The r-Major Index

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The r-major index is a new permutation statistic that is suggested by the work of Carlitz and Gansner on Foulkes' skew-hook rule for computing the r-Eulerian numbers. The new statistic (1) generalizes both the major index and the inversion number of a permutation and (2) leads to a q-analog of the r-Eulerian numbers.

1. INTRODUCTION

Let $G(n)$ denote the symmetric group on $\{1, 2,..., n\}$ and $W(n)$ the set of words $w = w_1 w_2 \cdots w_n$ with letters $w_i \in \{0, 1,..., i-1\}$. For an integer $r \geqslant 1$, the r-major index of a permutation $\sigma \in G(n)$, denoted by $r\,\text{maj}\,\sigma$, is defined to be the sum of the elements of the set

$$\{i: \sigma(i) \geqslant \sigma(i+1) + r, 1 \leqslant i \leqslant n-1\} \tag{1.1}$$

plus the cardinality of the set

$$\{(i, j): 1 \leqslant i < j \leqslant n, \sigma(i) > \sigma(j) > \sigma(i) - r\}. \tag{1.2}$$

Let $s(w)$ be the sum of the letters in the word w. The primary objective of this paper is to construct a bijection $\Gamma: G(n) \to W(n)$ with the property that

$$\text{if} \quad \Gamma(\sigma) = w \qquad \text{then} \quad r\,\text{maj}\,\sigma = s(w). \tag{1.3}$$

The correspondence Γ is referred to as the $r\,$maj coding of $G(n)$.

There are a number of motives for and consequences of defining the r-major index and constructing Γ. First, the r-major index generalizes both of the classic permutation statistics known as the major index and the inversion number. Indeed, the r-major index reduces to the major index when $r = 1$

and to the inversion number when $r \geqslant n$. Furthermore, if the q-analog and the q-factorial of a non-negative integer n are respectively defined as

$$\begin{aligned} &\text{(a)} \quad [n] = 1 + q + \cdots + q^{n-1} \\ &\text{(b)} \quad [n]! = [1]\,[2] \cdots [n], \end{aligned} \tag{1.4}$$

then it is immediate from (1.3) that

$$\sum_{\sigma} q^{r\operatorname{maj}\sigma} = [n]! \tag{1.5}$$

summed over $G(n)$. Rodriguez [15] derived (1.5) for the inversion case as early as 1839. Identity (1.5) also includes MacMahon's [11] observation that the major index and the inversion number have the same generating function. Incidentally, Foata [4] provided a combinatorial proof of MacMahon's discovery by constructing a bijection Φ: $G(n) \to G(n)$ with the property that the major index of $\sigma \in G(n)$ is equal to the inversion number of $\Phi(\sigma)$.

Second, the special cases $r = 1$ and $r = n$ of Γ are common and useful tools in working with the major index and the inversion number. In the inversion case, $\Gamma(\sigma)$ is known as the inversion table (see Knuth [10, p. 12]) or the Lehmer code of σ. The major index case of Γ is implicit in the work of Carlitz [2]. In fact, the labeling used in Section 4 of this paper to construct Γ was used by Carlitz for $r = 1$. The two cases $r = 1$ and $r = n$ of Γ were further considered and developed by Gerard Viennot in an unpublished work. Viennot combined the two cases to provide another combinatorial proof of the fact that the major index and the inversion number are identically distributed over $G(n)$.

Third, the $r$maj coding unifies and extends the work of Carlitz [3] and Gansner [7] on Foulkes' [6] skew-hook formula for computing the r-Eulerian numbers. The q-analog of Foulkes' formula obtained by Carlitz for $r = 1$ will be extended to all $r \geqslant 1$ using the r-major index. Also, the skew-hooks which appear in the bijection that Gansner developed will be constructed from the word $\Gamma(\sigma)$. In fact, Γ is a modification of Gansner's correspondence.

Finally, the r-major index also leads to a q-analog of Foulkes' observation that the skew-hook formula is a kind of scalar product of Stirling numbers. The q-analog of the Stirling numbers that arises in this connection agrees with the one introduced by Gould [9] and further considered by Milne [12] and Garsia [8].

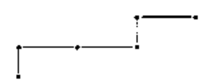
2. The r-Eulerian Numbers

The elements of set (1.1) are referred to as the r-descents of σ. The number of r-descents is denoted by $r\operatorname{des}\sigma$. Then, as in [5, 14], the r-Eulerian numbers $A(n, k; r)$ may be interpreted as the number of permutations in $G(n)$ having k r-descents. They satisfy the recurrence

$$A(n, k; r) = (k + r)\,A(n-1, k; r) + (n + 1 - k - r)\,A(n-1, k-1; r), \tag{2.1}$$

where $A(r, 0; r) = r!$.

Foulkes' skew-hook rule provides an alternative for computing $A(n, k; r)$. Let $H(n, k, r)$ denote the set of $\binom{n-r}{k}$ paths in an $(n + 1 - k - r)$ row by $(k + 1)$ column rectangle of nodes that proceed from the bottom left-hand corner to the top right-hand corner by a sequence of steps, either upward or to the right. In any such path, attach the label i to any horizontal step in the ith row from the bottom, and attach the label $i + r - 1$ to any vertical step in the ith column from the left. As an example, for $r = 3$ the labels of the steps in the path



(2.2)

of $H(8, 3, 3)$ are from bottom to top 3, 2, 2, 5, 3. Let $\prod(h)$ denote the product of the $n - r$ labels of the path $h \in H(n, k, r)$. Then Foulkes showed that

$$A(n, k; r) = r! \sum_{h} \prod (h) \tag{2.3}$$

summed over $H(n, k, r)$.

3. The (q, r)-Stirling Numbers

Let $P(n, k)$ denote the collection of partitions $p = (B_1, B_2, \ldots, B_k)$ of the set $\{1, 2, \ldots, n\}$ into k parts with $\min B_1 < \min B_2 < \cdots < \min B_k$. For $p \in P(n, k)$ define

$$m(p) = \sum_{i=1}^{k} (i-1)\,|B_i|, \tag{3.1}$$

where $|B_i|$ denotes the cardinality of B_i. Further, let $P_1(n, k, r)$ and

$P_2(n,k,r)$ respectively denote the subsets of $P(n,k+r)$ and $P(n,n+1-k-r)$ defined by

$$\begin{aligned}
&\text{(a)}\quad (B_1,B_2,\dots,B_{k+r})\in P_1(n,k,r) && \text{if } \ i\in B_i \text{ for } 1\leqslant i\leqslant r\\
&\text{(b)}\quad (B_1,B_2,\dots,B_{n+1-k-r})\in P_2(n,k,r) && \text{if } \ \{1,2,\dots,r\}\subset B_1 .
\end{aligned}\tag{3.2}$$

Then, for $i=1,2$, the (q,r)-Stirling numbers are defined as

$$S_i[n,k;r]=\sum_{p} q^{m(p)} \tag{3.3}$$

summed over $P_i(n,k,r)$.

As elements of $P_1(n,k,r)$ are either of the form $(p,\{n\})$ with $p\in P_1(n-1,k-1,r)$ or obtained by inserting n into a set of some $p\in P_1(n-1,k,r)$ it follows that

$$S_1[n,k;r]=q^{k+r-1}S_1[n-1,k-1;r]+[k+r]\,S_1[n-1,k;r] \tag{3.4}$$

with $S_1[r,0;r]=q^{\binom{r}{2}}$. Similarly,

$$S_2[n,k;r]=q^{n-k-r}S_2[n-1,k;r]+[n+1-k-r]\,S_2[n-1,k-1;r] \tag{3.5}$$

with $S_2[r,0;r]=1$. The case $r=1$ is discussed in [8, 9, 12].

4. The Insertion Lemma

The construction of Γ is based on the insertion proof of (2.1). To observe the effect that inserting n into a permutation $\theta\in G(n-1)$ has on the r-major index and the r-descent number, the insertion positions between the letters of the word $\theta=\theta(1)\,\theta(2)\cdots\theta(n-1)$ are labeled as follows. Using the labels $0, 1,\dots, n-1$ in order, first read from right to left and label the positions that will *not* result in the creation of a new r-descent. Then, reading back from left to right the positions that *will* create a new r-descent are labeled. For instance, if $r=3$ and $\theta=5176324\in G(7)$, then the labels in the top and bottom rows of

$$\begin{array}{ccccccccccccccc}
 & \cdot & & 4 & & 3 & & 2 & & 1 & & \cdot & & \cdot & & 0\\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & & & \downarrow\\
\theta= & & 5 & & 1 & & 7 & & 6 & & 3 & & 2 & & 4 & \\
 & \uparrow & & & & & & & & & & \uparrow & & \uparrow & & \\
 & 5 & & \cdot & & \cdot & & \cdot & & \cdot & & 6 & & 7 & & \cdot
\end{array} \tag{4.1}$$

respectively indicate the positions that will not and that will result in a new 3-descent.

Let (θ,l) denote the permutation obtained by inserting n into position l. From (4.1), $(\theta,6)=51763824\in G(8)$. Note that $3\operatorname{des}(\theta,6)=1+3\operatorname{des}\theta$ and that $3\operatorname{maj}(\theta,6)=6+3\operatorname{maj}\theta$. This demonstrates the

Insertion Lemma. *For $\theta\in G(n-1)$ and $0\leqslant l\leqslant n-1$*

$$\begin{aligned}
&\text{(a)}\quad r\operatorname{des}(\theta,l)=r\operatorname{des}\theta && \text{if}\quad 0\leqslant l\leqslant r-1+r\operatorname{des}\theta\\
&\qquad\qquad\qquad\quad\;=1+r\operatorname{des}\theta && \text{otherwise,}\\
&\text{(b)}\quad r\operatorname{maj}(\theta,l)=l+r\operatorname{maj}\theta. &&
\end{aligned}$$

Proof. For (a), note that there are $r+r\operatorname{des}\theta$ insertion positions that will not result in a new r-descent: preceeding any of the $r-1$ integers greater than $n-r$, in any of the r-descents, or at the extreme right end of θ. As these positions are labeled first, (a) is immediate.

For (b), let m be the number of r-descents and integers greater than $n-r$ that are to the right of position l. In the case $0\leqslant l\leqslant r-1+r\operatorname{des}\theta$, from (1.1) and (1.2) it follows that inserting n into position l will increase the r-major index by m. But by the labeling, $l=m$. For the case $r+r\operatorname{des}\theta\leqslant l\leqslant n-1$, note that there are $l-m-1$ integers in θ to the left of position l. As a new r-descent is created in this case, the r-major index will be increased by $(l-m)+m=l$.

5. The $r$maj Coding

To define $\Gamma(\sigma)=w$, note that for $\sigma\in G(n)$ there is a unique pair (θ,l) such that $\theta\in G(n-1)$, $0\leqslant l\leqslant n-1$, and $\sigma=(\theta,l)$. Let $w_n=l$. Induction then determines the remaining letters $w_1,w_2,\dots,w_{n-1}$ of w, and it follows from the insertion lemma that $r\operatorname{maj}\sigma=w_n+r\operatorname{maj}\theta=w_n+w_{n-1}+\cdots+w_1=s(w)$. From example (4.1), for $r=3$ and $\sigma=51763824\in G(8)$ one sees that $\theta=5176324\in G(7)$ and $w_8=6$. Iteration leads to

$$\begin{aligned}
\Gamma(\sigma)&=00103426\in W(8),\\
3\operatorname{maj}\sigma&=16=s(w).
\end{aligned}\tag{5.1}$$

Besides the r-major index, the r-descent number of σ may also be determined directly from w. As no r-descents are created by the first r insertions, the word w will be factorized as the juxtaposition product uv, where $u=w_1w_2\cdots w_r$ and $v=w_{r+1}w_{r+2}\cdots w_n$. Let $U(r)$ and $V(n,r)$ respectively denote the sets of such factors. Now for a word z and a letter l inductively define

$$
\begin{aligned}
c(zl) &= c(z) && \text{if} \quad 0 \leqslant l \leqslant r-1+c(z) \\
&= 1 + c(z) && \text{otherwise,}
\end{aligned}
\tag{5.2}
$$

where $c(\text{empty word}) = 0$. Comparison of (5.2) with (a) of the insertion lemma leads to the conclusion that $r\text{des}\,\sigma = c(v)$. In example (5.1), $u = 001$, $v = 03426$, and $c(v) = 3 = 3\,\text{des}\,\sigma$.

6. The (q, r)-Eulerian Numbers

Let $G(n, k, r) = \{\sigma \in G(n)\colon\ r\text{des}\,\sigma = k\}$ and $V(n, k, r) = \{v \in V(n, r)\colon c(v) = k\}$. Set

$$
\begin{aligned}
&\text{(a)} \quad A[n, k; r] = \sum_{\sigma} q^{r\,\text{maj}\,\sigma}, \\
&\text{(b)} \quad B[n, k; r] = \sum_{v} q^{s(v)}
\end{aligned}
\tag{6.1}
$$

summed respectively over $G(n, k, r)$ and $V(n, k, r)$. These polynomials satisfy the identities

$$
\begin{aligned}
&\text{(a)} \quad B[n, k; r] = [r+k]\,B[n-1, k; r] \\
&\qquad\qquad + q^{k+r-1}[n+1-k-r]\,B[n-1, k-1; r], \\
&\text{(b)} \quad A[n, k; r] = [r]!\,B[n, k; r], \\
&\text{(c)} \quad A[n, k; r] = [r+k]\,A[n-1, k; r] \\
&\qquad\qquad + q^{k+r-1}[n+1-k-r]\,A[n-1, k-1; r],
\end{aligned}
\tag{6.2}
$$

where $B[r, 0; r] = 1$ and $A[r, 0; r] = [r]!$. Part (c) gives a q-analog of (2.1).

Proof of (6.2). For (a), note that $V(n, k, r)$ is the disjoint union of the two sets $\{vl\colon v \in V(n-1, k, r),\ 0 \leqslant l \leqslant r-1+k\}$ and $\{vl\colon\ v \in V(n-1, k-1, r),\ \ r-1+k \leqslant l \leqslant n-1\}$. This observation, the fact that $s(vl) = s(v) + l$, and definition (b) of (6.1) imply (a). Part (b) follows from an application of Γ along with the calculation

$$
A[n, k; r] = \sum_{u} \sum_{v} q^{s(uv)} = \sum_{u} q^{s(u)} \sum_{v} q^{s(v)},
$$

the sums being respectively over $U(r)$ and $V(n, k, r)$. Part (c) is a consequence of combining (a) and (b).

7. The Skew-Hooks

To expose the relationship between (2.3) and (b) of (6.2), the skew-hook is now constructed from the factor v of w and q-labeled. For $v = w_{r+1} w_{r+2} \cdots w_n \in V(n, k, r)$ the lth step of the corresponding skew-hook $h \in H(n, k, r)$ is vertical if $c(w_{r+1} w_{r+2} \cdots w_l) = c(w_{r+1} w_{r+2} \cdots w_{l-1})$ and horizontal otherwise. In other words, the horizontal steps of h correspond to insertions that create r-descents. Note that the skew-hook corresponding to the word in (5.1) is given in (2.2).

Now for $h \in H(n, k, r)$ attach the label $[i]$ to any horizontal step in the ith row from the bottom, and attach $[i+r-1]$ to any vertical step in the ith column from the left. Further, attach the label $q^{\binom{r}{2}}$ to the left bottom node of h, to a node at the end of a horizontal step in the ith column attach q^{i+r-2}, and to a node at the end of a vertical step in the ith row attach q^{i-1}. For instance, the labels of the nodes and steps of the path in (2.2), in order from bottom to top, are q^3, $[3]$, q, $[2]$, q^3, $[2]$, q^4, $[5]$, q^2, $[3]$, q^5.

Respectively define the row, column, end horizontal, and end vertical polynomials of h to be

(a) $R[h] =$ product of horizontal step labels,

(b) $C[h] =$ product of vertical step labels,

(c) $EH[h] =$ product of node labels at the end of horizontal steps,

(d) $EV[h] =$ product of node labels at the end of vertical steps. (7.1)

For the trivial path consisting of a single node they are all set equal to 1. The polynomials of (7.1) are related to $B[n, k; r]$ and to the (q, r)-Stirling numbers by the identities

$$
\begin{aligned}
&\text{(a)} \quad B[n, k; r] = \sum_{h} R[h]\,C[h]\,EH[h], \\
&\text{(b)} \quad S_1[n, k; r] = q^{\binom{r}{2}} \sum_{h} C[h]\,EH[h], \\
&\text{(c)} \quad S_2[n, k; r] = \sum_{h} R[h]\,EV[h]
\end{aligned}
\tag{7.2}
$$

all summed over $H(n, k, r)$. Combining (a) of (7.2) with (b) of (6.2) yields a q-analog of the skew-hook formula (2.3). Furthermore, in view of (b) and (c) of (7.2), part (a) is a kind of scalar product of the (q, r)-Stirling numbers.

Proof of (7.2). For the initial case $n = r$, $H(r, 0, r)$ consists of a single node and $R[h] = C[h] = EH[h] = EV[h] = 1$. Thus, the initial conditions $B[r, 0; r] = 1$, $S_1[r, 0; r] = q^{\binom{r}{2}}$, and $S_2[r, 0; r] = 1$ are all satisfied in (7.2).

Since every $h \in H(n, k, r)$ is obtained by adding a vertical step to some $h_1 \in H(n-1, k, r)$ or horizontal step to some $h_2 \in H(n-1, k-1, r)$ it follows that

$$\sum_h R[h]\, C[h]\, EH[h] = [k+r] \sum_{h_1} R[h_1]\, C[h_1]\, EH[h_1]$$

$$+ q^{k+r-1}[n+1-k-r] \sum_{h_2} R[h_2]\, C[h_2]\, EH[h_2].$$

This recurrence and (a) of (6.2) imply (a) of (7.2). Similar reasoning along with recurrences (3.4) and (3.5) yields parts (b) and (c) of (7.2).

Remarks. In [13] the definition of the r-major index is extended from $G(n)$ to arbitrary finite sequences. This leads to a (q, r)-Simon Newcomb problem.

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