(nZ) \right)^2 + J_n^2(nZ) \right]^{1/2} \sin(n\theta + \phi_n(Z)). \quad (19)$$

The derivation of an analogous expansion for a truly cubic system is given in Ref. [18]. In the expansion (19), which contains only odd harmonics,  $E_n$  is the Weber function and  $J_n$  is the Bessel function. The result (19) represents an analog of the Bessel–Fubini solution for quadratic nonlinearity [2]. At small nonlinear distances  $Z$  it follows from (19) that:

$$V \approx \frac{4}{3\pi} Z \sin(3\theta), \quad p' \approx p_0^2 z \frac{4}{3\pi} \frac{\xi \omega}{c^3 \rho} \sin 3\omega\tau. \quad (20)$$

The second formula in (20) is written in physical dimensional variables. One can see that the third harmonic in a QC medium really is proportional to the squared amplitude of the fundamental harmonic  $\sim p_0^2$  and grows with distance as  $z^1$ .

An additional advanced research area is connected with the solid-state physics. Some solids like mica contain crystal planes of heavy cells with weak bonds between neighboring planes. These types of systems, shown in Fig. 1, can be described by similar mathematical models [19].

### 3 Self-similar solutions of the quadratically cubic equation

The summary of systematic analysis of Lie group symmetries for Burgers' equation is given by Ibragimov in Ref. [20]. The infinitesimal symmetries of this equation form 5D Lie algebra stretched over the five linearly independent operators. Unfortunately, the occurrence of a module in the QC equation eliminates most of these symmetries. Nevertheless, some of the remaining symmetries generate exact solutions which carry important physical meanings.

Let us at first consider the self-similar solution. Using the substitution following from the dilation symmetry group:

$$V = \sqrt{\frac{2\Gamma}{Z}} \Psi \left( \xi = \frac{\theta}{\sqrt{2\Gamma Z}} \right). \tag{21}$$

we can reduce (5) to an ordinary differential equation:

$$\frac{d^2\Psi}{d\xi^2} + 2|\Psi| \frac{d\Psi}{d\xi} + \xi \frac{d\Psi}{d\xi} + \Psi = 0. \tag{22}$$

Integrating once, we get

$$\frac{d\Psi}{d\xi} + |\Psi|\Psi + \xi\Psi + C = 0. \tag{23}$$

By transformation of variables  $|\Psi| = Y'/Y$  the Eq. (23) can be reduced to linear form:

$$Y'' + \xi Y' + CY \operatorname{sgn}(\Psi) = 0. \tag{24}$$

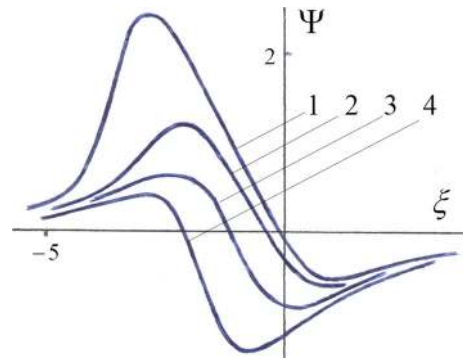
The solution of (24) at  $C = 0$  [2] describes a unipolar pulse for which  $|\Psi| = \Psi$ . Therefore, this solution satisfies Burgers equation as well and does not deal with any QC nonlinearity. The simplest non-trivial QC-specific solution corresponds to  $C = 1$ , where

$$\begin{aligned} Y_1 &= \exp\left(-\frac{\xi^2}{2}\right) \left( C_1 + \int_0^\xi \exp\left(\frac{t^2}{2}\right) dt \right), \\ &\Psi > 0, \\ Y_2 &= C_2 \xi + \exp\left(-\frac{\xi^2}{2}\right) + \xi \int_0^\xi \exp\left(-\frac{t^2}{2}\right) dt, \\ &\Psi < 0. \end{aligned} \tag{25}$$

By matching the two branches of solution (25) we can determine the arbitrary constants  $C_1$  and  $C_2$ . Let both branches (25) vanish in some point  $\xi_0$ , i.e.,  $Y_1(\xi_0) = Y_2(\xi_0) = 0$ . The derivatives at  $\xi = \xi_0$  must be equal to:

$$\begin{aligned} \left. \frac{d\Psi}{d\xi} \right|_{\xi_0} &= -C = -1, \\ \frac{1}{Y_1} \frac{d^2 Y_1}{d\xi^2} &= -\frac{1}{Y_2} \frac{d^2 Y_2}{d\xi^2} = -1. \end{aligned} \tag{26}$$

These matching conditions lead to the following relations between the constants  $C_1$ ,  $C_2$ , and  $\xi_0$ :



**Fig. 2** The shape of single pulses described by the self-similar solution. Curves 1–4 are constructed for  $\xi_0 = -0.01, -0.25, -1, -2$

$$\begin{aligned} C_1 &= \frac{1}{\xi_0} \exp\left(\frac{\xi_0^2}{2}\right) + \int_0^{-\xi_0} \exp\left(\frac{t^2}{2}\right) dt, \\ C_2 &= -\sqrt{\frac{\pi}{2}} \Phi\left(\frac{\xi_0}{\sqrt{2}}\right). \end{aligned} \tag{27}$$

Here

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-\frac{t^2}{2}\right) dt$$

is the error integral. By specifying the matching point  $\xi_0$ , we can calculate the constants  $C_1$  and  $C_2$  and construct the analytical solution (25), (27). This result is shown in Fig. 2. At small values of  $|\xi_0|$  (curve 1), the matching point is close to the origin of coordinates. The shape of the corresponding single pulse is similar to a non-symmetric N-wave. The positive area is bigger than the negative one and contains a smooth shock of compression at the leading front. With increase in  $|\xi_0|$ , the negative area becomes bigger, and a rarefaction front appears, which increases in steepness. The behavior and structure of the fronts are clarified by means of an exact solution given below.

The next Lie group symmetry passed on from Burgers' equation to the QC equation is the translation symmetry. This will generate a stationary wave conserving its shape during propagation.

#### 4 Stable and unstable shock waves of compression and rarefaction in a quadratically cubic medium

Some physically interesting solutions to Eq. (5) describe shock waves where the shock width is controlled by dissipation. These solutions can be obtained

from the invariance with respect to time translation group and can be sought for as

$$V(Z, \theta) = V(\theta_* = \theta + \alpha Z). \tag{28}$$

Substitution of (28) into (5) transforms it from a partial differential equation to an ordinary:

$$\Gamma \frac{dV}{d\theta_*} + \frac{1}{2}|V|V - \alpha V = \frac{1}{2}\alpha^2. \tag{29}$$

The constant  $\alpha = \sqrt{2} - 1 \approx 0.414$  is determined here from the boundary condition for the shock of compression:  $V(\theta_* \rightarrow \infty) \rightarrow 1$ . For negative values of  $V$  Eq. (29) takes form:

$$\Gamma \frac{dV}{d\theta_*} = \frac{1}{2}(V + \alpha)^2. \tag{30}$$

The second boundary condition must be  $V(\theta_* \rightarrow -\infty) \rightarrow -\alpha$ . Only these compressional shock waves can be stable. The solution to (30) which satisfies the condition  $V(\theta_* = \theta_0) = 0$ , where  $\theta_0$  is an indefinite constant, is

$$V = \alpha^2 \frac{\theta_* - \theta_0}{2\Gamma} \left[ 1 - \alpha \frac{\theta_* - \theta_0}{2\Gamma} \right]^{-1}, \quad -\infty < \theta_* < \theta_0. \tag{31}$$

For positive  $V$ , Eq. (29) takes the form:

$$\Gamma \frac{dV}{d\theta_*} + \frac{1}{2}(V - \alpha)^2 = \alpha^2. \tag{32}$$

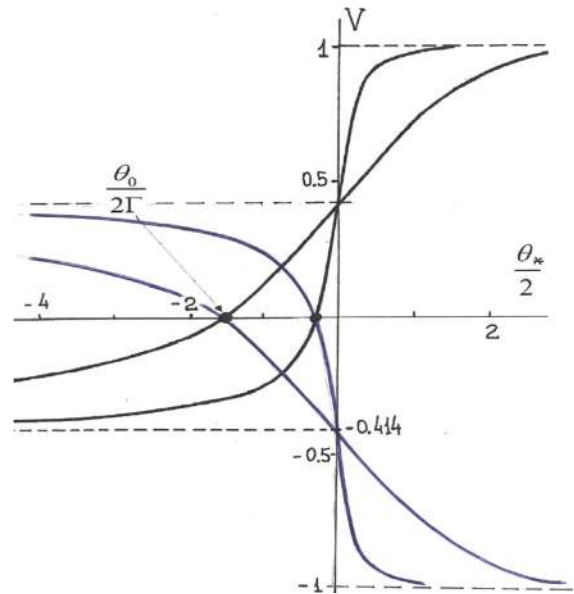
Its solution for the boundary condition  $V(\theta_* \rightarrow \infty) \rightarrow 1$  is:

$$V = \alpha \left[ 1 + \sqrt{2} \tanh \left( \alpha \sqrt{2} \frac{\theta_*}{2\Gamma} \right) \right], \quad \theta_0 < \theta_* < \infty. \tag{33}$$

The complete solution must be continuous at  $\theta_* = \theta_0$  and by matching its two branches (31) and (33), we determine the constant to be

$$\frac{\theta_0}{2\Gamma} = -\frac{1}{\alpha\sqrt{2}} \operatorname{arctanh} \frac{\sqrt{2}}{2} \approx -1.52. \tag{34}$$

It is interesting that the derivative is smooth at  $V = 0$ . And at  $\theta_* = \theta_0$ , both the function  $V(\theta_*)$  and its first derivative are continuous.



**Fig. 3** The internal structure of stable shocks of compression and rarefaction

The structure of the rarefaction shock wave can be calculated in a similar way. Instead of (29), we solve the following equation:

$$\Gamma \frac{dV}{d\theta_*} + \frac{1}{2}|V|V - \alpha V = -\frac{1}{2}\alpha^2. \tag{35}$$

The right-hand side of (35) is determined from the boundary condition  $V(\theta_* \rightarrow \infty) \rightarrow -1$ , and the analogs of Eqs. (30) and (32) are:

$$\Gamma \frac{dV}{d\theta_*} = -\frac{1}{2}(V - \alpha)^2, \quad V > 0, \quad \theta_* < \theta_0,$$

$$\Gamma \frac{dV}{d\theta_*} - \frac{1}{2}(V + \alpha)^2 = -\alpha^2 V < 0, \quad \theta_* > \theta_0, \tag{36}$$

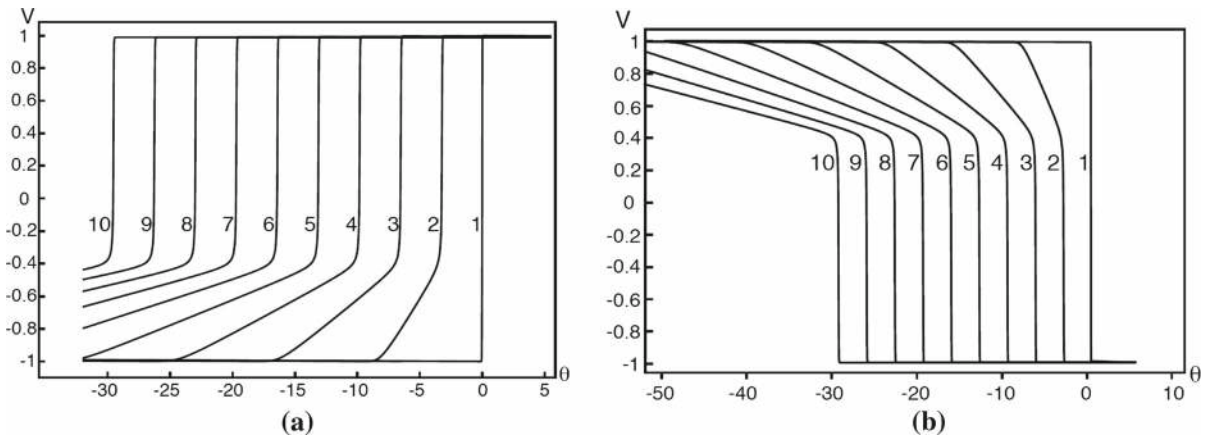
Their solutions are:

$$V = \alpha^2 \frac{\theta_* - \theta_0}{2\Gamma} \left[ 1 + \alpha \frac{\theta_* - \theta_0}{2\Gamma} \right]^{-1}, \quad -\infty < \theta_* < \theta_0.$$

$$V = -\alpha \left[ 1 + \sqrt{2} \tanh \left( \alpha \sqrt{2} \frac{\theta_*}{2\Gamma} \right) \right], \quad \theta_0 < \theta_* < \infty. \tag{37}$$

Stable shocks of both compression and rarefaction are shown in Fig. 3 for the values of  $\Gamma = 0.2$  and  $\Gamma = 1.0$ . The typical front width  $\Delta\theta \sim \Gamma$  increases with dissipation and decreases with nonlinearity.

The processes of shock waves approaching their steady-state form are shown in Fig. 4a for a compression shock, and in Fig. 4b for a rarefaction shock. Both



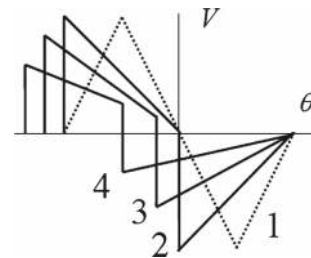
**Fig. 4** Formation of stable fronts of compression (a) and rarefaction (b) for  $\Gamma = 0.01$ . Curves 1–10 correspond to distances  $Z = 0.01, 8, 16, 24, 32, 40, 48, 56, 64, 72$

shocks initially have the shape of a symmetric jump between  $-1$  and  $+1$ . One can see the forerunner moving away from the shock front. As a result of this, the shock waves obtain asymptotic values providing stability. More exactly, the compression shock grows with time from  $-\alpha$  to  $1$ , and the rarefaction shock decreases from  $\alpha$  to  $-1$ .

### 5 An N-wave in a quadratically cubic medium

The so-called N-wave is an asymptotically universal form of any single pulse with zero linear momentum. During propagation the leading section of the wave is compressed and the tail section is stretched. An N-wave can be formed as result of explosion, or at supersonic flight at large distances from aircraft [3]. Both the leading and the tail shocks of a normal N-wave are compressional. A similar problem appears in QC media. For example, for medical diagnostics, it is necessary to calculate the shape of a pulsed shear wave in biological tissue caused by the radiation force of focused ultrasound [21].

For very weak linear dissipation ( $\Gamma \rightarrow 0$ ) the solution to the QC Eq. (5) can be constructed employing a graphic approach (see details in Ref. [2]). The result is shown in Fig. 5. Here the initial bipolar pulse consisting of two triangular regions is shown by the dotted line (curve 1). Curves 2, 3, and 4 correspond to increasing distances  $Z$ . At  $Z = 1$  (curve 2) two steep shocks have formed—a leading shock of compression and a tail shock of rarefaction. Further down the nonlinear



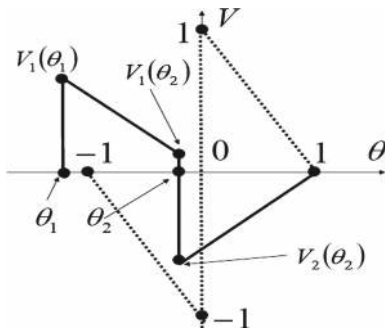
**Fig. 5** Nonlinear evolution of a bipolar pulse in a QC medium with infinitesimal dissipation

evolution (curves 3 and 4), two sections are formed. The positive pressure shape is trapezoidal, and the rarefaction shape is triangular. The two areas of the trapezium and the triangle are equal. The asymptotic form of the N-wave at  $\Gamma \rightarrow 0, Z > 1$  has a simple analytical representation:

$$\begin{aligned} \theta_1 &= -\sqrt{1+Z}, & \theta_2 &= -\frac{1}{2}(\sqrt{1+2Z}-1), \\ V_1(\theta_1) &= \frac{1}{\sqrt{1+Z}}, & V_1(\theta_2) &= \frac{\sqrt{1+2Z}-1}{2(1+Z)}, \\ V_2(\theta_2) &= -\frac{\sqrt{1+2Z}+1}{2(1+Z)}. \end{aligned} \tag{38}$$

The notations used here are shown in Fig. 6, with their positions marked by the large dots.

In Fig. 7 the evolution of the N-wave is shown for different values of the Goldberg number. Already the initial bipolar pulse (curves 1) contains two steep shocks. In Fig. 7a one can still trace the development of trape-



**Fig. 6** Notations to the asymptotic form of N-wave in a QC medium

zoidal and triangular regions, but as distinct from Fig. 5, the dissipation causes the shocks to have a finite width. In Fig. 7b the dissipation is stronger, and the nonlinear aspect manifests itself much less.

**6 An S-wave in a quadratically cubic medium**

The S-wave is another asymptotically universal form of a single pulse with zero linear momentum. Its behavior is opposite the N-wave in that the leading section is stretched and the tail section is compressed.

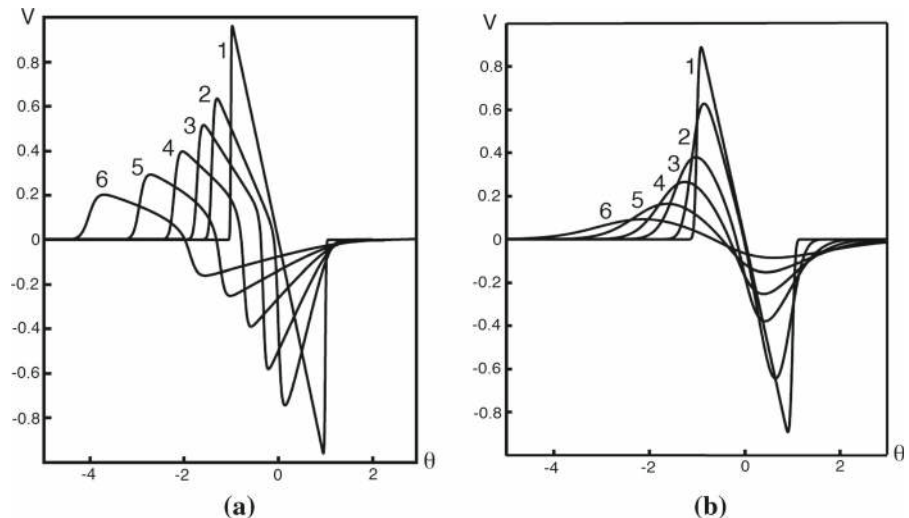
In Fig. 8 the evolution of an initial S-wave containing one shock (curves 1) is shown. Two shock fronts form at weak dissipation (Fig. 8a). The leading front is a rarefaction shock and the tail front is a compression shock. One can trace the development of trapezoidal and triangular regions, but as opposed to Figs. 5 and 7,

the rarefaction is ahead of the compression. In Fig. 8b a strong dissipation suppresses the nonlinear process. The analytical representation for the S-wave, analogous to (38) for the N-wave, can be derived fairly easily but is not presented here.

**7 An initially harmonic wave in a quadratically cubic medium**

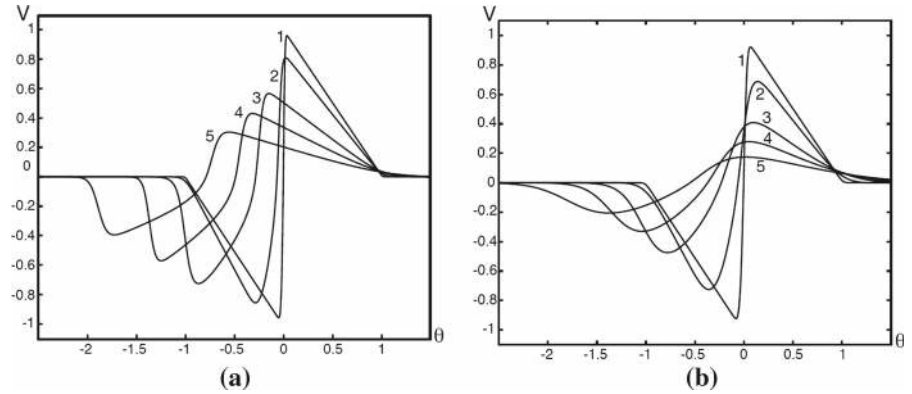
A continuous periodic wave having a sinusoidal shape at the input ( $Z = 0$ ) is fundamental for experiments and applications, because electromagnetic transducers usually generate single-frequency vibrations. Such waves are in quadratic nonlinear media governed by the usual Burgers equation, which describes the transformation of a harmonic time signal profile to sawtooth shapes. Each period has a triangular form and contains a compression shock at the leading front [2–5]. A similar evolution of one period of a continuous sinusoidal input wave in a QC medium is shown in Fig. 9. As for the periodic waves described by the Burgers equation, a universal sawtooth-shaped profile forms at large distances. However, for the QC wave each of the teeth of the saw has a trapezoidal form and contains two shock waves—one of compression and one of rarefaction. Two additional phenomena exist for a QC wave. First, the curves in Fig. 9 demonstrate the shift of profile to the left, which means that the wave propagation velocity is higher. This velocity dependence on intensity is known for quasi-harmonic waves in dispersive

**Fig. 7** N-wave dynamics in a QC nonlinear medium at finite dissipation:  $\Gamma = 0.01$  (a) and for distances: 0.01, 1, 2, 4, 8, 16 (curves 1–6). (b) is constructed for  $\Gamma = 0.1$  and distances 0.01, 0.2, 1, 2, 4, 8 (curves 1–6)

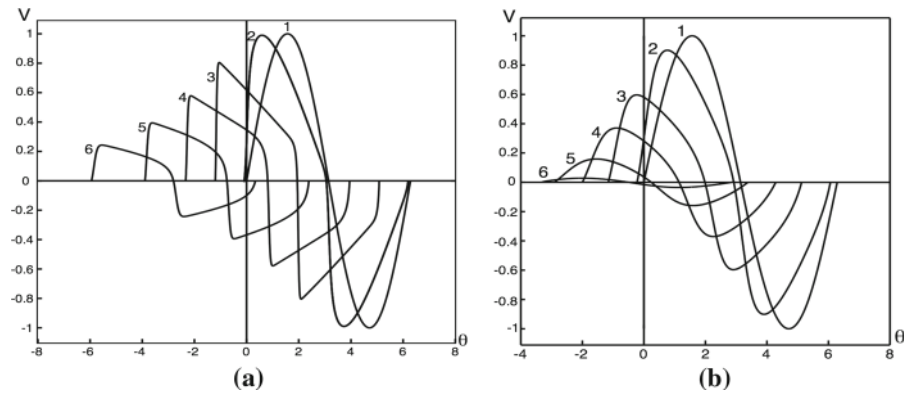




**Fig. 8** S-wave dynamics in a QC nonlinear medium at finite dissipation:  $\Gamma = 0.01$  (a) and for distances 0.01, 0.2, 1, 2, 4 (curves 1–5). **b** is constructed for  $\Gamma = 0.04$  and the same distances



**Fig. 9** Transformation of an initially harmonic wave (curve 1) to a periodic trapezoidal sawtooth-shaped wave in a QC medium taking place when the Goldberg number is  $\Gamma = 0.01$  (a), and the smoother wave obtained for when  $\Gamma = 0.1$  (b). The curves 1–6 correspond to distances  $Z = 0, 1, 4, 8, 16, 32$



media as a self-action effect. It exists for odd nonlinearities, and QC is one exotic example of this nonlinearity type. Secondly, the nonlinear energy loss in the shock fronts seen in Fig. 9 cannot exist in dispersive media.

### 8 Conclusion

In this paper the attention was focused on the possibility of an exact linearization of the quadratically cubic (QC) Eq. (5), on its exact solution, as well as on the behavior of shock fronts and single N-wave and S-wave pulses. All of these mathematical results for QC models have physical applications.

The profiles in Figs. 7, 8 and 9, for the dissipation parameter  $\Gamma \ll 1$ , can be constructed also by the matched asymptotic expansion method [22]. The solutions shown in Fig. 3 can be used as main terms of the internal expansion, and the solution of the quadratically cubic equation at  $\Gamma = 0$  [like formula (38)] can serve as the main term of the external expansion. After

the profiles have been described analytically, it is then possible to calculate the spectral content, the nonlinear loss of energy at the shock fronts, and other physical characteristics.

Finally, we emphasize that this work is intended to draw attention to the quadratically cubic equation model (5) in the belief that detailed studies of it will continue as other QC-modifications of well-known equations [6] are valuable from both mathematical and physical points of view.

One example is to extend the group analysis of these equations, similar to what has been done for integro-differential equations [23]. Wave transformations for integro-differential equations have been treated by similar methods [24, 25].

Another important modification can be done for nonlinear waves in grainy media [26]. When the shear nonlinearity is included, an inhomogeneous Burgers equation appears. It has quadratically cubic properties, exhibiting interesting mathematical descriptions of new physical phenomena. We intend on submitting these results in the nearest future.

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