International Journal of Modern Physics B
Vol. 17, No. 25 (2003) 4555–4592

THE QUANTUM WIGNER FUNCTION IN A MAGNETIC FIELD

TOMAS B. MATERDEY*

Department of Physics, Cornell University, Ithaca, NY 14853, USA
tomas.materdey@umb.edu

CHARLES E. SEYLER

School of Electrical and Computer Engineering,
Cornell University, Ithaca, NY 14853, USA
ces7@cornell.edu

Received 3 January 2002
Accepted 3 June 2003

The Wigner function is shown related to the quantum dielectric function derived from the quantum Vlasov equation (QVE), with and without a magnetic field, using a standard method in plasma physics with linear perturbations and a self-consistent mean field interaction via Poisson's equation. A finite-limit-of-integration Wigner function, with oscillatory behavior and negative values for free particles, is proposed. In the classical regimes, where the problem size is huge compared to the particle wavelength, these limits go to infinity, and for free particles, the Wigner function becomes a positive delta function as expected. For the harmonic oscillator potential, there is no distinction between finite and infinite limits of integration when these are larger than the eigenfunction localization length.

Keywords: Wigner function; dielectric function; Lindhard constant; quantum; magnetic field; Vlasov equation; Schrödinger equation; symmetric gauge; modified Wigner function; de Haas–van Alphen.

PACS numbers: 71.10.Ca; 71.45.Gm; 77.22.Ch

1. Introduction

The Weyl–Wigner correspondence associates each quantum mechanical operator with a function of position and momentum. It is known that the quantum mechanical (QM) average of an operator $\hat{A}$ can be expressed as a classical average of the Wigner–Weyl representation of $\hat{A}$ with the Wigner Function (WF) *playing the role* of the classical phase space distribution.[1] The WF is not a classical probability density function, just a quasi-probability density function since it can be

*Current address: Department of Physics, University of Massachusetts, Boston, 100 Morrissey Blvd., Boston, MA 02125, USA.

negative due of the uncertainty principle (no classically well-defined position and momentum allowed), but it helps describe a quantum particle in a way that is very similar to classical mechanics. As a quasi-probability density function, the Wigner function for $B = 0$ appears frequently in the literature, from a quantum solution to plasma physics,[2] a single-particle description of transport properties in semiconductors including quantum interference and intracollisional field effects,[3] a description of mesoscopic systems with nonlinear effects and self-consistency,[4] and transport properties,[5] to a kinetic solution for astrophysical collisionless self-gravitating bodies using a modified non-negative probability density function called the Husimi function.[6,7] The quantum nature of the WF has not been enough emphasized; why it has been given the only treatment of a useful math tool connecting quantum and classical mechanics is unknown, since the uncertainty principle prevents it from being a full classical distribution function and its classical limit is singular. In fact, the WF is the Weyl–Wigner representation of the density matrix. Furthermore, it can be expressed as the trace of a product of this density matrix and a quantum phase-space kernel (see Appendix A).

Bertrand and coworkers were closest to associate a physical meaning to the WF by showing an equation governing its dynamics that reduces to the classical Vlasov equation when $\hbar \to 0$. However the purpose was to develop quantum numerical methods for describing a classical Vlasov plasma,[2] and so their point of view as stated was "(the WF) is a useful mathematical tool in spite of its poor physical properties." It was not clear whether Rammer[3] gave more emphasis on the physical meaning or the mathematical tool aspect of the WF, he stated "the Heisenberg uncertainty principle excludes the existence of a probability distribution with such a physical interpretation, but not, however, the introduction of a function with a formal resemblance to it." No explicit distinction was found in the paper by Bordone and co-workers[4] regarding quantum transport of electrons in open nanostructures with the WF formalism.

The wave-particle duality allows one to represent a free electron as a plane wave where the wavelength depends on the particle's momentum via the de Broglie's formula:

$$\frac{2\pi}{\lambda} = k = \frac{p}{\hbar}\,. \tag{1.1}$$

In cold plasma, electrons are almost free, using the plane wave

$$\psi(x) = \frac{e^{-ik_0 x}}{\sqrt{2\pi}} \tag{1.2}$$

the WF is

$$f_W(x,k) = \int_{-\infty}^{\infty} ds e^{-iks} e^{ik_0(x-\frac{s}{2})} e^{-ik_0(x+\frac{s}{2})} = \delta(k+k_0)\,, \tag{1.3}$$

which expresses the uniformly streaming particles at a constant speed. If a harmonic oscillator potential is applied, the eigenfunctions are the Hermite functions

$$\begin{aligned} \psi_n(x) &= \left(\frac{\alpha^2}{\pi}\right)^{1/4}\left(\frac{1}{2^n n!}\right)^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_n(\alpha x)\,, \\ \alpha &\equiv \left(\frac{mw}{\hbar}\right)^{1/2} \end{aligned} \tag{1.4}$$

and the WF's are oscillatory Laguerre polynomials (Ref. 1)

$$\begin{aligned} f_W^{(n)}(x,k) &= \frac{\alpha}{\sqrt{\pi}}\frac{1}{2^n n!}e^{-\alpha^2 x^2}\int_{-\infty}^{\infty} dse^{2iks}e^{-\alpha^2 s^2}H_n(\alpha(x+s))H_n(\alpha(x-s)) \\ &= (-1)^n e^{-\frac{2H}{\hbar\omega}}L_n(4H/\hbar\omega)\,. \end{aligned} \tag{1.5}$$

$$H \equiv \frac{\hbar^2 k^2}{2m} + \frac{1}{2}mw^2 x^2$$

This is an example of the "poor physical properties" of the WF mentioned in Ref. 2 since it can take on negative values.

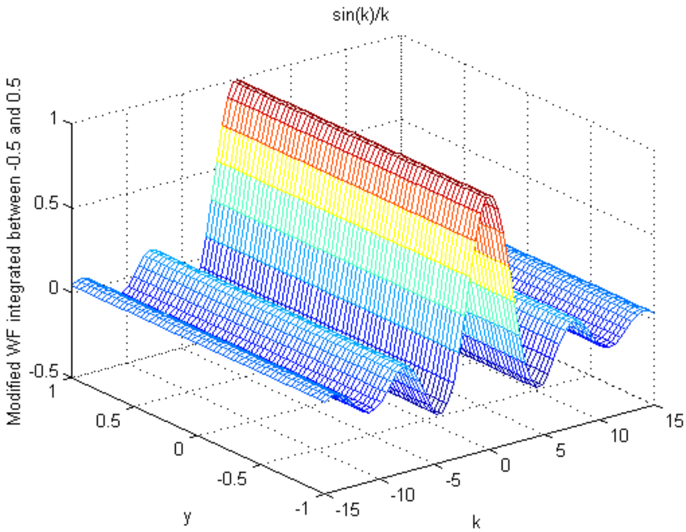
In this paper we will derive the quantum linear dielectric functions from a quantum Vlasov equation. This will show how these physically observable quantities are expressed in terms of the WF. These simple expressions suggest that all physical information should reside in the WF itself. We are then led to think that the WF's negative values are not representatives of poor physical properties but a consequence of the wave nature of the quantum particle. When we deal with dimensions much larger than the particle's de Broglie wavelength the particle behavior dominates resulting in a classical regime. This situation can be artificially created with smoothening/averaging techniques as, for example, used in Ref. 2. When the dimensions are comparable to the de Broglie's wavelength, the wave behavior is important and it is expected that the WF present negative values. This dimension/wavelength relation can be incorporated into the WF via the limits of integration. A plane wave can be used to represent a free particle moving inside a mesoscopic sample. Now we are not dealing with a plane wave running from $-\infty$ to $+\infty$ but between the finite limits of the sample, if finite limits of integration are used in the WF, it shows oscillatory behavior in momentum around the classical value as shown in Fig. 1. The sharp contrast between the primary and the secondary peaks increases as the limits are expanded leading to the classical one-momentum value when the sample is very large compared to the particle's wavelength. See Fig. 2 where the limits of integration have been tripled.

The proposed finite-limit-of-integration WF is

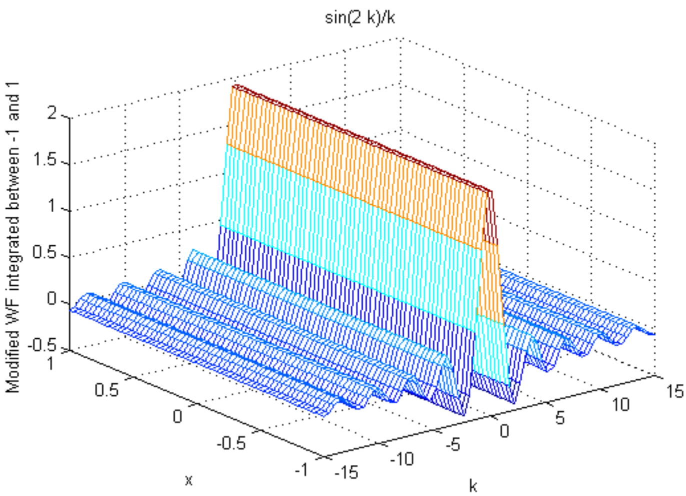
$$f_W^a(x,k) = \frac{1}{2\pi}\int_{-a}^{a} dse^{-iks}e^{ik_0(x-\frac{s}{2})}e^{-ik_0(x+\frac{s}{2})}\,. \tag{1.6}$$

For plane waves this looks like

$$f_W^a(x,k) = \frac{1}{2\pi}\left[\frac{e^{-i(k+k_0)s}}{-i(k+k_0)}\right]_{s=-a}^{s=a} = \frac{\sin(k+k_0)a}{\pi(k+k_0)}\,. \tag{1.7}$$



(a) Modified WF, integrated between -0.5 and 0.5

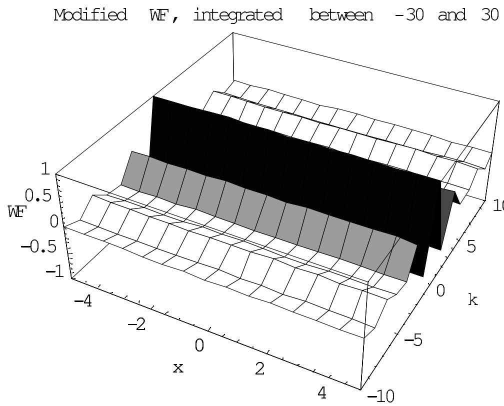


(b) Modified WF, integrated between -1 and 1

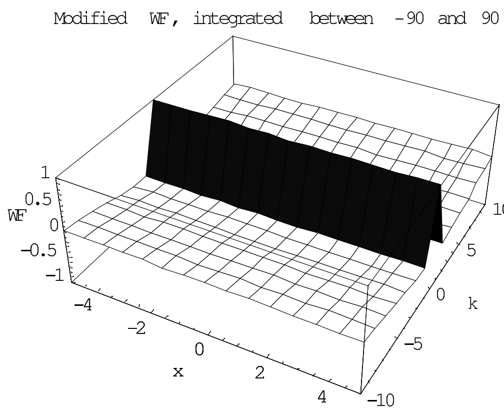
Fig. 1. Modified WF for plane waves: relative magnitude between the primary and secondary peaks increases as the limit of integration is doubled.

These WF's show oscillatory behavior versus k, and naturally with negative values as can be seen in Fig. 3 for $k_0 = k$ (a cut for constant x's from the previous plots).

A three-dimensional plot of the modified WF versus the limit of integration a and the wave number k shows that the main peak around $k = 0$ increases linearly with a (in units of $(2\pi)^{-1}$ the WF is $2a$ when $k = 0$ for plane waves), with the tendency of turning into a mere sinusoidal variation away from $k = 0$. (See Fig. 4.) This confirms our previous discussion related to Figs. 1 and 2.



(a) WF for plane waves with limits of integration −30 and 30.



(b) WF for plane waves with limits of integration −90 and 90.

Fig. 2. The modified WF shows oscillatory behavior for plane waves, it turns into the classical delta function when integration interval is much larger than the de Broglie's wavelength.

There is a mathematical advantage to the finite-limit-of-integration WF, in the case of plane waves, since the limit of these functions for $a \to \infty$ does not rigorously exist as a function but only as a distribution (see Ref. 8, Eq. (8.112) and discussion thereafter). That is,

$$\delta_a(k) \equiv \frac{\sin ka}{\pi k} \tag{1.8}$$

has no limit, but it is true that

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} dk \delta_a(k+k_0) f(k) = \int_{-\infty}^{\infty} dk \delta(k+k_0) f(k) = f(-k_0) \tag{1.9}$$

where $f(k)$ is any well-behaved function of k. The WF for plane waves is singular and positive while the modified WF is finite with oscillatory behavior and negative values.

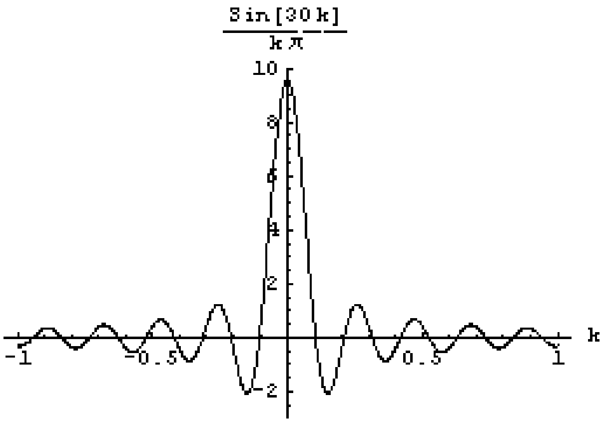


Fig. 3. WF versus k for plane waves with $a = 30$.

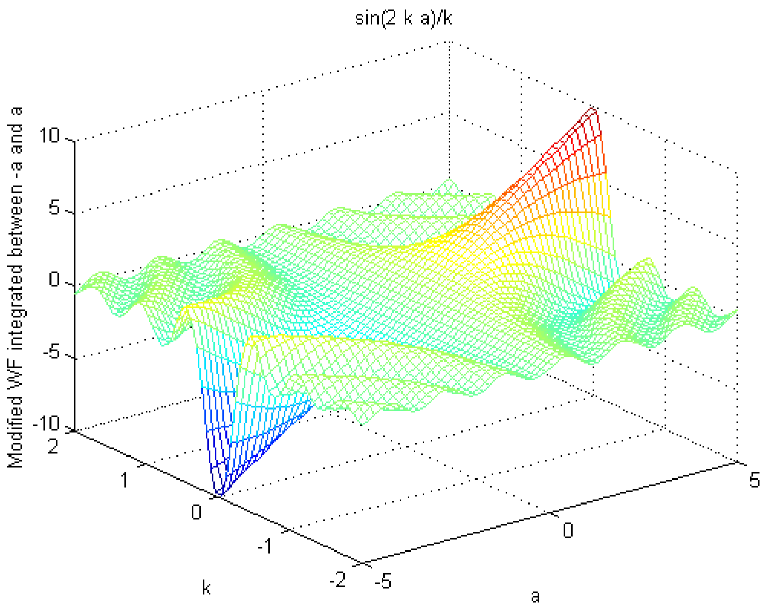


Fig. 4. Modified WF (in units of $(2\pi)^{-1}$) versus the limit of integration and the wave number.

In the case of the harmonic oscillator eigenfunctions, these are localized in space (see Fig. 5), and we always deal with finite limits actually that show oscillations; there is no difference between the finite and infinite-limit-of-integration WF's. (See Figs. 6–8.)

In the case of the harmonic oscillator potential, when the eigenfunctions are localized, the WF is unchanged when the limits of integration are ten times larger. The WF is not dependent on these limits as discussed.

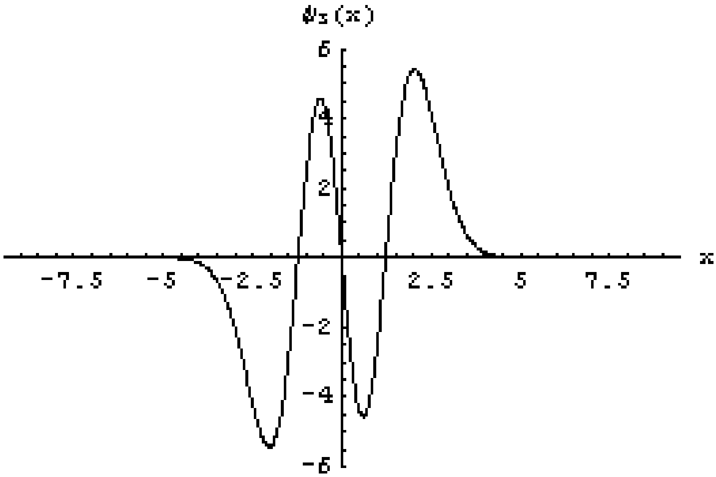


Fig. 5. Eigenfunction in an harmonic oscillator potential for $n = 3$.

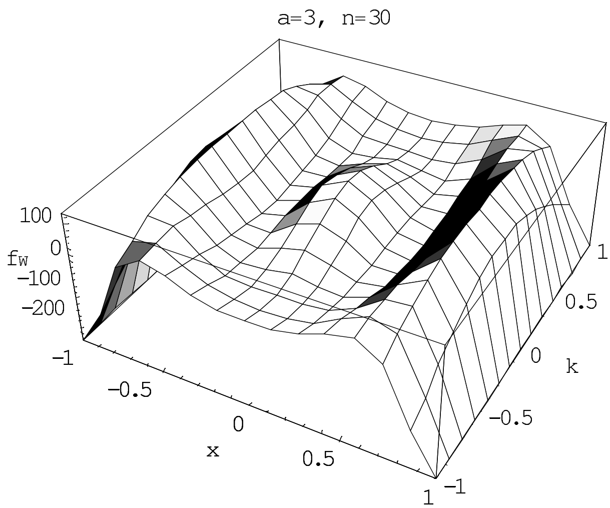


Fig. 6. Wigner function using the harmonic oscillator eigenfunction at $n = 3$; limits of integration = (−30, 30).

Motivated by the standard derivation of the classical dielectric constant from the classical Vlasov equation (CVE — the Vlasov equation is a collisionless homogeneous kinetic equation governing the behavior of a distribution function) using linear perturbations and a self-consistent mean field interaction via Poisson's equation (see e.g. Ref. 9), we will derive a quantum dielectric function from the quantum Vlasov equation (QVE), first for $B = 0$ and then for $B \neq 0$. Electronic properties such as conductivity can be easily obtained from the dielectric constant.

The difference between the QVE and the CVE is that the former contains the

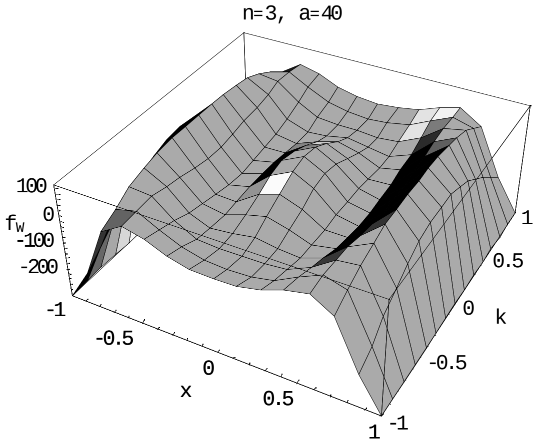


Fig. 7. Wigner function using the harmonic oscillator eigenfunction at $n = 3$; limits of integration $= (-40, 40)$.

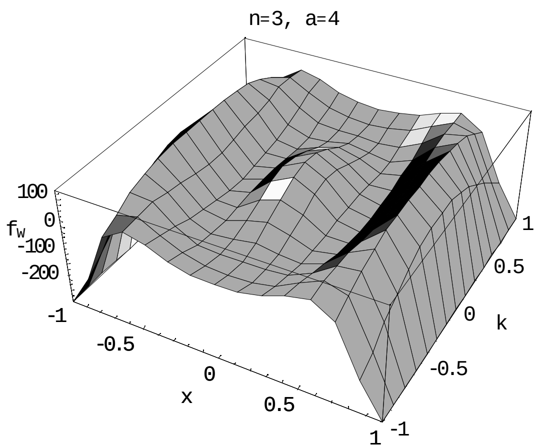


Fig. 8. Wigner function using the harmonic oscillator eigenfunction at $n = 3$; limits of integration $= (-4, 4)$.

Planck constant and a non-local term; it is a quantum kinetic equation that governs the behavior of the WF. The simultaneous appearance of momentum and position in the Weyl–Wigner representations is only superficial, and does not necessarily mean that the calculations involving these functions are classical. In this context, the derivation of the QVE from the Schrödinger equation (Section 2), and the derivation of the quantum dielectric function from the QVE (Section 3) are quantum derivations per se.

When $B = 0$, except for the expression in term of the WF this dielectric constant is the Lindhard constant [see Ashcroft and Mermin (Ref. 10) Eq. (17.60) or Madelung (Ref. 11) Eq. (3.58)] which is usually derived from the Schrödinger equation (e.g. Ref. 11).

The $B \neq 0$ quantum dielectric function in term of the magnetic WF is derived for the first time. The derivation starts with the quantum kinetic equation with a magnetic field as derived in Section 4. The method is similar to that used in the derivation of the $B = 0$ linear quantum dielectric function in Section 3. This result indicates that:

(i) Since properties such as conductivities are related to the dielectric function and so to the Wigner function, the physics is contained in the latter. A magnetic Wigner function shows de Haas–van Alphen oscillations as proved in Ref. 12.
(ii) An explicit manifestation that the structure of the kinetic equation gives rise to $\mathbf{D}$, not $\mathbf{\Pi}$ (see Appendix B for their definitions). This is related to the choice of $\mathbf{D}$ or $\mathbf{\Pi}$ in the definition of the magnetic Wigner function, and we argue for $\mathbf{D}$ for gauge invariance, see Ref. 13.

The derivation of the $B \neq 0$ QVE in the symmetric gauge will be presented for the first time in Section 4. Both the $B \neq 0$ QVE and the linear quantum dielectric function reduce to the appropriate limits when $B \to 0$ (see Sections 4 and 5, respectively). We have found related but not similar work in the papers by Kelly[14] and Harris.[15] Kelly obtained a dielectric tensor, not dielectric function, whose components did not present any de Haas–van Alphen oscillations. Harris gave an expression (2.69) for the dielectric function in terms of "a quantum mechanical distribution function, which is a Fourier transform of the density matrix… It is similar but not identical to the well-known distribution function of Wigner", which differs from ours by an extra summation over the indices of Bessel functions. It is not clear how this expression would reduce to the Lindhard equation at $B = 0$.

2. Derivation of the QVE from the SE

This equation was given in, e.g. the paper by Bertrand and co-workers in 1980.[2] It is called the quantum Vlasov equation (QVE) since it differs from the classical Vlasov equation (CVE), a well-known equation in plasma physics, in a non-local integral in the third and last term that reduces to the classical form when $\hbar \to 0$. A same quantum Vlasov equation (QVE) is obtained from the single-particle Schrödinger equation (SE) (a QVE for a purely single-particle density matrix) or *also* by taking the N-particle QVE (obtained from the N-particle SE), integrating out the variables corresponding to the other $N-1$ particles, arriving at a quantum kinetic equation for a single-particle density matrix that involves a two-particle density matrix (this new term is given in (2.20)), and then ignoring any two-particle correlation by approximating that two-particle density matrix as a simple product of two single-particle density matrices. At this point we have a QVE for a one-particle density matrix obtained by integrating out the variables of the other $N-1$ particles, which is the same as the QVE for a single-particle density matrix. Consequently, when the correlation between two particles is ignored, the purely single-particle density matrix and the single-particle density matrix obtained by integrating out all variables of the other $N-1$ particles are the same thing. No distinction is needed in this case.

In fact, when the correlation between two particles is ignored in a two-particle density matrix (DM), the single-particle DM constructed from a many-particle system is the same as a purely single-particle DM. This point has not been clearly discussed in the literature related to the Wigner function.

The integration over $N-1$ variables is a standard procedure in plasma physics known as the Born–Bogoliubov–Green–Kirkwood–Yvon (BBGKY) hierarchy reduction, as mentioned in I.A. The reader is referred to the paper by Bertsch[16] or Hillery and co-workers[1] for a complete BBGKY hierarchy reduction of the N-particle quantum kinetic equation. However, the relevant modification in the final single-particle equation is included in Eq. (2.20).

2.1. *Explicit derivation from the single-particle Schrödinger equation*

Given the Wigner function

$$f(x,v,t) = \frac{m}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-i\frac{mv\Delta}{\hbar}}\,\Psi^*\left(x-\frac{\Delta}{2},t\right)\Psi\left(x+\frac{\Delta}{2},t\right)d\Delta \tag{2.1}$$

by using the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + e\phi\Psi \tag{2.2}$$

we will show that it satisfies the quantum Vlasov equation (2.3):

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\int_{-\infty}^{+\infty} K(x,v,v',t)\frac{\partial f}{\partial v'}dv' = 0 \tag{2.3}$$

where

$$K(x,v,v',t) \equiv \frac{m}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-i\frac{m(v-v')\Delta}{\hbar}}\left[\phi\left(x-\frac{\Delta}{2},t\right)-\phi\left(x+\frac{\Delta}{2},t\right)\right]\frac{1}{\Delta}d\Delta\,. \tag{2.4}$$

To construct the special Wigner density matrix, we rewrite the Schrödinger equation and its complex conjugate in the following way

$$\xi \equiv x-\frac{\Delta}{2}\,,\qquad \xi' \equiv x+\frac{\Delta}{2}\,, \tag{2.5}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(\xi',t) = -\frac{\hbar^2}{2m}\nabla^2_{\xi'}\Psi(\xi',t) + e\phi(\xi')\Psi(\xi',t)\,, \tag{2.6}$$

$$-i\hbar\frac{\partial}{\partial t}\Psi^*(\xi,t) = -\frac{\hbar^2}{2m}\nabla^2_{\xi}\Psi^*(\xi,t) + e\phi(\xi)\Psi^*(\xi,t)\,. \tag{2.7}$$

Let us call the Wigner density matrix by

$$\rho(\xi,\xi',t) \equiv \Psi^*(\xi,t)\Psi(\xi',t) \tag{2.8}$$

The operation Eq. (2.7) $\times (-\psi) + (\psi^*) \times$ Eq. (2.6) gives

$$i\hbar\frac{\partial}{\partial t}\rho(\xi,\xi',t) = \frac{\hbar^2}{2m}(\nabla_\xi^2 - \nabla_{\xi'}^2)\rho(\xi,\xi',t) - e[\phi(\xi) - \phi(\xi')]\rho(\xi,\xi',t) \tag{2.9}$$

where ψ have been moved across Laplacian operators as ξ and ξ' are independent variables. Using the following properties

$$\frac{\partial}{\partial\Delta}\rho(\xi,\xi',t) \equiv \frac{\partial\rho}{\partial\xi}\frac{\partial\xi}{\partial\Delta} + \frac{\partial\rho}{\partial\xi'}\frac{\partial\xi'}{\partial\Delta} = \frac{1}{2}\left(-\frac{\partial\rho}{\partial\xi} + \frac{\partial\rho}{\partial\xi'}\right), \tag{2.10}$$

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial\Delta}\rho(\xi,\xi',t) &\equiv \frac{1}{2}\left[\frac{\partial}{\partial\xi}\left(-\frac{\partial\rho}{\partial\xi} + \frac{\partial\rho}{\partial\xi'}\right)\frac{\partial\xi}{\partial x} + \frac{\partial}{\partial\xi'}\left(-\frac{\partial\rho}{\partial\xi} + \frac{\partial\rho}{\partial\xi'}\right)\frac{\partial\xi'}{\partial x}\right] \\ &= -\frac{1}{2}\left[\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\xi'^2}\right]\rho(\xi,\xi',t),\end{aligned} \tag{2.11}$$

and multiplying Eq. (2.9) by $e^{-\frac{imv\Delta}{\hbar}}$ and integrating over Δ, we obtain

$$\begin{aligned} i\hbar\frac{2\pi\hbar}{m}\frac{\partial f}{\partial t} &= -\frac{\hbar^2}{m}\int_{-\infty}^{+\infty} d\Delta e^{-\frac{imv\Delta}{\hbar}}\nabla_x\cdot\nabla_\Delta\rho - e\int_{-\infty}^{+\infty} d\Delta e^{-\frac{imv\Delta}{\hbar}}[\phi(\xi) - \phi(\xi')]\rho \\ &= -i\hbar\frac{2\pi\hbar}{m}v\cdot\nabla_x f - e\int_{-\infty}^{+\infty} d\Delta e^{-\frac{imv\Delta}{\hbar}}[\phi(\xi) - \phi(\xi')]\rho \end{aligned} \tag{2.12}$$

where in order to obtain the second equality, an integration by part has been performed on the first integral. To rewrite the second integral, we view the Wigner function $f(x,v,t)$ [Eq. (2.1)] as the "Wigner transform" of $\rho(x-\Delta/2, x+\Delta/2, t)$ and define an "inverse Wigner transform" by

$$\rho\left(x - \frac{\Delta}{2}, x + \frac{\Delta}{2}, t\right) = \int_{-\infty}^{+\infty} dv' e^{\frac{imv'\Delta}{\hbar}} f(x,v',t). \tag{2.13}$$

As a check, Eq. (2.13) can be plugged into Eq. (2.1) to obtain an identity, by noting that

$$\int_{-\infty}^{+\infty} d\Delta e^{-\frac{im(v-v')\Delta}{\hbar}} = \frac{2\pi\hbar}{m}\delta(v-v'). \tag{2.14}$$

Using Eqs. (2.13) and (2.4) we can rewrite the second integral in Eq. (2.12) as

$$\begin{aligned} &-e\int_{-\infty}^{+\infty} dv' f(x,v',t)\int_{-\infty}^{+\infty} d\Delta e^{-im(v-v')\Delta/\hbar}[\phi(\xi) - \phi(\xi')]\frac{\Delta}{\Delta} \\ &= -e\int_{-\infty}^{+\infty} dv' f(x,v',t)\frac{2\pi\hbar}{m}\frac{\hbar}{im}\frac{\partial K}{\partial v'} \\ &= -e\frac{2\pi\hbar}{m}\frac{i\hbar}{m}\int_{-\infty}^{+\infty} dv'\frac{\partial f(x,v',t)}{\partial v'}K \end{aligned} \tag{2.15}$$

where an integration by parts has been performed in the last equality. Using Eq. (2.15) in Eq. (2.12) we obtain

$$\frac{\partial f(x,v,t)}{\partial t} = -v\frac{\partial}{\partial x}f(x,v,t) - \frac{e}{m}\int_{-\infty}^{+\infty} dv'\frac{\partial f}{\partial v'}K(x,v,v',t) \tag{2.16}$$

which is the quantum Vlasov equation (2.3).

2.1.1. *Classical limit*

In the definition of K (2.4), because of the highly oscillating nature of the exponential, when $\hbar \to 0$ contributions to the integral are non-vanishing when also $\Delta \to 0$, in which case

$$K(x, v, v', t) \to -\frac{m}{2\pi\hbar}\frac{\partial \phi}{\partial x}\int_{-\infty}^{+\infty} e^{-i\frac{m(v-v')\Delta}{\hbar}} d\Delta = -\frac{\partial \phi}{\partial x}\delta(v - v') . \tag{2.17}$$

When we use this in the quantum Vlasov equation (2.16), it reduces to

$$\frac{\partial f}{\partial t} = -v\frac{\partial}{\partial x} f - \frac{e}{m} E\frac{\partial f}{\partial v} \tag{2.18}$$

which is the one-dimensional classical Vlasov equation.

2.1.2. *Discussion*

Two observations are in order

(i) The different signs in the definitions of the Wigner functions are correlated for mathematical consistency, an equivalent definition is[16,17]

$$f(\mathbf{p}, \mathbf{r}, t) = \frac{1}{(2\pi\hbar)^3}\int d\boldsymbol{\xi} e^{\frac{i}{\hbar}\mathbf{p}\cdot\boldsymbol{\xi}}\Psi^*\left(\mathbf{r} + \frac{\boldsymbol{\xi}}{2}, t\right)\Psi\left(\mathbf{r} - \frac{\boldsymbol{\xi}}{2}, t\right) . \tag{2.19}$$

(ii) There is a missing minus sign in the exponential of [Ref. 2, Eq. (4)].

2.2. *From the N-particle Schrödinger equation with BBGKY hierarchy reduction*

The quantum Vlasov equation (2.3) has been derived by using the one-particle Schrödinger equation (2.2). Had we started with a N-particle Schrödinger equation and then reduce to an one-body density matrix equation equivalent to Eq. (2.9), we would have obtained a different quantum Vlasov equation that includes correlation effects between two particles. The last term in (2.12) would be replaced by

$$-e\int_{-\infty}^{+\infty} d\Delta e^{-\frac{imv\Delta}{\hbar}}\int_{-\infty}^{+\infty} d\kappa[\phi(\xi, \kappa) - \phi(\xi', \kappa)]\rho^{(2)}(\xi, \kappa; \xi', \kappa) \tag{2.20}$$

where $\rho^{(2)}$ is the two-particle density matrix function.

2.3. *Single-particle description from a many-particle approach*

If any two-particle correlation is ignored, i.e. the following approximation is made:

$$\rho^{(2)}(\xi_1, \xi_2; \xi_1', \xi_2') = \rho^{(1)}(\xi_1, \xi_1')\rho^{(1)}(\xi_2, \xi_2') . \tag{2.21}$$

Using the definition (2.8) for the single-particle density matrix, term (2.20) becomes

$$-e\int_{-\infty}^{+\infty} d\Delta e^{-\frac{imv\Delta}{\hbar}} \underbrace{\int_{-\infty}^{+\infty} d\kappa[\phi(\xi,\kappa)-\phi(\xi',\kappa)]|\Psi(\kappa)|^2}_{\phi(\xi)-\phi(\xi')} \rho^{(1)}(\xi,\xi') \tag{2.22}$$

which is identical to that obtained directly from the single-particle Schrödinger equation (2.12). So it turns out that the single-particle density matrix in a single-particle approach is the same as that obtained by integrating the N-particle density matrix over the variables dscribing the other $N-1$ particles, when the two-particle correlation is ignored.

3. The $B = 0$ Quantum Dielectric Function

In this section we derive a quantum dielectric function when $B = 0$. The quantum dielectric function (3.23) or the alternative form (3.35), in term of the Fourier transform (FT) of the Wigner function (WF) is just the standard Lindhard dielectric constant in term of the FT of a distribution function [see Ref. 10, Eq. (17.60) or Ref. 11 Eq. (3.58)]. We present an original derivation from the quantum Vlasov equation (QVE) — derived in Section 2, and the Madelung's derivation from the Schrödinger equation (SE), in Appendix C. Results from the two derivations are identical, which is shown in Section 3.1.3 via the alternative form. Our derivation was motivated by the similar standard derivation of the classical dielectric function from the classical Vlasov equation (CVE) using linear perturbations, described in plasma physics textbooks (see e.g. Ref. 9). In fact when $\hbar \to 0$ our derivation leads to the classical linear dielectric function, see Section 3.1.1.

The simultaneous appearance of momentum and position in the arguments of the Weyl–Wigner representations does not necessarily mean that the calculations involving these functions are classical, since the Weyl–Wigner correspondence associates each quantum mechanical operator with a function of position and momentum coordinates. It is in this context that the derivation of the QVE from the Schrödinger equation (Section 2), and that of the quantum dielectric function from the QVE are quantum derivations.

We recall that the QVE is a quantum kinetic equation for a single particle WF or a reduced many-particle WF with two-particle correlations ignored. It can be noted that the quantum linear dielectric function describes a single-particle that interacts with the mean field created by the rest via the self-consistently coupled Poisson's equation. However it includes the main important collective behavior obtained otherwise from a many-particle SE in a much more complicated derivation that uses a charge-screened and mass-renormalized electron to eliminate divergences due to the long-range Coulomb interaction.

3.1. *Quantum linear dielectric function from the quantum Vlasov–Poisson system*

We consider a 1D neutral quantum plasma — there is no preferred direction when $B = 0$, the modifications to include additional dimensions are trivial — in which the electron and ion distribution functions F_- and F_+ satisfy the quantum Vlasov equation (QVE). The electric field is related to these distribution functions via Gauss' law. We assume small perturbations from the equilibrium values with the ion distribution remains unperturbed due to their large mass:

$$\begin{aligned} F_- &= F_0^- + \varepsilon g\,, \qquad F_+ = F_0^+\,, \\ \mathbf{E} &= 0 + \varepsilon \mathbf{E}_1\,, \qquad \phi = 0 + \varepsilon \phi_1\,, \end{aligned} \tag{3.1}$$

where $\mathbf{E}$ and ϕ are the electric field and electrostatic potential, respectively.

Then the linearized QVE and Gauss' law read, respectively,

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} + \frac{e}{m}\int_{-\infty}^{\infty} dv' K_1 \frac{\partial F_0^-}{\partial v'} = 0 \tag{3.2}$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{E}_1 = -4\pi e \int g dv \tag{3.3}$$

in which from Eq. (2.4):

$$\begin{aligned} K_1(x, v, v', t) = \frac{m}{2\pi\hbar}\int_{-\infty}^{\infty} d\Delta e^{-\frac{i}{\hbar}m(v-v')\Delta} \\ \times \frac{1}{\Delta}\left[\phi_1\left(x - \frac{\Delta}{2}, t\right) - \phi_1\left(x + \frac{\Delta}{2}, t\right)\right]. \end{aligned} \tag{3.4}$$

From now on we will drop the super index '−' in the electron equilibrium distribution function.

We do Fourier transforms in space and time by substituting

$$g(x, v, t) = \frac{1}{(2\pi)^2}\int dk \int d\omega \bar{g}(\omega, k, v) e^{i(kx-\omega t)} \tag{3.5}$$

and

$$\phi_1(x, t) = \frac{1}{(2\pi)^2}\int dk \int d\omega \bar{\phi}_1(\omega, k) e^{i(kx-\omega t)}\,. \tag{3.6}$$

In this case

$$\begin{aligned} &\left[\phi_1\left(x - \frac{\Delta}{2}, t\right) - \phi_1\left(x + \frac{\Delta}{2}, t\right)\right] \\ &\quad = \frac{1}{(2\pi)^2}\int dk \int d\omega \bar{\phi}_1(\omega, k) e^{i(kx-\omega t)}\left(-2i \sin k\frac{\Delta}{2}\right). \end{aligned} \tag{3.7}$$

Then

$$K_1 = \frac{1}{(2\pi)^2}\int dk\int d\omega\bar{\phi}_1(\omega,k)e^{i(kx-\omega t)}\left(\frac{-im}{\pi\hbar}\right)\underbrace{\int_{-\infty}^{\infty} d\Delta e^{-\frac{i}{\hbar}m(v-v')\Delta}\frac{\sin k\frac{\Delta}{2}}{\Delta}}_{-2\tan^{-1}\frac{i\hbar k}{2m(v-v')}=-2i\tanh^{-1}\frac{\hbar k}{2m(v-v')}} \tag{3.8}$$

After inserting Eqs. (3.5), (3.6), and (3.8) into the linearized QVE (3.2) we get

$$-i\omega\bar{g} + ikv\bar{g} - \frac{2e}{\pi\hbar}\bar{\phi}_1(\omega,k)\int dv'\tanh^{-1}\left[\frac{\hbar k}{2m(v-v')}\right]\frac{\partial F_0}{\partial v'} = 0\,, \tag{3.9}$$

i.e.

$$\bar{g} = \frac{1}{\omega - kv}\left\{\frac{2ie}{\pi\hbar}\bar{\phi}_1(\omega,k)\int dv'\tanh^{-1}\left[\frac{\hbar k}{2m(v-v')}\right]\frac{\partial F_0}{\partial v'}\right\}. \tag{3.10}$$

We assume that the perturbation is due to an electron test charge. The generalization of Gauss' law to include this test charge is not different from the classical linear analysis:

$$\nabla\cdot\mathbf{E}_1 = -4\pi e\delta(x - v_0 t) - 4\pi e\int g dv \tag{3.11}$$

whose Fourier transform expression reads

$$i\mathbf{k}\cdot\bar{\boldsymbol{E}}_1 = -8\pi^2 e\delta(\omega - kv_0) - 4\pi e\int\bar{g}dv \tag{3.12}$$

where

$$\mathbf{E}_1(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int dk\int d\omega\bar{\mathbf{E}}_1(\omega,\mathbf{k})e^{i(k\cdot x-\omega t)}\,. \tag{3.13}$$

By substituting the Fourier transform of the perturbation distribution, expression (3.10), into (3.12) we obtain the quantum equation

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{E}_1 = {} & -8\pi^2 e\delta(\omega - kv_0) - \frac{8ie^2}{\hbar}\bar{\phi}_1(\omega,k) \\ & \times\int dv\frac{1}{\omega - kv}\int dv'\tanh^{-1}\left[\frac{\hbar k}{2m(v-v')}\right]\frac{\partial F_0}{\partial v'}\,. \end{aligned} \tag{3.14}$$

3.1.1. *Classical limit of the plasma dielectric function*

As $\hbar$ is small, to first order we note that

$$\tanh^{-1}\left[\frac{\hbar k}{2m(v-v')}\right] \approx \frac{\hbar k}{2m(v-v')}\,. \tag{3.15}$$

In the classical limit of $\hbar \to 0$, for nontrivial result we expect

$$2m(v-v')\xrightarrow{\hbar\to 0} 0 \quad\text{or}\quad v\to v' \tag{3.16}$$

which can be expressed rigorously by using the Plemelj's formula:

$$\lim_{\varepsilon\to 0}\frac{1}{v - v' - i\varepsilon} = \wp\left(\frac{1}{v-v'}\right) + i\pi\delta(v-v')\,. \tag{3.17}$$

As a consequence the quantum equation (3.14) becomes *in the classical limit*

$$i\mathbf{k}\cdot\mathbf{E}_1 = -8\pi^2 e\delta(\omega - kv_0) + \frac{8e^2}{2m}\underbrace{(-i)k\bar{\phi}_1(\omega,k)}_{=\mathbf{E}_1=\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{E}_1)}{k^2}}$$

$$\times\int dv\frac{1}{\omega - kv}\left[\underbrace{\left(\oint_{\Gamma_1} dv'\frac{1}{v-v'}\frac{\partial F_0}{\partial v'}\right)}_{-2\pi i\frac{\partial F_0}{\partial v}} + \left(i\pi\frac{\partial F_0}{\partial v}\right)\right] \tag{3.18}$$

where we assumed that only longitudinal waves exist in the perturbation and the path Γ_1 goes around the pole $v' = v$ in the clockwise direction. With

$$F_0 = n_0\bar{f}_0(v)\,, \qquad \omega_p^2 = \frac{4\pi n_0 e^2}{m_e}\,, \tag{3.19}$$

Eq. (3.18) now reads

$$i\mathbf{k}\cdot\mathbf{E}_1\underbrace{\left[1 + \frac{\omega_p^2}{k^2}\int dv\frac{\mathbf{k}\cdot\frac{\partial\bar{f}_0}{\partial v}}{\omega - kv}\right]}_{\varepsilon_{cl}(k,\omega)} = -8\pi^2 e\delta(\omega - kv_0)\,. \tag{3.20}$$

i.e. we obtain the classical plasma dielectric function $\varepsilon_{cl}(k,\omega)$. (see Refs. 15 and 18).

3.1.2. *Quantum plasma dielectric function*

The quantum plasma dielectric function can be obtained from the quantum equation (3.14) by first doing an integration by parts assuming that $\bar{f}_0$ goes to zero at $\pm\infty$:

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{E}_1 &= -8\pi^2 e\delta(\omega - kv_0) + \frac{8in_0e^2}{\hbar}\bar{\phi}_1(\omega,k) \\ &\quad\times\int dv\frac{1}{\omega - kv}\int dv'\frac{\frac{\hbar k}{2m}}{(v-v')^2 - (\frac{\hbar k}{2m})^2}\bar{f}_0 \\ &= -8\pi^2 e\delta(\omega - kv_0) - \frac{1}{\pi}\omega_p^2\underbrace{(-i)k\bar{\phi}_1(\omega,k)}_{=\mathbf{E}_1=\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{E}_1)}{k^2}} \\ &\quad\times\int dv\frac{1}{\omega - kv}\underbrace{\oint_{\Gamma_2} dv'\frac{\bar{f}_0}{(v-v')^2 - (\frac{\hbar k}{2m})^2}}_{I_2}\,. \end{aligned} \tag{3.21}$$

With a contour Γ_2 that deforms to include poles in the clockwise direction, the integral I_2 in Eq. (3.21) reads

$$I_2 = \oint_{\Gamma_2} dv'\frac{\bar{f}_0}{[v' - (v + \frac{\hbar k}{2m})][v' - (v - \frac{\hbar k}{2m})]}$$

$$= \pi i \left[\frac{\bar{f}_0(v + \hbar k/2m)}{\hbar k/m} + \frac{\bar{f}_0(v - \hbar k/2m)}{-\hbar k/m} \right]$$

$$= \frac{\pi i m}{\hbar k} \left[\bar{f}_0 \left(v + \frac{\hbar k}{2m} \right) - \bar{f}_0 \left(v - \frac{\hbar k}{2m} \right) \right]. \tag{3.22}$$

Then the quantum plasma dielectric function reads

$$\varepsilon_Q(k, \omega) = 1 + \frac{m\omega_p^2}{\hbar k^2} \int dv \frac{\bar{f}_0(v + \frac{\hbar k}{2m}) - \bar{f}_0(v - \frac{\hbar k}{2m})}{\omega - kv} . \tag{3.23}$$

This is the Lindhard equation for the dielectric constant [Ref. 10, p. 344 Eq. (17.60) or Ref. 11, p. 116]. An alternative form is being derived in the next section.

For the equilibrium distribution function $\bar{f}_0$ we could use the fermionic or the bosonic one. We first consider the Maxwellian distribution

$$\bar{f}_0(v') = \frac{e^{-v'^2/2C^2}}{\sqrt{2\pi} C} \tag{3.24}$$

where C is a constant. With it, Eq. (3.21) becomes

$$i\mathbf{k} \cdot \mathbf{E}_1 = -8\pi^2 e\delta(\omega - kv_0) + \frac{i}{C} \sqrt{\frac{1}{2\pi}} \frac{m\omega_p^2}{\hbar k^3} (\mathbf{k} \cdot \mathbf{E}_1)$$

$$\times \oint_{\Gamma_3} dv \frac{e^{-\frac{1}{2C^2}(v+\hbar k/2m)^2} - e^{-\frac{1}{2C^2}(v-\hbar k/2m)^2}}{v - \frac{\omega}{k}}$$

$$= -8\pi^2 e\delta(\omega - kv_0) - \frac{2}{C} \sqrt{2\pi} \frac{m\omega_p^2}{\hbar k^3} (\mathbf{k} \cdot \mathbf{E}_1)$$

$$\times e^{-\frac{1}{2C^2}[(\omega/k)^2 + (\hbar k/2m)^2]} \sinh \frac{\hbar\omega}{2mC^2} . \tag{3.25}$$

Consequently, the quantum plasma dielectric function for a Maxwellian distribution is

$$\varepsilon_Q(k, \omega) = 1 - \frac{2i}{C} \sqrt{2\pi} \frac{m\omega_p^2}{\hbar k^3} e^{-\frac{1}{2C^2}[(\omega/k)^2 + (\hbar k/2m)^2]} \sinh \frac{\hbar\omega}{2mC^2} . \tag{3.26}$$

If the Fermi–Dirac (FD) or Bose–Einstein (BE) distributions are used:

$$\bar{f}_0(v') = \frac{1}{C_\pm} \frac{1}{1 \pm e^{\frac{1}{k_B T}(\frac{1}{2}mv'^2 - \mu)}} \tag{3.27}$$

where the upper and lower signs correspond to FD and BE distributions, respectively, and C is a normalization constant. The corresponding quantum plasma dielectric functions read, respectively,

$$\varepsilon_Q(k, \omega) = 1 \mp \frac{4\pi i}{C_\pm} \frac{m\omega_p^2}{\hbar k^3} \frac{\sinh \frac{\hbar\omega}{2k_B T}}{e^{\frac{\mu}{k_B T} - \frac{m}{2k_B T}[(\frac{\omega}{k})^2 + (\frac{\hbar k}{2m})^2]} + e^{-\frac{\mu}{k_B T}} \pm 2\cosh \frac{\hbar\omega}{2k_B T}} . \tag{3.28}$$

3.1.3. *Quantum plasma dielectric function: alternative form*

To compare this result from the quantum Vlasov–Poisson equations with the one from the Schrödinger–Poisson equations, we first rename $k \to q$

$$\varepsilon_Q(q,\omega) = 1 + \frac{m\omega_p^2}{\hbar q^2}\int dv \frac{\bar{f}_0(v + \frac{\hbar q}{2m}) - \bar{f}_0(v - \frac{\hbar q}{2m})}{\omega - qv}\,. \tag{3.29}$$

With a change of variable $v' = v - \frac{\hbar q}{2m}$, it becomes

$$\varepsilon_Q(q,\omega) = 1 + \frac{m\omega_p^2}{\hbar q^2}\int dv' \frac{\bar{f}_0(v' + \frac{\hbar q}{m}) - \bar{f}_0(v')}{\omega - q(v' + \frac{\hbar q}{2m})}\,. \tag{3.30}$$

In order to change the variable of integration from v to k, we make the change $v' = \frac{\hbar k}{m}$, obtaining

$$\varepsilon_Q(q,\omega) = 1 + \frac{m\omega_p^2}{\hbar q^2}\int \frac{\hbar}{m} dk \frac{\bar{f}_0[\frac{\hbar}{m}(k+q)] - \bar{f}_0(\frac{\hbar}{m}k)}{\omega - q\frac{\hbar}{m}(k + \frac{q}{2})}\,. \tag{3.31}$$

To do the discussion on the rescaling of the integrand in the last equation we rewrite it as follows

$$\varepsilon_Q(q,\omega) = 1 + \frac{m\omega_p^2}{\hbar q^2}\Bigg\{ \underbrace{\int \frac{\hbar}{m} d(k+q) \frac{\bar{f}_0[\frac{\hbar}{m}(k+q)]}{\omega - q\frac{\hbar(k+q)}{m} + q^2\frac{\hbar}{2m}}}_{(a)} - \underbrace{\int \frac{\hbar}{m} dk \frac{\bar{f}_0(\frac{\hbar}{m}k)}{\omega - q\frac{\hbar k}{m} - q^2\frac{\hbar}{2m}}}_{(b)} \Bigg\}. \tag{3.32}$$

After the following rescaling

$$\begin{aligned} &\frac{\hbar}{m}(k+q) \to (k+q) \qquad &\text{for } (a) \\ &\frac{\hbar}{m}k \to k \qquad &\text{for } (b) \end{aligned}\,, \tag{3.33}$$

the last expression for the dielectric constant becomes

$$\varepsilon_Q(q,\omega) = 1 + \frac{m\omega_p^2}{\hbar q^2}\Bigg\{ \underbrace{\int d(k+q) \frac{\bar{f}_0[(k+q)]}{\omega - q\frac{\hbar}{m}(k+q) + q^2\frac{\hbar}{2m}}}_{(a)} - \underbrace{\int dk \frac{\bar{f}_0(k)}{\omega - q\frac{\hbar k}{m} - q^2\frac{\hbar}{2m}}}_{(b)} \Bigg\}. \tag{3.34}$$

Note that the denominators remain unchanged, since rescaling here can only means regrouping the $\frac{\hbar}{m}$ factor from $(k+q)$ or k to q, otherwise dimensional analysis is violated. Then the original dielectric function turns out to be

$$\begin{aligned}\varepsilon_Q(q,\omega) &= 1 + \frac{m\omega_p^2}{\hbar q^2}\int dk \frac{\bar{f}_0[(k+q)] - \bar{f}_0(k)}{\omega - q\frac{\hbar}{m}(k+\frac{q}{2})} \\ &= 1 - \frac{m\omega_p^2}{q^2}\int dk \frac{\bar{f}_0[(k+q)] - \bar{f}_0(k)}{\frac{\hbar^2}{2m}(2qk+q^2) - \hbar\omega} \\ &= 1 - \frac{m\omega_p^2}{q^2}\int dk \frac{\bar{f}_0[(k+q)] - \bar{f}_0(k)}{E(k+q) - E(k) - \hbar\omega}\,. \end{aligned} \tag{3.35}$$

This expression can readily be generalized to 3D by adding a vector sign over k and q:

$$\varepsilon_Q(\mathbf{q},\omega) = 1 - \frac{m\omega_p^2}{q^2}\int d\mathbf{k} \frac{\bar{f}_0[(\mathbf{k}+\mathbf{q})] - \bar{f}_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega}\,. \tag{3.36}$$

3.2. *Linear quantum dielectric function from the Schrödinger–Poisson system*

The quantum dielectric function from the Schrödinger–Poisson equations can be written in the form (see Appendix C)

$$\varepsilon(\mathbf{q},\omega) = 1 - \frac{m_e\omega_p^2}{q^2}\int d\mathbf{k} \frac{\bar{f}_0(\mathbf{k}+\mathbf{q}) - \bar{f}_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega}\,. \tag{3.37}$$

This is identical to the one derived above from the quantum Vlasov–Poisson system (3.36).

3.3. *Summary*

A linear analysis of the quantum Vlasov equation in 1D as been performed, the classical plasma dielectric function has been obtained in the classical limit. A quantum plasma dielectric function has been derived that agrees with those of Ref. 11 and 15. The linear quantum dielectric functions for the Maxwellian, Fermi–Dirac and Bose–Einstein distributions have been obtained.

On the other hand, starting from the Schrödinger–Poisson system of equations, a dielectric function defined as the ratio of the applied external potential over the total potential[11] (sum of the external and the internal potential given by the rearrangement of charges via the Poisson equation), whose derivation is included in Appendix C, is identical with that derived here from the quantum Vlasov–Poisson system of equation. We present this case as a proof of the equivalence between the two systems of equations.

4. Derivation of the $B \neq 0$ QVE from the SE

Ours is a $3D$ problem with a uniform magnetic field along the z-axis. Positions and momenta are coupled in the transverse XY plane via the mechanical momentum with the orbits being quantized. The motion along the z-axis is that of a free particle, with a z-dependence in the wave function via a plane wave. The z-component of the mechanical momentum is just equal to the z-component of the canonical momentum. If a plane wave along the z-axis is employed in the definition of the Wigner function (WF), it can be seen immediately that the latter does not depend on z and neither on p_z. We will derive a quantum Vlasov equation (QVE) in the transverse plane to a uniform magnetic field from the Schrödinger equation (SE), in analogy to the corresponding derivation for $B = 0$ (see Section 2). In this case the symmetric gauge is required. A key property is that a spatial derivative now has an extra contribution due to the position-momentum coupling via the mechanical momentum. This magnetic QVE will serve as the starting point in our derivation of the quantum linear dielectric function in a magnetic field in Section 5.

The magnetic QVE is a quantum kinetic equation that governs the dynamics of the magnetic WF. The difference with the classical equation resides in the fourth and last term that exhibits an integral indicating a nonlocal character, as happened in the $B = 0$ QVE. Again it will be shown that this term reduces to the appropriate classical limit when $\hbar \to 0$.

As when $B = 0$, ours is a single-particle description — that is the same as a many-particle description with BBGKY hierarchy reduction and two-particle correlations ignored — in a mean field created by other particles.

A QVE for an electron in a self-consistent magnetic field was given in Ref. 14. However in practical applications, the self-consistent magnetic field is negligible as compared to the external field.

4.1. *Quantum Vlasov equation (QVE) from the Schrödinger equation*

To derive the QVE from the Schrödinger equation with the magnetic field pointing along the z direction, we need the symmetric gauge

$$\mathbf{A} = -\frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{B_0}{2} x\hat{y} - \frac{B_0}{2} y\hat{x} \tag{4.1}$$

in which the canonical momentum and the vector potential commute

$$\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p} = 2\mathbf{A} \cdot \mathbf{p}\,. \tag{4.2}$$

The kinetic term in the Hamiltonian is

$$\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 = \mathbf{p}^2 + \frac{m^2\Omega^2}{4}(x^2 + y^2) - i\hbar m\Omega(y\partial_x - x\partial_y)\,. \tag{4.3}$$

If ϕ allows, the problem is translationally invariant in the z direction.

Given the symmetry, the following definitions are used

$$\begin{aligned}\mathbf{\Pi}_\perp &\equiv \mathbf{\Pi} = (\Pi_x, \Pi_y) = \left(p_x + \frac{m\Omega}{2}y, p_y - \frac{m\Omega}{2}y\right), \\ \boldsymbol{\nabla}_\perp &\equiv \boldsymbol{\nabla} = (\partial_x, \partial_y)\,,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\boldsymbol{\zeta} \equiv \mu\hat{x} + \eta\hat{y}\,, \\ &\boldsymbol{\xi} \equiv \mathbf{r} + \frac{\boldsymbol{\zeta}}{2} = \xi_x\hat{x} + \xi_y\hat{y}\,, \qquad \xi_x \equiv x + \frac{\mu}{2}\,, \qquad \xi_y \equiv y + \frac{\eta}{2}\,, \\ &\boldsymbol{\xi}' \equiv \mathbf{r} - \frac{\boldsymbol{\zeta}}{2} = \xi'_x\hat{x} + \xi'_y\hat{y}\,, \qquad \xi'_x \equiv x - \frac{\mu}{2}\,, \qquad \xi'_y \equiv y - \frac{\eta}{2}\,.\end{aligned} \tag{4.5}$$

Even though the quadratic terms proportional to $m\Omega^2/8$ in Eqs. (4.6) and (4.7) would not cancel each other immediately, the linear terms are much easier to work with if we write the Schrödinger equation and its complex conjugate at (ξ', t) and (ξ, t), respectively, instead of at the same point (x, t) and then putting it in terms of ξ' and ξ:

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{\xi}', t) = \Bigg\{&-\frac{\hbar^2}{2m}(\partial^2_{\xi'_x} + \partial^2_{\xi'_y}) + \frac{m\Omega^2}{8}(\xi'^2_x + \xi'^2_y) \\ &- \frac{i\hbar\Omega}{2}(\xi'_y\partial_{\xi'_x} - \xi'_x\partial_{\xi'_y}) + e\Phi(\boldsymbol{\xi}', t)\Bigg\}\Psi(\boldsymbol{\xi}', t)\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}-i\hbar\frac{\partial}{\partial t}\Psi^*(\boldsymbol{\xi}, t) = \Bigg\{&-\frac{\hbar^2}{2m}(\partial^2_{\xi_x} + \partial^2_{\xi_y}) + \frac{m\Omega^2}{8}(\xi^2_x + \xi^2_y) \\ &+ \frac{i\hbar\Omega}{2}(\xi_y\partial_{\xi_x} - \xi_x\partial_{\xi_y}) + e\Phi(\boldsymbol{\xi}, t)\Bigg\}\Psi^*(\boldsymbol{\xi}, t)\,.\end{aligned} \tag{4.7}$$

By doing Eq. (4.7) $\times\, [-\Psi(\xi', t)] + [\Psi^*(\xi, t)] \times$ Eq. (4.6) the following equation for the Wigner density matrix is obtained

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\rho(\boldsymbol{\xi}, \boldsymbol{\xi}', t) = \Bigg\{&\underbrace{+\frac{\hbar^2}{2m}[(\partial^2_{\xi_x} - \partial^2_{\xi'_x}) + (\partial^2_{\xi_y} - \partial^2_{\xi'_y})]}_{(\mathbf{I}a)} + \underbrace{\frac{m\Omega^2}{8}[(\xi'^2_x - \xi^2_x) + (\xi'^2_y - \xi^2_y)]}_{(\mathbf{I}b)} \\ &\underbrace{-\frac{i\hbar\Omega}{2}(\xi_y\partial_{\xi_x} - \xi_x\partial_{\xi_y} + \xi'_y\partial_{\xi'_x} - \xi'_x\partial_{\xi'_y})}_{(\mathbf{II})} \\ &\underbrace{-e[\Phi(\boldsymbol{\xi}, t) - \Phi(\boldsymbol{\xi}', t)]}_{(\mathbf{III})}\Bigg\}\rho(\boldsymbol{\xi}, \boldsymbol{\xi}', t)\,.\end{aligned} \tag{4.8}$$

Following the same procedures as for the free particle, i.e. doing

$$\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}(\Pi_x\mu+\Pi_y\eta)} \times \text{Eq. (4.8)}\,. \tag{4.9}$$

It is important to note that the procedures and properties that will be used in the derivation of the quantum and classical Vlasov equation for the Wigner function do not depend on whether its definition is gauge-invariant (i.e. with $\mathbf{d}$ in the exponent) or not (i.e. with $\mathbf{\Pi}$ in the exponent). Using Eq. (4.9) we will arrive at a quantum Vlasov equation in terms of the mechanical momentum $\mathbf{\Pi}$. It can be checked that a similar equation but in terms of $\mathbf{d}$ would be obtained had we used

$$\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}(d_x\mu+d_y\eta)} \times \text{Eq. (4.8)}\,. \tag{4.10}$$

A classical Vlasov equation in terms of $\mathbf{d}$ would be obtained if we exploit the $\mathbf{d}$-dependence of the Wigner function instead of what we will do with the $\mathbf{\Pi}$-dependence in the next section. The LHS of (4.9) is

$$i\hbar\frac{\partial}{\partial t}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}(\Pi_x\mu+\Pi_y\eta)}\rho(\boldsymbol{\xi},\boldsymbol{\xi}',t) = i\hbar\frac{\partial}{\partial t} f(\mathbf{r},\mathbf{\Pi},t)\,. \tag{4.11}$$

For the first term (Ia) in the RHS of (4.9), we use

$$(\partial^2_{\xi_x} - \partial^2_{\xi'_x})\rho = 2\partial_x\partial_\mu\rho\,, \qquad (\partial^2_{\xi_y} - \partial^2_{\xi'_y})\rho = 2\partial_y\partial_\eta\rho\,, \tag{4.12}$$

and notice that for the problem with a magnetic field

$$\partial_{\mathbf{r}} \to \partial_{\mathbf{r}} + (\partial_{\mathbf{r}}\mathbf{\Pi})\cdot\partial_{\mathbf{\Pi}} \tag{4.13}$$

which in the symmetric gauge means

$$\begin{gathered}\partial_a \to \partial_a + (\partial_a\Pi_{\bar a})\partial_{\Pi_{\bar a}}\,,\\ (a,\bar a) \equiv (x,y) \quad \text{or} \quad (y,x)\,,\end{gathered} \tag{4.14}$$

and that the density ρ depends on $\mathbf{\Pi}$,[12] after an integration by parts in μ and η for the two terms of (Ia), respectively,

$$\begin{aligned}-\frac{i\hbar}{m}\sum_{a=x,y}\Pi_a\Bigg\{\partial_a f + \frac{\partial\Pi_{\bar a}}{\partial a}\int\frac{d\mu d\eta}{(2\pi\hbar)^2}\big[\partial_{\Pi_{\bar a}}(e^{\frac{i}{\hbar}(\Pi_x\mu+\Pi_y\eta)}\rho(\boldsymbol{\xi},\boldsymbol{\xi}',t))\\ - (\partial_{\Pi_{\bar a}}e^{\frac{i}{\hbar}(\Pi_x\mu+\Pi_y\eta)})\rho(\boldsymbol{\xi},\boldsymbol{\xi}',t)\big]\Bigg\}\,.\end{aligned} \tag{4.15}$$

So the first term in the RHS of Eq. (4.9) or term (Ia) becomes

$$\begin{aligned}-\frac{i\hbar}{m}(\Pi_x\partial_x + \Pi_y\partial_y)f + \frac{i\hbar\Omega}{2}(\Pi_x\partial_{\Pi_y} - \Pi_y\partial_{\Pi_x})f\\ - \frac{i\hbar\Omega}{2}\int\frac{d\mu d\eta}{(2\pi\hbar)^2}\frac{i}{\hbar}(\Pi_x\eta - \Pi_y\mu)e^{\frac{i}{\hbar}\mathbf{\Pi}\cdot\boldsymbol{\zeta}}\rho\,.\end{aligned} \tag{4.16}$$

On the other hand

$$\xi_x'^2 + \xi_y'^2 - (\xi_x^2 + \xi_y^2) = -2(x\mu + y\eta) \tag{4.17}$$

and the second term in the RHS of Eq. (4.9) or term (Ib) quickly gives

$$-\frac{m\Omega^2}{4}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\mathbf{\Pi}\cdot\boldsymbol{\zeta}}(x\mu + y\eta)\rho\,. \tag{4.18}$$

The linear term in the RHS of Eq. (4.9) or term (II) is

$$-\frac{i\hbar\Omega}{2}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\mathbf{\Pi}\cdot\boldsymbol{\zeta}}(\xi_y\partial_{\xi_x} - \xi_x\partial_{\xi_y} + \xi_y'\partial_{\xi_x'} - \xi_x'\partial_{\xi_y'})\rho\,. \tag{4.19}$$

By noting

$$\begin{aligned} \Pi_x(\boldsymbol{\xi}) &= p_x + \frac{m\Omega}{2}\xi_y = \Pi_x + \frac{m\Omega}{4}\eta\,, \\ \Pi_y(\boldsymbol{\xi}) &= p_y - \frac{m\Omega}{2}\xi_x = \Pi_y - \frac{m\Omega}{4}\mu\,, \\ \mathbf{\Pi}(\boldsymbol{\xi})\cdot\boldsymbol{\zeta} &= \mathbf{\Pi}\cdot\boldsymbol{\zeta}\,, \end{aligned} \tag{4.20}$$

and similarly

$$\begin{aligned} \Pi_x(\boldsymbol{\xi}') &= p_x + \frac{m\Omega}{2}\xi_y' = \Pi_x - \frac{m\Omega}{4}\eta\,, \\ \Pi_y(\boldsymbol{\xi}') &= p_y - \frac{m\Omega}{2}\xi_x' = \Pi_y + \frac{m\Omega}{4}\mu\,, \\ \mathbf{\Pi}(\boldsymbol{\xi}')\cdot\boldsymbol{\zeta} &= \mathbf{\Pi}\cdot\boldsymbol{\zeta}\,. \end{aligned} \tag{4.21}$$

Equation (4.19) can be rewritten as

$$\begin{aligned} -\frac{i\hbar\Omega}{2}\Bigg\{ &\underbrace{\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\mathbf{\Pi}(\boldsymbol{\xi})\cdot\boldsymbol{\zeta}}(\xi_y\partial_{\xi_x} - \xi_x\partial_{\xi_y})\rho}_{\xi_x',\xi_y'\ const.} \\ &+ \underbrace{\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\mathbf{\Pi}(\boldsymbol{\xi}')\cdot\boldsymbol{\zeta}}(\xi_y'\partial_{\xi_x'} - \xi_x'\partial_{\xi_y'})\rho}_{\xi_x,\xi_y\ const.}\Bigg\}. \end{aligned} \tag{4.22}$$

Since

$$\begin{aligned} \mu &= \xi_x - \xi_x'\,, \\ \eta &= \xi_y - \xi_y'\,, \end{aligned} \tag{4.23}$$

and as for the spatial derivatives in a magnetic problem (4.14)

$$\left.\begin{aligned}\partial_{\xi_x} &\to \partial_\mu + \frac{\partial \Pi_y(\boldsymbol{\xi})}{\partial \mu}\partial_{\Pi_y(\boldsymbol{\xi})} = \partial_\mu - \frac{m\Omega}{4}\partial_{\Pi_y(\boldsymbol{\xi})}\\ \partial_{\xi_y} &\to \partial_\eta + \frac{\partial \Pi_x(\boldsymbol{\xi})}{\partial \eta}\partial_{\Pi_x(\boldsymbol{\xi})} = \partial_\eta + \frac{m\Omega}{4}\partial_{\Pi_x(\boldsymbol{\xi})}\end{aligned}\right\} \xi'_x, \xi'_y\, const\,,$$

$$\left.\begin{aligned}\partial_{\xi'_x} &\to -\partial_\mu - \frac{\partial \Pi_y(\boldsymbol{\xi}')}{\partial \mu}\partial_{\Pi_y(\boldsymbol{\xi}')} = -\partial_\mu - \frac{m\Omega}{4}\partial_{\Pi_y(\boldsymbol{\xi}')}\\ \partial_{\xi'_y} &\to -\partial_\eta - \frac{\partial \Pi_x(\boldsymbol{\xi}')}{\partial \eta}\partial_{\Pi_x(\boldsymbol{\xi}')} = -\partial_\eta + \frac{m\Omega}{4}\partial_{\Pi_x(\boldsymbol{\xi}')}\end{aligned}\right\} \xi_x, \xi_y\, const\,. \tag{4.24}$$

As all the derivatives are not independent of the variables of integrations μ and η, we just need to do integrations by parts. Grouping separately in Eq. (4.22) terms from the two types of derivatives:

$$\begin{aligned}-\frac{i\hbar\Omega}{2}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\boldsymbol{\Pi}\cdot\boldsymbol{\zeta}}[(\xi_y - \xi'_y)\partial_\mu - (\xi_x - \xi'_x)\partial_\eta]\rho\\ - \frac{i\hbar m\Omega^2}{8}\int \frac{d\mu d\eta}{(2\pi\hbar)^2}\{e^{\frac{i}{\hbar}\boldsymbol{\Pi}(\boldsymbol{\xi})\cdot\boldsymbol{\zeta}}[-\xi_y\partial_{\Pi_y(\boldsymbol{\xi})} - \xi_x\partial_{\Pi_x(\boldsymbol{\xi})}]\rho\\ + e^{\frac{i}{\hbar}\boldsymbol{\Pi}(\boldsymbol{\xi}')\cdot\boldsymbol{\zeta}}[-\xi'_y\partial_{\Pi_y(\boldsymbol{\xi}')} - \xi'_x\partial_{\Pi_x(\boldsymbol{\xi}')}]\rho\}\,.\end{aligned} \tag{4.25}$$

After an integration by parts

$$\begin{aligned}&\frac{i\hbar\Omega}{2}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\boldsymbol{\Pi}\cdot\boldsymbol{\zeta}}\frac{i}{\hbar}[\Pi_x \underbrace{(\xi_y - \xi'_y)}_{\eta} - \Pi_y \underbrace{(\xi_x - \xi'_x)}_{\mu}]\rho\\ &+ \frac{i\hbar m\Omega^2}{8}\int \frac{d\mu d\eta}{(2\pi\hbar)^2}\left\{e^{\frac{i}{\hbar}\boldsymbol{\Pi}(\boldsymbol{\xi})\cdot\boldsymbol{\zeta}}\frac{i}{\hbar}[-\xi_y\eta - \xi_x\mu]\rho + e^{\frac{i}{\hbar}\boldsymbol{\Pi}(\boldsymbol{\xi}')\cdot\boldsymbol{\zeta}}\frac{i}{\hbar}[-\xi'_y\eta - \xi'_x\mu]\rho\right\}.\end{aligned} \tag{4.26}$$

Using Eqs. (4.20) and (4.21) again for the second term in Eq. (4.26) it becomes

$$\begin{aligned}\frac{i\hbar\Omega}{2}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\boldsymbol{\Pi}\cdot\boldsymbol{\zeta}}\frac{i}{\hbar}[\Pi_x\eta - \Pi_y\mu]\rho\\ + \frac{m\Omega^2}{8}\int \frac{d\mu d\eta}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\boldsymbol{\Pi}\cdot\boldsymbol{\zeta}}[\underbrace{(\xi_y + \xi'_y)}_{2y}\eta + \underbrace{(\xi_x + \xi'_x)}_{2x}\mu]\rho\end{aligned} \tag{4.27}$$

where we can see that first term exactly cancels the last term in Eq. (4.16), and the second term cancels (4.18) or term (Ib).

The last term in the RHS of Eq. (4.9) or term (III) reads

$$\frac{e}{(2\pi\hbar)^2}\int d\boldsymbol{\zeta} e^{\frac{i}{\hbar}\boldsymbol{\Pi}\cdot\boldsymbol{\zeta}}[\phi(\boldsymbol{\xi}, t) - \phi(\boldsymbol{\xi}', t)]\rho(\boldsymbol{\xi}, \boldsymbol{\xi}', t)\,. \tag{4.28}$$

After introducing the $2D$ "inverse Wigner transform"

$$\rho(\boldsymbol{\xi}, \boldsymbol{\xi}', t) = \rho\left(\mathbf{r} + \frac{\boldsymbol{\varsigma}}{2}, \mathbf{r} - \frac{\boldsymbol{\varsigma}}{2}, t\right) = \int d\mathbf{\Pi}' e^{-\frac{i}{\hbar}\mathbf{\Pi}'\cdot\boldsymbol{\zeta}} f(\mathbf{r}, \mathbf{\Pi}', t) \tag{4.29}$$

and noting that

$$\begin{aligned} &\phi(\boldsymbol{\xi}, t) - \phi(\boldsymbol{\xi}', t) = \phi_x(\boldsymbol{\xi}, \xi_x', t) + \phi_y(\xi_y, \boldsymbol{\xi}', t)\,, \\ &\qquad \phi_x(\boldsymbol{\xi}, \xi_x', t) \equiv \phi\left(x + \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right), \\ &\qquad \phi_y(\xi_y, \boldsymbol{\xi}', t) \equiv \phi\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi\left(x - \frac{\mu}{2}, y - \frac{\eta}{2}, t\right). \end{aligned} \tag{4.30}$$

Equation (4.28) can be rewritten as

$$\frac{e}{(2\pi\hbar)^2} \int d\mathbf{\Pi}' f(\mathbf{r}, \mathbf{\Pi}', t) \int d\boldsymbol{\zeta} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\zeta}} \left[\mu\frac{\phi_x}{\mu} + \eta\frac{\phi_y}{\eta}\right]. \tag{4.31}$$

In analogy with the $1D$ free particle QVE derivation, by introducing the vector K with cartesian components defined as

$$K_a(\mathbf{r}, \mathbf{\Pi}, \mathbf{\Pi}', t) \equiv \frac{1}{(2\pi\hbar)^2} \int d\boldsymbol{\zeta} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\zeta}} \frac{\phi_a}{\zeta_a}\,, \qquad a = x, y\,. \tag{4.32}$$

Equation (4.31) can be rewritten as

$$e \int d\mathbf{\Pi}' f(\mathbf{r}, \mathbf{\Pi}', t) \left(-\frac{\hbar}{i}\right) \partial_{\Pi_a'} K_a\,, \qquad a = x, y \tag{4.33}$$

where summation over repeated index has been used. With an integration by parts, term (III) finally becomes

$$-i\hbar e \int d\mathbf{\Pi}' \partial_{\Pi_a'} f(\mathbf{r}, \mathbf{\Pi}', t) K_a = -i\hbar e \int d\mathbf{\Pi}' \frac{\partial f}{\partial \mathbf{\Pi}'} \cdot \mathbf{K}\,. \tag{4.34}$$

Combining Eqs. (4.11), (4.16), (4.18), (4.27), and (4.34), we have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{\Pi}, t) = &-\frac{i\hbar}{m}(\Pi_x\partial_x + \Pi_y\partial_y)f + \frac{i\hbar\Omega}{2}(\Pi_x\partial_{\Pi_y} - \Pi_y\partial_{\Pi_x})f \\ &- i\hbar e \int d\mathbf{\Pi}' \left(\frac{\partial f}{\partial \Pi_x'} K_x + \frac{\partial f}{\partial \Pi_y'} K_y\right) \end{aligned} \tag{4.35}$$

or

$$\frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{\Pi}) + \frac{1}{m}\mathbf{\Pi}\cdot\boldsymbol{\nabla} f - \frac{\Omega}{2}(\mathbf{\Pi}\times\boldsymbol{\nabla}_\Pi)\cdot\hat{z} f + e\int d\mathbf{\Pi}' \boldsymbol{\nabla}_{\Pi'} f \cdot \mathbf{K} = 0\,. \tag{4.36}$$

4.2. *Classical Vlasov equation (CVE)*

In a problem with a magnetic field the *generalized or mechanical momentum* $\mathbf{\Pi}$, following Sakurai's notation,[19] is form invariant under gauge transformation:

$$\mathbf{\Pi} = m\frac{d\mathbf{r}}{dt} = \mathbf{p} - \frac{e}{c}\mathbf{A} \tag{4.37}$$

where p is called the *canonical momentum*. The r introduced in the above equation can be referred to as the *generalized spatial coordinate*. It is reasonable to write the Wigner distribution function as

$$f = f(\mathbf{r}, \dot{\mathbf{r}}, t) = f(\mathbf{r}, \mathbf{\Pi}, t)\,. \tag{4.38}$$

Then the classical Vlasov equation can be written as

$$\frac{d}{dt}f(\mathbf{r}, \mathbf{\Pi}, t) = \left(\frac{\partial}{\partial t} + \frac{\mathbf{\Pi}}{m}\cdot\left(\frac{\partial}{\partial \mathbf{r}} + \left(\frac{\partial \mathbf{\Pi}}{\partial \mathbf{r}}\right)\cdot\frac{\partial}{\partial \mathbf{\Pi}}\right) + \dot{\mathbf{\Pi}}\cdot\frac{\partial}{\partial \mathbf{\Pi}}\right) f(\mathbf{r}, \mathbf{\Pi}, t) = 0\,. \tag{4.39}$$

Using the Ehrenfest's theorem[19]

$$\frac{d\mathbf{\Pi}}{dt} = e\left[\mathbf{E} + \frac{1}{2c}\left(\frac{d\mathbf{r}}{dt}\times\mathbf{B} - \mathbf{B}\times\frac{d\mathbf{r}}{dt}\right)\right]. \tag{4.40}$$

In our case as B is constant, pointing along the z-direction, and does not depend on any momentum coordinate, this equation reduces to the classical Lorentz force equation:

$$\frac{d\mathbf{\Pi}}{dt} = e\left[\mathbf{E} + \frac{1}{c}\frac{d\mathbf{r}}{dt}\times\mathbf{B}\right] = e\mathbf{E} + \Omega\mathbf{\Pi}\times\hat{z}\,. \tag{4.41}$$

Concentrating on the $2D$ problem in the transverse plane, i.e.

$$\mathbf{r} = (x, y) \tag{4.42}$$

we get

$$\begin{aligned}\frac{\mathbf{\Pi}}{m}\cdot\left(\frac{\partial \mathbf{\Pi}}{\partial \mathbf{r}}\right)\cdot\frac{\partial}{\partial \mathbf{\Pi}} &= \frac{1}{m}(\Pi_x, \Pi_y)\cdot\left(\frac{\partial \Pi_y}{\partial x}\partial_{\Pi_y}, \frac{\partial \Pi_x}{\partial y}\partial_{\Pi_x}\right)\\ &= \frac{\Omega}{2}(-\Pi_x\partial_{\Pi_y} + \Pi_y\partial_{\Pi_x})\,.\end{aligned} \tag{4.43}$$

And by using Eq. (4.41)

$$\dot{\mathbf{\Pi}}\cdot\frac{\partial}{\partial \mathbf{\Pi}} = e\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{\Pi}} + \Omega(\mathbf{\Pi}\times\hat{z})\cdot\frac{\partial}{\partial \mathbf{\Pi}} \tag{4.44}$$

the CVE (4.39) becomes

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \frac{1}{m}\left(\Pi_x\frac{\partial}{\partial x} + \Pi_y\frac{\partial}{\partial y}\right)\right.\\ &\qquad\left. + \frac{\Omega}{2}\left(\Pi_y\frac{\partial}{\partial \Pi_x} - \Pi_x\frac{\partial}{\partial \Pi_y}\right) + e\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{\Pi}}\right] f(\mathbf{x}, \mathbf{\Pi}, t) = 0\,.\end{aligned} \tag{4.45}$$

When we look back at the quantum Equation (4.36) and the definition of the vector $\mathbf{K}$ in Eq. (4.32), in the classical limit $\hbar \to 0$ the exponential in Eq. (4.32) oscillates very rapidly and the contributions are phased out unless also $\boldsymbol{\zeta} \to 0$ in which case $\phi_a/\zeta_a \to \partial\phi/\partial a$. Then $\mathbf{K}$ becomes

$$\mathbf{K}(\mathbf{r}, \boldsymbol{\Pi}, \boldsymbol{\Pi}', t) \equiv \frac{\boldsymbol{\nabla}\phi}{(2\pi\hbar)^2} \int d\boldsymbol{\zeta} e^{\frac{i}{\hbar}(\boldsymbol{\Pi}-\boldsymbol{\Pi}')\cdot\boldsymbol{\zeta}} = \boldsymbol{\nabla}\phi\delta^2(\boldsymbol{\Pi} - \boldsymbol{\Pi}')\,. \tag{4.46}$$

Then, in the classical limit the last term in the quantum equation (4.36) becomes

$$e\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}_{\Pi} f\,. \tag{4.47}$$

Then we see that QVE reduces to CVE in the classical limit for a particle in a magnetic field ($\mathbf{E} = -\boldsymbol{\nabla}\phi$).

5. The $B \neq 0$ Quantum Dielectric Function

In this section we use the quantum kinetic equation for a particle in a magnetic field derived in Section 4 and a similar procedure as that used in Section 3 for zero field, to derive a linear quantum dielectric function for a particle in a magnetic field in term of the Wigner function. The dielectric function, and the closely related conductivity, describes the particle collective behavior. The direct implication of the Wigner function in the dielectric function means that the former is not just a calculation tool but also carries the underlying physics. For example it has been shown[12] that the Wigner function show oscillations in term of B^{-1} that are consistent with de Haas–van Alphen effect (dHvA), in which the period of oscillation is related to the Fermi energy. It will be straightforward to note that the quantum linear dielectric function for $B \neq 0$ reduces to the Lindhard equation of Section 3 when $B \to 0$.

As a by-product, the derivation of this equation shows the internal structure of the quantum dielectric function sheds light on the correct sign for the gauge invariance of the Wigner function.

Kelly[14] obtained a dielectric tensor, not dielectric function, whose components did not present any oscillations. Harris gave an expression [Ref. 15 Eq. (2.69)] for the dielectric function in terms of "a quantum mechanical distribution function, which is a Fourier transform of the density matrix... It is similar but not identical to the well-known distribution function of Wigner", which differs with ours by an extra summation over the indices of Bessel functions, but it is not clear how this expression would reduce to the Lindhard equation at $B = 0$.

Starting with the quantum Vlasov equation (4.46)

$$\frac{\partial f}{\partial t} + \frac{1}{m}\boldsymbol{\Pi} \cdot \boldsymbol{\nabla} f - \frac{\Omega}{2}(\boldsymbol{\Pi} \times \boldsymbol{\nabla}_{\boldsymbol{\Pi}}) \cdot \hat{z} f + e \int d\boldsymbol{\Pi}' \boldsymbol{\nabla}_{\boldsymbol{\Pi}'} \cdot \mathbf{K} = 0 \tag{5.1}$$

or the expanded version in the $XY(2D)$ plane perpendicular to the field, Eq. (4.45)

$$\frac{\partial f}{\partial t} + \frac{1}{m}(\Pi_x\partial_x + \Pi_y\partial_y)f - \frac{\Omega}{2}(\Pi_x\partial_{\Pi_y} - \Pi_y\partial_{\Pi_x})f + e\int d\boldsymbol{\Pi}' \left(\frac{\partial f}{\partial \Pi_x'}K_x + \frac{\partial f}{\partial \Pi_y'}K_y\right) = 0 \tag{5.2}$$

with

$$K_a = \frac{1}{(2\pi\hbar)^2}\int d^2\boldsymbol{\xi} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\xi}}\frac{\phi_a}{\xi_a}\,, \qquad a = x, y\,, \qquad \boldsymbol{\xi} \equiv (\mu, \eta)\,, \tag{5.3}$$

and

$$\begin{aligned} \phi_x &= \phi\left(x + \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right), \\ \phi_y &= \phi\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi\left(x - \frac{\mu}{2}, y - \frac{\eta}{2}, t\right). \end{aligned} \tag{5.4}$$

As in Section 3 we assume small perturbations from the equilibrium values with the ion distribution remains unperturbed due to their large mass:

$$f_- = f_0^- + \varepsilon g\,, \qquad f_+ = f_0^+\,, \qquad \phi = 0 + \varepsilon\phi_1 \tag{5.5}$$

The perturbations for K_a are

$$K_{1x} = \frac{1}{(2\pi\hbar)^2}\int d^2\boldsymbol{\xi} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\xi}}\frac{1}{\mu}\left[\phi_1\left(x + \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi_1\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right)\right], \tag{5.6}$$

$$K_{1y} = \frac{1}{(2\pi\hbar)^2}\int d^2\boldsymbol{\xi} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\xi}}\frac{1}{\eta}\left[\phi_1\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi_1\left(x - \frac{\mu}{2}, y - \frac{\eta}{2}, t\right)\right]. \tag{5.7}$$

Now we go to the frequency domain by writing

$$g(\mathbf{r}, \mathbf{v}, t) = \frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega \bar{g}(\omega, \mathbf{k}, \mathbf{v}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{5.8}$$

$$\phi_1(\mathbf{r}, t) = \frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega \bar{\phi}_1(\omega, \mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,. \tag{5.9}$$

With these definitions, the appropriate combinations are

$$\begin{aligned} &\phi_1\left(x + \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi_1\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) \\ &\quad = \frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega \bar{\phi}_1(\omega, \mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} e^{ik_y\frac{\eta}{2}} 2i \sin k_x\frac{\mu}{2} \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} &\phi_1\left(x - \frac{\mu}{2}, y + \frac{\eta}{2}, t\right) - \phi_1\left(x - \frac{\mu}{2}, y - \frac{\eta}{2}, t\right) \\ &\quad = \frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega \bar{\phi}_1(\omega, \mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} e^{-ik_x\frac{\mu}{2}} 2i \sin k_y\frac{\eta}{2}\,. \end{aligned} \tag{5.11}$$

With these combinations of perturbations for ϕ we can go to the frequency domain for the perturbations in K with

$$K_{1x} = \frac{1}{(2\pi\hbar)^2}\frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega\bar{\phi}_1(\omega,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \times \underbrace{\int d^2\boldsymbol{\xi} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\xi}}e^{ik_y\frac{\eta}{2}}\frac{2i\sin k_x\frac{\mu}{2}}{\mu}}_{I_{1x}}\,. \tag{5.12}$$

The integral I_{1x} is done as follows

$$I_{1x} = \underbrace{\int d\mu e^{\frac{i}{\hbar}(\Pi_x-\Pi'_x)\mu}2i\frac{\sin k_x\frac{\mu}{2}}{\mu}}_{(2i)^2\tanh^{-1}\frac{\hbar k_x}{2(\Pi_x-\Pi'_x)}}\underbrace{\int d\eta e^{\frac{i}{\hbar}(\Pi_y-\Pi'_y)\eta}e^{ik_y\frac{\eta}{2}}}_{2\pi\hbar\delta(\Pi_y-\Pi'_y+\frac{\hbar k_y}{2})}\,. \tag{5.13}$$

Similarly,

$$K_{1y} = \frac{1}{(2\pi\hbar)^2}\frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega\bar{\phi}_1(\omega,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \times \underbrace{\int d^2\boldsymbol{\xi} e^{\frac{i}{\hbar}(\mathbf{\Pi}-\mathbf{\Pi}')\cdot\boldsymbol{\xi}}e^{-ik_x\frac{\mu}{2}}\frac{2i\sin k_y\frac{\eta}{2}}{\eta}}_{I_{1y}}\,. \tag{5.14}$$

The integral I_{1y} is done as follows

$$I_{1y} = \underbrace{\int d\mu e^{\frac{i}{\hbar}(\Pi_x-\Pi'_x)\mu}e^{-ik_x\frac{\mu}{2}}}_{2\pi\hbar\delta(\Pi_x-\Pi'_x-\frac{\hbar k_x}{2})}\underbrace{\int d\eta e^{\frac{i}{\hbar}(\Pi_y-\Pi'_y)\eta}2i\frac{\sin k_y\frac{\eta}{2}}{\eta}}_{(2i)^2\tanh^{-1}\frac{\hbar k_y}{2(\Pi_y-\Pi'_y)}}\,. \tag{5.15}$$

Before writing down the frequency domain quantum Vlasov equation for the linear perturbations we consider one issue. The derivatives with respect to the mechanical momentum in the third term are taken at fixed spatial positions. In the symmetric gauge, the mechanical momenta are defined as

$$\Pi_x = p_x + \frac{m\Omega}{2}y\,, \qquad \Pi_y = p_y - \frac{m\Omega}{2}x\,, \tag{5.16}$$

and for the third term in the quantum Vlasov equation:

$$\partial_{\Pi_y}|_x = \frac{1}{\hbar}\partial_{k_y}\,, \qquad \partial_{\Pi_x}|_y = \frac{1}{\hbar}\partial_{k_x}\,. \tag{5.17}$$

Then the frequency domain equation in the linear perturbation reads

$$-i\omega\bar{g} + \underbrace{\frac{i}{m}(\Pi_x k_x + \Pi_y k_y)\bar{g} - \frac{i\Omega}{2\hbar}(\Pi_x y - \Pi_y x)\bar{g}}_{T_{23}}$$
$$+ \frac{(2i)^2 e}{(2\pi\hbar)}\bar{\phi}_1\left\{\int d\Pi'_x \frac{\partial f_0}{\partial \Pi'_x}\bigg|_{\Pi'_y=\Pi_y+\frac{\hbar k_y}{2}}\tanh^{-1}\frac{\hbar k_x}{2(\Pi_x-\Pi'_x)}\right.$$

$$+ \int d\Pi'_y \frac{\partial f_0}{\partial \Pi'_y}\bigg|_{\Pi'_x = \Pi_x - \frac{\hbar k_x}{2}} \tanh^{-1} \frac{\hbar k_y}{2(\Pi_y - \Pi'_y)} \Bigg\} = 0\,. \tag{5.18}$$

Term T_{23} is rearranged into

$$\begin{aligned} T_{23} &= \left[\frac{i}{m\hbar}\Pi_x\left(\hbar k_x - \frac{m\Omega}{2}y\right) + \frac{i}{m\hbar}\Pi_y\left(\hbar k_y + \frac{m\Omega}{2}x\right)\right]\bar{g} \\ &\equiv (iV_xK_x + iV_yK_y)\bar{g} \end{aligned} \tag{5.19}$$

with

$$\begin{aligned} K_x &\equiv \frac{D_x}{\hbar}\,, \qquad D_x \equiv \hbar k_x - \frac{m\Omega}{2}y\,, \\ K_y &\equiv \frac{D_y}{\hbar}\,, \qquad D_y \equiv \hbar k_y + \frac{m\Omega}{2}x\,, \end{aligned} \tag{5.20}$$

and

$$V_a \equiv \frac{\Pi_a}{m}\,, \qquad a = x, y\,. \tag{5.21}$$

Here we pause to make an important discussion. From the derivation of the quantum Vlasov equation in Section 4, Π_a is replaced by D_a should we choose to do that in the definition of the Wigner function. So the exact definition of V_a in term of Π_a or D_a is arbitrary, depending on a choice in the definition of the Wigner function (if we ignore the fact that gauge invariance require the choice of D_a). However, the definition of K_a is not arbitrary, it comes from the internal structure of the quantum Vlasov equation, and it is related to D_a not Π_a!

Back to the frequency domain equation in the linear perturbation of the distribution function, it leads to

$$\begin{aligned} \bar{g} = \frac{4ie\bar{\phi}_1}{(2\pi\hbar)}\frac{1}{\omega - \mathbf{K}\cdot\mathbf{V}}\Bigg\{ &\int d\Pi'_x \frac{\partial f_0}{\partial \Pi'_x}\bigg|_{\Pi'_y = \Pi_y + \frac{\hbar k_y}{2}} \tanh^{-1}\frac{\hbar k_x}{2(\Pi_x - \Pi'_x)} \\ &+ \int d\Pi'_y \frac{\partial f_0}{\partial \Pi'_y}\bigg|_{\Pi'_x = \Pi_x - \frac{\hbar k_x}{2}} \tanh^{-1}\frac{\hbar k_y}{2(\Pi_y - \Pi'_y)}\Bigg\}\,. \end{aligned} \tag{5.22}$$

As with the derivation for $B = 0$ in Section 3, we assume that the perturbation is due to an electron test charge. The generalization of Gauss' law to include this test charge is

$$\boldsymbol{\nabla}\cdot\mathbf{E}_1 = -4\pi e\delta(\mathbf{r} - \mathbf{v}_0 t) - 4\pi e\int d\mathbf{V} g\,. \tag{5.23}$$

Using the last expression for $\bar{g}$ in the Fourier transform expression of the generalized Gauss' law:

$$i\mathbf{k}\cdot\bar{\boldsymbol{E}}_1 = -8\pi^2 e\delta(\omega - \mathbf{k}\cdot\mathbf{v}_0) - 4\pi e\int d\mathbf{V}\bar{g} \tag{5.24}$$

where

$$\mathbf{E}_1(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int d\omega \bar{\boldsymbol{E}}_1(\omega,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{5.25}$$

we arrive at

$$\begin{aligned} i\mathbf{k}\cdot\bar{\mathbf{E}}_1 &= -8\pi^2 e\delta(\omega-\mathbf{k}\cdot\mathbf{v}_0) + \frac{16\pi i n_0 e^2}{(2\pi\hbar)}\bar{\phi}_1\int d\mathbf{V}\frac{1}{\omega-\mathbf{K}\cdot\mathbf{V}} \\ &\quad\times\Bigg[\underbrace{\int d\Pi'_x\bar{f}_0(\Pi'_y=\Pi_y+\hbar k_y/2)\frac{\hbar k_x/2}{(\Pi_x-\Pi'_x)^2-(\hbar k_x/2)^2}}_{\equiv I_x(\hbar k_x/2)} \\ &\quad+\underbrace{\int d\Pi'_y\bar{f}_0(\Pi'_x=\Pi_x-\hbar k_x/2)\frac{\hbar k_y/2}{(\Pi_y-\Pi'_y)^2-(\hbar k_y/2)^2}}_{\equiv I_y(\hbar k_y/2)}\Bigg] \end{aligned} \tag{5.26}$$

where we also have used $f_0 = n_0\bar{f}_0$ and applied the result

$$\frac{\partial}{\partial\Pi'_a}\left[\tanh^{-1}\frac{\hbar k_a}{2(\Pi_a-\Pi'_a)}\right] = \frac{\hbar k_a/2}{(\Pi_a-\Pi'_a)^2-(\hbar k_a/2)^2}\,, \qquad a=x,y\,. \tag{5.27}$$

The next step is to use the residue theorem to evaluate the integrals

$$\begin{aligned} I_x &= \oint_{\Gamma_2} d\Pi'_x\frac{\bar{f}_0\left(\Pi'_y=\Pi_y+\dfrac{\hbar k_y}{2}\right)}{\left[\Pi'_x-\left(\Pi_x+\dfrac{\hbar k_x}{2}\right)\right]\left[\Pi'_x-\left(\Pi_x-\dfrac{\hbar k_x}{2}\right)\right]} \\ &= \frac{\pi i}{\hbar k_x}\left[\bar{f}_0\left(\Pi_x+\frac{\hbar k_x}{2},\Pi_y+\frac{\hbar k_y}{2}\right)-\bar{f}_0\left(\Pi_x-\frac{\hbar k_x}{2},\Pi_y+\frac{\hbar k_y}{2}\right)\right], \end{aligned} \tag{5.28}$$

$$\begin{aligned} I_y &= \oint_{\Gamma_2} d\Pi'_y\frac{\bar{f}_0\left(\Pi'_x=\Pi_x-\dfrac{\hbar k_x}{2}\right)}{\left[\Pi'_y-\left(\Pi_y+\dfrac{\hbar k_y}{2}\right)\right]\left[\Pi'_y-\left(\Pi_y-\dfrac{\hbar k_y}{2}\right)\right]} \\ &= \frac{\pi i}{\hbar k_y}\left[\bar{f}_0\left(\Pi_x-\frac{\hbar k_x}{2},\Pi_y+\frac{\hbar k_y}{2}\right)-\bar{f}_0\left(\Pi_x-\frac{\hbar k_x}{2},\Pi_y-\frac{\hbar k_y}{2}\right)\right], \end{aligned} \tag{5.29}$$

and also using the definition of the plasma frequency

$$\omega_p^2 = \frac{4\pi n_0 e^2}{m} \tag{5.30}$$

to arrive at

$$i\mathbf{k}\cdot\mathbf{E}_1 = -8\pi^2 e\delta(\omega-\mathbf{k}\cdot\mathbf{v}_0) - \frac{m\omega_p^2}{\hbar}\bar{\phi}_1\int d\mathbf{V}\frac{1}{\omega-\mathbf{K}\cdot\mathbf{V}}$$

$$\times \left\{ \bar{f}_0 \left(\Pi_x + \frac{\hbar k_x}{2}, \Pi_y + \frac{\hbar k_y}{2} \right) - \bar{f}_0 \left(\Pi_x - \frac{\hbar k_x}{2}, \Pi_y + \frac{\hbar k_y}{2} \right) \right.$$

$$\left. + \bar{f}_0 \left(\Pi_x - \frac{\hbar k_x}{2}, \Pi_y + \frac{\hbar k_y}{2} \right) - \bar{f}_0 \left(\Pi_x - \frac{\hbar k_x}{2}, \Pi_y - \frac{\hbar k_y}{2} \right) \right\}. \quad (5.31)$$

Since

$$i\mathbf{k} \cdot \mathbf{E}_1 \varepsilon(k, \omega) = -8\pi^2 e \delta(\omega - \mathbf{k} \cdot \mathbf{v}_0) \quad (5.32)$$

and

$$-\boldsymbol{\nabla}\phi_1 = \mathbf{E}_1 \to -i\mathbf{k}\bar{\phi}_1 = \bar{\mathbf{E}}_1 \to \bar{\phi}_1 = i\frac{\mathbf{k} \cdot \bar{\mathbf{E}}_1}{k^2} \quad (5.33)$$

finally, we have

$$\varepsilon(k, \omega) = 1 + \frac{m\omega_p^2}{\hbar k^2} \times \int d\mathbf{V} \frac{\bar{f}_0(\Pi_x + \hbar k_x/2, \Pi_y + \hbar k_y/2) - \bar{f}_0(\Pi_x - \hbar k_x/2, \Pi_y - \hbar k_y/2)}{\omega - \mathbf{K} \cdot \mathbf{V}}. \quad (5.34)$$

It can be noted that this equation reduces to that derived in Section 3 when $B = 0$.

6. Conclusions

We have derived a quantum dielectric function (QDF) from the quantum Vlasov equation (QVE), first for $B = 0$ (Section 3) and then for $B \neq 0$ (Section 5), using linear perturbations and a self-consistent mean field interaction via Poisson's equation. The QVE is a quantum kinetic equation that governs the behavior of the Wigner function (WF). The QVE was derived from the Scrodinger equation (SE) for $B = 0$ (Section 2) and for $B \neq 0$ (Section 4). The $B = 0$ QDF, except for its expression in term of the WF, is the Lindhard constant, which is usually derived from the SE in the literature. The $B \neq 0$ QDF was derived for the first time. This result indicated that

(i) Since properties such as conductivities are related to the dielectric function and so to the Wigner function, the physics is contained in the later. A magnetic Wigner function should show de Haas–van Alphen oscillations as proved in Ref. 12.
(ii) An explicit manifestation that the structure of the kinetic equation gives rise to $\mathbf{D}$, not $\boldsymbol{\Pi}$ (see Appendix B for their definitions). This is related to the choice of $\mathbf{D}$ or $\boldsymbol{\Pi}$ in the definition of the magnetic Wigner function, and we argue for $\boldsymbol{D}$ for gauge invariance.[13]

We have found related but not similar work in the papers by Kelly[14] and Harris.[15] Kelly obtained a dielectric tensor, not dielectric function, whose components did not present any de Haas–van Alphen oscillations. Harris gave an expression (2.69) for the dielectric function in terms of "a quantum mechanical distribution function, which is a Fourier transform of the density matrix... It is similar but not identical to the well-known distribution function of Wigner", which differs with ours by an extra summation over the indices of Bessel functions. It is not clear how this expression would reduce to the Lindhard constant at $B = 0$.

We have also proposed a modified WF with finite limits of integration. For free particles when the eigenfunctions are plane waves, the WF is singular and positive while the modified WF is finite with oscillatory behavior. For a harmonic oscillator potential when the eigenfunctions are localized in space, finite limits of integration are effective even in the original WF and there is no distinction with the modified one.

Appendix A. The Kernel for the Trace expression of WF

The Wigner function can be obtained from the density matrix ρ and a kernel Δ via a trace expression

$$f = hTr(\rho\Delta)\,. \tag{A.1}$$

To obtain an expression in the position representation we expand the trace as

$$f = h\int dp'\langle p'|\rho\Delta|p'\rangle\,. \tag{A.2}$$

When the kernel

$$\Delta_W(p-\hat{p}, q-\hat{q}) = \frac{1}{(2\pi)^2}\iint d\xi d\eta e^{i[\xi(p-\hat{p})+\eta(q-\hat{q})]} \tag{A.3}$$

is used in Eq. (A.2), the usual Wigner function is obtained,[20] with $\hat{p}, \hat{q}$ being the momentum and position operators, respectively. We can note that Δ has dimension of $[\text{action}]^{-1}$. The trace and ρ are dimensionless, so Eq. (A.1) corresponds to dimensionless distribution functions. In fact, by using the Baker–Campbell–Hausdoff formula in Eq. (A.3), inserting position-space completeness relations into Eq. (A.2), and using the resulting delta functions, we recover the usual expression for the Wigner distribution function

$$f_W(p,q) = \hbar\int d\xi e^{i\xi p}\left\langle q-\frac{\hbar\xi}{2}\right|\rho(p,q)\left|q+\frac{\hbar\xi}{2}\right\rangle \tag{A.4}$$

which is dimensionless as each position ket has a dimension of $[\text{length}]^{-1/2}$.

Here is the complete derivation:

$$\begin{aligned} f_W &= h\mathrm{Tr}[\rho(\hat{p},\hat{q})\Delta_W(p-\hat{p},q-\hat{q})] \\ &= \frac{h}{(2\pi)^2}\int dp'\langle p'|\rho(\hat{p},\hat{q})\iint d\xi d\eta\, \underbrace{e^{i[\xi(p-\hat{p})+\eta(q-\hat{q})]}}_{e^{i\eta(q-\hat{q})}e^{i\xi(p-\hat{p})}e^{\frac{1}{2}\xi\eta[\hat{q},\hat{p}]}}|q'\rangle\,. \end{aligned} \tag{A.5}$$

The Baker–Campbell–Hausdoff formula has been used in the last equality. Using

$$[\hat{q}, \hat{p}] = i\hbar\,, \tag{A.6}$$

and applying the momentum operator in the exponential to the momentum ket and taking all the number exponentials outside the expectation value, one obtains

$$f_W(p,q) = \frac{h}{(2\pi)^2} \iint d\xi d\eta e^{i(\eta q + \xi p)} e^{\frac{i\hbar}{2}\xi\eta} \int dp' e^{-i\xi p'} \underbrace{\langle p'|\rho(\hat{p},\hat{q}) e^{-i\eta\hat{q}}|p'\rangle}_{\iint dq' dq'' \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}p'(q''-q')} e^{-i\eta q''} \langle q'|\rho|q''\rangle} \,. \tag{A.7}$$

The last result has been obtained by inserting completeness relations in q' and q''

$$\int dq' |q'\rangle\langle q'|\,, \qquad \int dq'' |q''\rangle\langle q''| \tag{A.8}$$

immediately after the bra and before the ket, respectively, and using the plane wave definition

$$\langle p'|q'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}\mathbf{p}'\cdot\mathbf{q}'}\,. \tag{A.9}$$

Rearranging Eq. (A.7):

$$\begin{aligned} f_W(p,q) &= \frac{1}{(2\pi)^2} \iint d\xi d\eta e^{i(\eta q + \xi p)} e^{\frac{i\hbar}{2}\xi\eta} \iint dq' dq'' e^{-i\eta q''} \langle q'|\rho|q''\rangle \underbrace{\int dp' e^{\frac{i}{\hbar}p'(q''-q'-\hbar\xi)}}_{2\pi\hbar\delta[q''-(q'+\hbar\xi)]} \\ &= \frac{\hbar}{2\pi} \iiint d\xi d\eta dq' e^{i(\eta q+\xi p)} e^{\frac{i\hbar}{2}\xi\eta} e^{-i\eta(q'+\hbar\xi)} \langle q'|\rho|q'+\hbar\xi\rangle \\ &= \frac{\hbar}{2\pi} \iint d\xi dq' e^{i\xi p} \langle q'|\rho|q'+\hbar\xi\rangle \underbrace{\int d\eta e^{-i\eta(q'+\frac{\hbar\xi}{2}-q)}}_{2\pi\delta[q'-(q-\frac{\hbar\xi}{2})]} \\ &= \hbar \int d\xi e^{i\xi p} \left\langle q - \frac{\hbar\xi}{2} \middle| \rho \middle| q + \frac{\hbar\xi}{2} \right\rangle \\ &= \hbar \int d\xi e^{i\xi p} \psi^* \left(q + \frac{\hbar\xi}{2}\right) \psi \left(q - \frac{\hbar\xi}{2}\right). \end{aligned} \tag{A.10}$$

Appendix B. Definitions of D and Π

The form, which we will call the operator form (operators carry a hat)

$$f(\mathbf{p}) = \int d\mathbf{r}_0 e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}_0} \langle\psi| e^{-\frac{1}{2}\mathbf{r}_0\cdot\hat{\mathbf{D}}^+} \otimes e^{\frac{1}{2}\mathbf{r}_0\cdot\hat{\mathbf{D}}}|\psi\rangle \tag{B.1}$$

is gauge invariant,[21] where the d-derivative operator should be defined as:

$$\hat{\mathbf{D}} \equiv \frac{i}{\hbar}\left(\hat{\mathbf{p}} + \frac{e}{c}\hat{\mathbf{A}}\right) \tag{B.2}$$

whose Cartesian components commute with the free particle Hamiltonian in the transverse plane (the constant magnetic field points along z):

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{\Pi}}^2 \tag{B.3}$$

with the mechanical momentum defined as

$$\hat{\mathbf{\Pi}} \equiv \hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}\,, \tag{B.4}$$

i.e.

$$[\hat{H}_\perp, \hat{D}_x] = [\hat{H}_\perp, \hat{D}_y] = 0\,. \tag{B.5}$$

This property is not shared by the Cartesian components of the mechanical momentum itself

$$[\hat{H}_\perp, \hat{\Pi}_x] = -2i\hbar m\Omega\hat{\Pi}_y\,, \qquad [\hat{H}_\perp, \hat{\Pi}_y] = 2i\hbar m\Omega\hat{\Pi}_x\,. \tag{B.6}$$

The fact that Eq. (B.1) is invariant under a gauge transformation that leaves the mechanical momentum invariant

$$\hat{\mathbf{A}} \to \hat{\mathbf{A}} + \boldsymbol{\nabla}\chi(\hat{\mathbf{r}}, t)\,, \qquad \psi \to e^{\frac{ie}{\hbar c}\chi(\hat{\mathbf{r}},t)}\psi\,, \tag{B.7}$$

$$\hat{\mathbf{p}} \to \hat{T}_\chi^+\hat{\mathbf{p}}\hat{T}_\chi = \hat{\mathbf{p}} + \frac{e}{c}\boldsymbol{\nabla}\chi(\hat{\mathbf{r}}, t)\,, \qquad \hat{T}_\chi(t) = e^{\frac{ie}{\hbar c}\chi(\hat{\mathbf{r}},t)}\,, \tag{B.8}$$

but not the d-derivative

$$\hat{\mathbf{D}} \to \hat{\mathbf{D}} + 2\frac{ie}{\hbar c}\boldsymbol{\nabla}\chi(\hat{\mathbf{r}}, t) \tag{B.9}$$

is because

$$e^{-\frac{1}{2}\mathbf{r}_0\cdot\hat{\mathbf{D}}^+}e^{\frac{1}{2}\mathbf{r}_0\cdot\hat{\mathbf{D}}} \to e^{-\frac{1}{2}\mathbf{r}_0\cdot[\hat{\mathbf{D}}^+ - 2\frac{ie}{\hbar c}(\boldsymbol{\nabla}\chi)^+]}e^{\frac{1}{2}\mathbf{r}_0\cdot[\hat{\mathbf{D}} + 2\frac{ie}{\hbar c}(\boldsymbol{\nabla}\chi)]} = e^{-\frac{1}{2}\mathbf{r}_0\cdot\hat{\mathbf{D}}^+}e^{\frac{1}{2}\mathbf{r}_0\cdot\hat{\mathbf{D}}}\,. \tag{B.10}$$

Appendix C. Linear Quantum Dielectric Function from the Schrödinger–Poisson System

Starting from the Schrödinger equation (SE) and its complex conjugate

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle \quad \text{and} \quad -i\hbar\frac{\partial}{\partial t}\langle\Psi| = \langle\Psi|H\,, \tag{C.1}$$

when we differentiate the density matrix with respect to time, the Heisenberg equation is obtained:

$$i\hbar\frac{\partial\rho}{\partial t} = i\hbar\left(\frac{\partial|\Psi\rangle}{\partial t}\langle\Psi| + |\Psi\rangle\frac{\partial\langle\Psi|}{\partial t}\right) = H\rho - \rho H = [H, \rho]\,. \tag{C.2}$$

The density matrix is the statistical operator, which in term of the wave functions reads

$$\rho = \Psi\Psi^*\,. \tag{C.3}$$

It is observed that the bra-ket notation clearly indicates the right order for the wave functions.

Our SE is a single-particle mean-field equation with

$$H = H_0 + V \tag{C.4}$$

where H_0 is the free-particle Hamiltonian. V, which is considered a perturbation, can be decomposed into an applied external potential V_e plus an internal potential V_i.

$$V(\mathbf{r}, t) = V_e(\mathbf{r}, t) + V_i(\mathbf{r}, t)\,. \tag{C.5}$$

The internal potential is produced by the charged particle fluctuations

$$n = n_0 + \delta n\,, \qquad \rho = \rho_0 + \delta\rho\,, \tag{C.6}$$

via the Poisson equation:

$$\nabla^2 V_i(\mathbf{r}, t) = -4\pi e^2 \delta n(\mathbf{r}, t)\,. \tag{C.7}$$

The linearized Heisenberg equation reads

$$i\hbar \frac{\partial \delta\rho}{\partial t} = [H_0, \delta\rho] + [V, \rho_0]\,. \tag{C.8}$$

This equation in terms of the matrix elements in momentum space can be derived using the properties

$$H_0|k\rangle = E(k)|k\rangle\,, \qquad E(k) = \frac{\hbar^2 k^2}{2m}\,, \tag{C.9}$$

$$\rho_0|k\rangle = f_0(k)|k\rangle\,, \qquad f_0 = \mathit{fermi\ distribution}\,, \tag{C.10}$$

to be

$$i\hbar\langle k'|\delta\dot{\rho}|k\rangle = [E(\mathbf{k}') - E(\mathbf{k})]\langle k'|\delta\rho|k\rangle - [f_0(\mathbf{k}') - f_0(\mathbf{k})]\langle k'|V|k\rangle\,. \tag{C.11}$$

The matrix element for V is the q-component of the Fourier transform of $V(\mathbf{q} = \mathbf{k}' - \mathbf{k})$). In fact, by inserting position-space closure relations and using plane-wave definitions

$$\begin{aligned}\langle k'|V(\mathbf{r}, t)|k\rangle &= \iint d\mathbf{r}_1 d\mathbf{r}_2 \quad \underbrace{\langle k'|r_1\rangle}_{\frac{1}{(2\pi)^{3/2}}e^{-i\mathbf{k}'\cdot\mathbf{r}_1}} \quad \underbrace{\langle r_1|V(\mathbf{r}, t)|r_2\rangle}_{\delta(\mathbf{r}_1-\mathbf{r}_2)V(\mathbf{r}_2,t)} \quad \underbrace{\langle r_2|k\rangle}_{\frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{r}_2}} \\ &= \frac{1}{(2\pi)^3}\int d\mathbf{r}_1 e^{-i\mathbf{q}\cdot\mathbf{r}_1} V(\mathbf{r}_1, t) \equiv V_q(t)\,.\end{aligned} \tag{C.12}$$

We introduce the Fourier transform in space and time for V and $\delta\rho$:

$$V(\mathbf{r}, t) = \frac{1}{(2\pi)^3}\iint d\mathbf{q}' d\omega e^{i(\mathbf{q}'\cdot\mathbf{r}\omega t)} V(\mathbf{q}', \omega)\,. \tag{C.13}$$

Then the matrix elements become

$$\begin{aligned}\langle k'|V(\mathbf{r},t)|k\rangle = V_q(t) &= \frac{1}{(2\pi)^3}\int d\mathbf{r}_1 e^{-i\mathbf{q}\cdot\mathbf{r}_1}V(\mathbf{r}_1,t)\\ &= \frac{1}{(2\pi)^3}\iint d\mathbf{q}' d\omega e^{-i\omega t}V(\mathbf{q}',\omega)\underbrace{\int d\mathbf{r}_1 e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{r}_1}}_{(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')}\\ &= \int d\omega e^{-i\omega t}V(\mathbf{q},\omega)\,. \end{aligned} \tag{C.14}$$

Similarly,

$$\langle k'|\delta\rho(\mathbf{r},t)|k\rangle = \int d\omega e^{-i\omega t}\delta\rho(\mathbf{q},\omega)\,. \tag{C.15}$$

By using these Fourier transforms in the linearized Heisenberg equation (C.11) it gives

$$i\hbar(-i\omega)\delta\rho(\mathbf{q},\omega) = [E(\mathbf{k}') - E(\mathbf{k})]\delta\rho(\mathbf{q},\omega) - [f_0(\mathbf{k}') - f_0(\mathbf{k})]V(\mathbf{q},\omega) \tag{C.16}$$

which means

$$\delta\rho(\mathbf{q},\omega) = \frac{f_0(\mathbf{k}') - f_0(\mathbf{k})}{E(\mathbf{k}') - E(\mathbf{k}) - \hbar\omega}V(\mathbf{q},\omega)\,. \tag{C.17}$$

On the other hand, the particle fluctuation can be written in term of the statistical operator fluctuation as

$$\begin{aligned}\delta n(\mathbf{r}_0,t) = Tr\{\delta(\mathbf{r}-\mathbf{r}_0)\delta\rho\} &= \int d\mathbf{k}\langle k|\delta(\mathbf{r}-\mathbf{r}_0)\delta\rho|k\rangle\\ &= \iint d\mathbf{k}d\mathbf{k}' \underbrace{\langle k|\delta(\mathbf{r}-\mathbf{r}_0)|k'\rangle}_{\frac{1}{(2\pi)^3}\int d\mathbf{r}e^{i\mathbf{q}\cdot\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}_0)} \langle k'|\delta\rho|k\rangle\\ &= \frac{1}{(2\pi)^3}\iint d\mathbf{k}d\mathbf{k}' e^{i\mathbf{q}\cdot\mathbf{r}_0}\langle k'|\delta\rho|k\rangle\,. \end{aligned} \tag{C.18}$$

By using Eqs. (C.13), (C.15), and (C.18) in the linearized Poisson equation (C.7) it becomes

$$\frac{1}{(2\pi)^3}\iint d\mathbf{q}d\omega e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}(-q^2)V_i(\mathbf{q},\omega) = -4\pi e^2\frac{1}{(2\pi)^3}\iiint d\mathbf{k}d\mathbf{k}'d\omega e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\delta\rho(\mathbf{q},\omega) \tag{C.19}$$

which reduces, with $d\mathbf{k}' = d\mathbf{q}$, to

$$-q^2V_i(\mathbf{q},\omega) = -4\pi e^2\int d\mathbf{k}\delta\rho(\mathbf{q},\omega)\,. \tag{C.20}$$

When the final version of the linearized Poisson equation (C.20) is used in the final version of the linearized Heisenberg equation (C.17), it shows

$$V_i(\mathbf{q},\omega) = \frac{4\pi e^2}{q^2}\int d\mathbf{k}\frac{f_0(\mathbf{k}') - f_0(\mathbf{k})}{E(\mathbf{k}') - E(\mathbf{k}) - \hbar\omega}V(\mathbf{q},\omega)\,. \tag{C.21}$$

If we follow Ref. 11 in defining, the dielectric constant as

$$\varepsilon(\mathbf{q},\omega) \equiv \frac{V_a(\mathbf{q},\omega)}{V(\mathbf{q},\omega)} = 1 - \frac{V_i(\mathbf{q},\omega)}{V(\mathbf{q},\omega)}$$

$$= 1 - \frac{4\pi e^2}{q^2} \int d\mathbf{k} \frac{f_0(\mathbf{k}+\mathbf{q}) - f_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega}\,. \tag{C.22}$$

With the definitions

$$f_0 \equiv n_0 \bar{f}_0\,, \qquad \omega_p^2 \equiv \frac{4\pi n_0 e^2}{m_e}\,, \tag{C.23}$$

the quantum dielectric function from the Schrödinger–Poisson equations is finally written in the form

$$\varepsilon(\mathbf{q},\omega) = 1 - \frac{m_e \omega_p^2}{q^2} \int d\mathbf{k} \frac{\bar{f}_0(\mathbf{k}+\mathbf{q}) - \bar{f}_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega}\,. \tag{C.24}$$

This is identical to the one derived from the quantum Vlasov–Poisson system (3.36).

References

1. M. Hillery, R. F. O'Connell, M. O. Scully and E. P. Wigner, *Phys. Rep.* **106**, 121 (1984).
2. P. Bertrand, Nguyen Van Tuan, M. Gros, B. Izrar, M. Feix and J. Gutierrez, *J. Plasma Physics* **23**, 401 (1980).
3. J. Rammer, *Rev. Mod. Phys.* **63**, No. 4, 781 (1991).
4. P. Bordone, M. Pascoli, R. Brunetti, A. Bertoni and C. Jacobini, *Phys. Rev.* **B59**, 3060 (1999).
5. A. Haque and A. N. Khondker, *J. Appl. Phys.* **87**, 2553 (2000).
6. L. M. Widrow and N. Kaiser, *ApJ.* **416**, L71 (1993).
7. G. Davies and L. M. Widrow, *ApJ.* **485**, 484 (1997).
8. G. Arfken, *Mathematical Methods for Physicists* (Academic Press Inc., 1985).
9. D. R. Nicholson, *Introduction to Plasma Theory* (1983).
10. N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rhinehart & Winston Inc. 1976).
11. O. Madelung, *Introduction to Solid-State Theory* (Springer-Verlag, 1978), Chapter 3.
12. "Wigner Function in the Symmetric Gauge: de Haas–van Alphen Oscillations, Magnetic Field Localization and the Uncertainty Principle", to be published in *Int. J. Mod. Phys.* **B17** (2003).
13. 'Gauge Invariance of the Wigner Function. Geometrical Interpretations', in preparation.
14. D. C. Kelly, *Phys. Rev.* **134**, A641 (1964).
15. E. G. Harris, in *Advances in Plasma Physics Vol. 3*, edited by A. Simon and W. B. Thompson (1967), p. 167.
16. G. F. Bertsch, in *Les Houches Lectures 1977*, edited by Roger Balian and Manque Rho (North-Holland, 1978).
17. H. Neunzert, *Il Nuovo Cimento* **87A**, N.2, 151 (1985).
18. R. L. Liboff, *Kinetic Theory: Classical, Quantum, and Relativistic Description* (Wiley, 1998).
19. J. J. Sakurai, *Modern Quantum Mechanics*, Revised Ed. (Addison Wesley, 1994).
20. R. Kubo, *J. Phys. Soc. Japan* **19**, 2127 (1964).
21. D. Vasak, M. Gyulassy and H.-T. Elze, *Annals of Physics* **173**, 462 (1987).