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THE QUATERNIONIC KP HIERARCHY AND CONFORMALLY IMMERSED 2-TORI IN THE 4-SPHERE

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Abstract. The quaternionic KP hierarchy is the integrable hierarchy of p.d.e obtained by replacing the complex numbers with the quaternions in the standard construction of the KP hierarchy and its solutions; it is equivalent to what is often called the Davey-Stewartson II hierarchy. This article studies its relationship with the theory of conformally immersed tori in the 4-sphere via quaternionic holomorphic geometry. The Sato-Segal-Wilson construction of KP solutions is adapted to this setting and the connection with quaternionic holomorphic curves is made. We then compare three different notions of "spectral curve": the QKP spectral curve; the Floquet multiplier spectral curve for the related Dirac operator; and the curve parameterising Darboux transforms of a conformal 2-torus in the 4-sphere.

1. Introduction. There is a well established connection between the Davey-Stewartson (DS) hierarchy and conformal immersions of tori in $\boldsymbol{R}^3$ and $\boldsymbol{R}^4$ (a good survey can be found in [25]). One compelling reason for studying this correspondence is that the simplest "first integral" of this hierarchy represents the Willmore functional, and it may be that a strategy based upon integrable systems could resolve the Willmore conjecture (see [17] and [9] for two different perspectives on this).

But the original point of view makes it difficult to see the conformal invariance and presents other problems in $\boldsymbol{R}^4$ (see [24]). These difficulties are avoided by using the theory of quaternionic holomorphic curves, developed by the Berlin school [14, 5, 9, 1, 3, 2]. The conformal 4-sphere is thought of as $\boldsymbol{HP}^1$; a conformal immersion of a surface M becomes a quaternionic line bundle over M whose dual bundle possesses a canonical quaternionic holomorphic structure D, the quaternionic analogue of a $\bar{\partial}$-operator. This operator D is essentially the Dirac operator which plays the central role in the DS hierarchy.

The relationship between the DS hierarchy and the geometry of the immersed torus is still not very well understood. There is at present no direct definition of the flows as deformations of a torus (although, see [6] for an attempt at this); there is only an indirect construction for tori of finite type via the linear motion on the spectral curve. For the spectral curve, Taimanov [20] proposed the normalisation of the analytic curve of Floquet multipliers of the Dirac operator: generically this non-compact with "infinite genus". The quaternionic perspective [3, 2, 4] provides a conformally invariant construction, and also gives a geometric interpretation of the

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smooth points on this curve, as parameterising "Darboux transformations" of the immersed torus.

This article shows how to use the quaternionic point of view to give a more natural link to the integrable hierarchy. The hierarchy can be obtained by replacing the field $\boldsymbol{C}$ with the division algebra $\boldsymbol{H}$ in the standard constructions of the KP hierarchy and its solutions (particularly, following [26, 27, 18]). This point of view is sufficiently rewarding that I feel the new name "quaternionic KP" (QKP) hierarchy is justified. The plan of this article is to start with the QKP theory and work towards quaternionic holomorphic geometry, paying particular attention to the spectral data.

Properly thought of, the QKP hierarchy is a collection of Lax equations $L_t = [P, L]$ for a *pseudo-differential* operator with quaternionic coefficients

$$L = i\partial_y + U_0 + U_1\partial_y^{-1} + \cdots ,$$

and a differential operator P. These equations provide a commuting family of derivations on a real differential algebra. One of these provides a real derivation ∂_x for which

$$\mathcal{D} = \frac{1}{2}(\partial_x + i\partial_y + U_0) ,$$

plays the role of the Dirac operator.

This construction is purely algebraic: to reconnect with analysis, we adopt the Sato-Segal-Wilson [18] point of view and realise the QKP equations as flows on a manifold, the quotient of an infinite dimensional Grassmannian. This Grassmannian $\mathrm{Gr}(\boldsymbol{H})$ possesses pretty much all the properties that hold for KP, provided we are careful to distinguish between the cases where the Dirac potential U_0 is trivial or non-trivial. In particular, we show that:

(a) the QKP flows correspond to the action of an infinite dimensional abelian Lie group Γ_+ on $\mathrm{Gr}(\boldsymbol{H})$ and the QKP solutions are parameterized by a quotient manifold $\mathcal{M}$. This manifold is the disjoint union of two manifolds, $\mathcal{M}_{\mathrm{KP}}$, which is a copy of the ordinary KP phase space and parameterises QKP solutions for which U_0 is trivial, and $\mathcal{M}_{\mathrm{QKP}}$, which parameterizes solutions with non-trivial U_0.

(b) the points of $\mathcal{M}$ which have finite dimensional Γ_+-orbits correspond precisely to the solutions which can be constructed from spectral data (i.e., a complete algebraic curve X and other algebraic data on it), and the orbits themselves are open subvarieties of (generalised) Jacobi varieties. We call these "of finite type".

The object which mediates between Lax equations and points of $\mathrm{Gr}(\boldsymbol{H})$ is the (quaternionic) Baker function. This is an $\boldsymbol{H}$-valued solution to the spectral problem $L\psi = -\psi\zeta$ which admits a left $\boldsymbol{H}$ Fourier series expansion of the form

$$\psi(x, y, \zeta) = (1 + a_1(x, y)\zeta^{-1} + \cdots)\exp[(x + iy)\zeta] , \quad \zeta \in S^1 .$$

Points of the Grassmannian $\mathrm{Gr}(\boldsymbol{H})$ parameterise Baker functions which converge on $|\zeta| > 1$. For the QKP solutions of finite type there is a representative Baker function which extends to a globally meromorphic function on the spectral curve. We call this the global Baker function: it plays an important role in translating between the different notions of spectral curve.

The Baker function satisfies $\mathcal{D}\psi = 0$ for all ζ and can therefore be used to obtain homogeneous coordinates of a quaternionic holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{HP}^n$, by evaluating ψ at different values of ζ. When ψ is global these points give a divisor $\mathfrak{q}$ on X and we show that the (double) periodicity of such a curve is governed by the Jacobi variety of a singularisation $X_\mathfrak{q}$ of X. In that case f projects to a quaternionic holomorphic torus in $\boldsymbol{HP}^1$ under any homogeneous projection from $\boldsymbol{HP}^n$: we can think of f as a common "linear system" for each of these.

In this passage between QKP and quaternionic holomorphic curves two obstacles remain. The first is that the periodicity conditions on the QKP spectral data imply that the whole QKP operator L is "periodic" (the precise meaning of this is given in Theorem 3.9(c)). However, given a conformal torus then it seems one only obtains "periodicity" of the Dirac operator $\mathcal{D}$. There is a Baker-type function in the kernel of the Dirac operator when the multiplier spectral curve has finite type. But I was unable to show that the two Baker functions coincide without knowing *a priori* that the QKP operator L is "periodic" (Remark 4.10). Secondly, when the multiplier spectral curve is not of finite type it seems there can be no convergent multiplier Baker function, which raises the problem of how to make the passage between QKP and conformal tori at all. In particular, it is not clear how to describe integrable deformations in this case. This may be possible using the theory of infinite genus Riemann surfaces [8, 16], but that would take us out of the Grassmannian class of solutions discussed here.

In the case where the two Baker functions are the same, Section 6 relates three spectral curves: the QKP spectral curve, the multiplier spectral curve, and what we call the Darboux spectral curve. The Darboux spectral curve has a particularly neat characterisation via an Abel map image of a punctured version of the QKP spectral curve X: when X is smooth the punctured curve is just $X \setminus \mathfrak{q}$, whose Abel image lies in $\mathrm{Jac}(X_\mathfrak{q})$, which possesses a natural map to $\boldsymbol{CP}^{2n+1}$. The multiplier and Darboux spectral curves can both be obtained from $X_\mathfrak{q}$, which suggests that the singular curve $X_\mathfrak{q}$ is the correct object for classifying quaternionic holomorphic tori of finite type.

Acknowledgments. I would like to thank Christoph Bohle, Katrin Leschke, Franz Pedit and Ulrich Pinkall for letting me see preliminary versions of [3, 4] and for explaining their work on quaternionic holomorphic curves to me over a number of years. In particular, the idea that there might be a "quaternionic KP" explanation for the spectral curve of a conformally immersed torus is due to Ulrich Pinkall. Some of these results were announced in the conference proceedings [13].

2. Quaternionic holomorphic curves. We begin by summarising the correspondence between conformally immersed surfaces in $S^4 \simeq \boldsymbol{HP}^1$ (as the space of left quaternionic lines in $\boldsymbol{H}^2$) and quaternionic holomorphic curves (based on [5, 9]). Each immersed surface $f : M \to S^4$ can be equated with a smooth (quaternionic) line subbundle L of the trivial bundle $\underline{\boldsymbol{H}}^2 = M \times \boldsymbol{H}^2$ via $f(p) = L_p$.

The reader is warned at this point that we do not adopt the convention of [9] that all quaternionic bundles are right $\boldsymbol{H}$ bundles, since this does not fit well with our construction of the QKP hierarchy using differential operators, with coefficients over $\boldsymbol{H}$, acting on the left on $\boldsymbol{H}$-valued functions. Therefore L will be a left $\boldsymbol{H}$-bundle and its quaternionic dual $L^* = \mathrm{Hom}_{\boldsymbol{H}}(L, \boldsymbol{H})$ a right $\boldsymbol{H}$-bundle.

The theory of quaternionic holomorphic curves encodes the conformality of the map f as follows. The differential df of any immersed surface $f : M \to \boldsymbol{HP}^1$ can be represented, after the standard manner for maps into projective spaces, by $\delta f \in \Omega^1_M(\mathrm{Hom}(L, \underline{\boldsymbol{H}}^2/L))$ where

$$\delta f : \psi \mapsto d\psi \bmod L\,, \quad \psi \in \Gamma(L)\,.$$

Here $\Gamma(L)$ denotes the space of smooth sections of L. We also adopt the convention that, for any vector bundle valued 1-form ω, $*\omega = \omega \circ J_M$ where J_M is the complex structure on M. Now f is a conformal immersion if and only if there exists $J \in \mathrm{End}_{\boldsymbol{H}}(L)$ with $J^2 = -I$ for which $*\delta f = \delta f \circ J$. The pair (L, J) is called a quaternionic holomorphic line bundle.

Further, L^* inherits a complex structure, which we will also call J, and a canonical quaternionic holomorphic structure

$$D : \Gamma(L^*) \to \Omega^1(L^*)\,.$$

This is an $\boldsymbol{H}$-linear operator satisfying, for any $\Psi \in \Gamma(L^*)$ and $\mu : M \to \boldsymbol{H}$:

$$*D\Psi = -JD\Psi\,, \quad D(\Psi\mu) = (D\Psi)\mu + \frac{1}{2}(\Psi d\mu + J\Psi * d\mu)\,.$$

It is characterised by the property that the natural inclusion $(\boldsymbol{H}^2)^* \subset \Gamma(L^*)$ maps into the kernel of D, i.e., into the space $H^0_D(L^*)$ of quaternionic holomorphic sections (see [5, §4.3]).

The definitions above apply equally well to line subbundles of $\underline{\boldsymbol{H}}^{n+1}$ and provide the notion of a quaternionic holomorphic curve in $\boldsymbol{HP}^n$. Conversely, given a complex quaternionic line bundle (E, J) equipped with a quaternionic holomorphic structure D, define $H^0_D(E)$ to be the kernel of D. When M is compact this space is finite dimensional (indeed, a Riemann-Roch formula can be derived to calculate this dimension: see [9]). The “Kodaira construction” follows through to give a quaternionic holomorphic curve

$$f : M \to \boldsymbol{P}H^0_D(E)^*\,; \quad p \mapsto E^*_p\,. \tag{2.1}$$

A two dimensional $\boldsymbol{H}$-subspace $H \subset H^0_D(E)$ can be thought of as a linear system, in the sense of algebraic curve theory. Provided H is well-positioned (has no base points) the dual projection $H^0_D(E)^* \to H^*$ gives us a conformal immersion $f : M \to \boldsymbol{P}H^* \simeq \boldsymbol{HP}^1$.

2.1. The Dirac operator for a conformal torus in S^4 with flat normal bundle. For any complex quaternionic line bundle (L, J) we define $\hat{L}$ to be the complex line bundle whose fibres are

$$\hat{L}_P = \{\sigma \in L_P\,; J\sigma = i\sigma\}\,.$$

By definition, $\deg(L)$ is the degree of the complex line bundle $\hat{L}$. Bohle [1, p.19] showed that when M is a compact Riemann surface the degree of the normal bundle of f is $2\deg(L) - \deg(K_M)$, where K_M is the canonical bundle.

Suppose we have a complex (right) quaternionic line bundle (E, J) over a torus M, with quaternionic holomorphic structure D. Let us assume this has degree zero, i.e., $\deg(\hat{E}) = 0$. The complex bundle $\hat{E}$ inherits the complex holomorphic structure

$$\bar{\partial}_J = \frac{1}{2}(D - JDJ)\,.$$

The moduli space of degree zero holomorphic line bundles is isomorphic to the moduli space $H^1(M, S^1)$ of Hermitian line bundles possessing a flat Hermitian connexion, so $\bar{\partial}_J$ can be extended in a unique way to a flat connexion $\hat{\nabla}$ which is also Hermitian with respect to some Hermitian inner product on $\hat{E}$. Now, if we represent M as $\boldsymbol{C}/\Lambda$ for some lattice Λ and let $\pi : \boldsymbol{C} \to M$ denote the universal cover then we can trivialise $\pi^*\hat{E}$ with a smooth $\hat{\nabla}$-parallel section Φ which is therefore $\bar{\partial}_J$-holomorphic. This section is unique up to right multiplication by a complex constant and has unimodular monodromy $\mu \in \mathrm{Hom}(\Lambda, S^1)$, i.e., for each $\lambda \in \Lambda$ and all $z \in \boldsymbol{C}$

$$\Phi(z + \lambda) = \Phi(z)\mu(\lambda)\,.$$

For any $\Psi \in \Gamma(E)$ we therefore have some $\boldsymbol{H}$-valued function ψ on $\boldsymbol{C}$ for which $\Psi = \Phi\psi$ and we observe that ψ has monodromy μ^{-1} along the lattice Λ. It follows that

$$D\Psi = (D\Phi)\psi + \Phi\frac{1}{2}(d\psi + i * d\psi)\,.$$

Now recall the decomposition $D = \bar{\partial}_J + Q$, where $Q = (D + JDJ)/2$ is called the "Hopf field", and write $Q\Phi = \Phi d\bar{z}U$ for some $U : \boldsymbol{C} \to \boldsymbol{H}$. Since Q anti-commutes with J it follows that U anti-commutes with i. Then

$$D(\Phi\psi) = (\Phi d\bar{z})\mathcal{D}\psi\,, \quad \mathcal{D} = \partial/\partial\bar{z} + U\,. \tag{2.2}$$

We call $\mathcal{D}$ the Dirac operator. For simplicity let us define

$$\ker(\mathcal{D}) = \{\psi \in C^\infty(\boldsymbol{C}, \boldsymbol{H}); \mathcal{D}\psi = 0\}\,.$$

Then we have an identification

$$H^0_D(E) \simeq \{\psi \in \ker(\mathcal{D}); \psi(z + \lambda) = \mu^{-1}(\lambda)\psi(z) \text{ for all } \lambda \in \Lambda\}\,. \tag{2.3}$$

Notice that $\mathcal{D}$ does not in general preserve the space of functions on the torus M, since for any $\lambda \in \Lambda$ and all $z \in \boldsymbol{C}$

$$U(z + \lambda) = \mu(\lambda)^{-1}U(z)\mu(\lambda) = U(z)\mu(\lambda)^2\,. \tag{2.4}$$

Therefore $\mathcal{D}$ is doubly periodic if and only if $\hat{E}$ is a spin bundle. We know from [14] that this is the case which corresponds to immersions into $\boldsymbol{R}^3$ (see also [1]). By (2.4) $|U|$ is always a function on M itself and the L^2-norm of U over M is called the Willmore energy of $\mathcal{D}$: it is essentially the Willmore functional for the corresponding conformally immersed torus.

2.2. The multiplier spectrum and the Baker function. For any degree zero quaternionic holomorphic (right $\boldsymbol{H}$) line bundle (E, D) over a torus $\boldsymbol{C}/\Lambda$ let (π^*E, D) denote its pull-back to the universal cover $\pi : \boldsymbol{C} \to \boldsymbol{C}/\Lambda$. Using this we construct the multiplier spectrum of (E, D):

$$\mathrm{Sp}(E, D) = \{\chi \in \mathrm{Hom}(\Lambda, \boldsymbol{C}^\times); \exists \Psi \in H^0_D(\pi^*E),\ \Psi \neq 0,\ \lambda^*\Psi = \Psi\chi(\lambda) \text{ for all } \lambda \in \Lambda\}.$$

By fixing two generators for Λ we can identify $\mathrm{Sp}(E, D)$ with a subset of $(\boldsymbol{C}^\times)^2$. It has been shown (see [25] for a survey of results) that $\mathrm{Sp}(E, D)$ is a complex analytic curve. Right multiplication on $H^0_D(\pi^*E)$ by j induces on $\mathrm{Sp}(E, D)$ a real involution $\chi \mapsto \bar{\chi}$. Using the isomorphism (2.3) we can write

$$\begin{aligned} \mathrm{Sp}(E, D) = \{\chi \in \mathrm{Hom}(\Lambda, \boldsymbol{C}^\times); \exists \psi \in \ker(\mathcal{D}), \psi \neq 0, \\ \lambda^*\psi = \mu(\lambda)^{-1}\psi\chi(\lambda) \text{ for all } \lambda \in \Lambda\}. \end{aligned} \tag{2.5}$$

Here μ is the monodromy for the flat bundle L^*.

The asymptotic structure of this spectrum is quite well understood (see [3, 25]) and is described by comparison with the spectrum of the "vacuum" operator $D_0 = \bar{\partial}_J$, with vacuum Dirac operator $\mathcal{D}_0 = \partial/\partial\bar{z}$. Let us assume for the moment that E is trivial (i.e., $\mu = 1$). Taking $\psi(z, \zeta) = e^{z\zeta}$, for $\zeta \in \boldsymbol{C}$, we observe that $\mathcal{D}_0\psi = 0$ for all ζ and

$$\psi(z + \lambda, \zeta) = \psi(z, \zeta)e^{\lambda\zeta}, \qquad \text{for all } z \in \boldsymbol{C},\ \lambda \in \Lambda.$$

Hence $\mathrm{Sp}(E, D_0)$ contains an analytic curve of monodromies

$$C = \{(e^{\lambda_1\zeta}, e^{\lambda_2\zeta}) \in (\boldsymbol{C}^\times)^2; \zeta \in \boldsymbol{C}\}, \quad \Lambda = \boldsymbol{Z}\langle\lambda_1, \lambda_2\rangle.$$

The full multiplier spectrum of $\mathcal{D}_0$ is $C \cup \bar{C}$. These two branches of $\mathrm{Sp}(E, D_0)$ intersect infinitely often in double points. When the monodromy μ is nontrivial the structure is the same but with the branches C and $\bar{C}$ shifted by appropriate factors.

The spectrum $\mathrm{Sp}(E, D)$ is asymptotic to $\mathrm{Sp}(E, D_0)$ in the sense that outside a compact subset of $(\boldsymbol{C}^\times)^2$ the former is contained in an arbitrarily small tube around the latter. Away from the double points of $\mathrm{Sp}(E, D_0)$ the curve $\mathrm{Sp}(E, D)$ is a graph over $\mathrm{Sp}(E, D_0)$, while near each double point $\mathrm{Sp}(E, D)$ either has a double point itself or is annular: in the latter case $\mathrm{Sp}(E, D)$ resolves the double point into a handle. Now consider the curve C along $|\zeta| \to \infty$. Either for every $R > 0$ the domain $|\zeta| > R$ contains at least one handle, or there is some R for which $\mathrm{Sp}(E, D)$ only contains double points. In the latter case $\mathrm{Sp}(E, D)$ must have two intersecting branches, one of which can be parameterised by $\Delta = \{\zeta; |\zeta| > R\}$, thought of as a punctured parameter disc about the point at infinity of C. Thus we have a map

$$\chi : \Lambda \times \Delta \to \boldsymbol{C}^\times, \quad \chi(\cdot, \zeta) \in \mathrm{Hom}(\Lambda, \boldsymbol{C}^\times),$$

which is holomorphic in Δ for each λ. The following result is a direct consequence of [4, Theorem 4.1 and Lemma 5.1] (cf. [25, §4]).

THEOREM 2.1. *When* $\mathrm{Sp}(E, D)$ *has only double points along* Δ *there exists a function*

$$\psi : \boldsymbol{C} \times \Delta \to \boldsymbol{H}$$

satisfying:

(a) $\mathcal{D}\psi = 0$ *for all* $\zeta \in \Delta$,

(b) ψ *is holomorphic in* ζ *and* $\lim_{\zeta\to\infty} \psi(z,\zeta)e^{-z\zeta} = 1$,

(c) $\psi(z+\lambda,\zeta) = \mu(\lambda)^{-1}\psi(z,\zeta)\chi(\lambda,\zeta)$ *for all* $z \in \boldsymbol{C}, \lambda \in \Lambda$.

Further, ψ *is uniquely determined by* $\psi(0,\zeta)$.

We will call this function $\psi(z,\zeta)$ the *multiplier Baker function* for $\mathrm{Sp}(E,D)$. By the properties above it has the expansion (in left Fourier series)

$$\psi(z,\zeta) = \left(1 + \sum_{j=1}^{\infty} a_j(z)\zeta^{-j}\right)\exp(z\zeta), \quad a_j : \boldsymbol{C} \to \boldsymbol{H},\ |\zeta| > R.$$

By rescaling ζ we may as well assume Δ is the punctured disc $|\zeta| > 1$. In the next section we will introduce the QKP Baker function, and later on we will consider under what conditions we can show that the two agree.

The following example will help us understand later (Section 6) the difference between the multiplier spectrum and the QKP spectral curve. It comes from the study of Hamiltonian stationary Lagrangian (HSL) tori in $\boldsymbol{R}^4$.

EXAMPLE 2.2. Fix a torus $\boldsymbol{C}/\Lambda$ and equip $\boldsymbol{C}$ with its standard metric $\langle z, w\rangle = \frac{1}{2}(z\bar{w} + \bar{z}w)$. We use this to embed the dual lattice Λ^* in $\boldsymbol{C}$. Fix some non-zero $\beta_0 \in \Lambda^*$ and define $\beta(z) = 2\pi\langle\beta_0, z\rangle$. Now consider the complex quaternionic line bundle (E, J) where $\pi : E \to \boldsymbol{C}/\Lambda$ is the trivial right $\boldsymbol{H}$-bundle and $J\sigma = N\sigma$ for $N = e^{j\beta}i$. This has quaternionic holomorphic structure

$$D\sigma = \frac{1}{2}(d\sigma + N * d\sigma).$$

From [11] we know that $f : \boldsymbol{C}/\Lambda \to \boldsymbol{H}$ is HSL with Maslov class β_0 whenever $Df = 0$. It is easy to check that $\Phi = e^{j\beta/2}i$ is a parallel section of $\pi^*\hat{E}$ for the unique flat Hermitian connexion $\hat{\nabla}$ on $\hat{E}$ for which $\hat{\nabla}'' = \bar{\partial}_J$, and the Dirac operator given by (2.2) is

$$\mathcal{D} = \frac{\partial}{\partial\bar{z}} - \frac{\pi}{2}\beta_0 j. \tag{2.6}$$

Notice that $(\hat{E}, J)$ is trivial if $\beta_0/2 \in \Lambda^*$ but otherwise a spin bundle, since the monodromy of Φ is given by $\mu(\lambda) = e^{i\beta(\lambda)/2} = \pm 1$ for $\lambda \in \Lambda$.

We can explicitly calculate $\mathrm{Sp}(E,D)$ for this example by writing any solution of $\mathcal{D}\psi = 0$ in the form

$$\psi(z) = [\mu^{-1}(z)\varphi_1(z) + j\mu^{-1}(z)\varphi_2(z)]\exp[\pi i(\xi z + \eta\bar{z})],$$

where $\mu(z) = e^{i\beta(z)/2}$ and φ_m are Λ-periodic complex valued functions, and ξ, η are complex parameters which parameterise the logarithmic spectrum. Since the complex valued functions φ_m are both Λ-periodic they can be represented by Fourier series

$$\varphi_m = \sum_{\alpha\in\Lambda^*} \varphi_{m\alpha} e^{2\pi i\langle\alpha,z\rangle}.$$

There is a non-trivial solution to $\mathcal{D}\psi = 0$ if and only if there exists $\alpha \in \Lambda^*$ for which the linear system

$$\begin{pmatrix} \alpha + \eta + \beta_0/2 & -i\beta_0/2 \\ i\bar{\beta}_0/2 & \bar{\alpha} + \xi + \bar{\beta}_0/2 \end{pmatrix} \begin{pmatrix} \varphi_{1\alpha} \\ \varphi_{2\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.7}$$

has a non-trivial solution. Thus if we set

$$F_\alpha(\eta, \xi) = (\alpha + \eta + \beta_0/2)(\bar{\alpha} + \xi + \bar{\beta}_0/2) - |\beta_0|^2/4$$

we can describe the logarithmic spectrum $\tilde{\Sigma}$ of (E, D) as the union of infinitely many irreducible rational curves:

$$\tilde{\Sigma} = \bigcup_{\alpha \in \Lambda^*} C_\alpha, \quad C_\alpha = \{(\eta, \xi) \in \boldsymbol{C}^2; F_\alpha(\eta, \xi) = 0\}.$$

Even though each component C_α is smooth the curve $\tilde{\Sigma}$ possesses infinitely many singularities caused by the intersections of components. The dual lattice Λ^* acts on $\tilde{\Sigma}$ by $(\eta, \xi) \mapsto (\eta + \alpha, \xi + \bar{\alpha})$ and it is easy to see that this identifies all components C_α with one, say C_0. Thus

$$\mathrm{Sp}(E, D) \simeq \tilde{\Sigma}/\Lambda^* \simeq C_0/\sim$$

where $\sim$ is the equivalence relation on C_0 defined by

$$(\eta, \xi) \sim (\eta', \xi') \Leftrightarrow (\eta', \xi') = (\eta + \alpha, \xi + \bar{\alpha}) \quad \text{for some } \alpha \in \Lambda^*.$$

This identification creates the singularities of $\mathrm{Sp}(E, D)$, which are of two types.

(a) For each non-zero $\alpha \in \Lambda^*$ the points (η, ξ) and $(\eta + \alpha, \xi + \bar{\alpha})$ are identified whenever

$$\frac{\alpha}{\bar{\alpha}} Z^2 + \alpha Z + \frac{|\beta_0|^2}{4} = 0, \quad Z = \xi + \frac{\bar{\beta}_0}{2}.$$

When the discriminant (which is proportional to $|\alpha|^2 - |\beta_0|^2$) is non-zero this gives two distinct intersections between C_0 and C_α, which will both be double points provided no other component of $\tilde{\Sigma}$ intersects here. The possibility of more components meeting in one point, and hence higher order singularities, cannot be ruled out: the conditions are rather subtle and depend upon both β_0 and Λ. When the discriminant vanishes C_0 and C_α meet tangentially at one point. This introduces a cuspidal singularity.

(b) The point $(0, 0)$ is identified with every point in the set

$$S = \left\{(\alpha, \bar{\alpha}); \left|\alpha + \frac{\beta_0}{2}\right| = \left|\frac{\beta_0}{2}\right|, \alpha \in \Lambda^*\right\}.$$

This includes $(\beta_0, \bar{\beta}_0)$ and therefore one of the above cusps is folded into this singularity. This leads to the multiplier Baker function

$$\psi(z, \zeta) = \left(1 + j\frac{\pi\bar{\beta}_0}{2}\zeta^{-1}\right) \exp\left(-\frac{\pi^2|\beta_0|^2\bar{z}}{4}\zeta^{-1}\right) e^{z\zeta}, \tag{2.8}$$

which is uniquely determined by its initial value $\psi(0,\zeta) = 1 + j(\pi\bar{\beta}_0/2)\zeta^{-1}$. It follows that the space $H^0_D(E)$ of global quaternionic holomorphic sections is spanned by the sections $\Phi\psi$ obtained by evaluating ψ at each $\zeta = \pi i(\bar{\alpha} + \bar{\beta}_0/2)$ for $(\alpha, \bar{\alpha}) \in S$ (cf. [11, Theorem 2.9]).

3. The QKP hierarchy.

3.1. Formal construction of the QKP hierarchy. The QKP hierarchy is a real form of the two component KP hierarchy (and is referred to elsewhere, for example [20, 22], as the Davey-Stewartson II hierarchy). It is constructed by the formal dressing method, working entirely within a quaternionic framework, so that the comparison between KP and QKP is quite literally the replacement of $\boldsymbol{C}$ by $\boldsymbol{H}$. I will follow the purely algebraic approach given in two papers by George Wilson [26, 27].

We begin by fixing a real differential algebra $\mathcal{B}$, with derivation ∂_y, of the form

$$\mathcal{B} = \boldsymbol{R}[u^{(k)}_{\alpha\beta}], \quad \text{for } \alpha, k = 0, 1, 2, \dots, \quad \beta = 1, 2, 3, 4.$$

These generators are algebraically free but related under the derivation by

$$\partial_y u^{(k)}_{\alpha\beta} = u^{(k+1)}_{\alpha\beta}.$$

With this we construct a formal pseudo-differential operator with coefficients in $\mathcal{B} \otimes \boldsymbol{H}$

$$L = i\partial_y + U_0 + U_1\partial_y^{-1} + \cdots \tag{3.1}$$

where

$$U_0 = j(u_{03} + iu_{04}), \quad U_\alpha = u_{\alpha 1} + iu_{\alpha 2} + j(u_{\alpha 3} + iu_{\alpha 4}).$$

It is important for this construction that the leading coefficient i of L is regular for the Lie algebra structure on $\boldsymbol{H}$ (i.e., the commutator of i has minimal dimension 2). This means we may ignore the component of U_0 which commutes with i.

The construction provides an infinite family of independent derivations on $\mathcal{B}$, each of which commutes with ∂_y, via Lax equations. This is done via the formal dressing construction to produce a subalgebra of the commutative algebra $Z(L)$ of all pseudo-differential operators (over $\mathcal{B} \otimes \boldsymbol{H}$) which commute with L. The method is summarised in the following two theorems.

THEOREM 3.1 ([26]). *There exists a formal operator of the form*

$$K = 1 + \sum_{k\geq 1} a_k (i\partial_y^{-1})^k$$

such that $K^{-1}LK = i\partial_y$. *Moreover,* K *is unique up to right multiplication by operators of the form* $1 + \sum_{k\geq 1} c_k\partial_y^{-k}$ *where each* c_k *is a complex constant.*

Note that the components of the coefficients a_k do not belong to $\mathcal{B}$ but generate an extension algebra $\hat{\mathcal{B}}$.

THEOREM 3.2 ([26]). *Let* $Z_0(L)$ *denote the image of the* $\boldsymbol{R}$*-algebra homomorphism*

$$\boldsymbol{C}[\partial_y] \to Z(L); \quad P_0 \mapsto P = KP_0K^{-1}.$$

Then the coefficients of P all lie in $\mathcal{B}\otimes \boldsymbol{H}$ and there is a derivation ∂_P on $\mathcal{B}$, characterised by

$$\partial_P L = [L, P_+]\,, \quad [\partial_P, \partial_y] = 0\,, \tag{3.2}$$

where P_+ denotes the differential operator part of P. For any two $P, Q \in Z_0(L)$ the derivations ∂_P and ∂_Q commute and satisfy

$$\partial_P Q_+ = \partial_Q P_+ + [Q_+, P_+]\,.$$

We single out one of these derivations for special attention: the derivation ∂_L will be renamed ∂_x.

REMARK 3.3. In particular, this includes a family of p.d.e of the form

$$\partial_P L_+ = \partial_x P_+ + [L_+, P_+]\,. \tag{3.3}$$

Historically these are the equations referred to as the hierarchy, since these are the equations which occur in practical applications to physics and fluid dynamics (for example, the t_2 flow yields the Davey-Stewartson II equations). But this is misleading: the coefficients of P_+ cannot in general be expressed as differential polynomials in the coefficients of L_+ and therefore the equations (3.3) alone do not carry the information contained in the definition (3.2). Therefore we follow Sato's nomenclature, also used in [18], and call the system of equations (3.2) the QKP hierarchy.

These derivations also extend to $\hat{\mathcal{B}}$ via $\partial_P K = P_- K$ and we can introduce the formal Baker function. In the purely algebraic setting it is a formal power series

$$\psi = (1 + a_1\zeta^{-1} + \cdots)\exp(x\zeta + iy\zeta) = K\exp(x\zeta + iy\zeta)$$

where ζ is a formal parameter (to make sense of this we can take an appropriate extension of the differential algebra $\hat{\mathcal{B}}$). From the definition of L it follows that $L\psi = -\psi\zeta$. Therefore

$$\partial_x\psi + L_+\psi = L_-\psi + \psi\zeta - \psi\zeta - L_-\psi = 0\,.$$

Now we can extend the definition of ψ to

$$\psi = K\exp\left[\sum_{k\in \boldsymbol{N}}(s_k + it_k)\zeta^k\right]$$

for real variables s_k, t_k. It is not hard to see that for every $P_0 \in \boldsymbol{C}[\partial_y]$ we can find a sequence $t = (x, y, s_2, t_2, \ldots)$ for which, with the relabelling of ∂_P as ∂_t, we have $\partial_t\psi + P_+\psi = 0$.

3.2. The Grassmannian class of solutions. Following Segal and Wilson [18] we can construct an infinite dimensional Grassmannian $\mathrm{Gr}(\boldsymbol{H})$ whose points essentially parameterise the set of all formal Baker functions which actually converge for $|\zeta| = 1$. It is well-known that for the two component KP hierarchy this class of solutions "linearise" on a Grassmannian $\mathrm{Gr}(\boldsymbol{C}^2)$ of subspaces of the Hilbert space $H = L^2(S^1, \boldsymbol{C}^2)$. The QKP hierarchy is a real form of two component KP obtained by imposing a reality condition: this is achieved by fixing a left $\boldsymbol{H}$ action on H for which j acts conjugate linearly.

The quickest way to do this is to realise H as $L^2(S^1, \boldsymbol{H})$. We view $\boldsymbol{H}$ as having two complex structures: the first arises from left multiplication by $i \in \boldsymbol{H}$, the second comes from right multiplication by i. These structures are inherited by any space of $\boldsymbol{H}$-valued functions. We view H as a complex vector space with respect to the first complex structure. Then it has the Hermitian inner product

$$\langle f, g\rangle = \int_{S^1} (f\bar{g})_{\boldsymbol{C}}$$

where $\bar{g}$ is the quaternionic conjugate, and if $q = a + bj \in \boldsymbol{H}$ for $a, b \in \boldsymbol{C}$ then $q_{\boldsymbol{C}} = a$. The integral is normalised in the usual way so that $\langle 1, 1\rangle = 1$. To make calculations for differential operators acting on the left, we represent each element of H in its left Fourier series

$$f(\zeta) = \sum_{m\in\boldsymbol{Z}} (u_m + v_m j)\zeta^m\,, \quad |\zeta| = 1\,, \quad u_m, v_m \in \boldsymbol{C}\,.$$

Thus we identify $L^2(S^1, \boldsymbol{H})$ with $L^2(S^1, \boldsymbol{C}^2)$, so that

$$L^2(S^1, \boldsymbol{C}^2) \to L^2(S^1, \boldsymbol{H});\ (u, v) \mapsto u + j\bar{v}\,, \tag{3.4}$$

where $\bar{v}(\zeta) = \overline{v(\bar{\zeta})}$. Both left $\boldsymbol{H}$ multiplication and right complex multiplication preserve the polarization of H into orthogonal subspaces H_+ and H_- which consist of functions whose left Fourier series have, respectively, only non-negative and only negative powers of ζ.

Now define $\mathrm{Gr}(\boldsymbol{C}^2)$ to be the space of all complex subspaces $W \subset H$ (i.e., $iW = W$) for which the projections $\mathrm{pr}_+ : W \to H_+$ and $\mathrm{pr}_- : W \to H_-$ are respectively Fredholm (of index zero) and Hilbert-Schmidt. Then $\mathrm{Gr}(\boldsymbol{C}^2)$ is a complex Hilbert manifold [15]. Inside this we have the real submanifold

$$\mathrm{Gr}(\boldsymbol{H}) = \{W \in \mathrm{Gr}(\boldsymbol{C}^2);\ jW = W\}\,,$$

the fixed point subspace of the real involution $W \mapsto jW$. As with the KP equations, we can produce an abelian subgroup whose action on $\mathrm{Gr}(\boldsymbol{H})$ corresponds to the QKP equations.

Define $\Gamma \subset C^\omega(S^1, \boldsymbol{C}^*)$ to be the abelian subgroup of all non-vanishing analytic functions with winding number zero, and let $\mathcal{G}$ denote its Lie algebra $C^\omega(S^1, \boldsymbol{C})$. Γ acts $\boldsymbol{H}$-linearly on H by

$$\gamma : H \to H;\ f \mapsto f\gamma\,.$$

Thus we have a representation $\Gamma \subset GL_{\mathrm{res}}(\boldsymbol{H})$, hence Γ acts on $\mathrm{Gr}(\boldsymbol{H})$ as a real group: to emphasize the definition we will write this action as $\gamma \circ W = W\gamma$. The group Γ factorises into the product $\Gamma_-.\Gamma_+$ of two subgroups: Γ_+, whose elements extend holomorphically into the disc $|\zeta| < 1$ and are unimodular at $\zeta = 0$, and Γ_-, whose elements extend holomorphically into the disc $|\zeta| > 1$ and take a real positive value at $\zeta = \infty$. (This slightly unusual normalisation for Γ_- and Γ_+ makes our discussion of the Γ-orbits easier later on, since the real scaling action is trivial but the action of the unimodular scaling plays an important role.) We write $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ for the corresponding Lie algebra splitting. We will parameterise elements of Γ_+ by writing each in the form

$$\gamma(\mathbf{t}) = \exp\left[it_0 + \sum_{k\geq 1}(s_k + it_k)\zeta^k\right], \quad \mathbf{t} = (s_1, t_1, \ldots) \in \mathcal{G}_+\,.$$

We will tend to use x, y instead of s_1, t_1 below, and write $z = x + iy$.

In the same manner as [18], we can assign to $W \in \mathrm{Gr}(\boldsymbol{H})$ a convergent Baker function and thereby obtain solutions to the QKP equations. Such an assignment only works when the Γ_+-orbit of W meets the big cell, i.e., the open dense subset of $\mathrm{Gr}(\boldsymbol{C}^2)$ consisting of all $W \in \mathrm{Gr}(\boldsymbol{C}^2)$ for which $\mathrm{pr}_+ : W \to H_+$ is invertible. The following result shows that, like the KP case, this condition is always satisfied.

THEOREM 3.4. *Let* $W \in \mathrm{Gr}(\boldsymbol{H})$ *and define*

$$\Gamma_1 = \{\exp((x+iy)\zeta);\, x, y \in \boldsymbol{R}\} \simeq \boldsymbol{R}^2 .$$

Then the Γ_1*-orbit of W meets the big cell off a real analytic (proper) subvariety of* $\boldsymbol{R}^2$. *Consequently the* Γ_+*-orbit of W meets the big cell on the complement of a real analytic (proper) subvariety of* Γ_+.

The proof is a variation on the proof of the corresponding result in [18] and will be omitted here since this theorem does not play a central part in the subsequent discussion. As a consequence, to every $W \in \mathrm{Gr}(\boldsymbol{H})$ we can assign a convergent Baker function ψ_W as follows. Since Γ_+-orbits meet the big cell on open sets there is a function, defined for almost all $\mathbf{t} \in \mathcal{G}_+$,

$$\psi_W(\mathbf{t}) = (1 + a_1(\mathbf{t})\zeta^{-1} + \cdots)\gamma(\mathbf{t}) , \tag{3.5}$$

taking values in W: it is characterised by the property that

$$\mathrm{pr}_+(\psi_W(\mathbf{t})\gamma(\mathbf{t})^{-1}) = 1 .$$

Moreover, ψ_W uniquely determines W, since the set $\psi_W(\mathbf{0}), \psi'_W(\mathbf{0}), \psi''_W(\mathbf{0}), \ldots$ of all y derivatives spans an $\boldsymbol{H}$-subspace of H whose closure is W.

From now on we will set $z = x + iy$ and, for notational convenience, we will use, for example, $\gamma(t_0)$ to denote setting every parameter except t_0 equal to zero.

THEOREM 3.5. *To every* $W \in \mathrm{Gr}(\boldsymbol{H})$ *we can assign a formal pseudo-differential operator* L_W *of the form* (3.1) *satisfying* $L_W\psi_W = -\psi_W\zeta$. *Consequently, for* $t = s_k$ *or* $t = t_k$, $k \in \boldsymbol{N}$ *there exists a differential operator* P_+ *for which*

$$\partial\psi_W/\partial t + P_+\psi_W = 0 , \quad \textit{hence } L_t = [P_+, L] .$$

In particular, for each W we obtain a Dirac operator $\mathcal{D} = \partial/\partial\bar{z} + U_W$ *for which* $U_W = -(a_1 + ia_1 i)/2$, $\mathcal{D}\psi_W = 0$ *and the equations for* $\partial U_W/\partial t$ *are* (3.3).

The proof is identical to that for the KP hierarchy given in [18]. In particular, from ψ_W we extract

$$\tilde{\psi}_W(\mathbf{t}) = 1 + \sum_{k>0} a_k(\mathbf{t})\zeta^{-k} ,$$

from which we obtain $L_W = K_W i\partial_y K_W^{-1}$ using

$$K_W = 1 + \sum_{k>0} a_k(\mathbf{t})(i\partial_y^{-1})^k .$$

REMARK 3.6. Let $\mathrm{Gr}(\boldsymbol{C})$ to denote the Segal-Wilson Grassmannian for $L^2(S^1, \boldsymbol{C})$. We can embed this in $\mathrm{Gr}(\boldsymbol{H})$ using $V \mapsto V \oplus \bar{V}$, where the latter is the space $\{(u, \bar{v}); u, v \in V\} \subset \mathrm{Gr}(\boldsymbol{C}^2)$. Points of $\mathrm{Gr}(\boldsymbol{H})$ of this type yield solutions to the (complex scalar) KP hierarchy of equations, since it is clear that the complex Baker function ψ_V which Segal and Wilson assign to $V \in \mathrm{Gr}(\boldsymbol{C})$ is also our quaternionic Baker function ψ_W, for $W = V \oplus \bar{V}$. Therefore all calculations reduce to those of [18]. We will use $\mathrm{Gr}_{\mathrm{KP}}$ to denote the image of $\mathrm{Gr}(\boldsymbol{C})$ in $\mathrm{Gr}(\boldsymbol{H})$ and denote its complement by $\mathrm{Gr}_{\mathrm{QKP}}$. We can characterise the points of $\mathrm{Gr}_{\mathrm{KP}}$ as follows.

LEMMA 3.7. *$W \in \mathrm{Gr}_{\mathrm{KP}}$ if and only if $Wi = W$, equally, if and only if the Dirac potential U_W is trivial.*

PROOF. If $W \in \mathrm{Gr}_{\mathrm{KP}}$ then clearly we have both $Wi = W$ and $U_W = 0$. Now if $Wi = W$ then $W = V + jV$ where

$$V = \left\{ \frac{1}{2}(f - ifi); f \in W \right\}.$$

In particular, $\frac{1}{2}(\psi_W - i\psi_W i)$ belongs to V and by uniqueness of the Fourier expansion for Baker functions must equal ψ_W, hence ψ_W takes values in V. This means $V \in \mathrm{Gr}(\boldsymbol{C})$, since derivatives of ψ_W generate V over $\boldsymbol{C}$. Thus $W \in \mathrm{Gr}_{\mathrm{KP}}$.

Now suppose $U_W = 0$ and set $\partial = \partial/\partial z$. Then $\bar{\partial}\psi_W = 0$ and so if we write $\psi_W = \psi_1 + j\psi_2$, where both ψ_1, ψ_2 commute with i, we have $\bar{\partial}\psi_1 = 0$ and $\partial\psi_2 = 0$. We notice that, restricting ψ_W to x, y,

$$\psi_1 = (1 + p_1\zeta^{-1} + \cdots)\exp(z\zeta), \quad \psi_2 = (q_1\zeta^{-1} + \cdots)\exp(z\zeta),$$

for some complex valued functions $p_1, q_1, \ldots$. Now $\partial\psi_2 = 0$ means

$$(q_1 + (\partial q_1 + q_2)\zeta^{-1} + \cdots + (\partial q_k + q_{k+1})\zeta^{-k} + \cdots)e^{z\zeta} = 0.$$

So $\partial\psi_2 = 0$ if and only if $q_k = 0$ for all k, i.e., $\psi_2 = 0$. Therefore $\psi_W = \psi_1$ and since $\bar{\partial}\psi_1 = 0$ the closure V of the complex subspace of W generated by $\psi_W(0), \partial\psi_W(0), \ldots$ belongs to $\mathrm{Gr}(\boldsymbol{C})$. But V generates W over $\boldsymbol{H}$ and therefore $W = V + jV$, which is $V \oplus \bar{V}$ in our notation. □

3.3. QKP flows. An important consequence of the construction is that the action of Γ_+ generates the QKP flows, in the following very precise sense. Let $\mathbf{t}, \mathbf{t}' \in \mathcal{G}_+$. Then, by comparing Fourier series, we deduce that

$$\psi_{W\gamma(\mathbf{t}')}(\mathbf{t}) = \psi_W(\mathbf{t} - \mathbf{t}')\gamma(\mathbf{t}'), \tag{3.6}$$

and therefore, treating $\mathbf{t}$ as variable and $\mathbf{t}'$ as a constant, we deduce

$$L_{W\gamma(\mathbf{t}')}(\mathbf{t}) = L_W(\mathbf{t} - \mathbf{t}'). \tag{3.7}$$

Now, if our interest is purely in the QKP operator L_W then, as with the KP hierarchy [18], we observe that for any $\gamma \in \Gamma_-$ we have $\psi_{W\gamma} = \psi_W\gamma$, hence $L_{W\gamma} = L_W$. In fact we have the following result.

THEOREM 3.8. *The quotient space $\mathcal{M} = \mathrm{Gr}(\boldsymbol{H})/\Gamma_-$ is a manifold, and the map*

$$\mathcal{M} \to \{L_W;\, W \in \mathrm{Gr}(\boldsymbol{H})\}\,; \quad \Gamma_- \circ W \mapsto L_W$$

is bijective.

PROOF. It is easy to extrapolate from the proof of the analogous result in Segal and Wilson [18, 2.4] that the subgroup of elements of Γ_- of the form $1 + O(\zeta^{-1})$ acts freely on $\mathrm{Gr}(\boldsymbol{H})$, while the constant scalars act trivially, hence $\mathcal{M}$ is a manifold. By Theorem 3.1 L_W uniquely determines ψ_W up to right multiplication by an element of Γ_-, and ψ_W determines W, so the map is bijective. □

Since Γ is abelian the group Γ_+ acts on $\mathcal{M}$ and it is clear from (3.7) that the orbits correspond to the flows of the QKP hierarchy. The action of the circle subgroup $\Gamma_0 = \{\exp(it_0);\, t_0 \in \boldsymbol{R}\} \subset \Gamma_+$ turns out to be important. In the first place, the next theorem shows that its orbits are either points or circles depending upon whether or not $W \in \mathrm{Gr}_{\mathrm{KP}}$, and this distinction descends to the disjoint union $\mathcal{M} = \mathcal{M}_{\mathrm{KP}} \cup \mathcal{M}_{\mathrm{QKP}}$ obtained by taking the quotient of $\mathrm{Gr}_{\mathrm{KP}} \cup \mathrm{Gr}_{\mathrm{QKP}}$ by the action of Γ_-.

For ease of notation, for any $W \in \mathrm{Gr}(\boldsymbol{H})$ set $W(t_0, z) = We^{it_0 + z\zeta}$, with the usual convention that the absence of either variable denotes that it is zero.

THEOREM 3.9. (a) *For any $W \in \mathrm{Gr}(\boldsymbol{H})$*

$$L_{W(t_0)} = e^{-it_0} L_W e^{it_0}\,. \tag{3.8}$$

(b) *The group Γ_0 acts trivially on $\mathcal{M}_{\mathrm{KP}}$, while the space $\mathcal{M}_{\mathrm{QKP}}/\Gamma_0$ of Γ_0-orbits in $\mathcal{M}_{\mathrm{QKP}}$ is the quotient of $\mathcal{M}_{\mathrm{QKP}}$ by a free action of S^1.*

(c) *Let $\Lambda \subset \boldsymbol{C}$ be a lattice, then $L_W(z + \lambda) = \mu(\lambda)^{-1} L_W(z) \mu(\lambda)$ for some $\mu \in \mathrm{Hom}(\Lambda, S^1)$ if and only if the map*

$$\boldsymbol{C} \to \mathcal{M}/\Gamma_0\,; \quad z \mapsto [L_{W(z)}] \tag{3.9}$$

is Λ-periodic, where $[L_W]$ denotes the Γ_0-orbit of $L_W \in \mathcal{M}$.

PROOF. (a) First we observe that

$$e^{-it_0} \psi_W(z) e^{it_0} = (1 + e^{-it_0} a_1(z) e^{it_0} \zeta^{-1} + \cdots) \exp(z\zeta)\,,$$

and that this takes values in $W(t_0)$. By the uniqueness of the Baker function, we deduce that $\psi_{W(t_0)}(z) = e^{-it_0} \psi_W(z) e^{it_0}$. The formula (3.8) follows at once.

(b) The definition of the Γ_+ action, combined with (3.8), yields

$$e^{it_0} \circ L_W = L_{W(t_0)} = e^{-it_0} L_W e^{it_0}\,.$$

This is trivial if L_W is purely complex, which by Lemma 3.7 is the case precisely when $W \in \mathrm{Gr}_{\mathrm{KP}}$, so Γ_0 acts trivially on $\mathcal{M}_{\mathrm{KP}}$. Now observe that for any $q \in \boldsymbol{H}$, $e^{-i\pi} q e^{i\pi} = q$. Therefore if we define an S^1 action on $\mathcal{M}_{\mathrm{QKP}}$ by $e^{it_0} \cdot L_W = L_{W(t_0/2)}$ this action is free, since the points in the subspace $\mathcal{M}_{\mathrm{QKP}}$ are those for which ψ_W is not purely complex. Clearly the quotient of $\mathcal{M}_{\mathrm{QKP}}$ by this action equals $\mathcal{M}_{\mathrm{QKP}}/\Gamma_0$.

(c) Suppose $L_W(z+\lambda) = \mu(\lambda)^{-1} L_W(z)\mu(\lambda)$ for some $\mu \in \mathrm{Hom}(\Lambda, S^1)$ and all $z \in \boldsymbol{C}$. By (3.7) and (3.8) this means

$$L_{W(z+\lambda)}(z') = L_{W(z)}(z'-\lambda) = L_{W(z)\mu(-\lambda)}(z')$$

thinking of $\mu(\lambda) \in \Gamma_0$. But the last has the same Γ_0-orbit as $L_{W(z)}$, so $[L_{W(z+\lambda)}] = [L_{W(z)}]$.

Conversely, suppose that (3.9) is Λ-periodic. Let

$$G = \{\exp(it_0 + z\zeta) \in \Gamma\} = \Gamma_0\Gamma_1\,,$$

and let $\mathcal{S} \subset \mathcal{M}$ be the G-orbit of L_W. The projection $\mathcal{M} \to \mathcal{M}/\Gamma_0$ makes $\mathcal{S}$ a bundle over the Γ_1-orbit $M \simeq \boldsymbol{C}/\Lambda$ of $[L_W]$. When $L_W \in \mathcal{M}_{\mathrm{KP}}$ this projection is a bijection, so that $L_W(z)$ is Λ-periodic and μ is trivial. Otherwise $\mathcal{S} \to M$ is an S^1-bundle, by (b), with a natural flat connexion for which the action of $\Gamma_1 \subset G$ is horizontal. This makes $z \mapsto L_{W(z)}$ a flat section over the universal cover $\boldsymbol{C}$ of M and hence it has a monodromy $\mu' \in \mathrm{Hom}(\Lambda, S^1)$. The action of S^1 is via Γ_0 and therefore

$$L_{W(z+\lambda)} = \mu'(\lambda)^{-1} L_{W(z)}\mu'(\lambda)\,,$$

which implies

$$L_W(z'-\lambda) = \mu'(\lambda)^{-1} L_W(z')\mu'(\lambda)\,.$$

Taking $\mu = (\mu')^{-1}$ gives the required result. □

3.4. Solutions of finite type. By adapting the construction in [18] we can construct many points $W \in \mathrm{Gr}(\boldsymbol{H})$ corresponding to spectral data and thereby construct Baker functions (using, for example, Riemann θ-functions). Our spectral data will be a collection $(X, \rho, P, \zeta, \mathcal{L}, \varphi)$ of the following.

(a) X is a complete, reduced, algebraic curve of arithmetic genus g, with fixed-point free anti-holomorphic involution ρ. X need not be irreducible, but if it is reducible it must have no more than two irreducible components and these must be swapped by ρ.

(b) $P \in X$ is a smooth point and ζ^{-1} is a local parameter about P,

(c) $\mathcal{L}$ is a holomorphic line bundle of degree $g+1$ for which $\overline{\rho^*\mathcal{L}} \simeq \mathcal{L}$ and $\mathcal{L}(-P-\rho P)$ is non-special. This induces a unique, up to sign, conjugate linear isomorphism $\bar\rho^* : \mathcal{L} \to \mathcal{L}$ covering ρ and satisfying $(\bar\rho^*)^2 = -1$ (so $\bar\rho^*$ is a quaternionic involution).

(d) φ is a holomorphic trivialising section of $\mathcal{L}$ over the disc $\Delta_P = \{Q; |\zeta(Q)^{-1}| < 1\}$ and its boundary circle C_P, both of which we assume contain no singular points of X.

REMARK 3.10. (i) There is no requirement that X be smooth. In general we let X^{sm} denote the open variety of smooth points on X. It is possible for X to be disconnected but Example 3.14 shows that this only leads to points in $\mathrm{Gr}_{\mathrm{KP}}$. When X is singular the condition (c) is more strict than necessary. As explained in [18, p. 38], $\mathcal{L}$ need only be a maximal torsion free coherent sheaf of rank 1 with $\chi(\mathcal{L}) = 2$.

(ii) The QKP spectral curve is the natural quaternionic analogue of the pointed curve which appears in KP theory. It seems highly likely that $\mathrm{Gr}(\boldsymbol{H})$ plays the same role for the moduli space of such curves that $\mathrm{Gr}(\boldsymbol{C})$ plays for the moduli space of all pointed complete irreducible algebraic curves.

From this data we construct a point $W \in \mathrm{Gr}(\boldsymbol{H})$: the points constructed this way will be called *of finite type* (this terminology is justified by Theorem 3.16 below). To do this, first identify S^1 with the circle C_P (and also with ρC_P by the map $\zeta \mapsto \overline{\rho^*\zeta}$). Let X_0 denote the closed non-compact surface $X \setminus (\Delta_P \cup \rho(\Delta_P))$. We define

$$w : H^0(X_0, \mathcal{L}) \to H\,; \quad \sigma \mapsto (\sigma - j\bar{\rho}^*\sigma)/\varphi\,. \tag{3.10}$$

This is clearly $\boldsymbol{C}$-linear. Now define W to be the closure of the image of w. A simple generalisation of the Mayer-Vietoris argument in [18, §6] shows that $W \in \mathrm{Gr}(\boldsymbol{C}^2)$. Observe that since $\bar{\rho}^*$ is quaternionic on $H^0(X_0, \mathcal{L})$ we have

$$jw(\sigma) = w(\bar{\rho}^*\sigma)$$

and therefore $W \in \mathrm{Gr}(\boldsymbol{H})$. The non-speciality condition on $\mathcal{L}(-P - \rho P)$ ensures that W lies in the big cell (although this is not essential we may as well assume this since it happens in the Γ_+-orbit by Theorem 3.4).

The action of Γ on the spectral data is easily calculated. Clearly

$$w(\sigma)\gamma^{-1} = (\sigma - j\bar{\rho}^*\sigma)/(\varphi\gamma)\,.$$

This twists the trivialisation φ into $\varphi\gamma$, which we interpret as trivialisation for the line bundle $\mathcal{L} \otimes \ell(\gamma)$. Here $\ell(\gamma)$ is a degree zero line bundle obtained by glueing the trivial bundles over X_0 and $\mathrm{clos}(\Delta_P)$ together using γ as a transition function, and glueing that to the trivial bundle over $\mathrm{clos}(\rho(\Delta_P))$ using $\bar{\gamma}$ (after identifying ζ with $\overline{\rho^*\zeta}$). Since we will need to work with these trivialisations later on we will be more precise about this by introducing better notation (and then suppress this in most of what follows until we require it).

The construction of $\ell(\gamma)$ equips it with trivialising sections $\tau_P(\gamma)$ and $\tau_0(\gamma)$, over $\Delta_P \cup \rho\Delta_P$ and $X \setminus \{P, \rho P\}$ respectively, satisfying the transition relation

$$\tau_P(\gamma) = \gamma\tau_0(\gamma) \text{ over } \Delta_P \setminus \{P\}\,, \tag{3.11}$$

and the ρ^*-conjugate of this about ρP. So by "$\varphi\gamma$" we really mean the trivialisation $\varphi\tau_P(\gamma)$. Using this we obtain

$$H^0(X_0, \mathcal{L} \otimes \ell(\gamma)) \to C^\omega(S^1, \boldsymbol{C})\,; \quad \sigma \mapsto \sigma/\varphi\tau_P(\gamma)\,.$$

Now we observe that

$$\frac{\sigma}{\varphi\tau_P(\gamma)} = \frac{\sigma\tau_0(\gamma)^{-1}}{\varphi}\gamma^{-1}\,, \tag{3.12}$$

and $\sigma\tau_0(\gamma)^{-1} \in H^0(X_0, \mathcal{L})$, hence $W\gamma^{-1}$ corresponds to replacing $\mathcal{L}$ by $\mathcal{L} \otimes \ell(\gamma)$ and φ by $\varphi\tau_P(\gamma)$. We have therefore proved the following lemma.

LEMMA 3.11. *If W corresponds to the spectral data $(X, \rho, P, \zeta, \mathcal{L}, \varphi)$ and $\gamma \in \Gamma$ then $W\gamma^{-1}$ corresponds to the same data with $\mathcal{L}$ and φ replaced by $\mathcal{L} \otimes \ell(\gamma)$ and $\varphi\gamma$.*

Notice that when $\gamma \in \Gamma_-$ the two line bundles are isomorphic and only the trivialisation changes.

A consequence of this construction is the epimorphism of real groups $\ell : \Gamma \to J_{\boldsymbol{R}}(X)$, where the target here is the connected component of the identity of the real subgroup

$$\{L \in \mathrm{Jac}(X); \overline{\rho^* L} \simeq L\}$$

of the Jacobi variety of X. The restriction of ℓ to Γ_+ is still onto.

We can also characterise the Baker function in terms of the spectral data. Since $\mathcal{L}(-P - \rho P) \otimes \ell(\gamma)$ is almost always non-special (by Theorem 3.4) for almost all $\gamma \in \Gamma$ there is a unique $\sigma_\gamma \in H^0(X, \mathcal{L}(-\rho P) \otimes \ell(\gamma))$ for which $\sigma_\gamma/(\varphi\gamma)$ has value 1 at P: then $\sigma_\gamma/(\bar{\rho}^*\varphi\bar{\gamma})$ is locally holomorphic about ρP with a simple zero there. It follows from these properties that $\psi_W(\gamma) = w(\sigma_\gamma)$.

REMARK 3.12. This construction is at its most concrete in the case where the trivialising section φ extends to a globally holomorphic section on the whole of X. Such a choice is always possible for our spectral data (for example, using a non-zero section vanishing at ρP) and we can choose Δ_P so that it contains no zeroes of φ. Then $\psi_1 = \sigma_\gamma/\varphi$ and $\psi_2 = \bar{\rho}^*\sigma_\gamma/\varphi$ are both meromorphic functions on $X \setminus \{P, \rho P\}$. It follows that ψ_W extends to $X \setminus \{P, \rho P\}$ and we have

$$\psi_W : \Gamma_+ \times X \setminus \{P, \rho P\} \mapsto \boldsymbol{H}; \quad \psi_W = \psi_1 - j\psi_2 . \tag{3.13}$$

Notice that these conditions on φ do not uniquely specify it. For if D denotes its divisor of zeroes we are free to multiply φ by a rational function on X which does not vanish at P and whose divisor of poles lies in the linear system $|D|$. Thus we cannot talk of a unique global Baker function, but any two differ by such a rational function. Nevertheless, in what follows we will use the phrase "the global Baker function" to refer to any one of these.

REMARK 3.13. Every other trivialisation φ' of $\mathcal{L}$ over Δ_P is of the form $s\varphi$, where s is a non-vanishing holomorphic function on Δ_P. This can be factorised into $s = \gamma' e^{it_0}$ where $\gamma' \in \Gamma_-$. By previous observations we conclude that this change of trivialisation has the effect

$$\psi_W \mapsto e^{-it_0}\psi_W e^{it_0}\gamma' .$$

It follows that the quotient map $\mathrm{Gr}(\boldsymbol{H}) \to \mathcal{M}$ amounts to discarding from the spectral data almost all the information given by the choice of trivialisation: the remainder being the identification it gives between fibres of $\mathcal{L}$ over P and ρP. The further quotient $\mathcal{M} \to \mathcal{M}/\Gamma_0$ discards even this information. In particular, when dealing with QKP solutions we may as well work with global Baker functions.

EXAMPLE 3.14. Consider taking $X = Y \cup \bar{Y}$, a disjoint union of two copies of the same irreducible complete algebraic curve, with $\bar{Y}$ equipped with the opposite complex structure to Y. Then $\rho : (P_1, P_2) \to (P_2, P_1)$ is a fixed point free anti-holomorphic involution. It is not hard to see that if we equip Y with the spectral data described in [18], i.e., fix $P \in Y$ with local parameter ζ^{-1}, a line bundle $\mathcal{L}_Y$ over Y of degree equal to the genus of Y and a local trivialisation φ about P, then we automatically obtain our quaternionic spectral data, with $\mathcal{L}$ given by $\mathcal{L}|Y = \mathcal{L}_Y$ and $\mathcal{L}|\bar{Y} = \rho^*\mathcal{L}_Y$. It follows that the point $W \in \mathrm{Gr}(\boldsymbol{H})$ we obtain has the form $W = V \oplus \bar{V}$, where V is the point in the Grassmannian $\mathrm{Gr}(\boldsymbol{C})$ built from the data on Y. Therefore, by Remark 3.6, spectral curves of this type only result in solutions to the KP hierarchy.

Now let us consider the Γ_+-orbits in $\mathcal{M}_{\mathrm{QKP}}$ and $\mathcal{M}_{\mathrm{QKP}}/\Gamma_0$. Let $X_{\mathfrak{p}}$ be the singular curve obtained by identifying the two points P and $\rho(P)$ together to form an ordinary double point and define $J_{\boldsymbol{R}}(X_{\mathfrak{p}})$ to be the connected component of the identity of thereal group

$$\{L' \in \mathrm{Jac}(X_{\mathfrak{p}});\ \overline{\rho^* L'} \simeq L'\}\,.$$

LEMMA 3.15. *$J_{\boldsymbol{R}}(X_{\mathfrak{p}})$ is an S^1-bundle over $J_{\boldsymbol{R}}(X)$.*

PROOF. Think of L' as $L \in \mathrm{Jac}(X)$ equipped with an isomorphism between the fibres L_P and $L_{\rho(P)}$. An isomorphism between $\overline{\rho^* L'}$ and L' is a conjugate linear isomorphism of $\rho^* L$ with L which identifies the fibre isomorphisms. Therefore, given a fixed isomorphism $\overline{\rho^* L} \simeq L$ we can only vary the fibre identification by a unimodular scaling, hence the fibres of $J_{\boldsymbol{R}}(X_{\mathfrak{p}})$ are free S^1-orbits. □

As a corollary to Remark 3.13 and the previous lemma we see how the real groups $J_{\boldsymbol{R}}(X_{\mathfrak{p}})$ and $J_{\boldsymbol{R}}(X)$ sit geometrically with respect to the QKP phase space $\mathcal{M}_{\mathrm{QKP}}$. The equivalent theorem for the KP hierarchy is well-known and can be deduced from [18].

THEOREM 3.16. *Suppose $W \in \mathrm{Gr}_{\mathrm{QKP}}$ is of finite type, then the Γ_+-orbit of $L_W \in \mathcal{M}_{\mathrm{QKP}}$ can be identified with the real group $J_{\boldsymbol{R}}(X_{\mathfrak{p}})$, while the Γ_+-orbit of $[L_W] \in \mathcal{M}_{\mathrm{QKP}}/\Gamma_0$ can be identified with $J_{\boldsymbol{R}}(X)$. Moreover, L_W is of finite type if and only if it admits only finitely many independent non-stationary QKP flows.*

The last claim in this theorem is explained in §6.1 where we briefly discuss the reconstruction of the spectral curve from the QKP operator L_W via its ring of commuting differential operators.

REMARK 3.17. It is natural to ask whether there are any $W \in \mathrm{Gr}_{\mathrm{QKP}}$ whose Γ_+-orbit lies entirely in the big cell. When X is smooth and has genus $g \leq 2$ there are always QKP solutions which are globally defined. To see this, we need to find $\mathcal{L}$ of degree $g+1$ for which $\mathcal{L}(-P-\rho P) \otimes L$ is non-special for all $L \in J_{\boldsymbol{R}}(X)$. This is trivial for $g = 0$. More generally, this $J_{\boldsymbol{R}}(X)$-orbit lies in the real slice $\mathrm{Pic}_{g-1}(X)^\rho$ of $\mathrm{Pic}_{g-1}(X)$. For $g = 1$ the orbit must avoid the unique special line bundle, namely the trivial bundle. Since $\mathrm{Pic}_{g-1}(X)^\rho$ has two connected components for $g = 1$ we can take any $\mathcal{L}(-P-\rho P)$ to be any point lying on the component not containing the trivial bundle. For $g = 2$ the special line bundles of degree $g-1$ lie in the image of the map $X \to \mathrm{Pic}_{g-1}(X)$ which sends Q to $\mathcal{O}_X(Q)$. This image cannot intersect the real slice $\mathrm{Pic}_{g-1}(X)^\rho$, for if $\mathcal{O}_X(Q) \simeq \mathcal{O}_X(\rho Q)$ then $Q = \rho Q$, but ρ has no fixed points. Hence for $g = 2$ any choice of real line bundle $\mathcal{L}$ of degree 3 provides global solutions to the QKP hierarchy.

We finish this section by discussing the Γ_1-orbits. Let us identify Γ_1 with $\boldsymbol{C}$ by $e^{z\zeta} \mapsto z$. We wish to characterise the homomorphism $\ell : \boldsymbol{C} \to J_{\boldsymbol{R}}(X)$. This is straightforward: $\ell(z)$ is obtained from the trivial bundle by twisting it by $e^{z\zeta}$ about P and $\overline{\rho^* e^{z\zeta}}$ about $\rho(P)$. It follows that ℓ is uniquely determined by the property that

$$\left.\frac{\partial \ell}{\partial z}\right|_{z=0} = \left.\frac{\partial \mathcal{A}_P}{\partial \zeta^{-1}}\right|_{\zeta^{-1}=0} \tag{3.14}$$

where $\mathcal{A}_P : X^{\text{sm}} \to \text{Jac}(X)$ is the Abel map with base point P and we interpret this equation by identifying $T_e J_{\boldsymbol{R}}(X)^{\boldsymbol{C}}$ with $T_e \,\text{Jac}(X) \simeq T_e^{1,0}\,\text{Jac}(X)$. As a corollary to this and Theorem 3.9 we deduce the following.

THEOREM 3.18. *Suppose $W \in \text{Gr}_{\text{QKP}}$ arises from spectral data and its Γ_1-orbit lies entirely in the big cell. Then there is a lattice $\Lambda \subset \boldsymbol{C}$ for which $L_W(z+\lambda) = \mu(\lambda)^{-1} L_W(z)\mu(\lambda)$ for some $\mu \in \text{Hom}(\Lambda, S^1)$ if and only if $\ell(z)$ is Λ-periodic.*

REMARK 3.19. This monodromy $\mu \in \text{Hom}(\Lambda, S^1)$ corresponds to a flat S^1-bundle $\mathcal{S}$ over $\boldsymbol{C}/\Lambda$, the principal S^1-bundle for the dual L^* to our quaternionic holomorphic curve L. But we can also interpret this as follows. Using ℓ we can pull back the S^1-bundle $J_{\boldsymbol{R}}(X_{\mathfrak{p}})$ to an S^1-bundle over $\boldsymbol{C}/\Lambda$, which is a Lie group and isomorphic to $\mathcal{S}$. This comes equipped with a flat connexion as follows. By the construction above we have a natural homomorphism of real groups

$$\Gamma_+ \to J_{\boldsymbol{R}}(X_{\mathfrak{p}}) \to J_{\boldsymbol{R}}(X)\,.$$

When this is restricted to Γ_1 it gives ℓ and a lift $\ell^{\mathfrak{p}} : \boldsymbol{C} \to \mathcal{S}$ on its universal cover. This determines a flat connexion on $\mathcal{S}$ with monodromy which is, by Theorem 3.9, μ.

4. Construction of quaternionic holomorphic curves.

4.1. Construction via the Baker function. In this section we will suppose that $W \in \text{Gr}(\boldsymbol{H})$ has been chosen so that the Γ_1-orbit of $[L_W]$ is a torus $\boldsymbol{C}/\Lambda$: let $\mu \in \text{Hom}(\Lambda, S^1)$ be the corresponding monodromy of L_W. In that case there is a quaternionic holomorphic line bundle (E, J, D) over $\boldsymbol{C}/\Lambda$ whose Dirac operator $\mathcal{D}$ has Dirac potential U_W.

For any left $\boldsymbol{H}$-linear form $\alpha \in W^*$ the function $\psi : \boldsymbol{C} \to \boldsymbol{H}$; $\psi = \alpha(\psi_W)$ lies in the kernel of $\mathcal{D}$. It corresponds to a section in $H^0_D(E)$ whenever $\psi(z+\lambda) = \mu(\lambda)^{-1}\psi(z)$ for all $z \in \boldsymbol{C}, \lambda \in \Lambda$. The dimension of $H^0_D(E)$ will be bounded below by the number of independent $\alpha \in W^*$ possessing the same monodromy. In particular, with sufficiently many of these we obtain quaternionic holomorphic immersions of $\boldsymbol{C}/\Lambda$ in $\boldsymbol{HP}^n$.

Equivalently, to obtain a coordinate free perspective, let $V \subset W$ be an $\boldsymbol{H}$-subspace of $\boldsymbol{H}$-codimension $n+1$, and suppose it has the two properties:

(a) $\psi_W(z)$ does not belong to V for any z (we will say V is *well-positioned* in W),

(b) for every $\lambda \in \Lambda$ we have

$$\psi_W(z+\lambda) - \mu(\lambda)^{-1}\psi_W(z) \in V\,, \quad \text{for all } z \in \boldsymbol{C}\,.$$

Then we may define a map

$$f : \boldsymbol{C}/\Lambda \to \boldsymbol{P}(W/V) \simeq \boldsymbol{HP}^n ; \quad z + \Lambda \mapsto [\psi_W(z) + V] , \tag{4.1}$$

where the square brackets denote the corresponding left $\boldsymbol{H}$-line in W/V.

THEOREM 4.1. *The map* $f : \boldsymbol{C}/\Lambda \to \boldsymbol{HP}^n$ *given by* (4.1) *is a quaternionic holomorphic curve.*

The proof is just an instance of the general principle linking quaternionic holomorphic curves to quaternionic holomorphic sections of the dual bundle, but we will give it here since it makes explicit the complex structure J on L.

PROOF. Over $\boldsymbol{C}$ the map f has a global lift $\tilde{f} = \psi_W + V$: we can think of this as a section of the pullback to $\boldsymbol{C}$ of the line bundle L corresponding to f. This has complex structure J for which $J\tilde{f} = i\tilde{f}$. Now

$$*\delta f - \delta f \circ J : \tilde{f} \mapsto -2id\bar{z}\frac{\partial \tilde{f}}{\partial \bar{z}} \quad \text{mod } L .$$

But since $\partial\psi_W/\partial\bar{z} = -U_W\psi_W$ this is identically zero, whence f is a quaternionic holomorphic curve. □

REMARK 4.2. For every $\gamma \in \Gamma_-$ we know that $\psi_{W\gamma} = \psi_W\gamma$. Therefore the quaternionic holomorphic curve determined by (W, V) is congruent (i.e., equivalent up to the right action of $\boldsymbol{PGL}(n+1, \boldsymbol{H})$ on $\boldsymbol{HP}^n$, created by the choice of isomorphism $W/V \simeq \boldsymbol{H}^{n+1}$) to the one determined by $(W\gamma, V\gamma)$ when $\gamma \in \Gamma_-$. Similarly, since $\psi_{We^{it_0}} = e^{-it_0}\psi_W e^{it_0}$ the curve (4.1) is unchanged, up to congruence of $\boldsymbol{HP}^n$, by the action of Γ_0.

To study the periodicity conditions let us first consider the more general map

$$f : \Gamma \to \boldsymbol{P}(W/V) ; \quad \gamma \mapsto [\psi_W(\gamma) + V] . \tag{4.2}$$

Here we have extended ψ_W to a function on Γ by identifying Γ_+ with Γ/Γ_-, i.e., ψ_W is constant on Γ_- cosets. Now let

$$\Gamma_V = \{\gamma \in \Gamma ; W\gamma = W, V\gamma = V\} .$$

Then Γ_V acts on W/V and we define $\Gamma_V^{\boldsymbol{P}} \subset \Gamma_V$ to be the subgroup of those elements which fix $\boldsymbol{P}(W/V)$ pointwise. For example, when $W = H_+$ and $V = H_+\zeta^2$ we have

$$\Gamma_V = \{a_0 + a_1\zeta + \cdots \in \Gamma ; a_0 \in \boldsymbol{C}^\times\} , \quad \Gamma_V^{\boldsymbol{P}} = \{a_0 + a_2\zeta^2 + \cdots \in \Gamma ; a_0 \in \boldsymbol{R}^\times\} .$$

LEMMA 4.3. *The map* $f : \Gamma \to \boldsymbol{P}(W/V)$ *in* (4.2) *is constant on* $\Gamma_-\Gamma_0\Gamma_V^{\boldsymbol{P}}$ *cosets, and therefore it descends to* $\mathcal{J}_V = \Gamma/(\Gamma_-\Gamma_0\Gamma_V^{\boldsymbol{P}})$.

PROOF. It suffices to check the invariance of f along $\Gamma_0\Gamma_+^{\boldsymbol{P}}$ cosets. Let $\gamma' \in \Gamma$ and $\gamma \in \Gamma_+^{\boldsymbol{P}}$, then by (3.6) we have

$$\psi_W(\gamma'\gamma^{-1}) = \psi_{W\gamma}(\gamma')\gamma^{-1} .$$

Therefore, modulo V this is independent of γ. Similarly, by (3.6) and the proof of Theorem 3.9(a),

$$\psi_W(\gamma' e^{-it_0}) = e^{-it_0}\psi_W(\gamma') \tag{4.3}$$

and therefore the projective map f is constant along Γ_0-orbits. □

COROLLARY 4.4. *Let $\ell_V : \boldsymbol{C} \to \mathcal{J}_V$ be the homomorphism (of real groups) obtained by the composition $\boldsymbol{C} \simeq \Gamma_1 \to \Gamma \to \mathcal{J}_V$. Then $f : \boldsymbol{C} \to \boldsymbol{P}(W/V)$ factors through this. Hence a sufficient condition for* (4.1) *to be Λ-periodic is that ℓ_V be Λ-periodic.*

4.2. A construction for solutions of finite type. Let us now consider a more concrete version of the above construction which applies to solutions of finite type. The spectral data $(X, \rho, P, \zeta, \mathcal{L}, \varphi)$ provides us with W; we can choose a $\boldsymbol{H}$-codimension $n+1$ $\boldsymbol{H}$-subspace $W_\mathfrak{q}$ of W as follows. Fix a ρ-invariant divisor $\mathfrak{q}$ consisting of $2n+2$ distinct smooth points in $X \setminus \{P, \rho P\}$. We will write

$$\mathfrak{q} = Q_0 + \cdots + Q_n + \rho(Q_0) + \cdots + \rho(Q_n).$$

We may choose the local parameter disc Δ_P so that $\mathfrak{q} \subset X_0$. Let $W_\mathfrak{q}$ correspond to the subspace $H^0(X_0, \mathcal{L}(-\mathfrak{q}))$ of holomorphic sections of $\mathcal{L}$ which vanish on $\mathfrak{q}$: it is a left $\boldsymbol{H}$-subspace since $\mathfrak{q}$ is ρ-invariant. Since X_0 is a Stein manifold it follows, by calculating the cohomology of the sheaf exact sequence

$$0 \to \mathcal{L}(-\mathfrak{q}) \to \mathcal{L} \to \mathcal{L}_\mathfrak{q} \to 0, \tag{4.4}$$

that $\dim_{\boldsymbol{C}}(W/W_\mathfrak{q}) = \deg(\mathfrak{q}) = 2n+2$.

Now consider the dependence of f on the choice of trivialisation φ. By Remark 3.13 any change of trivialisation has the form $\varphi \mapsto \varphi e^{s_0+it_0}\gamma'$ where $\gamma' \in \Gamma_-$. By Remark 4.2 this only alters f by congruence. Further, a change of local parameter ζ amounts to rescaling the parameter z in the Dirac operator. Therefore it is enough to be given $(X, \rho, P, \mathcal{L}, \mathfrak{q})$ to have f well-defined up to congruence in $\boldsymbol{HP}^n$: we may as well take ψ_W to be the global Baker function. The condition above that f be well-defined is that ψ_W does not vanish on $\mathfrak{q}$ for any $z \in \boldsymbol{C}$. We can think of $W_\mathfrak{q}$ as the kernel of $n+1$ left $\boldsymbol{H}$-linear forms $\mathrm{ev}_{Q_0}, \ldots, \mathrm{ev}_{Q_n}$ on W, where ev_Q is obtained by extending the map "evaluation at Q" to all of W (it is straightforward to check that $\mathrm{ev}_{\rho Q} = \mathrm{ev}_Q u$ for some $u \in \boldsymbol{H}^\times$). In this case, in homogeneous coordinates, and up to congruence in $\boldsymbol{HP}^n$,

$$f(z) = [\psi_W(z, Q_0), \ldots, \psi_W(z, Q_n)]. \tag{4.5}$$

Further, for any $\boldsymbol{H}$-codimension two subspace V satisfying $W_\mathfrak{q} \subset V \subset W$ we obtain a projection of f onto $\boldsymbol{P}(W/V) \simeq \boldsymbol{HP}^1$, which will be a conformal immersion provided V is well-positioned in W. Equally, this arises from a well-positioned two $\boldsymbol{H}$-dimensional linear system in the dual space $(W/W_\mathfrak{q})^*$.

To study the periodicity conditions for f when it arises from this construction we begin by showing that when $V = W_\mathfrak{q}$ the group $\mathcal{J}_V$ appearing in Lemma 4.4 is isomorphic to a real subgroup of a generalised Jacobi variety. Let $X_\mathfrak{q}$ be the singularization of X obtained by identifying the points of $\mathfrak{q}$ together simply. The real involution ρ descends to $X_{\mathfrak{q}}$, so its generalised Jacobian $\mathrm{Jac}(X_{\mathfrak{q}})$ has a real subgroup $J_{\boldsymbol{R}}(X_{\mathfrak{q}})$ which is the connected component of the identity of the real subgroup of $\bar{\rho}^*$-fixed line bundles over $X_{\mathfrak{q}}$. Let $X_{\mathfrak{q}}^{\mathrm{sm}}$ denote the curve of smooth points on $X_{\mathfrak{q}}$ (this is just $X^{\mathrm{sm}} \setminus \mathfrak{q}$) and let $\mathcal{A}_P^{\mathfrak{q}} : X_{\mathfrak{q}}^{\mathrm{sm}} \to \mathrm{Jac}(X_{\mathfrak{q}})$ denote the Abel map for $X_{\mathfrak{q}}$, with base point P. Let $\ell^{\mathfrak{q}} : \boldsymbol{C} \to J_{\boldsymbol{R}}(X_{\mathfrak{q}})$ be the unique homomorphism of real groups determined by the property

$$\left.\frac{\partial \ell^{\mathfrak{q}}}{\partial z}\right|_{z=0} = \left.\frac{\partial \mathcal{A}_P^{\mathfrak{q}}}{\partial \zeta^{-1}}\right|_{\zeta^{-1}=0} . \tag{4.6}$$

LEMMA 4.5. *For a pair* $(W, W_{\mathfrak{q}})$ *given by spectral data* $(X, \rho, P, \mathcal{L}, \mathfrak{q})$ *in the manner above, we have* $\mathcal{J}_{W_{\mathfrak{q}}} \simeq J_{\boldsymbol{R}}(X_{\mathfrak{q}})$ *and* $\ell_{W_{\mathfrak{q}}} = \ell^{\mathfrak{q}}$.

PROOF. First we note that an equivalent way of describing $\ell^{\mathfrak{q}}$ above is that the line bundle $\ell^{\mathfrak{q}}(z)$ is obtained using transition functions $e^{z\zeta}$ and $e^{\overline{\bar{z}\rho^*\zeta}}$ to glue together trivial bundles as in (3.11), but where $X \setminus \{P, \rho P\}$ is replaced by its singularisation $X_{\mathfrak{q}} \setminus \{P, \rho P\}$. This extends naturally to a real homomorphism $\ell^{\mathfrak{q}} : \Gamma \to J_{\boldsymbol{R}}(X_{\mathfrak{q}})$ by using γ and $\overline{\rho^*\gamma}$ as transition functions.

Now γ lies in $\ker(\ell^{\mathfrak{q}})$ precisely when γ factorises into a product $\gamma = \alpha\beta$ where α extends holomorphically to Δ_P and β extends holomorphically to $(X_{\mathfrak{q}})_0$, i.e., β represents the boundary of a holomorphic function on X_0 which takes the same value at each point of $\mathfrak{q}$. Clearly $\Gamma_-\Gamma_0$ equals the group of all boundaries of the type α, while the boundaries of the type β are exactly those which, by multiplication, fix

$$H^0(X_0, \mathcal{L})/H^0(X_0, \mathcal{L}(-\mathfrak{q})) \simeq W/W_{\mathfrak{q}}$$

projectively, since they act by scaling. Hence $\ker(\ell^{\mathfrak{q}}) = \Gamma_-\Gamma_0\Gamma_{W_{\mathfrak{q}}}^{\boldsymbol{P}}$, whence the result. □

REMARK 4.6. One knows (from e.g., [19]) that $\mathrm{Jac}(X_{\mathfrak{q}})$ is a group extension of $\mathrm{Jac}(X)$:

$$1 \to K \to \mathrm{Jac}(X_{\mathfrak{q}}) \stackrel{\pi_{\mathfrak{q}}}{\to} \mathrm{Jac}(X) \to 1 , \tag{4.7}$$

where K is a linear algebraic group. In our case $K \simeq (\boldsymbol{C}^\times)^{2n+2}/\boldsymbol{C}^\times$. The real automorphism $\bar{\rho}^*$ acts on $\mathrm{Jac}(X_{\mathfrak{q}})$ and preserves K. Define $K_{\boldsymbol{R}} = K \cap J_{\boldsymbol{R}}(X_{\mathfrak{q}})$, then

$$K_{\boldsymbol{R}} \simeq (\boldsymbol{C}^\times)^{n+1}/\boldsymbol{R}^\times .$$

This is the kernel of the restriction of $\pi_{\mathfrak{q}}$ to $J_{\boldsymbol{R}}(X_{\mathfrak{q}})$.

We can now achieve a more precise understanding of how the map f factors through $\mathrm{Jac}(X_{\mathfrak{q}})$ by considering the natural twistor lift which every quaternionic holomorphic curve $f : M \to \boldsymbol{HP}^n$ possesses. This is a map $\hat{f} : M \to \boldsymbol{CP}^{2n+1}$ for which $T \circ \hat{f} = f$, where $T : \boldsymbol{CP}^{2n+1} \to \boldsymbol{HP}^n$ is the twistor projection assigning to every left $\boldsymbol{C}$-line in $\boldsymbol{H}^{n+1}$ its left $\boldsymbol{H}$-line. The lift arises as follows: the corresponding line bundle $L \subset \underline{\boldsymbol{H}}^{n+1}$ possesses a unique complex structure J for which $*\delta f = \delta f J$. The twistor lift $\hat{f}$ is given by the complex line subbundle $\hat{L} \subset L$ of i-eigenspaces of J. Now let $\tilde{f} : \boldsymbol{C} \to \boldsymbol{H}^{n+1}$ represent $\psi_W + W_{\mathfrak{q}}$

in some choice of coordinates on $W/W_{\mathfrak{q}}$ and write $f = \boldsymbol{H}\tilde{f}$. It follows from the proof of Theorem 4.1 that $\hat{f} = \boldsymbol{C}\tilde{f}$.

THEOREM 4.7. *Given spectral data as above, the natural twistor lift of* $f : \boldsymbol{C} \to \boldsymbol{HP}^n$ *is a composite of the form*

$$\hat{f} : \boldsymbol{C} \overset{\ell^{\mathfrak{q}}}{\to} \mathrm{Jac}(X_{\mathfrak{q}}) \overset{\theta}{\to} \boldsymbol{CP}^{2n+1}, \tag{4.8}$$

where θ *is a rational map.*

PROOF. We will show that $\hat{f}$ comes from a construction similar to the one given in [12] for harmonic tori. To this end, let $\mathcal{E} \to \mathrm{Jac}(X)$ denote the complex rank $2n+2$ vector bundle with fibres $\mathcal{E}_L = H^0(X, (\mathcal{L} \otimes L)_{\mathfrak{q}})$. It was shown in [12] that we can embed $\mathrm{Jac}(X_{\mathfrak{q}})$ into the bundle of (complex) projective frames of $\mathcal{E}$ and therefore the tautological section of the pullback $\pi_{\mathfrak{q}}^* \mathrm{Jac}(X_{\mathfrak{q}})$ of $\mathrm{Jac}(X_{\mathfrak{q}})$ over itself globally (and algebraically) trivialises bundle $\boldsymbol{P}\mathcal{E}'$ of complex projective space of $\mathcal{E}' = \pi_{\mathfrak{q}}^*\mathcal{E}$, by canonically identifying each fibre of $\boldsymbol{P}\mathcal{E}'$ with $\boldsymbol{P}H^0(X, \mathcal{L}_{\mathfrak{q}}) \simeq \boldsymbol{CP}^{2n+1}$. This works as follows: each point of $\mathrm{Jac}(X_{\mathfrak{q}})$ should be thought of as a line bundle $L \in \mathrm{Jac}(X)$ equipped with a trivialising section of $L_{\mathfrak{q}}$ determined up to scaling. This fixes a projective identification of $(\mathcal{L} \otimes L)_{\mathfrak{q}}$ with $\mathcal{L}_{\mathfrak{q}}$ by tensor product. We will denote this canonical trivialisation by

$$\tau : \boldsymbol{P}\mathcal{E}' \to \mathrm{Jac}(X_{\mathfrak{q}}) \times \boldsymbol{P}H^0(X, \mathcal{L}_{\mathfrak{q}}). \tag{4.9}$$

Now define

$$\mathcal{U} = \{L' \in \mathrm{Jac}(X_{\mathfrak{q}}); \mathcal{L}(-P - \rho P) \otimes \pi_{\mathfrak{q}}(L') \text{ is non-special}\}.$$

This is an affine open subvariety of $\mathrm{Jac}(X_{\mathfrak{q}})$ and it is a consequence of Theorem 3.4 that $J_{\boldsymbol{R}}(X_{\mathfrak{q}}) \cap \mathcal{U}$ is the complement of a real analytic subvariety of $J_{\boldsymbol{R}}(X_{\mathfrak{q}})$. For each $L \in \pi_{\mathfrak{q}}(\mathcal{U})$ the vector space $H^0(X, \mathcal{L}(-\rho P) \otimes L)$ is one-dimensional and therefore the bundle $\boldsymbol{P}\mathcal{E}'$ possesses an algebraic section $s : \mathcal{U} \to \boldsymbol{P}\mathcal{E}'$ corresponding to the injection

$$H^0(X, \mathcal{L}(-\rho P) \otimes L) \to H^0(X, \mathcal{L} \otimes L) \to H^0(X, (\mathcal{L} \otimes L)_{\mathfrak{q}}). \tag{4.10}$$

Now we define $\theta : \mathrm{Jac}(X_{\mathfrak{q}}) \to \boldsymbol{CP}^{2n+1}$ by $\theta = \tau \circ s$, having fixed some projective linear identification of $\boldsymbol{P}H^0(X, \mathcal{L}_{\mathfrak{q}})$ with $\boldsymbol{CP}^{2n+1}$. This is algebraic on the affine open subvariety $\mathcal{U}$ and therefore rational. The theorem is proved once we have shown that, via the identification

$$H^0(X, \mathcal{L}_{\mathfrak{q}}) = H^0(X_0, \mathcal{L})/H^0(X_0, \mathcal{L}(-\mathfrak{q})) \overset{w}{\to} W/W_{\mathfrak{q}},$$

derived from (3.10), the map $\hat{f}$ corresponds to $\theta \circ \ell^{\mathfrak{q}}$. To see this we note that $\ell^{\mathfrak{q}}(z)$ is defined using the transition relation (3.11) to glue together the trivial bundles over $X \setminus X_0$ and $(X_{\mathfrak{q}})_0$, and therefore can be thought of as $\ell(z)$ equipped with the trivialisation $\tau_0(z)$ restricted to $\mathfrak{q}$. But now we recall that the global Baker function in (3.13) is the result of applying (3.10) to the global section $\sigma_\gamma \in H^0(X, \mathcal{L}(-\rho P) \otimes \ell(\gamma))$ pulled back to a section of $\mathcal{L}$ over X_0 by $\sigma_z \to \sigma_z \tau_0(z)^{-1}$. When we restrict this to $\mathcal{L}_{\mathfrak{q}}$ we see that this generates the line $\tau \circ s \circ \ell^{\mathfrak{q}}(z)$ in $H^0(X, \mathcal{L}_{\mathfrak{q}})$. Hence $\hat{f} = \theta \circ \ell^{\mathfrak{q}}$. □

REMARK 4.8. From the previous proof we see that $f = T \circ \hat{f}$ is directly obtained from the $\boldsymbol{H}$-line subbundle of $\mathcal{E}'$ whose fibres are the left $\boldsymbol{H}$-lines $H^0(X, \mathcal{L} \otimes L) \subset H^0(X, (\mathcal{L} \otimes L)_{\mathfrak{q}})$. Notice that if $W_{\mathfrak{q}} \subset V \subset W$ for a well-positioned $\boldsymbol{H}$-codimension two subspace V and $f_V : \boldsymbol{C} \to \boldsymbol{HP}^1$ is the corresponding projection of f then the twistor lift of f_V is the projection of $\hat{f}$ onto $\boldsymbol{CP}^3$, i.e., the map

$$\hat{f}_V : \boldsymbol{C} \to \boldsymbol{P_C}(W/V); \quad z \mapsto \boldsymbol{C} \cdot (\tilde{f}(z) + V).$$

Now we return to the question of when the map f is doubly periodic.

THEOREM 4.9. *Let $f : \boldsymbol{C} \to \boldsymbol{HP}^n$ correspond to the spectral data above and suppose that it is linearly full (i.e., its image does not lie in some $\boldsymbol{HP}^{n-1}$). Then f and $\ell : \boldsymbol{C} \to \mathrm{Jac}(X)$ are simultaneously Λ-periodic if and only if $\ell^{\mathfrak{q}}$ is Λ-periodic.*

This implies that the monodromy of the corresponding Dirac potential (2.4) is that of the flat S^1-bundle $\mathcal{S}$ over $\boldsymbol{C}/\Lambda$ described in Remark 3.19.

PROOF. That the Λ-periodicity of $\ell^{\mathfrak{q}}$ is sufficient follows at once from Corollary 4.4 and Lemma 4.5. Now suppose both f and ℓ are Λ-periodic. Then $\hat{f}$ is Λ-periodic, and $\ell^{\mathfrak{q}}$ determines a homomorphism

$$h : \Lambda \to K_{\boldsymbol{R}} \subset \ker(\pi_{\mathfrak{q}}); \quad h(\lambda) = \ell^{\mathfrak{q}}(\lambda).$$

In this situation we can use [12, Lemma 1] to deduce that there is an injective homomorphism of $K_{\boldsymbol{R}}$ into $\boldsymbol{PGL}(n+1, \boldsymbol{H})$ which allows us to write $f(z+\lambda) = f(z)h(\lambda)$ for all $\lambda \in \Lambda$. Now since f is linearly full we can, after possibly a congruence, find homogeneous coordinates for $f(z)$ so that these are all non-zero at some z, hence h must be the identity, whence $\ell^{\mathfrak{q}}$ is Λ-periodic. □

REMARK 4.10. I was unable to find a way to remove the assumption that ℓ be Λ-periodic: it amounts to the difference between knowing that the Dirac potential has Λ-monodromy (2.4) (which follows from the periodicity of f) and knowing that the full QKP operator L_W has Λ-monodromy (i.e., $[L_W]$ is Λ-periodic). The former implies the latter if one knows *a priori* that the QKP Baker function has property (c) of Theorem 2.1. We will see later in the article that under the assumption that ℓ is Λ-periodic we do get agreement of the two Baker functions.

REMARK 4.11. In the case where X has genus 0 or 1 the map $\ell : \boldsymbol{C} \to \mathrm{Jac}(X)$ is not injective, so the periodicity condition on ℓ does not uniquely determine a lattice $\Lambda \subset \boldsymbol{C}$. But the arithmetic genus of $X_{\mathfrak{q}}$ is at least $2n + 1 \geq 3$ hence $\ell_{\mathfrak{q}}$ is always injective, therefore it uniquely determines the lattice Λ when such a lattice exists.

Finally, let us note a condition under which the conformal torus in $\boldsymbol{HP}^1$ can be immersed in $\boldsymbol{R}^3$, at least in the case where X is a smooth curve. Recall from the discussion at the end of §2.1 that $f : M \to S^4$ lies in some $\boldsymbol{R}^3$ if and only if $\hat{E}$ is a spin bundle, hence if and only if $\mathcal{S}$ is a spin bundle.

PROPOSITION 4.12. *When X is a smooth curve the bundle $\mathcal{S}$ above is a spin bundle if the divisor $P - \rho P$ has order two* (*in which case X is hyperelliptic*).

PROOF. One knows (e.g., from [19]) that pulling back line bundles along the Abel map gives an isomorphism between $\mathrm{Jac}(X)$ and its dual $\mathrm{Jac}(X)^*$ (the moduli space of flat line bundles on $\mathrm{Jac}(X)$). One also knows from [19, VII] that the flat line bundle over $\mathrm{Jac}(X)$ with principal bundle $\mathrm{Jac}(X_{\mathfrak{p}})$ is pulled back to $\mathcal{O}_X(P - \rho P)$. Hence $\mathcal{S}$ is a spin bundle whenever $\mathcal{O}_X(2P - 2\rho P)$ is trivial, i.e., when $P - \rho P$ is a divisor of order two. □

It can be shown that this condition obliges the "even" flows of the QKP hierarchy to be trivial, in which case we obtain the modified Novikov-Veselov hierarchy, as one expects from [22].

5. Darboux transformations. We will follow the notion of Darboux transformations introduced by Bohle et al. in [3]. It generalises the classical notion of a Darboux transform between isothermic surfaces in S^3, in which both surfaces envelope the same sphere congruence.

Let us recall first, from [5], that we can identify the set of all oriented round 2-spheres in S^4 with the set

$$\mathcal{Z} = \{S \in \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^2);\ S^2 = -I\}\,.$$

This identification gives to each $S \in \mathcal{Z}$ the 2-sphere $\{L \in \boldsymbol{HP}^1;\ SL = L\}$, which we will also denote by S. The orientation is given by the complex structure each 2-sphere inherits from S. Given a Riemann surface, M, a sphere congruence is a map $S : M \to \mathcal{Z}$. Now let $f : M \to S^4$ be a conformal map. We say f envelopes a sphere congruence $S : M \to \mathcal{Z}$ if $f(p) \in S(p)$ and the oriented tangent plane of $f(M)$ at $f(p)$ agrees with that of $S(p)$. In terms of the line subbundle $L \in \underline{\boldsymbol{H}}^2$ corresponding to f, these two conditions can be expressed as

$$SL = L\,, \quad \text{and} \quad *\,\delta f = S\delta f = \delta f S\,.$$

It is a classical result that two conformal maps which envelope the same sphere congruence must both be isothermic (see [1, Cor. 67, p. 78] for a modern proof), therefore Bohle et al. [3] (see also [1]) relax the condition slightly to achieve a broader class of transformations. We will say a conformal map $f : M \to S^4$ left-envelopes a sphere congruence $S : M \to \mathcal{Z}$ if $f(p) \in S(p)$ and their oriented tangent planes agree up to action of SU_2 on $T_{f(p)}S^4$ as the left component in the epimorphism $SU_2 \times SU_2 \to SO_4$ (the notion of a right-envelope is defined similarly). In terms of the line bundle L the property of being a left-envelope can be expressed as

$$SL = L\,, \quad \text{and} \quad *\,\delta f = S\delta f\,.$$

DEFINITION 5.1 ([3]). Let $f : M \to S^4$ be a conformal map of a Riemann surface M. Another conformal map $f^\sharp : M \to S^4$ is a Darboux transform of f if $f(p) \neq f^\sharp(p)$ for all $p \in M$ and there exists a sphere congruence $S : M \to \mathcal{Z}$ which is enveloped by f and left-enveloped by $f^\sharp$. In terms of the line bundles $L, L^\sharp$ this means

(5.1) $\underline{\boldsymbol{H}}^2 = L \oplus L^\sharp\,,\quad SL = L,\ SL^\sharp = L^\sharp\,,\quad *\delta f = S\delta f = \delta f S\,,\quad$ and $\quad *\,\delta f^\sharp = S\delta f^\sharp\,.$

Strictly speaking, we want to allow *singular* Darboux transformations of a torus: those for which $L\cap L^\sharp$ is trivial except at finitely many points (see [3]). We want to understand what Darboux transformations look like for maps arising from a pair (W, V) of the type above. First we invoke a simple result from [3] which gives a neat characterization for Darboux transforms.

LEMMA 5.2 ([3]). *Let $f, f^\sharp : M \to \boldsymbol{HP}^1$, with corresponding line bundles $L, L^\sharp$, be conformal immersions so that $\underline{\boldsymbol{H}}^2 = L \oplus L^\sharp$. Then $f^\sharp$ is a Darboux transform for f if and only if $*\delta f^\sharp = J\delta f^\sharp$ where $*\delta f = \delta f J$ and we identify $\underline{\boldsymbol{H}}^2/L^\sharp$ with L using projection along the splitting.*

It is easy to check that the sphere congruence which is enveloped by f and left-enveloped by $f^\sharp$ is given by $S|_L = J$ and $S|_{L^\sharp} = \tilde J$ where $*\delta f = \tilde J\delta f$, again using the splitting to identify $L^\sharp$ with $\underline{\boldsymbol{H}}^2/L$.

LEMMA 5.3. *Let $f : M \to \boldsymbol{P}(W/V) \simeq \boldsymbol{HP}^1$ be the conformal map defined by equation* (4.1). *Given $\lambda \in \boldsymbol{C}$, $0 < |\lambda| < 1$, define $W_\lambda = W(1-\lambda\zeta)$, $V_\lambda = V(1-\lambda\zeta)$ and let $f^\lambda : M' \to \boldsymbol{HP}^1$ be the map corresponding to the pair (W_λ, V_λ). Then f^λ is a Darboux transform of f over their common domain $M \cap M'$.*

PROOF. We assume, without loss of generality, that $M = M'$ is an open domain in $\boldsymbol{C}$. Let $\mathcal{V} = W_\lambda/V_\lambda$ and let $f, f^\lambda : M \to \boldsymbol{P}\mathcal{V}$ denote the conformal maps with lifts

$$\tilde f = \psi_W(1-\lambda\zeta) + V_\lambda\,,\quad \tilde{f^\lambda} = \psi_{W_\lambda} + V_\lambda\,,$$

and line subbundles $L, L^\lambda \subset M\times\mathcal{V}$. From the proof of Theorem 4.1 we know that $*\delta f = \delta f J$ for $J\tilde f = i\tilde f$. I claim that there is a function $b : M \to \boldsymbol{H}$ for which

$$\tilde{f^\lambda_z} = b\tilde{f^\lambda} - \frac{1}{\lambda}\tilde f\,. \tag{5.2}$$

Therefore

$$\delta f^\lambda(\tilde{f^\lambda}) = -dz\frac{1}{\lambda}\tilde f\,.$$

It follows that, since λ is complex, $*\delta f^\lambda = J\delta f^\lambda$. In light of the previous lemma, this proves the theorem.

It remains to verify equation (5.2). This is a direct result of an identity for Baker functions. Set

$$\psi = \psi_W(1-\lambda\zeta),\quad \psi^\lambda = \psi_{W_\lambda}\,.$$

These have respective Fourier series expansions

$$\psi = (-\lambda\zeta + (1-a\lambda) + \cdots)e^{z\zeta}\,,$$
$$\psi^\lambda = (1 + a^\lambda\zeta^{-1} + \cdots)e^{z\zeta}.$$

Therefore there is an $\boldsymbol{H}$-valued function b for which

$$\left(\psi_z^\lambda + \frac{1}{\lambda}\psi - b\psi^\lambda\right)e^{-z\zeta} \equiv 0 \bmod O(\zeta^{-1}).$$

Since $W_\lambda e^{-z\zeta}$ is transverse to H_- almost everywhere (Theorem 3.4), the right-hand side must be identically zero. Equation (5.2) follows. □

Notice that $(1-\lambda\zeta) \in \Gamma_+$, since its only zero is at λ^{-1}. It is clear that the proof above still works if we replace $(1-\lambda\zeta)$ by any representative in the coset $(1-\lambda\zeta)\Gamma_- \in \Gamma/\Gamma_-$. It follows that in the case where $W_\mathfrak{q} \subset V \subset W$ for some ρ-invariant divisor $\mathfrak{q} \subset X \setminus \{P, \rho P\}$ this Darboux transform acts on the spectral data by fixing everything except the pair $\mathcal{L}, \varphi$, which it transforms by

$$(5.3) \qquad (\mathcal{L}, \varphi) \mapsto (\mathcal{L}^Q, \varphi^Q), \quad \mathcal{L}^Q = \mathcal{L}(Q + \rho Q - P - \rho P),$$

where $\zeta(Q) = \lambda^{-1}$ and φ^Q is a local trivialising section of $\mathcal{L}^Q$ over Δ_P. In fact, provided Q is not in the support of $\mathfrak{q}$, the transform (5.3) always yields a Darboux transform.

THEOREM 5.4. *Let $f : \boldsymbol{C}/\Lambda \to \boldsymbol{HP}^1$ be a quaternionic holomorphic torus corresponding to the spectral data $(X, \rho, P, \mathcal{L}, \mathfrak{q})$ and a choice of well-positioned two dimensional linear system $H \subset (W/W_\mathfrak{q})^*$. Assume that $\ell : \boldsymbol{C} \to \mathrm{Jac}(X)$ is Λ-periodic. Let $X_\mathfrak{q}^{\mathrm{sm}}$ be the variety of smooth points in $X_\mathfrak{q}$. Then for every $Q \in X_\mathfrak{q}^{\mathrm{sm}} \setminus \{P, \rho P\}$ there is a Darboux transform f^Q of f arising from the spectral data obtained from the transformation* (5.3).

The linear system H is spanned by two linear forms each of which is a $\boldsymbol{H}$-linear combination of evaluation maps $\mathrm{ev}_{Q_0}, \ldots, \mathrm{ev}_{Q_n}$. Therefore f^Q corresponds to the linear system $H^Q \subset (W/W_\mathfrak{q})^*$ determined by the same combination of evaluations. Since both ℓ and f are assumed Λ-periodic Theorem 4.9 ensures that $\ell^\mathfrak{q}$ is Λ-periodic, and therefore the Γ_1-orbit of $\mathcal{L}^Q$ in $\mathrm{Pic}(X_\mathfrak{q})$ is isomorphic to $\boldsymbol{C}/\Lambda$. The proof now follows from Lemma 5.3 by choosing a coordinate disc on X centred at P and containing Q but not any points in $\mathfrak{q}$. The singularities of such a Darboux transform correspond to points where the Γ_1-orbit of W^Q, the point in $\mathrm{Gr}_{\mathrm{QKP}}$ corresponding to the transformed spectral data, leaves the big cell.

6. Spectral curves.

6.1. The QKP spectral curve. In the construction of $W \in \mathrm{Gr}_{\mathrm{QKP}}$ from spectral data we use the map w in (3.10). When this is restricted to the algebraic sections of $\mathcal{L}$ over X_0 (i.e., those which have only poles at $P, \rho P$) we obtain an open dense subspace $W^{\mathrm{alg}} \subset W$. The elements of W^{alg} are algebraic in the sense that their projections onto H_+ are polynomial in ζ. Now if $\mathcal{A}$ denotes the coordinate ring of $X \setminus \{P, \rho P\}$ its real subalgebra

$$\mathcal{A}^\rho = \{h \in \mathcal{A}; \overline{\rho^* h} = h\},$$

acts on W^{alg} by right multiplication, since $w(h\sigma) = w(\sigma)h$ (where on the right-hand side we restrict h to the circle C_P). Therefore

$$\mathcal{A}^\rho \hookrightarrow \mathcal{A}_W = \{h \in C^\omega(S^1, \boldsymbol{C}); W^{\mathrm{alg}} h \subset W^{\mathrm{alg}}\}.$$

In fact it is easily shown that this inclusion is onto, so that $\mathcal{A}^\rho \simeq \mathcal{A}_W$. Thus we recover $X \setminus \{P, \rho P\}$ as Spec$(\mathcal{A}_W^{\boldsymbol{C}})$, where $\mathcal{A}_W^{\boldsymbol{C}}$ is the complex subalgebra of $C^\omega(S^1, \boldsymbol{C})$ generated by $\mathcal{A}_W$. Now $(W^{\text{alg}})^{\boldsymbol{C}}$, the complexification $W^{\text{alg}} + W^{\text{alg}}i \subset H$, is a torsion free $\mathcal{A}_W^{\boldsymbol{C}}$-module and this recovers the rank one torsion free coherent sheaf $\mathcal{L}$ equipped with the trivialisation implicit in the inclusion $(W^{\text{alg}})^{\boldsymbol{C}} \subset H_+^{\text{alg}} \oplus H_-$.

Now, in analogy with the KP case, it is clear that we can assign a commutative algebra $\mathcal{A}_W$ to *any* $W \in \text{Gr}_{\text{QKP}}$, but in general this will not be very useful: typically $\mathcal{A}_W = \boldsymbol{R}$, and even if it is not so trivial we only obtain spectral data of the type we desire when $(W^{\text{alg}})^{\boldsymbol{C}}$ is locally rank one. Nevertheless, we can always obtain $\mathcal{A}^\rho$ as a commutative algebra of differential operators over $\boldsymbol{H}$, and this connects us to the stationary QKP flows. The proof of the following lemma is obtained *mutatis mutandis* from [18, Remark 6.4].

LEMMA 6.1. *Given $W \in \text{Gr}_{\text{QKP}}$ to each $h \in \mathcal{A}_W$ there is a unique pseudo-differential operator $P(h) \in Z_0(L_W)$ for which the differential operator part $P(h)_+$ satisfies $P(h)_+\psi_W = \psi_W h$. The $\boldsymbol{R}$-algebra $\{P(h)_+; h \in \mathcal{A}_W\}$ is isomorphic to $\mathcal{A}_W$. It follows that $[P(h)_+, L_W] = 0$ and therefore the QKP solution corresponding to W is stationary for every flow $\partial_{P(h)}$.*

One consequence of this lemma is that L_W admits only finitely many independent non-stationary QKP flows precisely when the algebra $\mathcal{A}_W$ possesses an element of every sufficiently high order, and therefore $(W^{\text{alg}})^{\boldsymbol{C}}$ is locally rank one. Hence L_W is a solution of finite type.

6.2. The curve of Darboux transforms. Assume we have a quaternionic holomorphic torus $f : \boldsymbol{C}/\Lambda \to \boldsymbol{HP}^n$ obtained from spectral data $(X, \rho, P, \mathcal{L}, \mathfrak{q})$ as described above. According to Bohle et al. [3] there will be a holomorphic curve in $\boldsymbol{CP}^3$ given by the Darboux transforms of any torus in $\boldsymbol{HP}^1$ obtained from f by projection. We are going to use the interpretation of the Darboux transform given in (5.3) to work directly with f and view the curve of Darboux transforms as an algebraic curve in $\boldsymbol{CP}^{2n+1}$. For this purpose, let f^Q be the quaternionic holomorphic torus obtained from the transformation (5.3). Any projection of f onto $\boldsymbol{HP}^1$ then has a Darboux transform by applying the same projection to f^Q (see Remark 4.8).

Since $f^Q = f^{\rho Q}$, a direct consequence of Theorem 5.4 is a geometric realisation of the Klein surface $X_{\mathfrak{q}}^{\text{sm}}/\rho$ (recall that $X_{\mathfrak{q}}^{\text{sm}} \subset X \backslash \mathfrak{q}$ is the subvariety of smooth points), since we obtain from it a map

$$F : (\boldsymbol{C}/\Lambda) \times (X_{\mathfrak{q}}^{\text{sm}}/\rho) \to \boldsymbol{HP}^n ; \quad (p, Q + \rho Q) \mapsto f^Q(p) . \tag{6.1}$$

Here we define $f^P = f$. An immediate consequence of Theorems 4.7 and 5.4 is that this factors through the generalised Jacobian Jac$(X_{\mathfrak{q}})$ via

$$(\boldsymbol{C}/\Lambda) \times (X_{\mathfrak{q}}^{\text{sm}}/\rho) \to \text{Jac}(X_{\mathfrak{q}}); \ (z, Q + \rho Q) \mapsto \ell^{\mathfrak{q}}(z) \otimes \mathcal{O}_{X_{\mathfrak{q}}}(Q + \rho Q - P - \rho P) .$$

Post-composition of this with $T \circ \theta : \mathrm{Jac}(X_{\mathfrak{q}}) \to \boldsymbol{HP}^n$ (cf. Remark 4.8) gives us F. For each $p \in \boldsymbol{C}/\Lambda$ let us define

$$\xi_p : X_{\mathfrak{q}}^{\mathrm{sm}}/\rho \to \boldsymbol{HP}^n ; \quad \xi_p(Q) = F(p, Q) = f^Q(p) .$$

This has a natural twistor lift, an algebraic map from $X_{\mathfrak{q}}^{\mathrm{sm}}$ into $\boldsymbol{CP}^{2n+1}$, which factors through the Abel map into the generalised Jacobian $\mathrm{Jac}(X_{\mathfrak{q}})$, as a consequence of Theorem 4.7.

Let $S^2 X_{\mathfrak{q}}^{\mathrm{sm}}$ denote the symmetric product of $X_{\mathfrak{q}}^{\mathrm{sm}}$ with itself; equally, think of it as the set of all divisors of degree two supported on $X_{\mathfrak{q}}^{\mathrm{sm}}$. Because $X_{\mathfrak{q}}^{\mathrm{sm}}$ excludes $\mathfrak{q}$ and all singularities of X, this symmetric product admits an Abel map

$$\mathcal{A}_{P+\rho P}^{\mathfrak{q}} : S^2 X_{\mathfrak{q}}^{\mathrm{sm}} \to \mathrm{Jac}(X_{\mathfrak{q}}) ; \quad A + B \mapsto \mathcal{O}_{X_{\mathfrak{q}}}(A + B - P - \rho P) . \tag{6.2}$$

We can embed $X_{\mathfrak{q}}^{\mathrm{sm}}$ algebraically in $S^2 X_{\mathfrak{q}}^{\mathrm{sm}}$ via $Q \mapsto Q + \rho P$; we can also embed $X_{\mathfrak{q}}^{\mathrm{sm}}/\rho$ real algebraically by thinking of it as the curve of pairs $Q + \rho Q$ for $Q \in X_{\mathfrak{q}}^{\mathrm{sm}}$. By post-composing each of these with the Abel map (6.2) we obtain

$$\alpha : X_{\mathfrak{q}}^{\mathrm{sm}} \to \mathrm{Jac}(X_{\mathfrak{q}}) ; \qquad Q \mapsto \mathcal{O}_{X_{\mathfrak{q}}}(Q - P) ,$$
$$\beta : X_{\mathfrak{q}}^{\mathrm{sm}}/\rho \to \mathrm{Jac}(X_{\mathfrak{q}}) ; \quad Q + \rho Q \mapsto \mathcal{O}_{X_{\mathfrak{q}}}(Q + \rho Q - P - \rho P) .$$

From the discussion above we see that, if $\mathcal{L}$ corresponds to the base point $p \in \boldsymbol{C}/\Lambda$,

$$\xi_p = T \circ \theta \circ \beta .$$

We define

$$\hat{\xi}_p = \theta \circ \alpha : X_{\mathfrak{q}}^{\mathrm{sm}} \to \boldsymbol{CP}^{2n+1} . \tag{6.3}$$

The image of $\hat{\xi}_p$ will be called the *Darboux spectral curve*. It is clearly an algebraic curve.

THEOREM 6.2. *$\hat{\xi}_p$ is a twistor lift of ξ_p.*

This can be summarised by the following commutative diagram, in which the top line is the algebraic map $\hat{\xi}_p$ and the bottom line is ξ_p.

$$\begin{array}{ccccc} X_{\mathfrak{q}}^{\mathrm{sm}} & \stackrel{\alpha}{\to} & \mathrm{Jac}(X_{\mathfrak{q}}) & \stackrel{\theta}{\to} & \boldsymbol{CP}^{2n+1} \\ \downarrow & & & & \downarrow T \\ X_{\mathfrak{q}}^{\mathrm{sm}}/\rho & \stackrel{\beta}{\to} & J_{\boldsymbol{R}}(X_{\mathfrak{q}}) & \stackrel{T\circ\theta}{\to} & \boldsymbol{HP}^n \end{array} . \tag{6.4}$$

PROOF. First we note that $\mathcal{O}_{X_{\mathfrak{q}}}(A + B - P - \rho P)$ can be thought of as the bundle $\mathcal{O}_X(A + B - P - \rho P)$ together with the fibre identification over $\mathfrak{q}$ uniquely determined by the rational section with divisor $A + B - P - \rho P$ (there is only one of these, up to scaling). Consequently the canonical trivialisation of the complex projective bundle $\boldsymbol{P}\mathcal{E}'$ over $\mathrm{Jac}(X_{\mathfrak{q}})$ (described in the proof of Theorem 4.7) works as follows over $\alpha(X_{\mathfrak{q}}^{\mathrm{sm}})$. Let σ_Q be a non-zero rational section of $\mathcal{O}_X(Q - P)$ with divisor $Q - P$. Recall from (4.4) that $\mathcal{L}_{\mathfrak{q}}$ denotes the skyscraper sheaf which is $\mathcal{L}$ restricted to $\mathfrak{q}$. The canonical trivialisation of $\boldsymbol{P}\mathcal{E}'$ identifies fibres over $\alpha(X_{\mathfrak{q}}^{\mathrm{sm}})$ via the isomorphism

$$\iota_1 : H^0(X, \mathcal{L}(Q - P)_{\mathfrak{q}}) \to H^0(X, \mathcal{L}_{\mathfrak{q}}) ; \quad s \mapsto s/\sigma_Q ,$$

thinking of σ_Q restricted to $\mathfrak{q}$. Similarly, over $\beta(X_{\mathfrak{q}}^{\mathrm{sm}})$ we have an isomorphism

$$\iota_2 : H^0(X, \mathcal{L}_{\mathfrak{q}}^Q) \to H^0(X, \mathcal{L}_{\mathfrak{q}})\,; \quad s \mapsto s/(\sigma_Q \bar{\rho}^* \sigma_Q)\,.$$

Now recall from (4.10) that θ is the result of applying this trivialisation to the line subbundle of $\mathcal{E}'$ which picks out the complex line

$$H^0(X, \mathcal{L}(-\rho P) \otimes L) \subset H^0(X, (\mathcal{L} \otimes L)_{\mathfrak{q}}) = \mathcal{E}'_{L'}\,, \quad L = \pi_{\mathfrak{q}}(L')\,, \tag{6.5}$$

while its twistor projection $T \circ \theta$ corresponds to the quaternionic line

$$H^0(X, \mathcal{L} \otimes L) \subset H^0(X, (\mathcal{L} \otimes L)_{\mathfrak{q}})\,.$$

The statement of the theorem is that

$$T \circ \theta \circ \alpha(Q) = T \circ \theta \circ \beta(Q + \rho Q)\,. \tag{6.6}$$

To prove this, consider

$$\iota_2^{-1} \circ \iota_1(s) = s \bar{\rho}^* \sigma_Q\,,$$

when s is a non-zero globally holomorphic section of $\mathcal{L}(Q - P)$ with a zero at ρP. The result is a globally holomorphic section of $\mathcal{L}^Q$, since the simple pole of $\bar{\rho}^* \sigma_Q$ at ρP is cancelled by the zero of s. Thus $\iota_1(s)$ lies in the $\boldsymbol{H}$-subspace of $H^0(X, \mathcal{L}_{\mathfrak{q}})$ given by the image of $H^0(X, \mathcal{L}^Q)$ under ι_2. This proves (6.6). □

6.3. The multiplier spectrum. Here we will examine how the multiplier spectrum $\mathrm{Sp}(L^*, D)$ is related to the spectral curve X in the case of a quaternionic holomorphic curve L arising from spectral data in the manner of Section 4.2.

Suppose that we have a conformally immersed torus $f : \boldsymbol{C}/\Lambda \to \boldsymbol{HP}^1$ of finite type, with spectral data $(X, \rho, P, \mathcal{L}, \mathfrak{q})$, i.e., we suppose that the map $\ell^{\mathfrak{q}} : \boldsymbol{C} \to J_{\boldsymbol{R}}(X)$ is Λ-periodic. We may as well assume $\mathfrak{q}$ is the largest ρ-invariant divisor of distinct smooth points which we can choose with this property. We will show how the multiplier spectrum $\mathrm{Sp}(L^*, D)$ arises from the sections of $\pi^* L^*$ obtained by evaluating the global Baker function at different points of $X \setminus \{P, \rho P\}$.

To X we can assign the subgroup

$$\Gamma_X = \{\gamma \in \Gamma;\, \gamma \text{ extends hol. to } h : X \setminus \{P, \rho P\} \to \boldsymbol{C}^\times,\ \overline{\rho^* h} = h\}\,.$$

Notice that $\Gamma_- \cap \Gamma_X = \boldsymbol{R}^+$ while $\Gamma_0 \cap \Gamma_X = \{1\}$. Clearly we have an exact sequence

$$1 \to \Gamma_- \Gamma_0 \Gamma_X \to \Gamma \xrightarrow{\ell} J_{\boldsymbol{R}}(X) \to 1\,.$$

In particular this gives us homomorphisms

$$\mu : \ker(\ell) \to \Gamma_0\,; \quad \chi : \ker(\ell) \to \Gamma_X$$

defined by the unique factorisation

$$\gamma = \gamma_- \mu(\gamma)^{-1} \chi(\gamma)\,, \quad \gamma \in \ker(\ell)\,,$$

where γ_- is normalised by $\gamma_- = 1 + O(\zeta^{-1})$.

LEMMA 6.3. *The global Baker function ψ_W for this spectral data satisfies*

$$\psi_W(\mathbf{t}' + \mathbf{t}) = \mu(\gamma)^{-1}\psi_W(\mathbf{t}')\chi(\gamma)\,, \tag{6.7}$$

whenever $\gamma = \gamma(\mathbf{t}) \in \Gamma_+ \cap \ker(\ell)$.

PROOF. The transformation due to Γ_0 comes directly from equation (4.3). So let us assume γ has trivial Γ_0 factor and that $\ell(\gamma) \simeq \mathcal{O}_X$. Since we are dealing with a global Baker function there is a trivialisation φ over P which extends holomorphically to X. Let $\varphi(\gamma)$ be the trivialisation for $\mathcal{L} \otimes \ell(\gamma)$ obtained by twisting the 1-cocycle for $(\mathcal{L}, \varphi)$ by γ. Then

$$\psi_W(\gamma) = \left(\frac{\sigma}{\varphi(\gamma)} - j\frac{\bar{\rho}^*\sigma}{\varphi(\gamma)}\right)\gamma\,,$$

where σ is the global section of $\mathcal{L}(-\rho P)$ normalised by $\sigma|_P = \varphi|_P$. Since $\gamma = \gamma_-\chi(\gamma)$ by assumption, the isomorphism $\mathcal{L} \otimes \ell(\gamma) \simeq \mathcal{L}$ equates $\varphi\gamma_-$ with $\varphi(\gamma)$, hence

$$\psi_W(\gamma) = \left(\frac{\sigma}{\varphi} - j\frac{\bar{\rho}^*\sigma}{\varphi}\right)\gamma_-^{-1}\gamma = \psi_W(1)\chi(\gamma)\,.$$

Equation (6.7) follows easily from this by replacing $\mathcal{L}$ with $\mathcal{L} \otimes \ell(\gamma')$. □

Now let us restrict ℓ to $\Gamma_1 \simeq \boldsymbol{C}$. Here it has kernel Λ and we obtain homomorphisms

$$\mu : \Lambda \to S^1\,; \quad \chi : \Lambda \to \Gamma_X\,.$$

We may think of χ as a function on $\Lambda \times X \setminus \{P, \rho P\}$. Thus to each point $Q \in X \setminus \{P, \rho P\}$ we get a function $\psi_W(z, Q) : \boldsymbol{C} \to \boldsymbol{H}$ with the properties

$$\mathcal{D}\psi_W(z, Q) = 0\,, \quad \psi_W(z + \lambda, Q) = \mu(\lambda)^{-1}\psi_W(z, Q)\chi(\lambda, Q)\,.$$

Hence $\chi(\lambda, Q) \in \mathrm{Sp}(L^*, D)$.

THEOREM 6.4. *Let $f : \boldsymbol{C}/\Lambda \to \boldsymbol{HP}^1$ be a non-constant conformal immersion of finite type from* $\mathrm{Gr}_{\mathrm{QKP}}$, *with global Baker function $\psi(z, Q)$ on $\boldsymbol{C} \times X \setminus \{P, \rho P\}$, and for which the flat bundle L^* has monodromy* $\mu \in \mathrm{Hom}(\Lambda, S^1)$. *Define*

$$\chi : \Lambda \times X \setminus \{P, \rho P\}\,; \quad \chi(\lambda, Q) = \psi(0, Q)^{-1}\mu(\lambda)\psi(\lambda, Q)\,.$$

For any pair of generators λ_1, λ_2 of Λ the holomorphic map

$$X \setminus \{P, \rho P\} \to \mathrm{Sp}(L^*, D)\,; \quad Q \mapsto (\chi(\lambda_1, Q), \chi(\lambda_2, Q)) \tag{6.8}$$

is surjective onto $\mathrm{Sp}(L^*, \mathcal{D})$. *Moreover, this map factors through the covering map $X \setminus \{P, \rho P\} \to X_{\mathfrak{q}} \setminus \{P, \rho P\}$.*

PROOF. This map is clearly holomorphic and non-constant when f is non-constant, and therefore the image is an analytic subvariety of $\mathrm{Sp}(L^*, D)$. The image also contains annuli about each of P, ρP, by the symmetry of the real involution. Since $\mathrm{Sp}(L^*, D)$ has at most two irreducible components, one about each of P, ρP, the image must agree with $\mathrm{Sp}(L^*, D)$. At each point of $Q \in \mathfrak{q}$ we know from Theorem 4.9 that $\psi(z, Q)$ corresponds to a quaternionic holomorphic section of L^*, i.e., it has trivial multiplier. Hence the map (6.8) factors through $X_{\mathfrak{q}}$. □

6.4. A comparison of spectral curves. Taimanov [22] initially proposed that, when it has finite genus, the normalisation Σ of $\mathrm{Sp}(L^*, D)$ should be the spectral curve, and this is the definition used in [3, 4]. But Σ lacks the subtlety necessary to be useful, because it throws away crucial information. It does this at two levels: (i) it throws away the information contained in the divisor $\mathfrak{q}$, (ii) unless X is smooth, which it need not be, it throws away the information of singularities in X. Examples of the latter case have been discussed by Taimanov himself, in the context of the generalised Weierstrass representation, in [21, 23, 25].

The virtue of the $\mathrm{Sp}(L^*, D)$ is that it is directly constructed from the Dirac operator, equally, the quaternionic holomorphic structure (L^*, D). This means that it is more properly an invariant of the quaternionic holomorphic curve $f : \boldsymbol{C}/\Lambda \to \boldsymbol{HP}^n$ given by (2.1) (where $n+1 = \dim_{\boldsymbol{H}} H^0_D(E)$). Provided we take $\mathfrak{q}$ to be the maximal divisor on which the multiplier of the Baker function is trivial, $X_{\mathfrak{q}}$ is likewise an invariant of this quaternionic holomorphic curve. Congruence of f in $\boldsymbol{HP}^n$, which is the natural equivalence relation on such curves, gives a broader equivalence than congruence of any of the projections of f into S^4, but it is clear from all the discussions above that this is the correct notion of equivalence from the point of view of spectral data. The virtue of $X_{\mathfrak{q}}$ over $\mathrm{Sp}(L^*, D)$ is that it is algebraic and has no spurious singularities.

For example, in the case of Example 2.2 earlier, $\mathfrak{q}$ is a divisor of distinct points on $X \simeq \boldsymbol{C}_\infty$, and therefore $X_{\mathfrak{q}}$ is a partial resolution of $\mathrm{Sp}(L^*, D)$: it keeps the essential information about where the multipliers are trivial but discards the singularities caused by its immersion into the plane by (6.8).

This prompts the question of how to describe $X_{\mathfrak{q}}$ as a cover of $\mathrm{Sp}(L^*, D)$ without passing through the QKP construction. I suspect the answer is something like the following: the kernel of the Dirac operator should determine over $\mathrm{Sp}(L^*, D)$ a coherent analytic sheaf $\mathcal{E}$ whose sections represent functions $\psi(z, \zeta)$ which satisfy $\mathcal{D}\psi = 0$ for each ζ, are holomorphic in ζ, and have the appropriate multiplier at ζ. The sheaf of algebras $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ would be the model for the structure sheaf of $X \setminus \{P, \rho P\}$. Notice that this is consistent with the construction of X as the compactification of $\mathrm{Spec}(\mathcal{A}_W^{\boldsymbol{C}})$ since $\mathcal{A}_W$ is isomorphic to an algebra of operators preserving the kernel of $\mathcal{D}$. This instantly makes $\mathcal{E}$ a maximal sheaf over $X \setminus \{P, \rho P\}$ which should be both rank one and torsion free.

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