THE QUANTUM LEFSCHETZ PRINCIPLE FOR VECTOR BUNDLES AS A MAP BETWEEN GIVENTAL CONES

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Abstract. Givental has defined a Lagrangian cone in a symplectic vector space which encodes all genus-zero Gromov–Witten invariants of a smooth projective variety X. Let Y be the subvariety in X given by the zero locus of a regular section of a convex vector bundle. We review arguments of Iritani, Kim–Kresch–Pantev, and Graber, which give a very simple relationship between the Givental cone for Y and the Givental cone for Euler-twisted Gromov–Witten invariants of X. When the convex vector bundle is the direct sum of nef line bundles, this gives a sharper version of the Quantum Lefschetz Hyperplane Principle.

1. Gromov–Witten Invariants and Twisted Gromov–Witten Invariants

Given a smooth projective variety X, one can define *Gromov–Witten invariants* of X [17, 18]:

$$\left\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n}\right\rangle^X_{g,n,d} := \int_{[X_{g,n,d}]^{\mathrm{vir}}} \prod_{i=1}^{i=n} \mathrm{ev}_i^\star \gamma_i \cup \psi_i^{k_i} \tag{1}$$

Notation here is by now standard; a list of notation and definitions can be found in Appendix A. Given a class $A \in H^\bullet(X_{g,n,d};\mathbb{Q})$, we can include it in the integral (1), writing:

$$\left\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n};A\right\rangle^X_{g,n,d} := \int_{[X_{g,n,d}]^{\mathrm{vir}}} A\cup\prod_{i=1}^{i=n} \mathrm{ev}_i^\star \gamma_i \cup \psi_i^{k_i} \tag{2}$$

In particular, we can consider *twisted Gromov–Witten invariants* [8]. Let $E \to X$ be a vector bundle, and let $\boldsymbol{c}(\cdot)$ be an invertible multiplicative characteristic class. We can evaluate $\boldsymbol{c}$ on classes in K-theory by setting $\boldsymbol{c}(A \ominus B) = \frac{\boldsymbol{c}(A)}{\boldsymbol{c}(B)}$. The twisting class $E_{g,n,d} \in K^0(X_{g,n,d})$ is defined by $E_{g,n,d} = \pi_! \,\mathrm{ev}^\star E$, where

$$\begin{tikzcd} C \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & X \\ X_{g,n,d} \end{tikzcd}$$

is the universal family over the moduli space of stable maps. $(\boldsymbol{c}, E)$-twisted Gromov–Witten invariants of X are intersection numbers of the form:

$$\left\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n};\boldsymbol{c}(E_{g,n,d})\right\rangle^X_{g,n,d} \tag{3}$$

Consider the S^1-action on vector bundles $V \to B$ which rotates the fibers of V and leaves the base B invariant. The S^1-equivariant Euler class $\boldsymbol{e}(\cdot)$ is invertible over the field of fractions $\mathbb{Q}(\lambda)$ of $H^\bullet_{S^1}\big(\{\text{point}\}\big) = \mathbb{Q}[\lambda]$. Taking $\boldsymbol{c} = \boldsymbol{e}$, we refer to twisted Gromov–Witten invariants (3) as Euler-twisted Gromov–Witten invariants.

1991 *Mathematics Subject Classification.* 14N35 (Primary); 53D45 (Secondary).

Key words and phrases. Gromov–Witten invariants, quantum cohomology, Quantum Lefschetz Hyperplane Principle, Givental's Lagrangian cone, convex vector bundle.

Givental has defined a Lagrangian cone $\mathcal{L}_X$ in a symplectic vector space $\mathcal{H}_X$ which encodes all genus-zero Gromov–Witten invariants of X [13, 14]. Fix a basis $\{\phi_\epsilon\}$ for $H^\bullet(X;\mathbb{Q})$, and let $\{\phi^\epsilon\}$ denote the dual basis with respect to the Poincaré pairing $(\cdot,\cdot)$ on $H^\bullet(X)$, so that $(\phi_\mu, \phi^\nu) = \delta_\mu^\nu$. Let Λ_X denote the Novikov ring of X; this is defined in Appendix A. Consider the vector space (or rather, free Λ_X-module):

$$\mathcal{H}_X := H^\bullet(X;\Lambda_X) \otimes \mathbb{C}(\!(z^{-1})\!)$$

equipped with the symplectic form (or rather, Λ_X-valued symplectic form):

$$\Omega_X(f,g) := \mathrm{Res}_{z=0}\, \big(f(-z), g(z)\big)\, dz$$

Let $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots$, where $t_i \in H^\bullet(X;\Lambda_X)$. A general point on Givental's Lagragian cone $\mathcal{L}_X \subset \mathcal{H}_X$ has the form:

$$\mathbf{J}_X(\mathbf{t}) := -z + \mathbf{t}(z) + \sum \frac{Q^d}{n!} \left\langle t_{k_1}\psi_1^{k_1}, \ldots, t_{k_n}\psi_n^{k_n}, \phi^\epsilon \psi_{n+1}^m \right\rangle_{0,n+1,d}^X \phi_\epsilon\, (-z)^{-m-1} \tag{4}$$

where the sum runs over non-negative integers n and m, multi-indices $k = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$, degrees $d \in H_2(X;\mathbb{Z})$, and basis indices ϵ. Knowing the Lagrangian submanifold $\mathcal{L}_X$ is equivalent to knowing all genus-zero Gromov–Witten invariants (1) of X.

A similar Lagrangian cone encodes all genus-zero Euler-twisted Gromov–Witten invariants of X. Consider the twisted Poincaré pairing $(\alpha,\beta)_{\boldsymbol{e}} = \int_X \alpha \cup \beta \cup \boldsymbol{e}(E)$, and the twisted symplectic form:

$$\Omega_{\boldsymbol{e}}(f,g) := \mathrm{Res}_{z=0}\, \big(f(-z), g(z)\big)_{\boldsymbol{e}}\, dz$$

on $\mathcal{H}_X$. Let $\{\phi_{\boldsymbol{e}}^\epsilon\}$ denote the basis dual to $\{\phi_\epsilon\}$ with respect to the twisted Poincaré pairing, so that $(\phi_\mu, \phi_{\boldsymbol{e}}^\nu)_{\boldsymbol{e}} = \delta_\mu^\nu$. A general point on the Lagrangian cone $\mathcal{L}_{\boldsymbol{e}} \subset \big(\mathcal{H}_X, \Omega_{\boldsymbol{e}}\big)$ has the form:

$$\mathbf{J}_{\boldsymbol{e}}(\mathbf{t}) := -z + \mathbf{t}(z) + \sum \frac{Q^d}{n!} \left\langle t_{k_1}\psi_1^{k_1}, \ldots, t_{k_n}\psi_n^{k_n}, \phi_{\boldsymbol{e}}^\epsilon \psi_{n+1}^m ; \boldsymbol{e}(E_{0,n+1,d}) \right\rangle_{0,n+1,d}^X \phi_\epsilon\, (-z)^{-m-1} \tag{5}$$

where the sum runs over the same set as above. Knowing $\mathcal{L}_{\boldsymbol{e}}$ is equivalent to knowing all genus-zero Euler-twisted Gromov–Witten invariants of X. In this expository note, we describe a close relationship, in the case where the vector bundle E is convex, between Euler-twisted invariants of X and Gromov–Witten invariants of the subvariety $Y \subset X$ defined by a regular section of E. We prove:

Theorem 1.1. *Let X be a smooth projective variety. Let $E \to X$ be a convex vector bundle, let Y be the subvariety in X defined by a regular section of E, and let $i\colon Y \to X$ be the inclusion map. Let $\mathbf{J}_{\boldsymbol{e}}$ denote the general point* (5) *on the Lagrangian cone $\mathcal{L}_{\boldsymbol{e}}$ for Euler-twisted Gromov–Witten invariants of X. Let $\mathbf{J}_Y$ denote the general point on the Lagrangian cone $\mathcal{L}_Y$ for genus-zero Gromov–Witten invariants of Y, as in* (4)*. Then the non-equivariant limit $\mathbf{J}_{\boldsymbol{e}}\big|_{\lambda=0}$ is well-defined and satisfies:*

$$i^\star \mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0} = \mathbf{J}_Y(i^\star \mathbf{t})$$

In particular, $i^\star \mathcal{L}_{\boldsymbol{e}}\big|_{\lambda=0} \subset \mathcal{L}_Y$.
Throughout here we have applied the homomorphism $Q^\delta \mapsto Q^{i_\star \delta}$ to the Novikov ring of Y.

Remark 1.2. A vector bundle $E \to X$ is called convex if and only if $H^1(C, f^\star E) = 0$ for all stable maps $f\colon C \to X$ such that the curve C has genus zero. Globally generated vector bundles are automatically convex, as are direct sums of nef line bundles.

Remark 1.3. If the dimension of Y is at least 3 then, by the Lefschetz theorem, the homomorphism of Novikov rings $\Lambda_Y \to \Lambda_X$ given by $Q^\delta \mapsto Q^{i_\star \delta}$ is an isomorphism.

Remark 1.4. In the non-equivariant limit, the map $i^\star\colon \mathcal{H}_X \to \mathcal{H}_Y$ becomes symplectic: it satisfies $i^\star \Omega_{\boldsymbol{e}}\big|_{\lambda=0} = \Omega_Y$. Thus Theorem 1.1 fits neatly into a general story that encompasses the Crepant Resolution Conjecture [9, 10], Brown's toric bundle theorem [2], and so on: geometrically-natural operations in Gromov–Witten theory give rise to symplectic transformations of Givental's symplectic space that preserve the Lagrangian cones.

Key Remark 1.5. Only the statement of Theorem 1.1 is new. As we will see, the proof is a very minor variation of an argument by Iritani [15, Proposition 2.4]. Iritani's result in turn builds on arguments by Kim–Kresch–Pantev [16] and Graber [21, §2].

Remark 1.6. Theorem 1.1 improves upon [8, formula 19], which roughly speaking, in the special case where E is the direct sum of nef line bundles, relates $\mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0}$ to $i_\star \mathbf{J}_Y(i^\star \mathbf{t})$. The improved version determines invariants of Y with one insertion (that at the last marked point) involving an arbitrary cohomology class on Y, whereas the original version determined only invariants of Y such that all insertions are pullbacks of cohomology classes on X. When combined with the Lee–Pandharipande reconstruction theorem [19] this determines, under moderate hypotheses on Y, the big quantum cohomology of Y. This should be compared with §0.3.2 of ibid., which gives a reconstruction result for Gromov–Witten invariants of Y such that all insertions are pullbacks of cohomology classes on X. One can use the same approach together with the Abelian/Non-Abelian Correspondence with bundles [4, §6.1] to determine the genus-zero Gromov–Witten invariants of many subvarieties of flag manifolds and partial flag bundles.

Remark 1.7. The formulation in Theorem 1.1 is well-suited to proving mirror theorems for toric complete intersections or subvarieties of flag manifolds. One first obtains a family $t \mapsto I_{\boldsymbol{e}}(t,z)$ of elements of $\mathcal{L}_{\boldsymbol{e}}$, by combining the Mirror Theorem for toric varieties or toric Deligne–Mumford stacks [3,5,12] with the Quantum Lefschetz theorem [8] or the Abelian/Non-Abelian Correspondence with bundles [4, §6.1]. After taking the non-equivariant limit $\lambda \to 0$ and applying Theorem 1.1, one can then argue as in [8, §9] or [6, Example 9].

2. The Proof of Theorem 1.1

2.1. **The Non-Equivariant Limit Exists.** For the remainder of this note, we consider only stable maps of genus zero. Since E is convex, we have that $R^1\pi_\star \operatorname{ev}^\star E = 0$ and hence that $E_{0,n+1,d}$ is a vector bundle. The fiber of $E_{0,n+1,d}$ over a stable map $f\colon C \to X$ is $H^0(C, f^\star E)$, and thus there is an exact sequence of vector bundles:

$$0 \longrightarrow E'_{0,n+1,d} \longrightarrow E_{0,n+1,d} \xrightarrow{\operatorname{ev}_{n+1}} \operatorname{ev}_{n+1}^\star E \longrightarrow 0 \tag{6}$$

This implies that $\boldsymbol{e}(E_{0,n+1,d}) = \boldsymbol{e}(E'_{0,n+1,d})\boldsymbol{e}(\operatorname{ev}_{n+1}^\star E)$. The Projection Formula, together with the fact that $\phi^\epsilon = \phi^\epsilon_{\boldsymbol{e}} \boldsymbol{e}(E)$, gives that:

$$\mathbf{J}_{\boldsymbol{e}}(\mathbf{t}) = -z + \mathbf{t}(z) + \sum \frac{Q^d}{n!} (\operatorname{ev}_{n+1})_\star \left[[X_{0,n+1,d}]^{\mathrm{vir}} \cap \boldsymbol{e}(E'_{0,n+1,d}) \cup \psi_{n+1}^m \cup \prod_{i=1}^n \operatorname{ev}_i^\star t_{k_i} \cup \psi_i^{k_i} \right] (-z)^{-m-1}$$

This makes it clear that the non-equivariant limit $\mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0}$ exists. Let us write $e(\cdot)$ for the non-equivariant Euler class, noting that $e(\cdot)$ is the non-equivariant limit of $\boldsymbol{e}(\cdot)$.

2.2. **A Comparison of Virtual Fundamental Classes.** Consider the diagram:

$$
\begin{tikzcd}
\coprod_{\delta : i_\star \delta = d} Y_{0,n+1,\delta} \arrow[r, "G"] \arrow[d, "\mathrm{ev}"] & Z \arrow[r, "F"] \arrow[d, "\mathrm{ev}"] & X_{0,n+1,d} \arrow[d, "\mathrm{ev}"] \\
Y^{n+1} \arrow[r, "g"] \arrow[dr, "r"'] & X^n \times Y \arrow[r, "f"] \arrow[d, "q"] & X^{n+1} \arrow[d, "p"] \\
& Y \arrow[r, "i"] & X
\end{tikzcd}
\tag{7}
$$

where p, q, and r are projections onto the last factor of their domains (which are products); f and g are induced by the inclusion $i \colon Y \to X$; the maps ev in the first and third columns are the evaluation maps $\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_{n+1}$; the upper right-hand square is Cartesian; the composition $G \circ F$ is the union of canonical inclusions $Y_{0,n+1,\delta} \to X_{0,n+1,d}$; and the map G is defined by the universal property of the fiber product Z. The stack Z consists of those stable maps in $X_{0,n+1,d}$ such that the last marked point lies in Y; it is the zero locus of the section $\mathrm{ev}_{n+1}^\star s \in \Gamma(X_{0,n+1,d}, \mathrm{ev}_{n+1}^\star E)$. The map ev in the second column is also given by $\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_{n+1}$.

Proposition 2.1. *With notation as above, we have:*

(A)

$$f^!\Big(e(E'_{0,n+1,d}) \cap [X_{0,n+1,d}]^{\mathrm{vir}}\Big) = \sum_{\delta : i_\star \delta = d} G_\star [Y_{0,n+1,\delta}]^{\mathrm{vir}}$$

(B) *For any* $(k_1, \ldots, k_{n+1}) \in \mathbb{N}^{n+1}$*:*

$$f^\star \mathrm{ev}_\star \Big(\psi_1^{k_1} \cup \cdots \cup \psi_{n+1}^{k_{n+1}} \cup e(E'_{0,n+1,d}) \cap [X_{0,n+1,d}]^{\mathrm{vir}}\Big) = \sum_{\delta : i_\star \delta = d} g_\star \mathrm{ev}_\star \Big(\psi_1^{k_1} \cup \cdots \cup \psi_{n+1}^{k_{n+1}} \cap [Y_{0,n+1,\delta}]^{\mathrm{vir}}\Big)$$

Proof. Let $0_X \colon X_{0,n+1,d} \to E_{0,n+1,d}$, $0'_X \colon X_{0,n+1,d} \to E'_{0,n+1,d}$, $0'_Z \colon Z \to E'_{0,n+1,d}\big|_Z$ denote the zero sections. Consider the Cartesian diagram:

$$
\begin{tikzcd}
\coprod_{\delta : i_\star \delta = d} Y_{0,n+1,\delta} \arrow[r, "G"] \arrow[d, "G"] & Z \arrow[r, "F"] \arrow[d, "\tilde{s}|_Z"] & X_{0,n+1,d} \arrow[dd, "\tilde{s}"] \\
Z \arrow[r, "0'_Z"] \arrow[d, "F"] & E'_{0,n+1,d}\big|_Z \arrow[d] & \\
X_{0,n+1,d} \arrow[r, "0'_X"] & E'_{0,n+1,d} \arrow[r, "j"] & E_{0,n+1,d}
\end{tikzcd}
$$

where j is the inclusion from (6) and $\tilde{s}$ is the section of $E_{0,n+1,d}$ induced by the section $s\colon X \to E$ that defines Y. Note that, on the bottom row, $0'_X \circ j = 0_X$. We have:

$$
\begin{aligned}
\sum_{\delta: i_\star \delta = d} G_\star [Y_{0,n+1,\delta}]^{\mathrm{vir}} &= \sum_{\delta: i_\star \delta = d} G_\star 0^!_X [X_{0,n+1,d}]^{\mathrm{vir}} && \text{(functoriality [16])} \\
&= \sum_{\delta: i_\star \delta = d} G_\star (0'_X)^! j^! [X_{0,n+1,d}]^{\mathrm{vir}} && \text{(functoriality [11, Theorem 6.5])} \\
&= \sum_{\delta: i_\star \delta = d} (0'_X)^\star (\tilde{s}|_Z)_\star j^! [X_{0,n+1,d}]^{\mathrm{vir}} && \text{(by [11, Theorem 6.2])} \\
&= e\big(E'_{0,n+1,d}|_Z\big) \cap j^! [X_{0,n+1,d}]^{\mathrm{vir}} \\
&= j^! \Big(e(E'_{0,n+1,d}) \cap [X_{0,n+1,d}]^{\mathrm{vir}} \Big) \\
&= f^! \Big(e(E'_{0,n+1,d}) \cap [X_{0,n+1,d}]^{\mathrm{vir}} \Big)
\end{aligned}
$$

This proves (A). Since $f^\star \operatorname{ev}_\star = \operatorname{ev}_\star f^!$ [11, Theorem 6.2] and $g_\star \operatorname{ev}_\star = \operatorname{ev}_\star G_\star$, and since the classes ψ_i on Z and on $Y_{0,n+1,\delta}$ are pulled back from the class ψ_i on $X_{0,n+1,d}$, (A) implies (B). $\square$

2.3. **Applying the Projection Formula.** We now deduce Theorem 1.1 from Proposition 2.1. This amounts to repeated application of the Projection Formula. Recall the diagram (7). The non-equivariant limit $\mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0}$ is equal to:

$$
\begin{aligned}
&-z + \mathbf{t}(z) + \sum \frac{Q^d}{n!} (\operatorname{ev}_{n+1})_\star \left[[X_{0,n+1,d}]^{\mathrm{vir}} \cap e(E'_{0,n+1,d}) \cup \psi_{n+1}^m \cup \prod_{i=1}^n \operatorname{ev}_i^\star t_{k_i} \cup \psi_i^{k_i} \right] (-z)^{-m-1} \\
&\quad = -z + \mathbf{t}(z) + \sum \frac{Q^d}{n!(-z)^{m+1}} p_\star \left[\operatorname{ev}_\star \left([X_{0,n+1,d}]^{\mathrm{vir}} \cap e(E'_{0,n+1,d}) \cup \psi_{n+1}^m \cup \prod_{i=1}^n \psi_i^{k_i} \right) \cup \bigotimes_{i=1}^n t_{k_i} \right]
\end{aligned}
$$

Using $i^\star p_\star = q_\star f^\star$, we see that the pullback $i^\star \mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0}$ is:

$$
-z + i^\star \mathbf{t}(z) + \sum \frac{Q^d}{n!(-z)^{m+1}} q_\star \left[f^\star \operatorname{ev}_\star \left([X_{0,n+1,d}]^{\mathrm{vir}} \cap e(E'_{0,n+1,d}) \cup \psi_{n+1}^m \cup \prod_{i=1}^n \psi_i^{k_i} \right) \cup \bigotimes_{i=1}^n t_{k_i} \right]
$$

Proposition 2.1(B) now gives:

$$
i^\star \mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0} = -z + i^\star \mathbf{t}(z) + \sum{}' \frac{Q^{i_\star \delta}}{n!(-z)^{m+1}} q_\star \left[g_\star \operatorname{ev}_\star \left([Y_{0,n+1,\delta}]^{\mathrm{vir}} \cup \psi_{n+1}^m \cup \prod_{i=1}^n \psi_i^{k_i} \right) \cup \bigotimes_{i=1}^n t_{k_i} \right]
$$

where the sum $\sum'$ runs over non-negative integers n and m, multi-indices $k = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$, degrees $\delta \in H_2(Y;\mathbb{Z})$, and basis indices ϵ. Applying the Projection Formula again, we see that:

$$
\begin{aligned}
i^\star \mathbf{J}_{\boldsymbol{e}}(\mathbf{t})\big|_{\lambda=0} &= -z + i^\star \mathbf{t}(z) + \sum{}' \frac{Q^{i_\star \delta}}{n!(-z)^{m+1}} q_\star \left[g_\star \operatorname{ev}_\star \left([Y_{0,n+1,\delta}]^{\mathrm{vir}} \cup \psi_{n+1}^m \cup \prod_{i=1}^n \psi_i^{k_i} \right) \cup \bigotimes_{i=1}^n t_{k_i} \right] \\
&= -z + i^\star \mathbf{t}(z) + \sum{}' \frac{Q^{i_\star \delta}}{n!(-z)^{m+1}} q_\star g_\star \left[\operatorname{ev}_\star \left([Y_{0,n+1,\delta}]^{\mathrm{vir}} \cup \psi_{n+1}^m \cup \prod_{i=1}^n \psi_i^{k_i} \right) \cup g^\star \bigotimes_{i=1}^n t_{k_i} \right] \\
&= -z + i^\star \mathbf{t}(z) + \sum{}' \frac{Q^{i_\star \delta}}{n!(-z)^{m+1}} r_\star \left[\operatorname{ev}_\star \left([Y_{0,n+1,\delta}]^{\mathrm{vir}} \cup \psi_{n+1}^m \cup \prod_{i=1}^n \psi_i^{k_i} \right) \cup \bigotimes_{i=1}^n i^\star t_{k_i} \right] \\
&= \mathbf{J}_Y(i^\star \mathbf{t})\big|_{Q^\delta \mapsto Q^{i_\star \delta}}
\end{aligned}
$$

The Theorem is proved. $\square$

Remark 2.2. Let X be a smooth Deligne–Mumford stack with projective coarse moduli space, let $E \to X$ be a convex vector bundle, let Y be the substack in X defined by a regular section of E, and let $i\colon IY \to IX$ be the map of inertia stacks induced by the inclusion $Y \to X$. The analog of Theorem 1.1 holds in this context, with the same proof: cf. [15, Proposition 2.4]. Note that a convex line bundle on a Deligne–Mumford stack is necessarily the pullback of a line bundle on the coarse moduli space [7].

Acknowledgements

I thank Alessio Corti, Tom Graber, Hiroshi Iritani, Yuan-Pin Lee, and Andrea Petracci for useful conversations.

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Appendix A. Notation

What follows is a list of notation and definitions: first for symbols in Roman font, then for Greek symbols, then for miscellaneous symbols.

$\boldsymbol{c}$	an invertible multiplicative characteristic class
$\boldsymbol{e}$	the S^1-equivariant Euler class; see page 1 for the definition of the S^1-action
e	the non-equivariant Euler class
E	a convex vector bundle over X
$E_{g,n,d}$	the twisting class $E_{g,n,d} \in K^0(X_{g,n,d})$; see page 1
$E'_{0,n+1,d}$	a sub-bundle of $E_{0,n+1,d}$; see page 3
ev_i	the evaluation map $X_{g,n,d} \to X$ at the ith marked point
$\mathcal{H}_X$, $\mathcal{H}_Y$	Givental's symplectic vector space; see page 2
$\mathcal{L}_{\boldsymbol{e}}$	Givental's Lagrangian cone for Euler-twisted invariants of X; see page 2
$\mathcal{L}_X$, $\mathcal{L}_Y$	Givental's Lagrangian cone for X, Y; see page 2
i	the inclusion map $Y \to X$
j	the inclusion map $E'_{0,n+1,d} \to E_{0,n+1,d}$
$\mathbf{J}_{\boldsymbol{e}}(\mathbf{t})$	a general point on $\mathcal{L}_{\boldsymbol{e}}$; see (5)
$\mathbf{J}_X(\mathbf{t})$	a general point on $\mathcal{L}_X$; see (4)
k_i	a non-negative integer
Q^d	the representative of $d \in H_2(X;\mathbb{Z})$ in the Novikov ring Λ_X
$\mathbf{t}$	$\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots$ where $t_i \in H^\bullet(X)$
t_i	a cohomology class on X
X	a smooth projective variety
$X_{g,n,d}$	the moduli space of stable maps to X, from genus-g curves with n marked points, of degree $d \in H_2(X;\mathbb{Z})$ [17,18]
$[X_{g,n,d}]^{\mathrm{vir}}$	the virtual fundamental class of the moduli space of stable maps to X [1,20]
Y	a subvariety of X cut out by a regular section of E
$Y_{g,n,d}$	the moduli space of stable maps to Y, from genus-g curves with n marked points, of degree $d \in H_2(Y;\mathbb{Z})$ [17,18]
$[Y_{g,n,d}]^{\mathrm{vir}}$	the virtual fundamental class of the moduli space of stable maps to Y [1,20]
γ_i	a cohomology class on X
λ	the generator of $H^\bullet_{S^1}\big(\{\text{point}\}\big)$ given by the first Chern class of $\mathcal{O}(1) \to \mathbb{C}P^\infty \cong BS^1$
Λ_X	the Novikov ring of X; this is a completion of the group ring $\mathbb{Q}\big[H_2(X;\mathbb{Z})\big]$ with respect to the valuation $v(Q^d) = \int_d \omega$, where Q^d is the representative of $d \in H_2(X;\mathbb{Z})$ in the group ring and ω is the Kähler form on X
ϕ_ϵ	an element of the basis $\{\phi_\epsilon\}$ for $H^\bullet(X;\mathbb{Q})$
ϕ^ϵ	an element of the dual basis $\{\phi^\epsilon\}$ for $H^\bullet(X;\mathbb{Q})$, so that $(\phi_\mu, \phi^\nu) = \delta_\mu^\nu$
ψ_i	the first Chern class of the universal cotangent line bundle $L_i \to X_{g,n,d}$ at the ith marked point
Ω_X, $\Omega_{\boldsymbol{e}}$, Ω_Y	the symplectic forms on $\mathcal{H}_X$, $\mathcal{H}_X$, and $\mathcal{H}_Y$ respectively; see page 2
0_X, $0'_X$, $0'_Z$	zero section maps; see page 4
$(\cdot,\cdot)$	the Poincaré pairing on $H^\bullet(X)$, $(\alpha,\beta) = \int_X \alpha \cup \beta$
$(\cdot,\cdot)_{\boldsymbol{e}}$	the twisted Poincaré pairing on $H^\bullet(X)$, $(\alpha,\beta) = \int_X \alpha \cup \beta \cup \boldsymbol{e}(E)$
$\big\langle \cdots \big\rangle^X_{g,n,d}$	Gromov–Witten invariants or twisted Gromov–Witten invariants of X; see (1–3)

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