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THE QUENCHING BEHAVIOR OF A SEMILINEAR HEAT EQUATION WITH A SINGULAR BOUNDARY OUTFLUX

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Abstract. In this paper, we study the quenching behavior of the solution of a semilinear heat equation with a singular boundary outflux. We prove a finite-time quenching for the solution. Further, we show that quenching occurs on the boundary under certain conditions and we show that the time derivative blows up at a quenching point. Finally, we get a quenching rate and a lower bound for the quenching time.

1. Introduction. In this paper, we study the quenching behavior of solutions of the following semilinear heat equation with a singular boundary outflux:

$$\begin{cases} u_t = u_{xx} + (1-u)^{-p}, \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) = 0, \ u_x(1,t) = -u^{-q}(1,t), \ 0 < t < T, \\ u(x,0) = u_0(x), \ 0 \le x \le 1, \end{cases} \tag{1}$$

where p, q are positive constants and $T \le \infty$. The initial function $u_0 : [0,1] \to (0,1)$ satisfies the compatibility conditions

$$u_0'(0) = 0, \ u_0'(1) = -u_0^{-q}(1).$$

Throughout this paper, we also assume that the initial function u_0 satisfies the inequalities

$$u_{xx}(x,0) + (1-u(x,0))^{-p} \ \ge \ 0, \tag{2}$$

$$u_x(x,0) \ \le \ 0. \tag{3}$$

Our main purpose is to examine the quenching behavior of the solutions of problem (1) having two singular heat sources. A solution $u(x,t)$ of problem (1) is said to quench

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if there exists a finite time T such that

$$\lim_{t\to T^-} \max\{u(x,t) : 0 \le x \le 1\} \to 1 \text{ or } \lim_{t\to T^-} \min\{u(x,t) : 0 \le x \le 1\} \to 0.$$

From now on, we denote the quenching time of problem (1) with T.

Since 1975, quenching problems with various boundary conditions have been studied extensively (cf. the surveys by Chan [1,2] and Kirk and Roberts [14] and [3,4,6–9,11–13, 15–18]). In the literature, quenching problems have been less studied with two nonlinear heat sources. We give as examples two of these papers. Chan and Yuen [5] considered the problem

$$\begin{aligned}
&u_t = u_{xx}, \text{ in } \Omega,\\
&u_x(0,t) = (1-u(0,t))^{-p},\ u_x(a,t) = (1-u(a,t))^{-q},\ 0 < t < T,\\
&u(x,0) = u_0(x),\ 0 \le u_0(x) < 1, \text{ in } \bar{D},
\end{aligned}$$

where $a,p,q > 0, T \le \infty, D = (0,a), \Omega = D \times (0,T)$. They showed that $x = a$ is the unique quenching point in finite time if u_0 is a lower solution, and that u_t blows up at quenching. Further, they obtained criteria for nonquenching and quenching by using the positive steady states. Zhi and Mu [19] considered the problem

$$\begin{aligned}
&u_t = u_{xx} + (1-u)^{-p},\ 0 < x < 1,\ 0 < t < T,\\
&u_x(0,t) = u^{-q}(0,t),\ u_x(1,t) = 0,\ 0 < t < T,\\
&u(x,0) = u_0(x),\ 0 < u_0(x) < 1,\ 0 \le x \le 1,
\end{aligned}$$

where $p,q > 0$ and $T \le \infty$. They showed that $x = 0$ is the unique quenching point in finite time if u_0 satisfies $u_0''(x) + (1-u_0(x))^{-p} \le 0$ and $u_0'(x) \ge 0$. Further, they obtained the quenching rate estimate which is $(T-t)^{1/2(q+1)}$ if T denotes the quenching time.

Here in this paper, a quenching problem with two types of singularity terms, namely, a source term $(1-u)^{-p}$ and the boundary outflux term $-u^{-q}$, is considered. In Section 2, we first show that quenching occurs in finite time under condition (2). Then, we show that the only quenching point is $x = 0$ under conditions (2) and (3). Further, we show that u_t blows up at quenching time. In Section 3, we get a quenching rate and a lower bound for quenching time.

2. Quenching on the boundary and blow-up of u_t.

REMARK 1. We assume that the conditions (2) and (3) are proper. Namely, we can easily construct such an initial function satisfying (2), (3) and compatibility conditions. Let $u_0(x) = 0.9 - \frac{2}{3}x^{4.5}$. For example, for $p = 9$ and $q = \log_{30/7} 3$, $u_0(x)$ satisfies (2), (3) and compatibility conditions.

REMARK 2. If u_0 satisfies (3), then we get $u_x < 0$ in $(0,1] \times (0,T)$ by the maximum principle. Thus we get $u(0,t) = \max_{0\le x\le 1} u(x,t)$.

LEMMA 1. If u_0 satisfies (2), then $u_t(x,t) \ge 0$ in $[0,1] \times [0,T)$.

Proof. We give the proof by utilizing Lemma 3.1 in [10]. Let $v = u_t(x,t)$. Then $v(x,t)$ satisfies

$$\begin{aligned} &v_t = v_{xx} + p\,(1-u)^{-p-1}\,v,\ 0 < x < 1,\ 0 < t < T,\\ &v_x\,(0,t) = 0,\ v_x\,(1,t) = qu^{-q-1}(1,t)v(1,t),\ 0 < t < T,\\ &v\,(x,0) = u_{xx}\,(x,0) + (1-u\,(x,0))^{-p} \geq 0,\ 0 \leq x \leq 1. \end{aligned}$$

For any fixed $\tau \in (0,T)$, let

$$\begin{aligned} L &= \max_{0\leq x\leq 1,\ 0\leq t\leq \tau} \left(\frac{1}{2} qu^{-q-1}(x,t)\right),\\ M &= 2L + 4L^2 + \max_{0\leq x\leq 1,\ 0\leq t\leq \tau} \left(p\,(1-u(x,t))^{-p-1}\right). \end{aligned}$$

Set $w(x,t) = e^{-Mt-Lx^2}v(x,t)$. Then w satisfies

$$\begin{aligned} &w_t = w_{xx} + 4Lxw_x + cw,\ 0 < x < 1,\ 0 < t \leq \tau,\\ &w_x(0,t) = 0,\ w_x(1,t) = d(t)w(1,t),\ 0 < t \leq \tau,\\ &w(x,0) \geq 0,\ 0 \leq x \leq 1, \end{aligned}$$

where

$$c = c(x,t) = 4L^2(x^2-1) + p\,(1-u(x,t))^{-p-1} - \max_{0\leq x\leq 1,\ 0\leq t\leq \tau} \left(p\,(1-u(x,t))^{-p-1}\right) \leq 0$$

and

$$d(t) = -\max_{0\leq x\leq 1,\ 0\leq t\leq \tau} \left(qu^{-q-1}(x,t)\right) + qu^{-q-1}(1,t) \leq 0.$$

By the maximum principle and the Hopf lemma, we obtain that $w \geq 0$ in $[0,1] \times [0,\tau]$. Thus, $u_t(x,t) \geq 0$ in $[0,1] \times [0,T)$. $\square$

THEOREM 1. If u_0 satisfies (2), then there exists a finite time T such that the solution u of problem (1) quenches at time T.

Proof. Assume that u_0 satisfies (2). Then we get

$$\omega = -u^{-q}\,(1,0) + \int_0^1 (1-u\,(x,0))^{-p}\,dx > 0.$$

Introduce a mass function: $m\,(t) = \int_0^1 (1-u\,(x,t))\,dx, 0 < t < T$. Then

$$m'\,(t) = u^{-q}\,(1,t) - \int_0^1 (1-u\,(x,t))^{-p}\,dx \leq -\omega,$$

by Lemma 1. Thus, $m\,(t) \leq m(0) - \omega t$, which means that $m\,(T_0) = 0$ for some $T_0 (0 < T \leq T_0)$, which means that u quenches in a finite time. $\square$

THEOREM 2. If u_0 satisfies (2) and (3), then $x = 0$ is the only quenching point.

Proof. Define

$$J(x,t) = u_x + \varepsilon\,(b_2 - x)\ \text{ in } [b_1,b_2] \times [\tau,T),$$

where $b_2 \in (0,1]$, $b_1 \in (0,b_2)$, $\tau \in [0,T)$ and ε is a positive constant to be specified later. Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = p(1-u)^{-p-1}u_x < 0 \text{ in } (b_1,b_2) \times [\tau,T),$$

since $u_x(x,t) < 0$ in $(0,1] \times [0,T)$. Thus, $J(x,t)$ cannot attain a positive interior maximum by the maximum principle. Further, if ε is small enough, $J(x,\tau) < 0$ since $u_x(x,t) < 0$ in $(0,1] \times [0,T)$. Furthermore, if ε is small enough,

$$\begin{aligned} J(b_1,t) &= u_x(b_1,t) + \varepsilon\,(b_2 - b_1) < 0, \\ J(b_2,t) &= u_x(b_2,t) < 0, \end{aligned}$$

for $t \in (\tau,T)$. By the maximum principle, we obtain that $J(x,t) < 0$, i.e., $u_x < -\varepsilon\,(b_2 - x)$ for $(x,t) \in [b_1,b_2] \times [\tau,T)$. Integrating this with respect to x from b_1 to b_2, we have

$$u(b_2,t) < u(b_1,t) - \frac{\varepsilon(b_2-b_1)^2}{2} < 1 - \frac{\varepsilon(b_2-b_1)^2}{2} < 1.$$

So u does not quench in $(0,1]$. The theorem is proved. $\square$

THEOREM 3. If $p \geq 1$, then u_t blows up at the quenching point $x = 0$.

Proof. Suppose that u_t is bounded on $[0,1] \times [0,T)$. Then, there exists a positive constant M such that $u_t < M$. That is,

$$u_{xx} + (1-u)^{-p} < M.$$

Multiplying this inequality by u_x, and integrating with respect to x from 0 to x, we have

$$\ln\left[1 - u(0,t)\right] > \frac{-1}{2}u_x^2 + \ln\left[1 - u(x,t)\right] + M\left[u(x,t) - u(0,t)\right]$$

for $p = 1$ and

$$\frac{(1-u(0,t))^{-p+1}}{-p+1} > \frac{-1}{2}u_x^2 + \frac{(1-u(x,t))^{-p+1}}{-p+1} + M\left[u(x,t) - u(0,t)\right]$$

for $p \neq 1$. We have, as $t \to T^-$ and $p \geq 1$, that the left-hand side tends to negative infinity, while the right-hand side is finite. This contradiction shows that u_t blows up at the quenching point $x = 0$. $\square$

3. A quenching rate and a lower bound for the quenching time. In this section, we get a quenching rate and a lower bound for the quenching time. Throughout this section, we assume that

$$\begin{aligned} u_x(x,0) &\leq -xu^{-q}(x,0), 0 \leq x \leq 1, && (4) \\ u_t(0,t) &= u_{xx}(0,t) + (1-u(0,t))^{-p}, 0 < t < T. && (5) \end{aligned}$$

THEOREM 4. If u_0 satisfies $(2), (3), (4)$ and (5), then there exists a positive constant C_1 such that

$$u(0,t) \geq 1 - C_1(T-t)^{1/(p+1)},$$

for t sufficiently close to T.

Proof. Define $J(x,t) = u_x + xu^{-q}$ in $[0,1] \times [0,T)$. Then, $J(x,t)$ satisfies

$$J_t - J_{xx} = \left[p(1-u)^{-p-1} + 2qu^{-q-1}\right]u_x - qxu^{-q-1}(1-u)^{-p} - q(q+1)xu^{-q-2}u_x^2,$$

since $u_x < 0$, $J(x,t)$ cannot attain a positive interior maximum. On the other hand, $J(x,0) \le 0$ by (4) and

$$J(0,t) = 0, \ J(1,t) = 0,$$

for $t \in (0,T)$. By the maximum principle, we obtain that $J(x,t) \le 0$ for $(x,t) \in [0,1] \times [0,T)$. Therefore

$$J_x(0,t) = \lim_{h\to 0^+} \frac{J(h,t) - J(0,t)}{h} = \lim_{h\to 0^+} \frac{J(h,t)}{h} \le 0.$$

From (5), we get

$$\begin{aligned} J_x(0,t) &= u_{xx}(0,t) + u^{-q}(0,t) \\ &= u_t(0,t) - (1-u(0,t))^{-p} + u^{-q}(0,t) \le 0 \end{aligned}$$

and

$$u_t(0,t) \le (1-u(0,t))^{-p}.$$

Integrating for t from t to T we get

$$u(0,t) \ge 1 - C_1(T-t)^{1/(p+1)},$$

where $C_1 = (p+1)^{1/(p+1)}$. □

REMARK 3. We can calculate a lower bound for the quenching time. From Theorem 4, a lower bound is $(1-u_0(0))^{p+1}/(p+1)$ for quenching time T. If we choose, as in Remark 1, $u_0(x) = 0.9 - \frac{2}{3}x^{4.5}$, then we have $T = 10^{-11}$ for $p = 9$.

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