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The Quantum States of a Graph

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Abstract: Quantum codes are crucial building blocks of quantum computers. With a self-dual quantum code is attached, canonically, a unique stabilised quantum state. Improving on a previous publication, we show how to determine the coefficients on the basis of kets in these states. Two important ingredients of the proof are algebraic graph theory and quadratic forms. The Arf invariant, in particular, plays a significant role.

Keywords: quantum state; graph; self-dual quantum code; Eulerian graph

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1. Introduction

1.1. General Conventions

We assume that the reader is familiar with the established theory of additive (or stabiliser) quantum codes, as found for example in [1,2]. We will be using, freely, the following notations.

Notation

1. q is a prime power.
2. $\mathbf{F}_q$ is the finite field of order q.
3. $\mathrm{V}(n,q)$ is n-dimensional vector space over $\mathbf{F}_q$.
4. $\mathrm{PG}(r,q)$ is r-dimensional projective space over $\mathbf{F}_q$.

Some standard quantum notations are also required.

(1) $|\, y \,\rangle$ is a type of column vector, a *ket* in Hilbert space H_n. These kets form a basis for this vector space of dimension 2^n over the complex numbers $\mathbb{C}$, as y is a general subset of $N := \{1, \ldots, n\}$. There is a normalisation, dividing by the norm of the vector, so that every element of H_n has norm 1; this can be thought of as a kind of projective space of dimension $2^n - 1$. Thus, H_n does not contain the sum of kets where the coefficients are all zero.

(2) A *phase* is an element of $\mathbb{C}$ of norm 1 which can multiply any element of H_n. Physically, the phase is not measurable, but is important in some situations such as probabilistic mixtures of states or superpositions.

(3) A *quantum state* having n qubits is just an element of H_n of norm 1. In quantum mechanics, there are various operators called *observables*. The main ones we use are

the Pauli operators σ_x and σ_z: here, σ_x is the flip at a particular qubit of a ket, taking $0 \leftrightarrow 1$, while σ_z is a controlled phase operator. If the control value at a qubit of a ket is 1, then the phase of the ket is multiplied by -1; otherwise the phase stays the same. In addition, $\langle \sigma_x, \sigma_z \rangle$ is the multiplicative Pauli group of order 16 with identity $\sigma_0 = 1$. The superscript (j) denotes that a Pauli operator affects only the jth qubit.

In general, a quantum stabiliser code $[[n, k, d]]$ is defined as the set of elements of H_n fixed by a collection of $n - k$ independent and commuting Hermitian operators, known as *observables*. It may be assumed that they are of type $(\gamma_1, \ldots, \gamma_n)$, where each γ_i operates on the ith component of H_n, and $\gamma_i \in \{1, \sigma_x, \sigma_y, \sigma_z\}$, which are four Pauli operators.

Consider any simple graph $G = (V, E)$ having n vertices, with $N(i)$ the collection of neighbours of vertex $i \in V = \{1, \ldots, n\}$, and E is a set of edges, which are unordered pairs $\{i, j\}$ of vertices. There are two main quantum stabiliser codes on n qubits derived from G. If G has an $n \times n$ adjacency matrix A, symmetric with zero diagonal over the finite field $\mathbf{F}_2 := \{0, 1 \mid 1 + 1 = 0\}$, then each row i of A corresponds to a vertex $i \in V$, and to Hermitian operators S_i and T_i acting on the Hilbert space H_n defined as follows.

Definition 1. *For a graph G on n vertices,*

$$\begin{aligned} QS &= \langle S_i \mid i \in V \rangle, \text{ where } S_i := \sigma_x^{(i)} \Pi_{j \in N(i)} \sigma_z^{(j)}, \\ QT &= \langle T_i \mid i \in V \rangle, \text{ where } T_i := \sigma_z^{(i)} \Pi_{j \in N(i)} \sigma_x^{(j)}. \end{aligned}$$

1.2. Contribution

These are two abelian stabiliser groups giving self-dual quantum codes, *qcodes*, with parameters $[[n, 0, d]]$, for some minimal distance d, locally equivalent under the involution $\sigma_x \leftrightarrow \sigma_z$ on each qubit. The second parameter $k = 0$ of a self-dual quantum code denotes a code having a basis of $2^0 = 1$ states; that is, there is a unique quantum state, up to phase, stabilised by the code.

Each qcode or group stabilises a unique (up to overall phase) quantum state, with the first being the *graph state*, which has been investigated in many papers, including [3,4]. The equivalent qcode QT also stabilises a unique graph state and was investigated in the work leading up to [1]. However, although the quantum codes are equivalent, having the same distance properties, the actual coefficients within each state are different. The second state may contain many fewer ($2^r, r \leq n$) non-zeros than the first, which has the maximum 2^n non-zeros. Having a smaller number of zeros may be advantageous for practical reasons, such as having a stronger signal when measuring the state. After some preliminary investigations about graphs and quadrics, this QT state is calculated.

Although a formula for QT appeared in [1], many details and some not very obvious calculations were omitted. The calculation of QS is simpler, and some observations are made in Section 2. In [5], the quantum states with all coefficients ± 1 were studied in detail. The case of coefficients in $\{0, \pm 1\}$ was left as an open problem in [5] (Problem 8.23).

1.3. Relevance to Coding Theory

Here are some observations about optimality. From the theory of quantum codes [1], a graph G corresponds to a *self-dual* stabiliser code $[[n, 0, d]]$ of length n, where d is a well-defined distance. This distance is calculated differently to the case of qcodes $[[n, k, d]]$, where $k > 0$. When $k = 0$, the quantum distance d of G, which is the same for both QS and QT, is the following: consider any subset y of vertices. Then, the Hamming weight $W(y)$ in the qcode is the size $|y|$, plus the number of vertices of $V \backslash \{y\}$ joined to an odd number of vertices in y. Then,

$$d = \min\{W(y) \mid y \subseteq V, y \neq \emptyset\}.$$

For the best qcodes, the distance is maximised. Consideration of singleton subsets y gives the bound $d \geq v(G) + 1$, where $v(G)$ is the minimum valency of a vertex in G.

For example, here are some distance-optimal self-dual qcodes, fixing length n and having largest distance [1,5].

1. $[[2,0,2]]$ from the graph K_2 with one edge;
2. $[[3,0,2]]$ corresponding to the two connected graphs on three vertices;
3. $[[4,0,2]]$ qcode corresponding to any connected graph on four vertices;
4. $[[5,0,3]]$ qcode corresponding, for example, to the 5-cycle;
5. $[[6,0,4]]$ hexacode corresponding to the wheel graph or to the triangular prism on 6 vertices;
6. $[[12,0,6]]$ dodecacode corresponding to certain graphs on 12 vertices, one of which is regular of valency 5.

To also minimise the number of non-zeros in the corresponding graph state, it is necessary to find a graph of minimal rank that generates the graph code. The set of all graphs that gives a particular graph code is generated by an operation at a vertex called *local complementation* or *vertex neighbourhood complementation, vnc*; see [1,3–5]. Thus, within that orbit, it could be good to find graphs of least (even binary) rank. For example, the $[[6,0,4]]$ hexacode has one graph of rank 6 and one of rank 4, the triangular prism, in its vnc orbit. Constructing the QT state from the latter graph could be best.

2. The Quantum State for QS

In this section, some results and observations are given without proof. They have been well studied in [3–5]. For future use, we point out some connections with quadrics in finite geometry and other areas.

Theorem 1. *Let G be a graph (V,E) on n vertices.*

1. *The QS state is given by*
$$|QS\rangle = \Pi_{e\in E}\, U^e\, |+\rangle^{\otimes V},$$
where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and U^e is the controlled-Z interaction for the two qubits on that edge.
2. *If a ket has 1 for both these two qubits, then the phase for that ket is multiplied by -1.*

The interpretation is that to prepare the state QS, a standard state $|+\rangle^{\otimes V}$ has to be made first; this is the same as $1/2^{n/2}\,\Sigma_{y\subseteq V}\,|y\rangle$. This state is then changed by the use of $|E|$ controlled-Z operations. This converts the coefficient of the ket $|y\rangle$ from 1 into $(-1)^{g(y)}$, where $g(y)$ is the number of edges in the induced subgraph of G with the vertices in y.

The geometrical interpretation of $g(y)$ is that it is the value of the quadratic form corresponding to G at the point y. This quadratic form is constructed as follows. Let A be the $n\times n$ adjacency matrix of G; then, $Q(x_1,\ldots,x_n) := \Sigma_{i<j}\, a_{ij}x_ix_j$. This corresponds to a quadric in the projective space $\mathrm{PG}(n-1,2)$ of $n-1$ dimensions over the finite field $\mathbf{F}_2$. Thus, the number of 1s in the coefficients of this state equals the number of points in this quadric, plus one. The addition of one comes from the zero of the vector space, which is not defined as a point in the projective space.

Example 1. *Path on three vertices.*

The path P_3 having two edges $\{1,2\}$ and $\{2,3\}$ corresponds to the quadric $x_1x_2 + x_2x_3 = 0$ in the plane $\mathrm{PG}(2,2)$. Since this is the product of the two lines with equations $x_1 + x_3 = 0$ and $x_2 = 0$, the set of points of this plane quadric is the union of these two lines having $3+3-1 = 5$ points. Thus, the number of 1s in the state $|QS\rangle$ is 6, and the number of -1 is $2^3 - 6 = 2$. The state is

$$1/2\sqrt{2}\,(|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle).$$

3. Preliminary Definitions and Results about Graphs

Given the simple graph G on n vertices, let r be the rank of its adjacency matrix A over $\mathbf{F}_2$. Thus, it is the size of a basis for the row-space R of A in $V(n,2)$. Since A is skew-symmetric over this field, r is even. The null-space $R^\perp$ of A is the subspace of $V(n,2)$ orthogonal to R. The rank of $R^\perp$ is $n-r$.

In the natural way, using indicator functions, take the subsets consisting of vertices in G to be the same as vectors in $V(n,2)$.

Lemma 1. *A vector of the null-space $R^\perp$ of A corresponds to a subset V_0 of vertices that induce a Eulerian subgraph such that*

1. *Each vertex is on an even number of edges in this subgraph;*
2. *Each vertex outside V_0 is joined to an even number of edges within V_0.*

Proof. This follows from the fact that the row space of A is generated by the neighbourhood subsets of each vertex. Thus, a set of vertices V_0 orthogonal to each such neighbourhood intersects the neighbourhood in an even number. □

Definition 2.

1. *For any set V_0 of vertices in V, let $g(V_0)$ be the number mod 2 of edges of the graph G induced by V_0.*
2. *If $V_1, V_2 \subset V$, then $V_1 + V_2$ denotes their symmetric difference.*

Lemma 2.

1. *g is an additive homomorphism from $R^\perp$ to $V(n,2)$.*
2. *For any $\alpha, \beta \subset V$,*
$$g(\alpha) + g(\beta) + g(\alpha+\beta) = \alpha^t A \beta.$$
3. *For any α, $\beta \subseteq V$ if α and β are in the same coset of $R^\perp$, then*
$$g(\alpha) + g(\beta) = g(\alpha+\beta)$$

Proof. We prove the statements in the order (2), (1), (3).

(2) Write $A = T + T^t$, where T is the upper-triangular part of A and T^t is the lower-triangular part. Then, the number of edges induced by α in G is $g(\alpha) = \alpha^t T \alpha$, and similarly for β. Then

$$\begin{aligned} g(\alpha+\beta) &= (\alpha+\beta)^t T(\alpha+\beta) = g(\alpha) + \alpha^t T\beta + \beta^t T\alpha + g(\beta) \\ &= g(\alpha) + g(\beta) + \alpha^t T\beta + (\beta^t T\alpha)^t = g(\alpha) + g(\beta) + \alpha^t T\beta + \alpha^t T^t \beta \\ &= g(\alpha) + g(\beta) + \alpha^t (T+T^t)\beta = g(\alpha) + g(\beta) + \alpha^t A \beta. \end{aligned}$$

(1) If $\beta \in R^\perp$, then $A\beta = 0 \implies \alpha^t A \beta = 0$. Then, without even assuming that $\alpha \in R^\perp$, from (2), $g(\alpha+\beta) = g(\alpha) + g(\beta)$.

(3) In the proof of (1), replace β by $\alpha - \beta \in R^\perp$. Then,

$$g(\alpha) + g(\beta - \alpha) = g(\alpha + \beta - \alpha);$$

that is, $g(\alpha) + g(\beta) = g(\alpha+\beta)$. □

Theorem 2. *There is a unique coset $C = R + c$ of the row space R such that $g(\alpha) = c \cdot \alpha$ for all $c \in C$ and $\alpha \in R^\perp$.*

Proof. From Lemma 2(1), g is an additive homomorphism from $R^\perp$ to $\mathbf{F}_2$. Thus, $ker(g)$ is either the whole of $R^\perp$, and it may be assumed that $C = R$, or else, as a subgroup of $R^\perp$, $ker(g)$ has index 2. In that case, $R^\perp$ is a subspace of V$(n,2)$ of rank $n-r$, while $ker(g) \subseteq R^\perp$

and is of rank $n-r-1$. Then, $ker(g)$ is determined by various linear functionals, $c \cdot \alpha = 0$. There are 2^r possible cs that differ by vectors in R forming a coset C of R. □

Definition 3. *The coset $C = R + c$ is the* invariant coset *of the graph.*

Example 2. Path on three vertices. *Consider the path P_3 of Example 1 on three vertices. It has rank 2, since $R = \langle (0,1,0),(1,0,1) \rangle$. As $R^\perp$ is generated by the vector $(1,0,1)$; thus, $R^\perp = \{(0,0,0),(1,0,1)\}$. Then, $g(1,0,1) = 0$ over $\mathbf{F}_2$, and $C = R$, so that $g(\alpha) = c \cdot \alpha = 0$, for all $\alpha \in R^\perp$. Hence c can be chosen to be any of the four vectors in the coset $C = R$.*

Example 3. Complete graph on three vertices. *Consider the graph K_3. It also has rank 2, since $R = \langle (0,1,1),(1,0,1) \rangle$. $R^\perp$ is generated by the vector $(1,1,1)$ and thus $R^\perp = \{(0,0,0),(1,1,1)\}$. Then, $g(1,1,1) = 3 = 1$ over $\mathbf{F}_2$, and $C = R + (1,0,0)$, so that $g(\alpha) = c \cdot \alpha = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_1 + \alpha_2 + \alpha_3$, for all $\alpha \in R^\perp$. Again, c can be chosen to be any of the four vectors in the coset C.*

4. The *T*-Action on Quantum States

Recall from Definition 1 that, for each vertex i of the graph G, the operator

$$T_i := \sigma_z^{(i)} \Pi_{j \in N(i)} \, \sigma_x^{(j)}.$$

Hence, a single vertex i, or row of the adjacency matrix A, induces on action on each ket by this operator, that is a phase change controlled by the diagonal qubit, and bit flips in the positions where the edges occur on that vertex. For a general subset of vertices, the action will be the product of the single vertex actions within the subset. The order in which these actions occur is irrelevant since the operator group is commutative. Here, the action of the operator $T_\alpha := \Pi_{i \in \alpha} \, T_i$ is calculated for a general subset α. It uses the mapping $g : V \to \mathbf{F}_2$ as in Definition 2.

Lemma 3. *Let α be a subset of vertices of the graph G, and let x be another subset of vertices, equivalently a vector in $\mathrm{V}(n,2)$. Then,*

$$T_\alpha |x\rangle = (-1)^{g(\alpha)+x\cdot\alpha} \, |\alpha A + x\rangle.$$

Proof. The formula proves to be independent of the order of the elements of α, which is consistent since the operator group of the stabiliser code generated by the vertices is abelian. We can use induction on the Hamming weight $|\alpha|$ = number of 1s of the vector, or number of elements of α.

For $\alpha = 0$, as a zero vector, or $\emptyset$ as a subset of V, $T_0 = 1$, and the formula holds. When the Hamming weight is 1, then $T_i|x\rangle = (-1)^{e_i \cdot x}|x + a_i\rangle$, where $a_i = e_i^t A$ is the i-th row of the adjacency matrix A of the graph G.

Suppose now that the formula is true for α of Hamming weight $|\alpha| = m \geq 1$, and consider $\beta = \alpha + e_i$, where $\alpha \cdot e_i = 0$, so that $|\beta| = m + 1$. Note that the action of T_i can be assumed to arise after that of T_α, as in any product of elements T_j, there is always a final one. Hence,

$$\begin{aligned}
T_\beta |x\rangle &= T_i T_\alpha |x\rangle \\
&= T_i (-1)^{g(\alpha)+x\cdot\alpha} \, |\alpha A + x\rangle \\
&= (-1)^{g(\alpha)+x\cdot\alpha} T_i |\alpha A + x\rangle \\
&= (-1)^{g(\alpha)+x\cdot\alpha} (-1)^{e_i \cdot (\alpha A + x)} |\alpha A + x + a_i\rangle \\
&= (-1)^{g(\alpha)+e_i\cdot\alpha A + x\cdot(\alpha+e_i)} |(\alpha + e_i)A + x\rangle \\
&= (-1)^{g(\beta)+x\cdot\beta} |\beta A + x\rangle,
\end{aligned}$$

where the main thing to observe in the last line is that $g(\alpha) + e_i \cdot \alpha A$ counts, mod 2, the number of edges in the induced graph $G|\alpha$ plus the number of edges from the vertex i to α. Since $i \notin \alpha$, it is the number of edges in $G|\beta$ where $\beta = \alpha \cup \{i\}$. Hence, the formula holds by induction. □

5. The Quantum State for QT

From Lemma 3, the operator T_α maps the ket of a certain vector $x \in V$ to the ket of another member of the same coset of the row space R. Thus, we can consider $\Pi_{\alpha \in V} T_\alpha |x\rangle = \Sigma_{y \in R+x} \gamma_y |y\rangle$, for some $\gamma_y \in \{\pm 1\}$. This is always fixed by every T_α. The problem then is to determine if it is zero or not. A non-zero sum would give the invariant state. Related to that problem is the following.

Lemma 4. *For $\alpha, \beta \in V$, $T_\alpha|x\rangle = T_\beta|x\rangle$ if and only if both* 1. *and* 2. *hold*:

1. $\alpha - \beta \in R^\perp$;
2. $g(\alpha) + g(\beta) = x \cdot (\alpha + \beta) = g(\alpha + \beta)$.

Proof.

$$\begin{aligned} T_\alpha|x\rangle = T_\beta|x\rangle \quad &\iff \quad (-1)^{g(\alpha)+x\cdot\alpha}\,|\alpha A + x\rangle = (-1)^{g(\beta)+x\cdot\beta}\,|\beta A + x\rangle \\ &\implies \quad |\alpha A + x\rangle = |\beta A + x\rangle \implies (\alpha - \beta)A = 0 \implies \alpha - \beta \in R^\perp. \end{aligned}$$

In addition,

$$(-1)^{g(\alpha)+x\cdot\alpha} = (-1)^{g(\beta)+x\cdot\beta} \iff g(\alpha) + g(\beta) \equiv x \cdot (\alpha + \beta) \pmod 2.$$

From Lemma 2, $g(\alpha) + g(\beta) = g(\alpha + \beta)$. Thus, $g(\alpha + \beta) \equiv x \cdot (\alpha + \beta) \pmod 2$, with $\alpha - \beta \in R^\perp$. □

Theorem 3 (after [1] §3.10)**.** *The quantum state induced by the stabiliser code GT of the graph G, having adjacency matrix A and invariant coset $C = R + c$, is given by the formula*

$$|GT\rangle = 2^{-r/2} \Sigma_{y = \alpha^t A + c \in C} \, (-1)^{\alpha \cdot c + g(\alpha)} \mid y \,\rangle.$$

Proof. If $y = \alpha^t A + c = \alpha'^t A + c$, then $\alpha - \alpha' \in R^\perp$ and, by Theorem 2,

$$g(\alpha - \alpha') = g(\alpha) - g(\alpha') = (\alpha - \alpha') \cdot y = (\alpha - \alpha') \cdot c.$$

Hence, $g(\alpha) + \alpha \cdot c = g(\alpha') + \alpha' \cdot c$. Thus, $g(\alpha) + \alpha \cdot c$ is independent of the representative α in its coset modulo $R^\perp$.

Since R is a subspace of V of rank r, it has 2^r elements. Thus, the size of the coset C of R is also 2^r. The sum of kets in the unnormalised state therefore has 2^r coefficients that are ± 1, and thus the norm is $\sqrt{2^r}$. Thus, to normalise, divide on the left by $2^{r/2}$. Note that r is even, and thus, this is an integer.

This state is kept invariant by T_β, where β is an arbitrary subset of vertices in V:

$$\begin{aligned} & T_\beta (-1)^{c\cdot\alpha + g(\alpha)} \mid y \,\rangle \\ &= (-1)^{c\cdot\alpha + g(\alpha)} T_\beta \mid y \,\rangle \\ &= (-1)^{c\cdot\alpha + g(\alpha)} (-1)^{g(\beta) + y\cdot\beta} \mid \beta^t A + y \,\rangle, \text{(from the } T\text{-action in Lemma 3)} \\ &= (-1)^{c\cdot(\alpha+\beta) + g(\alpha) + g(\beta) + \alpha^t A \beta} \mid \beta^t A + \alpha^t A + c \,\rangle, \text{(using } y = \alpha^t A + c\text{)} \\ &= (-1)^{c\cdot(\alpha+\beta) + g(\alpha+\beta)} \mid (\alpha + \beta)^t A + c \,\rangle, \text{(using Lemma 2(2)).} \end{aligned}$$

□

Corollary 1. *If the adjacency matrix A of the graph G with n vertices is of full rank $n = r$ over $\mathbf{F}_2$ and is therefore non-singular, then*

$$|GT\rangle = 2^{-r/2}\Sigma_{y=\alpha^t A\in R}\ (-1)^{g(\alpha)}\ \big|\ \alpha^t A\ \big\rangle.$$

Proof. The null space of A is $\{0\}$ and thus the invariant subspace $C = R$, and it can be assumed that $c = 0$. □

Remark 1. *When the graph G has a non-singular adjacency matrix A modulo two, the quantum states are related: $|GS(A)\rangle = \big|GT(A^{-1})\big\rangle$.*

This can be seen, either directly from the formulae, or by noting that the multiplying by A^{-1} on the left of the generator matrix for $GS(A)$ yields the corresponding generator matrix for $QT(A^{-1})$.

Although, because of the necessity to find c, the formula for the quantum state is more complicated in the case of singular graphs; fortunately, there is a way to evaluate the formula once any maximal non-singular subgraph is found.

As before, the invariant state of the graph is only defined up to an overall phase, which is a complex number of norm 1.

Lemma 5. *If B is any symmetric matrix of rank r over a field, and if the first r rows of B form a basis for its row space, then the restriction of B to its first r rows and r columns induces a non-singular $r \times r$ submatrix. In particular, there is always a maximal non-singular submatrix of a symmetric matrix situated symmetrically on the leading diagonal.*

Proof. Let

$$B = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where W is $r \times r$. Since B is symmetric, $W^t = W$, $Y^t = X$, $Z^t = Z$. Since the first r rows generate the row space of B, there exists a matrix L such that $LW = Y$ and $LX = Z$. Thus, $X = Y^t = W^t L^t = WL^t$, and $(W\ X) = (W\ WL^t) = W(I\ L^t)$. The first r rows of B form a basis and thus are independent. Hence, there are no non-trivial linear combinations of the first r rows that are zero. The same holds for W, and W is non-singular. □

Let the $n \times n$ adjacency matrix A of the graph G have even rank $r \leq n$ over $\mathbf{F}_2$. By Lemma 5, let the first r vertices induce a maximal non-singular subgraph G'. This makes it easier to find the invariant coset and to determine a formula for the quantum state more precisely.

Let $A = B$ as above, where now W is the adjacency matrix of G'. Then, W is $r \times r$, $W = W^t$, X is $r \times (n-r)$, $Y = X^t$, Z is $(n-r) \times (n-r)$, and $Z = Z^t$. If the first r rows form a basis for the row space R, then Lemma 5 shows that W is non-singular.

Since $R = \langle W\ X\rangle = \big\langle I\ W^{-1}X\big\rangle$, and these are $r \times n$ generator matrices for a binary $[n, r, d]$ code, $R^\perp = \big\langle YW^{-1}\ I\big\rangle$. This equation for $R^\perp$ implies that a basis for it can be translated into subsets of V as $\alpha_i \cup \{i + r\}$, for sets $\alpha_i \subseteq \{1, \ldots, r\}$ corresponding to row i of YW^{-1}. Now c can be assumed, since it is in a coset of R, to be zero in the first r components. If so, it is unique. Hence, $c \subseteq \{r + 1, \ldots, n\}$. The intersection of c with a set α in $R^\perp$ determines the parity of $g(\alpha)$, since Theorem 2 indicates that $g(\alpha) = c.\alpha$. From a basis element of $R^\perp$, it follows that $g(\alpha_i \cup \{i + r\}) = c.(\alpha_i \cup \{i + r\}) = c_{i+r}$. Hence c can be constructed:

$$c_1 = \ldots = c_r = 0, c_{r+i} = g(\alpha_i \cup \{r + i\}).$$

This can be further simplified using Lemma 1, since the subgraph of G induced by $\alpha_i \cup \{i + r\})$ is Eulerian, and so $g'(\alpha_i) = g(\alpha_i \cup \{i + r\})$, where g' is the function on the vertices of G' mapping a subset of vertices to the number of edges in the induced subgraph. Hence the following has been shown.

Lemma 6. *The invariant vector c, or subset of V, can be assumed to be the vector of length n over $\mathbf{F}_2$ with $(r+i)$th entry $g'(\alpha_i)$, where α_i, for $1 \leq i \leq n-r$, is defined as the i-th row of the $(n-r) \times r$ matrix YW^{-1}, and g' is the restriction of g to the subgraph G'.*

Using Theorem 3, and since $(\alpha\ 0) \cdot c = 0$, and assuming the adjacency matrix of G is

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where W is the adjacency matrix of a maximal non-singular subgraph, this leads to the following result.

Theorem 4. *If a maximal non-singular subgraph G' is contained in the first r vertices of G, where g' is the restriction of g to G', then*

$$|QT(G)\rangle = 2^{-r/2} \Sigma_{\alpha \in V(r,2)}\ (-1)^{g'(\alpha)} \left| (\alpha^t W\ \alpha^t X) + c \right\rangle.$$

This shows that a quantum state coming from a singular graph is closely related to the quantum state of any maximal non-singular subgraph G', and in particular, it has the same number of non-zero ± 1 entries as $|QT(G')\rangle$, or $|QS(G')\rangle$. This explains why a closer investigation in Section 6 of the non-singular case is warranted.

Assuming the setup of Theorem 4 with a maximal non-singular subgraph G' in the first r vertices of G, here is a concise preparation of the QT state. As before, let the adjacency matrix

$$A = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}.$$

The ket $|a, b\rangle$ of H_n is split into its first r qubits and the remaining $n-r$ ancillary qubits. Assume the following:

(1) D^i is a controlled bit-flip operator, $i \in V' := \{1, \ldots, r\}$, that takes qubit $|a, b\rangle$ to $|a, b + a_i x_i\rangle$, where x_i is the i-th row of X. It could be implemented by a collection of $|x_i|$ quantum controlled-NOT (CNOT or quantum XOR) gates, where $|x_i|$ is the Hamming weight of x_i; see [2] (pp. 51, 83, 148).
(2) $\left|QS(W^{-1})\right\rangle = |QT(W)\rangle$ is the prepared QT state for G' in H_r, using Theorem 1 and Remark 1.
(3) $\left|QS(W^{-1}) \otimes 0\right\rangle$ is thus embedded into H_n, by $n-r$ ancillary qubits labelled 0. This is the juxtaposition or tensor product of the QS state and a quantum disconnected trivial 0 state with one ket of length $n-r$; see [2] (p. 35).
(4) C is the product of $|c|$ bit-flips σ_x corresponding to the invariant vector c, as in Lemma 6, on the ancillary qubits taking $|a, b\rangle \mapsto |a, b + c\rangle$.

Theorem 5.

$$|QT\rangle = C\Pi_{i \in V'}\ D^i \left| QS(W^{-1}) \otimes 0 \right\rangle.$$

Proof. This is an application of Theorem 4. □

6. The Non-Singular Case

Let G be a graph on an even number r of vertices with non-singular $r \times r$ adjacency matrix $A = A^t$ over $\mathbf{F}_2$. Then, A corresponds to a non-singular quadric in $\mathrm{PG}(r-1, 2)$, given by $Q(x) = \Sigma_{i<j}\ a_{ij} x_i x_j = 0$ with symplectic polarity π, taking a point $x \in \mathrm{PG}(r-1, 2)$ to its polar hyperplane $[Ax]$ in dual coordinates. Note that $x^t A x = 0$ indicates that the polarity π is symplectic, since a point is always on its polar hyperplane. The group of homographies fixing Q in $\mathrm{PG}(r-1, 2)$ is the orthogonal group $PO_+(r, 2)$ if Q is hyperbolic, having Arf invariant $\alpha = 0$, or $PO_-(r, 2)$ if elliptic, having Arf invariant $\alpha = 1$. Note that the Arf invariant is preserved under taking inverse matrices, since the

inverse matrix corresponds to the dual quadric, in dual hyperplane coordinates, under the symplectic polarity.

The number of 1s in the quantum states, both $|QS\rangle$ and $|QT\rangle$, corresponding to G, is the same as the number of points on Q, plus one for the zero vector. Hence, the formula for the size of quadrics in [6] gives Table 1.

Table 1. Composition of $|QS\rangle$.

	Number of 1s	**Number of −1s**
Q hyperbolic	$2^{r-1}+2^{r/2}$	$2^{r-1}-2^{r/2}$
Q elliptic	$2^{r-1}-2^{r/2}$	$2^{r-1}+2^{r/2}$

The group fixing each state is the symplectic group $PSp(r,2)$, which commutes with π, and can be constructed explicitly using Q as follows.

The group PSp contains $PO = PO_+$, or PO_-, as a subgroup, and each coset of PO is either PO itself or the subset of elements of PSp that take Q to $Q + t^2$, where $t = P^\pi \cdot x$. For the orders of the groups, see [6] (Chapter 1). Thus, the group of the quantum state corresponds to the homographies that take Q to $Q + t^2$, where t is the linear form $P^\pi \cdot x$, $P \in Q$ or $P = 0$.

Such a homography α satisfies $Q(\alpha(x)) = Q(x) + t^2$; since $t^2 = t$ over $\mathbf{F}_2$, equivalently $Q(\alpha(x)) + t = Q(x)$. This is an automorphism of the state since α is a permutation of the coordinates (qubits), while adding t is essentially a flip on the qubits where $t = 1$, as in flipping the slices of the hypercube in various directions. Note that the phase shift induced by multiplying the state by -1 is induced by going from hyperbolic to elliptic graphs. This is related to pencils of quadrics containing hyperbolic, elliptic, and hyperplane types, as well as to the fact that the complement of the hypercube corresponding to an elliptic quadric is hyperbolic, and conversely. Here is the proof of the latter.

Theorem 6. *Let Q be non-singular quadratic form in an even number n of variables $x_1, \ldots, x_n$. Let*

1. $V = \mathrm{V}(n,2)$;
2. $P(Q) = \{x \in V \mid Q(x) = 0\}$;
3. $\bar{P}(Q) = \{x \in V \mid Q(x) = 1\}$.

Then, there exists a unique non-singular quadratic form Q' in the same n variables and a vector $k \in V$ such that $Q(x) + 1 = Q'(x + k)$ for all $x \in V$. In this situation, $Arf(Q) + Arf(Q') = 1$; that is, one of Q and Q' is elliptic and the other is hyperbolic.

Proof. Suppose that Q is hyperbolic. Take

$$Q(x) = x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n.$$

Then,

$$\begin{aligned} 1 + Q(x) &= (x_1^2 + x_2^2) + (x_1 + 1)(x_2 + 1) + x_3x_4 + \ldots + x_{n-1}x_n \\ &= (x_1 + 1)^2 + (x_2 + 1)^2 + (x_1 + 1)(x_2 + 1) + x_3x_4 + \ldots + x_{n-1}x_n. \end{aligned}$$

Hence,

$$Q' = y_1^2 + y_2^2 + y_1y_2 + y_3y_4 + \ldots + y_{n-1}y_n,$$

where $y_1 = x_1 + 1, y_2 = x_2 + 1, y_3 = x_3, \ldots, y_n = x_n$. The latter is a non-singular linear transformation of V that preserves the elliptic or hyperbolic character of a quadric. However, Q' is elliptic and Q is hyperbolic, since these are the standard forms for these non-singular quadrics; see [6] (Chapter 1), [7] (Chapter 5) or [8] for further results. □

The elliptic and hyperbolic quadrics have complementary numbers of points. The zero vector must be included as an extra point since the vector space is the union of the

projective space plus the zero vector. As indicated previously, for n even, there are two types of non-singular quadrics in PG$(n-1,2)$. : see Table 2.

Table 2. Types of non-singular quadrics.

Type	Notation	Number of Points
hyperbolic	$\mathcal{H}_{n-1}$	$2^{n-1}+2^{(n-2)/2}-1$
elliptic	$\mathcal{E}_{n-1}$	$2^{n-1}-2^{(n-2)/2}-1$

Corollary 2. *The number of $+1$s in a hyperbolic quantum state corresponding to a hyperbolic non-singular graph on an even number of vertices n is $2^{n-1}+2^{(n-2)/2}$, and the number of -1s is $2^{n-1}-2^{(n-2)/2}$. These numbers are reversed for an elliptic state.*

Proof. The zero point adds 1 to the number of points on the corresponding quadric. □

The interesting thing is that all non-singular graphs on r vertices give equivalent states, but the self-dual quantum code that is induced by the abelian group of order 2^r of vertex operators depends on the graph, and thus, each graph gives a different code.

For graphs of sizes less than 4, they all have Arf $= 0$; that is, they are hyperbolic. Then, for size 4, the only elliptic graph is the complete graph K_4.

Another observation that the flip operations on the quantum state, that is, the hypercube of size 2^n with entries 0 or ± 1) corresponds to the Abelian group of linear operators $x \mapsto xa$, where $x, a \in V(n,2)$. This follows, since a single flip, for example in the ith direction, corresponds to $a = e_i$. This is important since it means that the binary quadrics can be assumed to have no terms x_i^2. In the formula for the Arf invariant, it only needs to be known for the binary quadrics with no x_i^2, and this corresponds directly to the graph case.

7. Arf Invariant: The Determination of Elliptic or Hyperbolic

The aim of this section is to work out a formula to determine the number of ± 1 in the quantum state. As before, it is the same as first determining the rank of the adjacency matrix and then determining whether the chosen maximal non-singular quadric is hyperbolic or elliptic. However, it is not straightforward to determine the latter. In the theory of quadrics over fields of characteristic two, here $\mathbf{F}_2$, it is the same as working out the Arf invariant.

Theorem 7 ([6] (Lemma 1.1))**.** *Let A be the $n \times n$ symmetric, adjacency matrix over the rational number field of a graph G with an even number n of vertices, and let B be the anti-symmetric matrix constructed from A by multiplying all the elements below the main diagonal by -1. If $|A| \neq 0 \pmod 2$, then the binary rank of A is n, and if $\alpha = (|B| - (-1)^{n/2}|A|)/4$, then $\alpha \in \mathbb{Z}$ and $Arf(G) \equiv \alpha \pmod 2$.*

Note that as a polynomial in the $n(n-1)/2$ possible variables $a_{ij}, i < j$, the Arf invariant will be of degree n, and since B is skew-symmetric, $|B| = \mathrm{Pf}(B)^2$, where Pf is the Pfaffian of degree $n/2$ with each term, having a sign ± 1, corresponding to a perfect matching of $n/2$ disjoint edges of G. In the sum or difference with $|A|$, these all cancel. Thus, the terms with coefficient ± 2 in $|B|$ correspond to the product of two different perfect matchings of G, and these appear in $|A|$.

In [9], Theorem 7 was analysed closely and was explained in another way using graph theory. The following general theorem about properties of a graph is shown.

Theorem 8. *Given any true/false property P about any unlabelled simple graph G on n vertices, or set of such graphs, there is a unique set $f(P)$ of graphs on n vertices such that a graph $G \in P$ if and only if the number of edge-induced subgraphs of G in $f(P)$ is odd.*

Then, the set or property P was assumed to be $\{G \mid \mathrm{Arf}(G).\det(G) = 1\}$. Informally, $P(\mathrm{Arf} = 1)$ is this property. Then, $f(P)$ was found for small even n as follows.

(1) For $n = 2$, there is only one hyperbolic graph K_2.
(2) For $n = 4$, all graphs are hyperbolic except for K_4. Thus, $P = f(P) = \{K_4\}$.
(3) For $n = 6$, $f(P)$ is a set containing four subcubic graphs, which have 7, 7, 8 and 9 edges. These are (a) K_4 plus an independent edge, (b) two triangles connected by an edge, (c) K_4 with one edge replaced by a path of length three, and (d) the cubic triangular prism graph, having 2-rank 4. The first three graphs are all in P, being non-singular elliptic.

It was conjectured in [9] that $f(\text{Arf} = 1)$ contains only subcubic graphs, which are those having every vertex of valency, at most three.

8. The Minimal *QT* State of the $[[6, 0, 4]]$ Hexacode

It is known that there are two graphs on six vertices giving the hexacode: the triangular prism and the wheel. The 2-ranks of the adjacency matrices of these graphs are 4 and 6; thus, previously, the weights of the corresponding QT states are $2^4 = 16$ and $2^6 = 64$. Here, the minimal weight state for the triangular prism is calculated. First, this graph G_P is represented by $V := \{1, \ldots, 6\}$, and $E := \{12, 13, 23, 15, 34, 26, 45, 46, 56\}$. This is set up so that the first four vertices induce a maximal non-singular subgraph $G' = K_3$ plus an edge joined to it. Then, the adjacency matrix of G_P is

$$A = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where W is the adjacency matrix of G'. Then,

$$W = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Y = X^t = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the first $r = 4$ rows form a basis for the row space R, then Lemma 5 shows that W is non-singular.

Now, Lemma 6 is used to calculate the invariant vector c. Here,

$$W^{-1} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad YW^{-1} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \alpha_1 &= (1, 0, 1, 1) = \{1, 3, 4\} \quad \text{as a subset of vertices of } G', \\ \alpha_2 &= (0, 1, 1, 1) = \{2, 3, 4\} \quad \text{inside } G'. \end{aligned}$$

The number of edges of the subgraph on α_1 is $g'(\alpha_1) = 2 \equiv 0 \pmod 2$ and $g'(\alpha_2) = 2 \equiv 0$. Hence, $c_{4+1=5} = c_{4+2=6} = 0$. Since $c_1 = \ldots = c_4 = 0$, $c = 0$.

Now, the state can be calculated. This is given in the following table. In the ket column, the vector of length 6 is split into the G' part and the G/G' part.

Table: QT State of the Hexacode

From this table, there are 10 positive and 6 negative coefficients. This agrees with the fact that G' is a hyperbolic graph with Arf = 0. The corresponding quadric in $\text{PG}(3, 2)$ is a hyperbolic quadric and is a grid of three lines crossed over by another three lines. This has $3^2 = 9$ points, and plus one for the zero vector gives 10. If there were an elliptic maximal subgraph of G_P, this would correspond to an elliptic quadric having five points in $\text{PG}(3, 2)$. However, this does not happen for G_P since the only elliptic graph on four vertices is K_4,

which is not a subgraph of the triangular prism G_P. In general, a graph could have maximal non-singular subgraphs of both types, elliptic and hyperbolic. Then, the algorithm would give quantum states that are negatives of each other depending on the Arf character.

Another property of the above 16×6 table of non-zero kets of the QT state is that in any three columns, each of the possible $2^3 = 8$ binary vectors appears twice. This means, combinatorially, that the table is a t-$(v, k, \lambda) = 3 - (2, 6, 2)$ orthogonal array (OA). A similar property holds for the table of non-zero kets of any QT (or QS) : see Table 3 below.

Table 3. Kets of states.

$\boldsymbol{\alpha}$	$(-\mathbf{1})^{g'(\alpha)}$	$\lvert\boldsymbol{\alpha}(\boldsymbol{W},\boldsymbol{X})+\boldsymbol{c}\rangle$
0000	1	0000 00
0001	1	0010 11
0010	1	1101 00
0011	−1	1111 11
0100	1	1010 01
0101	1	1000 10
0110	−1	0111 01
0111	1	0101 10
1000	1	0110 10
1001	1	0100 01
1010	−1	1011 10
1011	1	1001 01
1100	−1	1100 11
1101	−1	1110 00
1110	−1	0001 11
1111	1	0011 00

From the standard definition (see [10]) an OA is a matrix with the property that each of its rows corresponds to a *run*, each column of the OA corresponds to a *factor*, the entries come from a set $\{0, \ldots, v-1\}$ of v *levels*, and the *strength* t is the number, usually maximal, of columns that always contain each possible row vector a constant *index* λ times.

Theorem 9. *Given a graph G on n vertices, of binary rank r and with corresponding QT state, the table K of non-zero kets in this state forms a $t-(v,k,\lambda) = t-(2, n, 2^{r-t})$ OA, where $t+1$ is the minimum number of distinct columns of the adjacency matrix A of G that are dependent. If A is non-singular, then $t = n$.*

Remark 2. *The parameter $t+1$ is the same as the Hamming distance of binary code $R^{\perp}$ orthogonal to the row space R of the generator matrix A over $\mathbf{F}_2$. Alternatively, it is the number of vertices of a minimal non-empty Eulerian subgraph E of G, with each vertex of $G \backslash E$ joined to an even number of vertices in E; see Lemma 1.*

Proof. Table K has the following properties. It is $2^r \times n$ with entries from $\{0, 1\}$. Thus, $v = |\{0, 1\}| = 2$. For K to be an OA, in any t columns of K, the rows should contain each vector of length t from $\{0, 1\}$ a constant number λ times. This is the number of rows of K divided by 2^t. Hence, $\lambda = 2^r/2^t = 2^{r-t}$.

From the construction of K in Theorem 3, its rows are formed from a coset of the row space of A in the vector space $\mathrm{V}(n, 2)$. Adding a constant vector c to every row vector in K does not affect the OA property. Hence, it may be assumed that the rows of K contain all 2^r vectors in the row space R of A. This is a linear binary code $[n, r, d]$, for a certain minimum distance d. The dual code generated by the vectors of $R^{\perp}$ has distance $t+1$, where t is the maximum number of columns of A that are always linearly independent. The restriction of K to t columns then induces a surjective homomorphism from $\mathrm{V}(r, 2) \to \mathrm{V}(t, 2)$. The kernel of this homomorphism has size $\lambda = 2^{r-t}$. □

Note that a $t-(2,n,2^{r-t})$ OA is also, by a similar proof, an $s-(2,n,2^{r-s})$ OA for all $1 \leq s \leq t$. In particular, for $s = 1 \leq t$, assume that there are no zero columns of A, or equivalently, there are no disconnected vertices, or that the dual code has a distance of at least two. Then, there will be a constant number 2^{r-1} of 0s and 1s in each column of N or in each non-zero qubit of the QT state.

For the QS state, every possible binary vector appears in a ket with non-zero $\pm$ coefficient, and thus, by standard definition, the corresponding OA is trivial.

9. Conclusions

This paper was concerned with the computation of a quantum state canonically attached to a self-dual quantum code. The abelian group of quantum operators generating a self-dual quantum code on n qubits gives the passage from a graph on n vertices to a stabilised quantum state. We have shown how to find the coefficients in the sum of the 2^n kets in these states; sometimes there are many zeros. The rank over $\mathbf{F}_2$ of the adjacency matrix of the graph basically the size of a maximal induced non-singular subgraph, plays a role. The elliptic or hyperbolic nature of a quadratic form, known as the Arf invariant, was also helpful in understanding these states. We have explained how the elliptic state is the negative of the hyperbolic state.

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References

1. Glynn, D.G.; Gulliver, T.A.; Maks, J.G.; Gupta, M.K. The Geometry of Additive Quantum Codes, 2004. Available online: www.researchgate.net/profile/David_Glynn3 (accessed on 1 April 2022).
2. Lomonaco, S.J., Jr. (Ed.) *Quantum Computation: A Grand Mathematical Challenge for the Twenty-First Century and the Millennium*; Proceedings of Symposia in Applied Mathematics; American Mathematical Society: Washington, DC, USA, 2002; Volume 58.
3. den Nest, M.V.; Dehaene, J.; Moor, B.D. Graphical description of the action of local Clifford transformations on graph states. *Phys. Rev. A* **2004**, *70*, 034302. [CrossRef]
4. Schlingemann, D. stabiliser codes can be realized as graph codes. *Quantum Inf. Comput.* **2002**, *2*, 307–323.
5. Danielsen, L.E. On Self-Dual Quantum Codes, Graphs, and Boolean Functions. Master's Thesis, Department Informatics, University Bergen, Bergen, Norway, 2005.
6. Hirschfeld, J.W.P.; Thas, J.A. *General Galois Geometries*, 2nd ed.; Springer: London, UK, 2016; p. xiv+409.
7. Hirschfeld, J.W.P. *Projective Geometries over Finite Fields*, 2nd ed.; Oxford University Press: Oxford, UK, 1998; p. xiv+555.
8. Hirschfeld, J.W.P. Quadrics over finite fields. In *Combinatorica*; Proceedings of Symposia in Pure Mathematics; Academic Press: London, UK, 1986; Volume 28, pp. 53–87.
9. Alahmadi, A.N.; Alkenani, A.N.; Hijazi, R.A.; Muthana, N.M.; Glynn, D.G. A transform of graph properties. 2019, *draft*
10. Hedayat, A.S.; Sloane, N.J.A.; Stufken, J. *Orthogonal Arrays: Theory and Applications*; Springer: New York, NY, USA, 1999; p. xxiv+416.