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Research Article

The Quality of Ranking during Simulated Pairwise Judgments for Examined Approximation Procedures

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An overview of current debates and contemporary research devoted to the modeling of decision-making processes and their facilitation directs attention to the quality of priority ratios estimation through pairwise comparisons. At the core of the process are various approximation procedures for a pairwise comparison matrix which, in a sense, reflects preferences of decision-makers. Certainly, when judgments regarding these preferences are perfectly consistent (cardinally transitive), all approximation procedures coincide and the quality of the prioritization process is exemplary. However, human judgments are very rarely consistent, and thus the quality of priority ratios estimation may significantly vary. Obviously, the range of these variations depends on the applied approximation procedure for a pairwise comparison matrix. Although there are many approximation procedures which can be applied in the prioritization process, it has been promoted for many decades that only one should be applied and no others qualify. This paper suggests this opinion is a fallacy. Research results argue that a genuine, commonly applied approximation procedure for a pairwise comparison matrix may deteriorate the quality of priority ratios estimation. Thus, a number of solutions are also proposed which can improve the process of priority ratios estimation. In order to provide credible and high quality results, the problem is studied via a properly designed and coded seminal simulation algorithm, executed in *Wolfram Mathematica 8.0.*

1. Introduction

The world is a complex system of interacting elements. Thus, people must be supported in expressing their preferences, ordering their priorities, and making tradeoffs to be able to serve the greatest common interest [1, 2]. That is why many support methodologies that have been elaborated to make this process efficient and more credible. Indeed, numerous psychological experiments [3], including the well-known Miller study [4], put forth the notion that humans are not capable of accurately dealing with more than about seven (±2) items at a time.

Humans learn about anything by two means. The first involves examining and studying some phenomenon from the perspective of its various properties and then synthesizing findings and drawing conclusions. The second entails studying some phenomenon in relation to other similar phenomena and correlating them by making comparisons [5]. The latter method leads directly to the essence of the matter, i.e., judgments regarding a phenomenon.

Judgments can be relative and absolute. A relative judgment deals with the identification of some relation between two stimuli both present to the observer [6]. It is stated that humans can make much better relative than absolute judgments [7].

For detailed knowledge, the mind structures complex reality into its constituent parts and in turn into their elements. The number of parts usually ranges between five and nine. By breaking down reality into homogeneous clusters and subdividing those into smaller units, humans can integrate large amounts of information into the structure of a problem and form a more comprehensive picture of a whole system. Abstractly, this process entails structuring a system into a hierarchy which is a complex reality model. Thus, a hierarchy constitutes a structure of multiple levels where the first level is the objective followed successively by levels of factors, criteria, subcriteria, and so on, down to a bottom level of alternatives. The goal of this hierarchy is to evaluate the influence of higher level elements on those of a lower level or alternatively and the contribution of elements in the lower level to the importance or fulfillment of the elements in the levels above. In this context, these latter elements serve as criteria and are called properties.

Generally, a hierarchy can be functional or structural. The latter closely relates to the way a human brain analyzes complexity by breaking down the objects perceived by the senses into clusters, subclusters, and so on. Thus, in structural hierarchies, complex systems are structured into their constituent parts in descending order according to their structural properties. In contrast, in functional hierarchies, complex systems are reduced to their constituent parts according to their essential relationships.

A large number of hierarchies in application are available in literature [8]. It has been argued that the hierarchical classification is the most powerful method applied by the human mind during intellectual reasoning and ordering of information and/or observations.

There are multiple criteria, decision-making support methodology which applies the hierarchical classification described above, i.e., the analytic hierarchy process (AHP). It was developed by Thomas Saaty [9], and although it is a very popular and widely implemented theory of choice, it is also controversial. It should be mentioned that the AHP—as a multicriteria decision-making support system—has received criticism [10–12] usually concerning one of its most controversial aspects, i.e., the phenomenon of alternative rank reversal which will be addressed in this paper.

The conventional procedure of priority ranking in AHP is grounded on the well-defined mathematical structure of consistent matrices and their associated right principal eigenvector's (REV) ability to generate true or approximate values of weights. However, the REV method which the AHP applies constitutes the crucial reason of its criticism which is the main point of reference for this research paper.

A number of authors have noticed that the REV does not optimize any objective function; what entails its results cannot be interpreted in statistical or optimization fashions [13–15]. In particular, solutions obtained with the application of REV cannot be compared to others received from the application of many commonly known methods [15–17]. Besides, unlike many optimization models, REV does not allow decision-makers to introduce additional constraints that may be deemed necessarily according to particular points of view [10, 18–20]. Moreover, REV is supposed to operate only with reciprocal pairwise comparison matrix (PCM)—at least as long as Saaty's consistency index is applied—which entails a limited range of applications and an increase of estimation errors. As a result, in practice, the reciprocity of PCM is a very popular requirement, although many authors argue that it is an artificial condition which impoverish the PCM about information concerning the unknown priority vector that otherwise could have been revealed [14, 21–24]. Furthermore, it was proved that the REV does not satisfy the condition of order preservation (COP) [10] which constituted, for a very long time, the barrier for acceptance of other existing methods, in particular those applying optimization procedures, which were rejected for producing nonunique answers.

Facing the above problems, it seems that debates concerning AHP issues will continue, maintaining the rather high popularity of the AHP, in this way making it prone to validation and valuation from the perspective of its applicability.

Undeniably, sustainable growth of the AHP applications' number (e.g., [2, 25–30]) entails a necessity for research dedicated to problematic issues of the AHP. Certainly, they should be examined, and the questions they provoke should be addressed.

The examination in this paper considers two very crucial issues within the AHP as follows:

(1) Is the right principal eigenvector, as the primary approximation procedure within the AHP, necessary for the priorities ratios estimation during the pairwise comparisons process?

(2) Is the reciprocity of the pairwise comparison matrix a condition which leads to improvement or deterioration of the priority ratios estimation quality?

2. Background

The AHP seems to be the most widely used multiple criteria decision-making approach in the world today. Actual applications in which the AHP results were accepted and used by competent decision-makers and can be found in [1, 5, 7, 22, 31, 32]. Regardless of AHP popularity, the genuine methodology is also undeniably the most valuated, developed, and perfected contemporary methodology, for example, [33–38].

The AHP allows decision-makers to set priorities and make choices on the basis of their objectives, knowledge, and experience in a way that is consistent with their intuitive thought process. AHP has substantial theoretical and empirical support encompassing the study of the human judgmental process by cognitive psychologists. It uses the hierarchical structure of the decision problem, pairwise relative comparisons of the elements in the hierarchy, and a series of redundant judgments. This approach reduces errors and provides a measure of judgment consistency. The process permits accurate priorities to be derived from verbal judgments even though the words themselves may not be very explicit. Thus, it is possible to use words for comparing qualitative factors and then deriving ratio-scale priorities that can be combined with quantitative factors.

To make a proposed solution possible, i.e., derive ratio-scale priorities on the basis of verbal judgments, a scale is utilized to evaluate the preferences for each pair of items. Apparently, the most popular is Saaty's numerical scale which comprises the integers from one (equivalent to a verbal judgment—"equally preferred") to nine (equivalent to a verbal judgment—"extremely preferred") and their reciprocals. However, in conventional AHP applications, it may be desirable to utilize other scales also, i.e., a geometric or any other numerical scale. The former usually consists of the numbers computed in accordance with the formula $2^{n/2}$, where n comprises the integers from minus eight to eight. The latter may, for example, involve arbitrary integers from one to n and their reciprocals.

The first step in using AHP is to develop a hierarchy by breaking a problem down into its primary components. The basic AHP model includes a goal (a statement of the overall objective), criteria (the factors to be considered in reaching the ultimate decision), and alternatives (feasible alternatives that are available to achieve the said ultimate goal). Although the most common and basic AHP structure consists of a goal-criteria-alternatives sequence (Figure 1), AHP can easily support more complex hierarchies. A variety of basic hierarchical structures include the following:

(i) Goal, criteria, subcriteria, scenarios, and alternatives
(ii) Goal, players, criteria, subcriteria, and alternatives
(iii) Goal, criteria, levels of intensities, and many alternatives

The conventional procedure of priority ranking in AHP is grounded on the well-defined mathematical structure of consistent matrices and their associated right-eigenvector's ability to generate true or approximate values of weights.

Oskar Perron, a German mathematician, proved in 1907 that, if $\mathbf{A} = (w_{ij})$, $w_{ij} > 0$, where $i, j = 1, \ldots, n$, and then $\mathbf{A}$ has a simple positive eigenvalue $\lambda_{\max}$ called the principal eigenvalue of $\mathbf{A}$ and $\lambda_{\max} > |\lambda_k|$ for the remaining eigenvalues of $\mathbf{A}$. Furthermore, the principal eigenvector $\mathbf{w} = [w_1, \ldots, w_n]^T$ that is a solution of $\mathbf{A}\mathbf{w} = \lambda_{\max} w$ has $w_i > 0$, $i = 1, \ldots, n$. Thus, the conventional concept of AHP can be presented as follows:

$$\begin{bmatrix} w_1/w_1 & w_1/w_2 & w_1/w_3 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & w_2/w_3 & \cdots & w_2/w_n \\ w_3/w_1 & w_3/w_2 & w_3/w_3 & \cdots & w_3/w_n \\ \vdots & \vdots & \vdots & & \vdots \\ w_n/w_1 & w_n/w_2 & w_n/w_3 & \cdots & w_n/w_n \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \lambda_{\max} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix}. \quad (1)$$

If the relative weights of a set of activities are known, they can be expressed as a pairwise comparison matrix (PCM) as shown above $\mathbf{A}(w)$. Now, knowing $\mathbf{A}(w)$ but not $\mathbf{w}$ (vector of priority ratios), Perron's theorem can be applied to solve this problem for $\mathbf{w}$. The solution leads to n unique values for λ, with an associated vector $\mathbf{w}$ for each of the n values.

PCMs in the AHP reflect relative weights of considered activities (criteria, scenarios, players, and alternatives), so the matrix $\mathbf{A}(w)$ has a special form. Each subsequent row of that matrix is a constant multiple of its first row. In this case, a matrix $\mathbf{A}(w)$ has only one nonzero eigenvalue, and since the sum of the eigenvalues of a positive matrix is equal to the sum of its diagonal elements, the only nonzero eigenvalue in such a case equals the size of the matrix and can be denoted as $\lambda_{\max} = n$.

The norm of the vector $\mathbf{w}$ can be written as $\|\mathbf{w}\| = \mathbf{e}^T\mathbf{w}$ where $e = [1, 1, \ldots, 1]^T$ and $\mathbf{w}$ can be normalized by dividing it by its norm. For uniqueness, $\mathbf{w}$ is referred to in its normalized form.

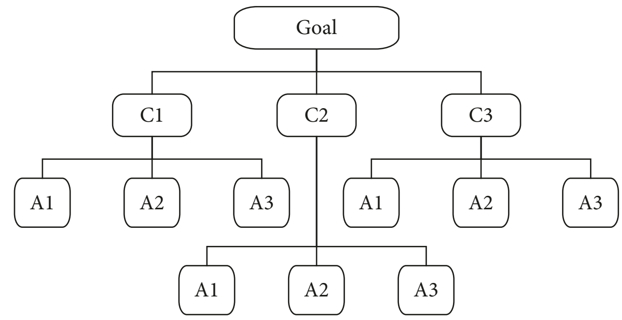

Figure 1: Example of a fundamental three-level hierarchy encompassing three criteria and three alternatives under each criterion.

Theorem 1. *A positive n by n matrix has the ratio form $\mathbf{A}(w) = (w_i/w_j)$, $i, j = 1, \ldots, n$, if, and only if, it is consistent.*

Theorem 2. *The matrix of ratios $\mathbf{A}(w) = (w_i/w_j)$ is consistent if and only if n is its principal eigenvalue and $\mathbf{A}\mathbf{w} = n\mathbf{w}$. Further, $w > 0$ is unique to within a multiplicative constant.*

Definition 1. If the elements of a matrix $\mathbf{A}(w)$ satisfy the condition $w_{ij} = 1/w_{ji}$ for all $i, j = 1, \ldots, n$ then the matrix $\mathbf{A}(w)$ is called *reciprocal*.

Definition 2. The matrix $\mathbf{A}(w)$ is called *ordinal transitive* if the following conditions hold: (A) if for any $i = 1, \ldots, n$, an element a_{ij} is not less than an element a_{ik} then $a_{ij} \geq a_{ik}$ for $i = 1, \ldots, n$, and (B) if for any $i = 1, \ldots, n$, an element a_{ji} is not less than an element a_{ki} then $a_{ji} \geq a_{ki}$ for $i = 1, \ldots, n$.

Definition 3. If the elements of a matrix $\mathbf{A}(w)$ satisfy the condition $w_{ik}w_{kj} = w_{ij}$ for all $i, j, k = 1, \ldots, n$, and the matrix is *reciprocal*, then it is called *consistent* or *cardinal transitive*.

Certainly, in real conditions when AHP is utilized, there is not an $\mathbf{A}(w)$ which would reflect weights given by the vector of priority ratios. As was stated earlier, the human mind is not a reliable measurement device. Assignments, such as "Compare—applying a given ratio scale—your feelings concerning alternative 1 versus alternative 2", do not produce accurate outcomes. Thus, $\mathbf{A}(w)$ is not established but only its estimate $\mathbf{A}(x)$ contains intuitive judgments, more or less close to $\mathbf{A}(w)$ in accordance with experience, skills, specific knowledge, personal taste, and even temporary mood or overall disposition. In such cases, consistency property does not hold, and the relation between elements of $\mathbf{A}(x)$ and $\mathbf{A}(w)$ can be expressed as follows:

$$x_{ij} = e_{ij}w_{ij}, \quad (2)$$

where e_{ij} is a perturbation factor fluctuating near unity. In the statistical approach, e_{ij} reflects a realization of a random variable with a given probability distribution.

It has been shown that, for any matrix, small perturbations in the entries imply similar perturbations in the eigenvalues which is why in the estimation of a true priority vector $\mathbf{w}$, conventional AHP utilizes Perron's theorem. The solution of the matrix equation $\mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w}$ gives us $\mathbf{w}$ as the right principal eigenvector (REV) associated with $\lambda_{\max}$. In practice, the REV solution is obtained by raising the matrix $\mathbf{A}(x)$ to a sufficiently large power; then, the rows of $\mathbf{A}(x)$ are totaled, and the resulting vector is normalized to produce $\mathbf{w}$.

The genuine concept of the AHP undeniably attracts attention, and thus, it evolves (see, e.g., [33–38]). At the same time, it is argued that so long as inconsistency in pairwise judgments is tolerated, the REV is the basic theoretical concept for deriving priority ratios and no other approximation procedure (AP) for priority ratios derivation qualifies. Concurrently, during the last three decades, numerous alternative APs have been proposed: beginning with the most popular, the geometric mean procedure [15, 39, 40], and other methods based on constrained optimization models, e.g., [20, 41, 42], including least squares method [24, 43], and various versions of goal programming methods, e.g., [13, 44–46], through methods based on statistical concepts, e.g., [21, 23, 47], ending with methods based on fuzzy preference description, e.g., [48], and heuristic algorithms, e.g., [49].

3. Facet of the Problem

It has been promoted that the REV approximation procedure (AP) is necessary and sufficient to uniquely establish the ratio-scale rank order inherent in inconsistent pairwise comparison judgments [40]. However, there are alternative APs devised to cope with this problem. Many of them are optimization based and seek a vector w, as a solution of the minimization problem given by the formula

$$\min \mathbf{D}(\mathbf{A}(x), \mathbf{A}(w)), \tag{3}$$

subject to some assigned constraints such as positive coefficients and normalization condition. Because the distance function $\mathbf{D}$ measures an interval between matrices $\mathbf{A}(x)$ and $\mathbf{A}(w)$, differing definitions lead to various prioritization concepts and prioritization results. As an example, Choo and Wedley [16] describes and compares eighteen APs for ranking purposes although some authors suggest there are only fifteen of them that are distinct. Furthermore, since the publication of the above article, a few additional procedures have been introduced to the literature, e.g., [50].

Certainly, when the PCM is consistent, all known procedures coincide. However, in real-life situations, as was discussed earlier, human judgments produce inconsistent PCMs. The inconsistency is a natural consequence of human brain dynamics described earlier and also a consequence of the questioning methodology, mistaken entering of judgment values, and scaling procedure, i.e., rounding errors. It seems crucial to emphasize here that even perfectly consistent PCMs are not error-free only because of rounding errors what can be illustrated on the basis of the following hypothetic example (Example 1).

Example 1. The genuine priority vector $\mathbf{w} = [7/20, 1/4, 1/4, 3/20]$ is considered and derived from it, and $\mathbf{A}(w)$ can be presented as follows:

$$\mathbf{A}(w) = \begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}. \tag{4}$$

Now it is considered that $\mathbf{A}(x)$ produced by a hypothetic decision-maker (DM), whose judgments are perfectly consistent. Even if it is assumed that the selected DM is very trustworthy and can express judgments very precisely, DM is still somehow limited by the necessity of expressing judgments on a scale (the example utilizes Saaty's scale). As such, the DM will produce the PCM ($\mathbf{A}(x)$) which is not error-free because the entries must be, in this case, rounded to the closest values of Saaty's scale. Since $\mathbf{A}(x)$ must be reciprocal (the fundamental requirement of the AHP), the PCM appears as follows:

$$\mathbf{A}(x) = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}. \tag{5}$$

It may be noticed that the above PCM is perfectly consistent, so this construct seems to be exemplary. However, the hypothetical DM, despite best intentions, is burdened with inescapable estimation errors. In the above situation, the priority vector (PV) derived from $\mathbf{A}(x)$ by any AP provides the following priority ratios (PRs): $\mathbf{x} = [2/7, 2/7, 2/7, 1/7]$ which are not equal to those considered exemplary and $\mathbf{w} = [7/20, 1/4, 1/4, 3/20]$. Obviously, the deviation between those PVs can also be expressed by their mean absolute error (MAE), for instance, established by the following formula:

$$\text{MAE}(w, x) = \frac{1}{n}\sum_{i=1}^{n}\left|w_i - x_i\right|, \tag{6}$$

where n is the number of elements within the particular PV. Noticeably, in the above example, MAE equals 1/28. Certainly, this error concerns any AP applied in this case, so it also concerns the REV.

Another exemplary scenario can be considered with the hypothetic case provided earlier (Example 1). Thus, the same genuine PV is reconsidered, $\mathbf{w} = [7/20, 1/4, 1/4, 3/20]$ and $\mathbf{A}(w)$ is derived from that PV as before.

Example 2. However, two PCMs are now considered, i.e., $\mathbf{R}(x)$ and $\mathbf{A}(x)$ produced by a hypothetical DM, whose judgments are rounded to Saaty's scale. In the first scenario, entries of $\mathbf{A}(w)$ are rounded to Saaty's scale and the entries are made reciprocal (a principal condition for a PCM in the AHP) producing

$$\mathbf{R}(x) = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}. \tag{7}$$

In the second scenario, only entries of $\mathbf{A}(w)$ are rounded to Saaty's scale (nonreciprocal case) producing

$$\mathbf{A}(x) = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}. \tag{8}$$

It should be noted that $\mathbf{R}(x)$ is perfectly consistent and $\mathbf{A}(x)$ is not. Tables 1 and 2 present that PVs derived from $\mathbf{R}(x)$ and $\mathbf{A}(x)$ with application of the REV and two other arbitrarily selected APs, mean absolute errors (MAEs) (Formula (6)), and Spearman rank correlation coefficients (SRCs) among $\mathbf{w}*$ (AP) and the genuine $\mathbf{w}$ for the case.

Surprisingly, a very interesting phenomenon can be noted on the basis of information provided in Tables 1 and 2. Apparent are smaller values of MAEs and perfect correlation of ranks between estimated and genuine PV for nonreciprocal versions of an analyzed PCM. Certainly, this conclusion concerns all analyzed approximation procedures.

Regardless of these facts, when only reciprocal PCMs are taken into consideration, Saaty and Hu [40] provide an example of a situation where variability in ranks does not occur for each individual judgment matrix, but it occurs in the overall ranking of the final alternatives due to the application of different APs and the multicriteria process itself. They argue that only the REV possesses a sound mathematical background directly dealing with the question of inconsistency. Furthermore, as they state, only the REV captures the rank order inherent in the inconsistent data in a unique manner. Thus, these statements will be verified with the application of the Monte Carlo simulations.

For that purpose, apart from the REV, four different APs have been arbitrarily selected which were ranked as the best within AHP methodology [16, 42, 43] (Table 3).

Table 1: PV estimates derived from $\mathbf{R}(x)$ with application of the particular AP and two characteristics of PV estimates quality in relation to the genuine PV for the case.

AP	Estimates	Performance measures	
		MAE	SRC
REV	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0357143	0.8164966
LUA	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0357143	0.8164966
LLSM	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0357143	0.8164966

REV: Right Principal Eigenvector; LUA: Logarithmic Utility Approach; LLSM: Logarithmic Least Squares Method.

Table 2: PVs estimates derived from $\mathbf{A}(x)$ with application of the particular AP and two characteristics of PVs estimates quality in relation to the genuine PV for the case.

AP	Estimates	Performance measures	
		MAE	SRC
REV	$[0.309401, 0.267949, 0.267949, 0.154701]^T$	0.0202995	1
LUA	$[0.306135, 0.268645, 0.268645, 0.156576]^T$	0.0219326	1
LLSM	$[0.314288, 0.264284, 0.264284, 0.157144]^T$	0.0178559	1

REV: Right Principal Eigenvector; LUA: Logarithmic Utility Approach; LLSM: Logarithmic Least Squares Method.

Table 3: Formulae for the approximation procedures.

The approximation procedure	Formula for the approximation procedure
Logarithmic utility approach (LUA)	$w_{(\mathrm{LUA})} = \min\sum_{i=1}^{n} \ln^2 \left(\sum_{j=1}^{n} a_{ij} w_j / n w_i\right)$
Sum of squared relative differences method (SRDM)	$w_{(\mathrm{SRDM})} = \min\sum_{i=1}^{n} \left((1/n w_i)\sum_{j=1}^{n} a_{ij} w_j - 1\right)^2$
Logarithmic least squares method (LLSM)	$w_{(\mathrm{LLSM})} = \min\sum_{i=1}^{n}\sum_{j=1}^{n} \ln^2 \left(a_{ij} \left(w_j / w_i\right)\right)$
Simple normalized column sum (SNCS)	$w_{i(\mathrm{SNCS})} = (1/n)\sum_{j=1}^{n} \left(a_{ij} / \sum_{k=1}^{n} a_{kj}\right)$

4. Methodology and Research Results

Taking into account the exemplary study of Saaty and Hu [40], it seems that the best way to evaluate the problem is to examine whether different APs are in fact inferior in the estimation of true PVs whose intent is accurate estimation. From that perspective, only computer simulations can illuminate the question, for it is possible to elaborate an algorithm which enables simulation of different kinds of errors which may occur during the process of judgment, and enable assessment which one from the selected APs delivers the better estimates (from a given perspective) of the genuine PV. Firstly, the process will be depicted in Example 3.

Example 3. It is assumed that the hierarchy consists of three levels: goal, criteria, and alternatives. Then, in order to compare the accuracy of estimations obtained by selected APs, different situations related to various sources of the PCM inconsistency are simulated [42, 50]. Primarily, an inconsistency usually results from errors caused by the nature of human judgments—that can be represented as a realization of some random process in accordance with Formula (2)—and technical errors connected with a realization of the comparison procedure, i.e., rounding errors and errors resulting from the forced reciprocity requirement.

In the example provided, only rounding errors (Saaty's scale) and forced reciprocity are taken into consideration. Let us consider the following ideal model of the AHP framework adopted from [42] with three levels (four criteria and four alternatives):

(1) With respect to the goal:

$$\begin{array}{c} \\ c1 \\ c2 \\ c3 \\ c4 \end{array}\begin{array}{c} \begin{array}{cccc} c1 & c2 & c3 & c4 \end{array} \\ \begin{bmatrix} 1 & 1.4 & 3.5 & 1.16667 \\ 0.714286 & 1 & 2.5 & 0.833333 \\ 0.285714 & 0.4 & 1 & 0.333333 \\ 0.857143 & 1.2 & 3 & 1 \end{bmatrix} \end{array}\begin{array}{c} \text{dPVc} \\ \begin{bmatrix} 0.35 \\ 0.25 \\ 0.10 \\ 0.30 \end{bmatrix} \end{array} \quad (9)$$

(2) With respect to criteria $c1$-$c2$:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array}\begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 1.4 & 2.33333 & 1.4 \\ 0.714286 & 1 & 1.66667 & 1 \\ 0.428571 & 0.6 & 1 & 0.6 \\ 0.714286 & 1 & 1.66667 & 1 \end{bmatrix} \end{array}\begin{array}{c} \text{dPVa} \\ \begin{bmatrix} 0.35 \\ 0.25 \\ 0.15 \\ 0.25 \end{bmatrix} \end{array} \quad (10)$$

(3) With respect to criteria $c3$-$c4$:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array}\begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 0.666667 & 0.285714 & 0.25 \\ 1.5 & 1 & 0.428571 & 0.375 \\ 3.5 & 2.33333 & 1 & 0.875 \\ 4 & 2.66667 & 1.14286 & 1 \end{bmatrix} \end{array}\begin{array}{c} \text{dPVa} \\ \begin{bmatrix} 0.10 \\ 0.15 \\ 0.35 \\ 0.40 \end{bmatrix} \end{array} \quad (11)$$

where "dPVc" and "dPVa" denote, respectively, *designated priority vector of criteria* and *designated priority vector of alternatives*.

After application of a standard AHP synthesis process, the following result is obtained: dCPV = $[0.25, 0.21, 0.23, 0.31]^{\mathrm{T}}$, where dCPV stands for the *designated cumulative priority vector*. For the purpose of scenario illustration, each element of the designated PCM is rounded consecutively to Saaty's numerical scale, and its reciprocity is imposed. Then, on the basis of such PCMs, their respective priority vectors are computed, in the example, with application of the REV, and the CPV (*cumulative priority vector*) is calculated as earlier.

After all these transformations, the model presents itself as follows:

(1) With respect to the goal:

$$\begin{array}{c} \\ c1 \\ c2 \\ c3 \\ c4 \end{array}\begin{array}{c} \begin{array}{cccc} c1 & c2 & c3 & c4 \end{array} \\ \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1/3 & 1/2 & 1 & 1/3 \\ 1 & 1 & 3 & 1 \end{bmatrix} \end{array}\begin{array}{c} \text{PVc}_{\text{REV}} \\ \begin{bmatrix} 0.304999 \\ 0.276859 \\ 0.113143 \\ 0.304999 \end{bmatrix} \end{array} \quad (12)$$

(2) With respect to criteria $c1$-$c2$:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array}\begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1/2 & 1/2 & 1 & 1/2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \end{array}\begin{array}{c} \text{PVa}_{\text{REV}} \\ \begin{bmatrix} 0.285714 \\ 0.285714 \\ 0.142857 \\ 0.285714 \end{bmatrix} \end{array} \quad (13)$$

(3) With respect to criteria $c3$-$c4$:

$$\begin{array}{c} \\ a1 \\ a2 \\ a3 \\ a4 \end{array}\begin{array}{c} \begin{array}{cccc} a1 & a2 & a3 & a4 \end{array} \\ \begin{bmatrix} 1 & 1/2 & 1/4 & 1/4 \\ 2 & 1 & 1/2 & 1/3 \\ 4 & 2 & 1 & 1 \\ 4 & 3 & 1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \text{PVa}_{\text{REV}} \\ \begin{bmatrix} 0.0887547 \\ 0.1611320 \\ 0.3550190 \\ 0.3950950 \end{bmatrix} \end{array} \quad (14)$$

After synthesis, the following result is obtained: $\text{CPV}_{\text{REV}} = [0.2034, 0.2336, 0.2316, 0.3315]^{\mathrm{T}}$, which do not coincide with dCPV = $[0.25, 0.21, 0.23, 0.31]^{\mathrm{T}}$. Moreover, a rank reversal situation can be noticed: dCPV = {2, 4, 3, 1} versus CPV_{REV} = {4, 2, 3, 1}. Thus, even from that singular perspective, very reasonable and desired is a search for other APs that could at least partially eliminate discrepancies depicted here during this illustrative prioritization process.

Thus, the following simulation algorithm was constructed. Assuming that the decisional problem can be presented in the form of a three-level hierarchy (goal, criteria, and alternatives; Figure 1). In order to emulate the problem presented in [40], the hypothetical hierarchy is also designed as a four criteria and four alternatives structure, i.e., $n = 4$ and $m = 4$. In agreement with these assumptions, it is possible to elaborate and execute the simulation algorithm **SA |1|** comprising the following steps:

Step 1. Randomly generate a priority vector $\mathbf{k} = [k_1, \ldots, k_n]^{\mathrm{T}}$ of assigned size $[n \times 1]$ for criteria and related perfect PCM(k) = $\mathbf{K}(k)$.

Step 2. Randomly generate n priority vectors $\mathbf{a_n} = [a_{n,1}, \ldots, a_{n,m}]$ of the assigned size $[m \times 1]$ for alternatives under each criterion and related perfect PCMs(a) = $\mathbf{A_n}(a)$.

Step 3. Compute a total priority vector $\mathbf{w}$ of the size $[m \times 1]$ applying the following procedure: $w_z = k_1 a_z(k_1) + k_2 a_z(k_2) + \ldots + k_n a_z(k_n)$, where $z \in \{1, 2, 3, \ldots, m-1, m\}$.

Step 4. Randomly choose a number e from the assigned interval $[\alpha; \beta]$ on the basis of assigned probability distribution π.

Step 5. Apply separately *Step 5A* and *Step 5B*.

Step 5A. The case of PCM forced reciprocity implementation; replace all elements a_{ij} for $i < j$ of all $\mathbf{A_n}(a)$ with ea_{ij} and all elements k_{ij} for $i < j$ of $\mathbf{K}(k)$ with ek_{ij}.

Step 5B. The case of arbitrary PCM acceptance; replace all elements a_{ij} for $i \neq j$ of all $\mathbf{A_n}(a)$ with ea_{ij}, and all elements k_{ij} for $i \neq j$ of $\mathbf{K}(k)$ with ek_{ij}.

Step 6. Apply separately Step 6A and Step 6B.

Step 6A. When Step 5A is performed, round all values of elements a_{ij} for $i < j$ of all $\mathbf{A_n}(a)$ and all values of elements k_{ij} for $i < j$ of $\mathbf{K}(k)$ to the closest values from a considered scale, and then replace all elements a_{ij} for $i > j$ of all $\mathbf{A_n}(a)$ with $1/a_{ij}$ and all elements k_{ij} for $i > j$ of $\mathbf{K}(k)$ with $1/k_{ij}$.

Step 6B. When Step 5B is performed, round all values of elements a_{ij} for $i \neq j$ of all $\mathbf{A_n}(a)$, and all values of elements k_{ij} for $i \neq j$ of $\mathbf{K}(k)$ to the closest values from a considered scale.

Step 7. On the basis of all perturbed $\mathbf{A_n}(a)$ denoted as $\mathbf{A_n}(a)^*$ and perturbed $\mathbf{K}(k)$ denoted as $\mathbf{K}(k)^*$, compute their respective priorities vectors $\mathbf{a_n^*}$ and $\mathbf{k}^*$ with application of assigned approximation procedure (AP), i.e., REV, LUA, SRDM, LLSM, and SNCS.

Step 8. Compute a total priority vector $\mathbf{w} * (\mathrm{AP})$ of the size $[m \times 1]$ applying the following procedure: $w_z^* = k_1^* a_z^* (k_1^*) + k_2^* a_z (k_2^*) + \ldots + k_n^* a_z (k_n^*)$, where: $z \in \{1, 2, 3, \ldots, m-1, m\}$.

Step 9. Calculate *Spearman rank correlation coefficients*—$\mathrm{SRC}_{\gamma,\chi} (\mathbf{w} * (\mathrm{AP}), \mathbf{w})$ between all $\mathbf{w} * (\mathrm{AP})$ and $\mathbf{w}$, as well-designated estimation precision characteristics, i.e., mean relative errors:

$$RE_{\gamma,\chi} (w * (AP), w) = \frac{1}{m} \sum_{i=1}^{m} \frac{\left| w_i - w_i * (AP) \right|}{w_i}, \tag{15}$$

along with mean relative ratios:

$$RR_{\gamma,\chi} (w * (AP), w) = \frac{1}{m} \sum_{i=1}^{m} \frac{w_i * (AP)}{w_i}. \tag{16}$$

Step 10. Repeat Steps 4 to 9, χ times, where χ denotes a size of the sample.

Step 11. Repeat Steps 1 to 9, γ times, where γ denotes a number of considered AHP models.

Step 12. Return arithmetic average values of all $\mathrm{SRC}_{\gamma,\chi}(\mathbf{w} * (\mathrm{AP}), \mathbf{w})$, $\mathrm{RE}_{\gamma,\chi}(\mathbf{w} * (\mathrm{AP}), \mathbf{w})$, and $\mathrm{RR}_{\gamma,\chi}(\mathbf{w} * (\mathrm{AP}), \mathbf{w})$ computed during all runs in Steps 10 and 11, i.e.,

$$\mathrm{MSRC} (w * (AP), w) = \frac{1}{\gamma \times \chi} \sum_{i=1}^{\gamma \times \chi} \mathrm{SRC}_i (w * (AP), w), \tag{17}$$

$$\mathrm{MRE} (w * (AP), w) = \frac{1}{\gamma \times \chi} \sum_{i=1}^{\gamma \times \chi} RE_i (w * (AP), w), \tag{18}$$

$$\mathrm{MRR} (w * (AP), w) = \frac{1}{\gamma \times \chi} \sum_{i=1}^{\gamma \times \chi} RR_i (w * (AP), w), \tag{19}$$

where $\mathrm{MSRC}(\mathbf{w} * (\mathrm{AP}), \mathbf{w})$, $\mathrm{MRE}(\mathbf{w} * (\mathrm{AP}), \mathbf{w})$, and $\mathrm{MRR}(\mathbf{w} * (\mathrm{AP}), \mathbf{w})$ denotes *mean Spearman rank correlation coefficient*, *average mean relative error*, and *average mean relative ratio*, respectively.

In the first experiment, the probability distribution π attributed in Step 4 to the perturbation factor e is selected arbitrarily to be the *gamma* or *uniform* distribution. These are two of the distribution types most frequently considered in literature for various implementation purposes [24, 43, 50, 51]. Usually recommended are types such as *gamma*, *log-normal*, *truncated normal*, or *uniform*. Apart from these most popular π, one can find applications of the Couchy, Laplace, or *triangle* or *beta* probability distributions (e.g., [52]).

The first simulation scenario also assumes that the perturbation factor e will be drawn from the interval $e \in [0.01; 1.99]$ with its expected value $\mathrm{EV}(e) = 1$. The latter condition comes from a reasonable assumption about human judgments, which undeniably intend to be perfect, although they are not perfect permanently.

Furthermore, the number of alternatives and criteria in a single AHP model will be assigned randomly. "Randomly" without any other explicit specification is meant hereafter "uniformly distributed." All simulation scenarios also assume application of the rounding procedure which in the first scenario operates according to the *geometric scale* described earlier in this paper.

Finally, the first scenario also takes into account the obligatory assumption in conventional AHP applications, i.e., the PCM reciprocity condition. In such cases, only judgments from the upper triangle of a given PCM are taken into account and those from the lower triangle are replaced by the inverses of the former.

The outcomes—mean characteristics—for 30,000 cases ($\chi = 15$ and $\gamma = 2{,}000$) of the first simulation scenario are presented in Table 4.

It may be noticed from Table 4, that the REV can be undeniably classified as the worst AP from the perspective of PRs derived from ranks established on the basis of three different prioritization quality measures, i.e., MRE, MSRC, and MRR. The best two APs, from the viewpoint of this classification, are LLSM (known also as geometric mean procedure (GM)), and LUA. Certainly, the first scenario experiment was designed only to contrast the results presented in [40]. It is the intention to establish wider and more fundamental relationships among the selected APs.

The second simulation scenario was designed to encompass new assumptions not yet taken into account in literature. First of all, taken into consideration were results obtained not only on the basis of reciprocal PCM but also the simulation outcomes of nonreciprocal PCM. Secondly, it was decided to implement into the simulations new intervals for random errors and apply their new probability distribution. As is known, many simulation analyses presented in literature assume very nonsymmetric intervals for a perturbation factor (considering its influence on the particular element of PCM). For example, consider the interval for the perturbation factor applied in the first simulation scenario, i.e., $e \in [0.01; 1.99]$. Under this assumption, it becomes apparent that if some entry of PCM is modified *in plus* by the perturbation factor from that particular interval, it is multiplied maximally by 1.99, so if the original entry is 3, the modified value will be around 6. However, if some entry of PCM is modified *in minus* by the perturbation factor from that particular interval, it may result that some entry will be multiplied by the number 0.01, so in fact the entry will be divided by 100. Thus, in the situation where the original

Table 4: Mean performance measures of arbitrarily selected APs for 30,000 cases.

Scenario details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean rank
Geometric scale	Gamma distribution	FR-PCM	LLSM	0.438438	1	0.682300	2	1.21242	1	1.3 (3)
			REV	0.452614	5	0.668380	5	1.22051	4	4.6 (6)
			LUA	0.447349	2	0.673067	3	1.21792	2	2.3 (3)
			SRDM	0.448759	3	0.671380	4	1.21870	3	3.3 (3)
			SNCS	0.450734	4	0.692453	1	1.24398	5	3.3 (3)
	Uniform distribution	FR-PCM	LLSM	0.288608	1	0.804860	2	1.12813	1	1.3 (3)
			REV	0.302346	4	0.792580	5	1.13530	4	4.3 (3)
			LUA	0.298401	2	0.795767	3	1.13350	2	2.3 (3)
			SRDM	0.299400	3	0.794820	4	1.13400	3	3.3 (3)
			SNCS	0.303463	5	0.808333	1	1.15450	5	3.6 (6)

Note: FR-PCM denotes *forced reciprocity* applied to PCM during simulations.

entry is 9, the modified value will be 0.09, which can be rounded to 1/9 on Saaty's scale. It may be noticed that this modification practically reverses the preference of DM from, for example, extremely preferred A over B, to extremely preferred B over A (applying the Saaty scale).

It is obvious that this very common assumption is imposed by another very crucial and logical assumption which states that the expected value of e in every particular simulation scenario should equal one, i.e., $\mathrm{EV}(e) = 1$. It is quite easy to fulfill that requirement on the basis of an asymmetric interval for the perturbation factor (from the perspective of its influence on a particular element of PCM). However, it is rather a challenge to have this assumption implemented with a symmetric interval for the perturbation factor. That is why commonly applied simulation scenarios minimize the range for the perturbation factor in order to achieve at least the delusion of symmetry for $e \in [0.5;\ 1.5]$. Nevertheless, that objective has been attained with the present research, yet to be achieved by other researchers. Presently, it seems reasonable to apply symmetric intervals to simulations for the perturbation factor because they better reflect true life situations. Thus, different kinds of probability distributions (PDs) were experimented with, and it was discovered that Fisher–Snedecor PD possesses the feature that can be useful in the present analysis. It occurs that, for $n_1 = 14$ and $n_2 = 40$ degrees of freedom for one thousand randomly generated numbers on the basis of this PD, their mean equals 1.03617, so it is very close to unity, and these numbers fluctuate from 0.174526 to 5.57826. So, with these assumptions, we have $e \in [0.174526;\ 5.57826]$, which gives a very symmetric distribution for the perturbation factor, and $\mathrm{EV}(e) \approx 1$. The results of prioritization quality for different selected APs and assumed prioritization quality measures, i.e., MSRC, MRE, and MRR, obtained on the basis of the earlier described simulation scenario, are presented in Table 5.

As can be noticed from Table 5, the REV again is not the dominant AP from the perspective of all simulation scenarios under prescribed frameworks (it takes the third place in the total classification order). Certainly, apparent differences in the PV estimation quality in relation to the selected AP are noticeable for nonreciprocal PCMs. Then, the LUA and SRDM or LLSM dominate over the rest of the selected APs, especially from the perspective of rank correlations which are the crucial issue from the viewpoint of rank preservation phenomena.

This phenomenon will now be addressed further from the perspective of this study's research objectives (preservation of preferences intensity during the prioritization process). Bana e Costa and Vansnick [10] provide the following definition: for all alternatives A1, A2, A3, and A4, A1 dominates A2 and A3 dominates A4, and the extent to which A1 dominates A2 is greater than the extent of which A3 dominates A4, we have not only $w_1 > w_2$ and $w_3 > w_4$ but also $w_1/w_2 > w_3/w_4$ for the derived PV.

Thus, the following scenario, provided in [10], is considered. When the PCM is given as

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 9 \\ 1/2 & 1 & 2 & 4 & 9 \\ 1/3 & 1/2 & 1 & 2 & 8 \\ 1/5 & 1/4 & 1/2 & 1 & 7 \\ 1/9 & 1/9 & 1/8 & 1/7 & 1 \end{bmatrix}, \tag{20}$$

following a common linguistic interpretation for AHP, A1 *strongly* dominates A4 (A1/A4 = 5), and A4 *very strongly* dominates A5 (A4/A5 = 7). That implies A1/A4 < A4/A5. However, the PV derived from the REV provides $[0.4262, 0.2809, 0.1652, 0.1008, 0.0269]^{\mathrm{T}}$ and yields the ratios A1/A4 = 4.218 > A4/A5 = 3.741, which violates the condition of order preservation (COP). On the contrary, the PV derived, e.g., from LUA provides $[0.434659, 0.282449, 0.163602, 0.097671, 0.021620]^{\mathrm{T}}$ and yields the ratios A1/A4 = 4.450245 < A4/A5 = 4.517668 which, contrary to REV, satisfy the COP.

5. Research Breakthrough

As was said by the creator of AHP: . . . *"there is a well-known principle in mathematics that is widely practiced, but seldom enunciated with sufficient forcefulness to impress its importance. A necessary condition that a procedure for solving a problem be a good one is that if it produces desired results, and we perturb the variables of the problem in some small sense, it gives us results that are 'close' to the original ones. (. . .) An extension of this philosophy in problems where order relations*

Table 5: Mean performance measures of arbitrarily selected five different approximation procedures for various uniformly drawn 100,000 AHP models–1,000 hypothetic decisional problems perturbed 100 times each.*

Scenario details			Procedure	MRE	Rank	MSRC	Rank	MRR	Rank	Mean rank
Geometric scale	$n,\ m \in \{3, 4, \ldots, 7\}$	FR-PCM	LLSM	0.123288	4	0.916281	1	1.04646	3	2.6 (6)
			REV	0.123030	1	0.915056	5	1.04546	1	2.3 (3)
			LUA	0.123044	3	0.915489	2	1.04699	4	3
			SRDM	0.123038	2	0.915476	3	1.04567	2	2.3 (3)
			SNCS	0.132926	5	0.915228	4	1.05865	5	4.6 (6)
		NR-PCM	LLSM	0.100511	1	0.930242	4	1.02953	4	3
			REV	0.101523	4	0.930164	5	1.02938	3	4
			LUA	0.100658	2	0.930965	2	1.02926	2	2
			SRDM	0.101310	3	0.930510	3	1.02925	1	2.3 (3)
			SNCS	0.108689	5	0.931026	1	1.04315	5	3.6 (6)
	$n,\ m \in \{8, 9, \ldots, 12\}$	FR-PCM	LLSM	0.079748	4	0.931396	1	1.03319	4	3
			REV	0.079110	1	0.928266	5	1.03116	1	2.3 (3)
			LUA	0.079321	3	0.928817	2	1.03173	3	2.6 (6)
			SRDM	0.079286	2	0.928769	4	1.03166	2	2.6 (6)
			SNCS	0.086223	5	0.928799	3	1.03935	5	4.3 (3)
		NR-PCM	LLSM	0.063936	4	0.943393	3	1.02252	4	3.6 (6)
			REV	0.062735	3	0.942399	5	1.02070	1	3
			LUA	0.061757	1	0.944593	1	1.02109	3	1.6 (6)
			SRDM	0.061852	2	0.944314	2	1.02105	2	2
			SNCS	0.068981	5	0.942764	4	1.02879	5	4.6 (6)
Saaty's scale	$n,\ m \in \{3, 4, \ldots, 7\}$	FR-PCM	LLSM	0.143650	4	0.911381	1	1.06578	4	3
			REV	0.142967	1	0.911151	4	1.06498	1	2
			LUA	0.143069	3	0.911347	2	1.06520	3	2.6 (6)
			SRDM	0.143054	2	0.911320	3	1.06517	2	2.3 (3)
			SNCS	0.155694	5	0.910735	5	1.07850	5	5
		NR-PCM	LLSM	0.116095	1	0.927455	1	1.04681	3	1.6 (6)
			REV	0.116994	4	0.926955	4	1.04705	4	4
			LUA	0.116337	2	0.927129	3	1.04657	1	2
			SRDM	0.116962	3	0.926532	5	1.04658	2	3.3 (3)
			SNCS	0.127154	5	0.927397	2	1.06051	5	4
	$n,\ m \in \{8, 9, \ldots, 12\}$	FR-PCM	LLSM	0.100279	4	0.917231	1	1.04856	4	3
			REV	0.098084	1	0.915833	4	1.04630	1	2
			LUA	0.098648	3	0.916245	2	1.04695	3	2.6 (6)
			SRDM	0.098569	2	0.916193	3	1.04687	2	2.3 (3)
			SNCS	0.106674	5	0.915633	5	1.05424	5	5
		NR-PCM	LLSM	0.078464	4	0.938192	3	1.03563	4	3.6 (6)
			REV	0.077002	3	0.937837	4	1.03422	1	2.6 (6)
			LUA	0.076762	1	0.939669	1	1.03469	3	1.6 (6)
			SRDM	0.076789	2	0.939415	2	1.03464	2	2
			SNCS	0.084307	5	0.937796	5	1.04125	5	5
Average mean rank			LLSM = 2.958	REV = 2.792	LUA = 2.292	SRDM = 2.417	SNCS = 4.542			
Order			4	3	1	2	5			

Note: *AHP models drawn randomly (uniformly) for the assigned set of criteria and alternatives. The scenario assumes application of both the perturbation factor drawn with F-Snedecor probability for $n_1 = 14$ and $n_2 = 40$ degrees of freedom and rounding errors associated with a given scale (geometric or Saaty's). It assumes calculation of performance measures either for reciprocal PCMs (FR-PCMs) or nonreciprocal PCMs (NR-PCMs).

between the variables are important is that on small perturbations of the variables, the procedure produces close, order preserving results [[53], p.18].

With the said notion in mind, an effort was undertaken to verify the statement of followers of the REV, boldly spreading the idea that so long as inconsistency is accepted, the REV is the paramount theoretical basis for deriving a scale and no other concepts qualify.

It is a fact that in order to support some theory, one must verify it through many experiments to validate its reliability. On the contrary, one needs only one example showing it does not work in order to abolish its credibility. Thus, numerous examples were provided indicating that the REV concedes to other devised APs to determine alternative rankings. Although data obtained during experiment simulations are unequivocal, scientific verification of their meaning on the basis of the statistical hypothesis testing theory (HTT) will now be carried out.

If $\mathrm{MSRC}_{\mathrm{AP}}$ and $\mathrm{MSRC}_{\mathrm{REV},}$ respectively, are denoted as mean SRC for a selected AP and mean SRC for the REV, their difference significance can be tested using "t" statistics defined by the following formula:

$$t = R\sqrt{\frac{n-2}{1-R^2}}, \tag{21}$$

where R is the difference between particular MSRCs.

This statistic has a distribution of *t-student* with n minus 2 degrees of freedom (df), where n equals the size of the sample.

The following hypothesis was tested.

$$\begin{aligned} &H_0\text{: } \mathrm{MSRC}_{\mathrm{AP}} - \mathrm{MSRC}_{\mathrm{REV}} = 0, \\ &\quad\text{versus} \\ &H_1\text{: } \mathrm{MSRC}_{\mathrm{AP}} - \mathrm{MSRC}_{\mathrm{REV}} > 0. \end{aligned} \tag{22}$$

In order to conform to the example presented in [40], the data gathered in Table 4 were considered. The simulation framework of that case is $df = 29{,}998$. Thus, for assumed levels of significance $\alpha = 0.01$, $\alpha = 0.02$, or $\alpha = 0.03$, the critical values of *t–student* statistics equal consecutively $t_{0.01} = 2.326472$, $t_{0.02} = 2.053838$, or $t_{0.03} = 1.880865$.

In the situation when a level of tested *t-student* statistics is higher than its critical value for the assumed level of significance, the hypothesis H_0 must be rejected in favor of alternative hypothesis H_1. In the opposite situation, there are no foundations to reject H_0. The selected statistics and their values for the problem evaluation are presented in Table 6.

Clearly, the results of the simulation scenario, designed in accordance with the framework presented in [40], indicate two APs which on the basis of statistical HTT always perform better than the REV, regardless of the preselected PD. It remains to be noted that the performance of selected APs is examined from the perspective of rank preservation phenomena which is reflected in this research by the MSRC between genuine and perturbed PV. It should be evident that the above conclusions, unlike any other before, are the effect of sound statistical reasoning (rigorous significance level) based on the seminal approach toward AHP methodology evaluation grounded on precisely planned and performed simulation study.

In order to develop the concept further, it was decided to expand the simulation program. The results of this endeavor are presented in Table 5. They should be considered as surprising, especially when one realizes that the AP embedded in the AHP is merely placed third in the overall performance ranking. The ranking takes into account not only MSRC, but also MRE and MRR, and the latter is never taken into consideration in previous simulation research. The MRR will now be examined to expand its concept and highlight its innovation.

Lets consider a vector $\mathbf{k}$ of values to be estimated, $\mathbf{k} = [3, 3, 3, 3]$, and three of its estimates, $\mathbf{k_1} = [2, 4, 2, 4]$, $\mathbf{k_2} = [2, 2, 2, 2]$, and $\mathbf{k_3} = [4, 4, 4, 4]$. It may be noted that the MREs of all the estimates (given by formula (18)) are the same and equal 1/3. However, MRRs of the estimates (given by formula (19)) are not the same and equal, respectively, $\mathrm{MRR}_1(k, k_1) = 1$, $\mathrm{MRR}_2(k, k_2) = 2/3$, and $\mathrm{MRR}_3(k, k_3) = 4/3$. Obviously, the goal of estimation is both, i.e., to minimize MREs and maintain the MRRs close to unity. This prerequisite is of great importance when one deals with PVs, i.e., vectors normalized to unity, as in the case of AHP. Certainly, one can encounter the following three estimation scenarios.

Scenario 1. Consider a vector $\mathbf{w}$ of genuine PRs trying to estimate $\mathbf{w} = [0.25, 0.25, 0.25, 0.25]$ and its estimate $\mathbf{w_1} = [0.01, 0.49, 0.05, 0.45]$. This scenario gives a rather high MRE of 0.88, which indicates a mean of 88% volatility for estimated PRs in relation to their primary values and MRR = 1.

Scenarios 2 and 3. Consider a vector $\mathbf{p}$ of genuine PRs trying to estimate $\mathbf{p} = [0.1, 0.2, 0.3, 0.4]$ and its two estimates $\mathbf{p_1} = [0.15, 0.3, 0.25, 0.3]$ and $\mathbf{p_2} = [0.05, 0.1, 0.35, 0.5]$. This situation entails a moderate MRE of 0.35425 for both estimates and two MRRs, i.e., $\mathrm{MRR}_1(p, p_1) = 1.145$ and $\mathrm{RR}_2(p, p_2) = 0.85425$, for the second and third scenario, respectively.

Obviously, during the PRs estimation process, it is desirable to avoid situations exemplified by the first and second scenario. Noticeably, they both have something in common. Apart from estimation discrepancies, they lead to rank reversal of the initial priorities (emphasis added).

Turning back to Table 5, having in mind the imposed simulation scenario, with an F-Snedecor PD mean value of a perturbation factor $\mathrm{EV}(e) = 1.03617$, we can conclude as follows:

(1) The applied measures (MRE, MSRC, and MRR) reflecting the PRs estimation process quality within the simulation framework are always better for nonreciprocal PCMs in relation to their reciprocal equivalents

(2) The applied measures of the PRs estimation quality within the simulation framework indicate better estimation results for a relatively higher number of alternatives

(3) Both MRE and MRR values indicate that the PRs estimation quality within the simulation framework is better when a geometric scale is implemented instead of Saaty's scale for preferences expression of DMs (MRR is then more often less than 1.03617, which indicates less risk of rank reversal)

(4) Last, but not least, the REV procedure is not a dominating procedure during PRs estimation in the simulated framework of the AHP

6. Conclusions

The objective of the research was to generate answers to the following questions:

(1) Is the right principal eigenvector, as the primary approximation procedure within the AHP, necessary for the priorities ratios estimation during the pairwise comparisons process?

(2) Is the reciprocity of the pairwise comparison matrix a condition which leads to improvement or deterioration of the priority ratios estimation quality?

Table 6: MSRC values and principal statistics for the performance test of the REV versus other selected APs.

Scenario details			Procedure	MSRC	R	R^2	t-value	α level*
Geometric scale	Gamma distribution	FR-PCM	LLSM	0.682300	0.01392	0.00019	2.411167969	0.007954
			REV	0.668380	—	—	—	—
			SNCS	0.692453	0.02407	0.00058	4.170635557	0.000015
			LUA	0.673067	0.00469	0.00002	0.811794069	0.208458
			SRDM	0.671380	0.00300	0.00001	0.519600260	0.301673
	Uniform distribution	FR-PCM	LLSM	0.804860	0.01228	0.00015	2.127047876	0.016712
			REV	0.792580	—	—	—	—
			SNCS	0.808333	0.01575	0.00025	2.728747286	0.003181
			LUA	0.795767	0.00319	0.00001	0.551988995	0.290480
			SRDM	0.794820	0.00224	0.00001	0.387967421	0.349021

Note: *the closest significance level providing the ground to reject a tested hypothesis.

The examination involved Monte Carlo simulations which were designed and executed in Wolfram Mathematica Software because only computer simulations provide the opportunity to analyze and evaluate research problems as in this examination. Indeed, only simulations make possible to apply an algorithm which enables evaluation of different kinds of errors which may occur during a process of judgment and enable assessment of selected APs from a given perspective, i.e., which of them delivers better estimates of the genuine PV.

The problem of this research was initially approached from the perspective of the case study presented in [40] and then expanded to other scenarios. The second simulation scenario was designed to encompass new assumptions not yet taken into account by the current literature. First of all, taken into consideration were results obtained not only on the basis of reciprocal PCM but also nonreciprocal PCM simulation outcomes. Secondly, it was decided to implement new intervals for random errors into the simulations and apply their new probability distribution. Commonly, many simulation analyses presented in current literature assume significant nonsymmetric intervals for a perturbation factor (considering its influence on the particular element of PCM). It is obvious that this very common assumption is imposed by another very crucial and logical assumption which states that the expected value of a perturbation factor (e) in every particular simulation scenario should equal one, i.e., $\mathrm{EV}(e) = 1$. It is quite easy to fulfill that requirement on the basis of an asymmetric interval for the perturbation factor (from the perspective of its influence on a particular element of PCM). However, it is rather a challenge to have this assumption implemented with a symmetric interval for the perturbation factor. That is why commonly applied simulation scenarios minimize the range for the perturbation factor in order to achieve at least the illusion of symmetry for $e \in [0.5; 1.5]$. Nevertheless, that objective has been attained with the present research, yet to be achieved by other researchers. Contemporarily, it seems reasonable to apply symmetric intervals to simulations for the perturbation factor because they better reflect real conditions.

Concluding the thorough and seminal investigation which significantly upgrades the methodology of the prioritization process provides the following answers to questions stated in this examination:

(1) The REV as the approximation procedure is neither necessary nor sufficient within the priority ratios estimation process through pairwise comparisons. Moreover, this research reveals two approximation procedures which outperform the REV from a perspective of several conditions, including the condition of order preservation.

(2) The reciprocity of PCM in the prioritization process is the artificial condition which directly leads to the deterioration of the priority ratios estimation quality.

6.1. Further Research. Certainly, there is a necessity for further research in this area; e.g., other approximation procedures can be taken into consideration for the simulation scenario provided in this paper, perhaps other probability distributions for perturbation factors could be considered and studied, and last but not least, the characteristics of the approximation procedures tested herein may also be examined during research concerning the real condition experiments regarding human behavior dynamics in decision-making.

6.2. Proposed. Withhold the PCM reciprocity requirement within the priority ratios estimation process through pairwise comparisons, and consider the replacement of the REV as the approximation procedure within the priority ratios estimation process through pairwise comparisons in favor of the approximation procedure which prevailed over the REV.

To recapitulate in conjunction with other contemporary and seminal research papers, e.g., [2, 34–38, 50, 54, 55], the results of this scientific research enriches the state of knowledge regarding the true value of the priority ratios estimation process through pairwise comparisons which is widely recognized and applied in many MCDM support systems. Hopefully, the results of this authentic, freshly finished examination will improve the quality of human's prospective choices.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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