TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 2, February 2007, Pages 701–740
S 0002-9947(06)03912-2
Article electronically published on August 24, 2006

THE RACK SPACE

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Abstract. The main result of this paper is a new classification theorem for links (smooth embeddings in codimension 2). The classifying space is the rack space and the classifying bundle is the first James bundle.

We investigate the algebraic topology of this classifying space and report on calculations given elsewhere. Apart from defining many new knot and link invariants (including generalised James-Hopf invariants), the classification theorem has some unexpected applications. We give a combinatorial interpretation for π_2 of a complex which can be used for calculations and some new interpretations of the higher homotopy groups of the 3-sphere. We also give a cobordism classification of virtual links.

0. Introduction

The main result of this paper is a classification theorem for links (smooth embeddings of codimension 2):

Classification Theorem. *Let X be a rack. Then the rack space BX has the property that $\pi_n(BX)$ is in natural bijection with the set of cobordism classes of framed submanifolds L of $\mathbb{R}^{n+1}$ of codimension 2 equipped with a homomorphism of the fundamental rack $\Gamma(L \subset \mathbb{R}^{n+1})$ to X.*

Moreover, there is a smooth mock bundle $\zeta^1(BX)$ over BX which plays the rôle of classifying bundle.

It is important to note that this is a totally new type of classification theorem. Classical cobordism techniques give a bijection between cobordism classes of framed submanifolds of $\mathbb{R}^{n+1}$ and $\pi_n(\Omega(S^2))$. But these techniques are unable to cope with the extra geometric information given by the homomorphism of fundamental rack. For readers unfamiliar with the power of the rack concept (essentially a rack is a way of encapsulating the fundamental group and peripheral group system in one simple piece of algebra) here is a weaker result phrased purely in terms of the fundamental group:

Corollary. *Let π be a group. There is a classifying space $BC(\pi)$ such that $\pi_n(BC(\pi))$ is in natural bijection with the set of cobordism classes of framed submanifolds L of $\mathbb{R}^{n+1}$ of codimension 2 equipped with a homomorphism $\pi_1(\mathbb{R}^{n+1} - L) \to \pi$.*

Received by the editors August 1, 2003 and, in revised form, November 24, 2004.

2000 *Mathematics Subject Classification.* Primary 55Q40, 57M25; Secondary 57Q45, 57R15, 57R20, 57R40.

Key words and phrases. Classifying space, codimension 2, cubical set, James bundle, link, knot, π_2, rack.

Proof. Let $C(\pi)$ be the conjugacy rack of π [8, Example 1.1.1, page 349]. Then a homomorphism $\Gamma(L \subset Q \times \mathbb{R}) \to C(\pi)$ is equivalent to a homomorphism $\pi_1(Q \times \mathbb{R} - L) \to \pi$; see [8, Corollaries 2.2 and 3.3, pages 354 and 361]. $\square$

There are several ingredients of the proof of the classification theorem. The classifying space (the rack space) BX is defined in [10]. In addition, we need the geometry of $\square$-sets developed in [12], and, in particular, the James bundles of a $\square$-set. The compression theorem [26] is needed to reduce the codimension 2 problem to the codimension 1 problem of classifying diagrams up to cobordism and, finally, we need to develop a theory of smooth transversality to, and smooth mock bundles over, a $\square$-set. This is contained in the present paper.

Here is an outline of this paper:

In section 1 "*Basic definitions*" we recall the definitions of $\square$-sets, $\square$-maps, the rack space and the associated James complexes of a $\square$-set. In section 2 "*Mock bundles and transversality*" we define the concept of a mock bundle over a $\square$-set (cf. [1]) and observe that the James complexes of a $\square$-set C define mock bundles $\zeta^i(C)$ which embed as framed mock bundles in $|C| \times \mathbb{R}$. These are the James bundles of C. We define transversality for a map of a smooth manifold into a $\square$-set and prove that any map can be approximated by a transverse map. Mock bundles pull back over transverse maps to yield mock bundles whose total spaces are manifolds and, in particular, the first James bundle pulls back to give a self-transverse immersion of codimension 1 and the higher James bundles pull back to give the multiple point sets of this immersion.

The transversality theorem leads to our first classification theorem in section 3 "*Links and diagrams*" namely that a $\square$-set C is the classifying space for cobordism classes of link diagrams labelled by the cubes of C. In the key example in which C is the rack space BX, there is a far simpler description and we deduce the classification theorem stated above which interprets the homotopy groups of BX as bordism classes of links with representation of fundamental rack in X. There are similar interpretations for sets of homotopy classes of maps of a smooth manifold in C and BX and for the bordism groups of C and BX.

In section 4 "*The classical case*" we look in detail at the lowest non-trivial dimension ($n = 2$) where the cobordism classes can be described as equivalence classes under simple moves. This gives a combinatorial description of $\pi_2(C)$ which can be used for calculations. To illustrate this we translate the Whitehead conjecture [30] into a conjecture about coloured link diagrams. We finish by classifying virtual links (Kauffman [19], see also Kuperberg [21]) up to cobordism, in terms of the 2-dimensional homology of the rack space.

The theory of James bundles gives invariants for knots and links for the following reason. If Γ is the fundamental rack of a link L, then any invariant of the rack space $B\Gamma$ is *a fortiori* an invariant of L. In particular, any of the classical algebraic topological invariants of rack spaces are link invariants. Further invariants are obtained by considering representations of the fundamental rack in a small rack and pulling back invariants from the rack space of this smaller rack. In section 5 "*The algebraic topology of rack spaces*" we concentrate on calculating invariants of rack spaces. We describe all the homotopy groups of BX where X is the fundamental rack of an irreducible (non-split) link in a 3-manifold (this is a case in which the rack completely classifies the link [8]). This description leads to the new geometric descriptions for the higher homotopy groups of the 3-sphere mentioned earlier. We also calculate π_2 of BX where X is the fundamental rack of a general link in S^3. We show that BX is always a simple space and we compute the homotopy type of BX in the cases when X is a free rack and when X is a trivial rack with n elements. We also report on further calculations given elsewhere [14, 15, 31].

It is important to note that there are invariants of the rack space which are not homotopy invariants, but combinatorial ones. These include the James-Hopf invariants (defined by the James bundles) and the characteristic classes and associated generalised cohomology theories constructed in [12]. Although not homotopy invariants, these all yield invariants of knots and links. Now the homotopy type of the rack space does not contain enough information to reconstruct the rack; there are examples where the fundamental rack is a classifying invariant but the homotopy type of the rack space does not classify. See the remarks following Theorem 5.4. However, the combinatorics of the rack space contain all the information needed to reconstruct the rack, so in principle combinatorial invariants should give a complete set of invariants.

This program is explored further in [13] where we prove that in the classical case of links in S^3 the rack together with the canonical class in $\pi_2(B\Gamma)$ determined by any diagram is a complete invariant for the link. This leads to computable invariants which can effectively distinguish different links.

This paper appeared in preliminary form as part of our 1996 preprint [11] and many of the results were announced with outline proofs in 1993 in [9]. Since this early work of ours, other authors have investigated rack and quandle cohomology, notably J. Scott Carter et al. [2]. Rack cohomology is the cohomology of the rack space and quandle cohomology is a quotient, see Litherland and Nelson [23].

1. Basic definitions

We give here a minimal set of definitions for $\square$-sets. For more detail, other definitions and examples, see [10, sections 2 and 3] and [12, section 1].

The category $\square$. The *n-cube* I^n is the subset $[0,1]^n$ of $\mathbb{R}^n$.

A *p-face* of I^n is a subset defined by choosing $n-p$ coordinates and setting some of these equal to 0 and the rest to 1. In particular, there are $2n$ faces of dimension $n-1$ determined by setting $x_i = \epsilon$ where $i \in \{1, 2, \ldots, n\}$ and $\epsilon \in \{0, 1\}$.

A 0-face is called a *vertex* and corresponds to a point of the form $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ where $\epsilon_i = 0$ or 1 and $i \in \{1, 2, \ldots, n\}$. The 1-faces are called *edges* and the 2-faces are called *squares*.

Let $p \leq n$ and let J be a p-face of I^n. Then there is a canonical *face map* $\lambda\colon I^p \to I^n$, with $\lambda(I^p) = J$, given in coordinate form by preserving the order of the coordinates $(x_1, \ldots, x_p)$ and inserting $n-p$ constant coordinates which are either 0 or 1. If λ inserts only 0's (resp. only 1's) we call it a *front* (resp. *back*) face map. Notice that any face map has a unique *front-back* decomposition as $\lambda\mu$, say, where λ is a front face map and μ is a back face map. There is also a unique *back-front* decomposition. There are $2n$ face maps defined by the $(n-1)$-faces which are denoted $\delta_i^\epsilon\colon I^{n-1} \to I^n$, and given by

$$\delta_i^\epsilon(x_1, x_2, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \epsilon, x_i, \ldots, x_{n-1}), \qquad \epsilon \in \{0, 1\}.$$

The following relations hold:

$$\delta_i^\epsilon \delta_{j-1}^\omega = \delta_j^\omega \delta_i^\epsilon, \qquad 1 \leq i < j \leq n,\ \epsilon, \omega \in \{0, 1\}. \tag{1.1}$$

Definition. The category $\square$ is the category whose objects are the n-cubes I^n for $n = 0, 1, \dots$ and whose morphisms are the face maps.

$\square$-sets and their realisations. A set $\square$-*set* is a functor $C : \square^{op} \to Sets$ where $\square^{op}$ is the opposite category of $\square$ and $Sets$ denotes the category of sets.

A $\square$-*map* between $\square$-sets is a natural transformation.

We write C_n for $C(I^n)$, λ^* for $C(\lambda)$ and we write ∂_i^ϵ for $C(\delta_i^\epsilon) = (\delta_i^\epsilon)^*$.

The *realisation* $|C|$ of a $\square$-set C is given by making the identifications $(\lambda^* x, t) \sim (x, \lambda t)$ in the disjoint union $\coprod_{n\geq 0} C_n \times I^n$.

We shall call 0-cells (resp. 1-cells, 2-cells) of $|C|$ vertices (resp. edges, squares) and this is consistent with the previous use for faces of I^n, since I^n determines a $\square$-set with cells corresponding to faces, whose realisation can be identified in a natural way with I^n.

Notice that $|C|$ is a CW complex with one n-cell for each element of C_n and that each n-cell has a canonical characteristic map from the n-cube. However, not every CW complex with cubical characteristic maps comes from a $\square$-set—even if the cells are glued by isometries of faces. In $|C|$, where C is a $\square$-set, cells are glued by face *maps*; in other words, by canonical isometries of faces.

There is also a notion of a $\square$-*space*, namely a functor $X\colon \square^{op} \to Top$ (where Top denotes the category of topological spaces and continuous maps) and its realisation $|X|$ given by the same formula as above.

Notation. We shall often omit the mod signs and use the notation C for both the $\square$-set C and its realisation $|C|$. We shall use the full notation whenever there is any possibility of confusion.

Key example (The rack space)**.** A *rack* is a set R with a binary operation written a^b such that $a \mapsto a^b$ is a bijection for all $b \in R$ and such that the *rack identity*

$$a^{bc} = a^{cb^c}$$

holds for all $a, b, c \in R$. (Here we use the conventions for order of operations derived from exponentiation in arithmetic. Thus, abc means $(a^b)^c$ and a^{cb^c} means $a^{c(b^c)}$.)

For examples of racks see [8].

If R is a rack, the *rack space* is the set $\square$-set denoted BR and defined by $BR_n = R^n$ (the n-fold cartesian product of R with itself).

$$\partial_i^0(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$
$$\partial_i^1(x_1, \dots, x_n) = ((x_1)^{x_i}, \dots, (x_{i-1})^{x_i}, x_{i+1}, \dots, x_n) \quad \text{for } 1 \leq i \leq n.$$

More geometrically, we can think of BR as the $\square$-set with one vertex, with (oriented) edges labelled by rack elements and with squares which can be pictured as part of a link diagram with arcs labelled by a, b and a^b (Figure 1).

The higher dimensional cubes are determined by the squares; roughly speaking, a cube is determined by its 2-skeleton, for more details see [12, Example 1.4.3].

Notice that the rack space of the rack with one element has precisely one cube in each dimension. This is a description of the *trivial* $\square$-*set*.

Associated James complexes of a $\square$-set.

Projections. An $(n + k, k)$-*projection* is a function $\lambda\colon I^{n+k} \to I^k$ of the form

$$\lambda\colon (x_1, x_2, \dots, x_{n+k}) \mapsto (x_{i_1}, x_{i_2}, \dots, x_{i_k}), \tag{1.2}$$

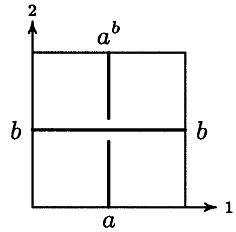


FIGURE 1. Diagram of a typical 2-cell of the rack space

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n+k$.

Let P_k^{n+k} denote the set of $(n+k, k)$-projections. Note that P_k^{n+k} is a set of size $\binom{n+k}{k}$.

Let $\lambda \in P_k^{n+k}$ and let $\mu\colon I^l \to I^k$, $l \leq k$ be a face map. The projection $\mu^\sharp(\lambda) \in P_l^{n+l}$ and the face map $\mu_\lambda\colon I^{n+1} \to I^{n+k}$ are defined uniquely by the following pull-back diagram:

$$\begin{array}{ccc} I^{n+l} & \xrightarrow{\mu_\lambda} & I^{n+k} \\ {\scriptstyle \mu^\sharp(\lambda)}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ I^l & \xrightarrow{\mu} & I^k \end{array} \tag{1.3}$$

Definition. Let C be a $\square$-set. The nth *associated James complex* of C, denoted $J^n(C)$ is the $\square$-set defined as follows. The k-cells are given by

$$J^n(C)_k = C_{n+k} \times P_k^{n+k}$$

and face maps by

$$\mu^*(x, \lambda) = (\mu_\lambda^*(x), \mu^\sharp(\lambda))$$

where $\mu\colon I^l \to I^k$, $l \leq k$ is a face map.

Notation. Let $\lambda \in P_k^{n+k}$ and $c \in C_{n+k}$. Then we shall use the notation c_λ for the k-cube $(c, \lambda) \in J^n(C)$. When necessary, we shall use the full notation $(\lambda_1, \dots, \lambda_n)$ for the projection λ (given by formula (1.2)) where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\{\lambda_1, \dots, \lambda_n\} = \{1, \dots, k+n\} - \{i_1, \dots, i_k\}$. In other words, we index cubes of J^n by the n directions (in order) which are collapsed by the defining projection.

Picture for James complexes. We think of $J^n(C)$ as comprising all the codimension n central subcubes of cubes of C. For example, a 3-cube c of C gives rise to the three 2-cubes of $J^1(C)$ which are illustrated in Figure 2.

In Figure 2 we have used full notation for projections. Thus, for example, $c_{(2)}$ corresponds to the projection $(x_1, x_2, x_3) \mapsto (x_1, x_3)$, ($x_2$ being collapsed).

This picture can be made more precise by considering the section $s_\lambda\colon I^k \to I^{n+k}$ of λ given by

$$s_\lambda(x_1, x_2, \dots, x_k) = (\tfrac{1}{2}, \dots, \tfrac{1}{2}, x_1, \tfrac{1}{2}, \dots, \tfrac{1}{2}, x_2, \tfrac{1}{2}, \dots, \tfrac{1}{2}, x_k, \tfrac{1}{2}, \dots)$$

where the non-constant coordinates are in places $i_1, i_2, \dots, i_k$ and λ is given by (1.2).

For example, in the picture the image of s_λ where $\lambda\colon (x_1, x_2, x_3) \mapsto (x_1, x_3)$ is the 2-cube labelled $c_{(2)}$.

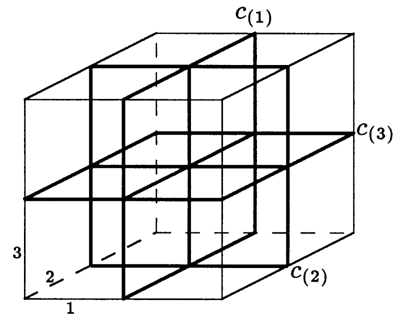


FIGURE 2.

Now the commuting diagram (1.3) which defines the face maps implies that the s_λ's are compatible with faces and hence they fit together to define a map

$$p_n\colon |J^n(C)| \to |C| \quad \text{given by} \quad p_n[c_\lambda, t] = [c, s_\lambda(t)].$$

Here $|\cdot|$ denotes realisation as usual and $|c_\lambda, t|$ denotes the equivalence class of $(c_\lambda, t) \in J^n(C)_k \times I^k$. In the next section we will see that p_n is a mock bundle projection.

2. Mock bundles and transversality

In this section we define mock bundles over smooth CW complexes, which include $\square$-sets, and prove a transversality theorem for $\square$-sets. This material is similar to material in [1, Chapters 2 and 7]. However, [1] is set entirely in the PL category and deals only with transversality with respect to a transverse CW complex. Here we shall need to extend the work to the smooth category and prove transversality with respect to a $\square$-set (which is not quite a transverse CW complex). However, many of the proofs are similar to proofs in [1] and, therefore, we omit details when appropriate. Similar material can also be found in [6] and [24].

The main technicalities concern manifolds with corners, which is where we start. We shall use *smooth* to mean C^∞.

Definition (Manifold with corners)**.** For background material on smooth manifolds with corners, see Cerf or Douady [4, 5]. In particular, these references contain an appropriate version of the tubular neighbourhood theorem. There is a uniqueness theorem for these tubular neighbourhoods. The proof can be obtained by adapting the usual uniqueness proof.

A *smooth n-manifold with corners* M is a space modelled on $\mathbb{E}^n$ where $\mathbb{E} = [0, \infty)$. In other words, M is equipped with a maximal atlas of charts from open subsets of $\mathbb{E}^n$ such that overlap maps are smooth. There is a natural stratification for such a manifold. Define the *index* of a point p to be the minimum q such that a neighbourhood of p in M is diffeomorphic to an open subset of $\mathbb{E}^q \times \mathbb{R}^{n-q}$. The *stratum of index* q denoted $M_{(q)}$ comprises all points of index q. Note that the dimension of $M_{(q)}$ is $n - q$ and that $M_{(0)}$ is the interior of M, $M_{(1)}$ is an open codimension 1 subset of ∂M and, in general, $M_{(i)}$ is an $(n - i)$-manifold lying in the closure of $M_{(i-1)}$.

Examples. I^n and $\mathbb{E}^n$ are manifolds with corners. If M and Q are manifolds with corners, then so is $M \times Q$.

Definition (Maps of manifolds with corners)**.** Let M and Q be manifolds with corners, then a *stratified map* is a smooth map $f\colon M \to Q$ such that $f(M_{(q)}) \subset Q_{(q)}$ for each q. This is the analogue for manifolds with corners of a proper map for manifolds with boundary.

An *embedding of manifolds with corners* is a smooth embedding $i\colon M \subset Q$ such that the pair is locally like the inclusions of $\mathbb{E}^p \times \mathbb{E}^t$ in $\mathbb{E}^p \times \mathbb{R}^t \times \mathbb{R}^s$ where $p + t = \dim(M)$ and $p + t + s = \dim(Q)$. Thus, an embedding of manifolds with corners which is proper (i.e., takes boundary to boundary and interior to interior) must be a stratified embedding. However, in general, an embedding of manifolds with corners allows corners on M not at corners of Q; see the examples below.

A *face map of manifolds with corners* $\lambda\colon M \to Q$ is a smooth embedding which is a proper map of topological spaces (preimage of compact is compact) such that $\lambda(M_{(q)}) \subset Q_{(t+q)}$ for each q where $t = \dim(Q) - \dim(M)$. Thus a face map is a diffeomorphism of M onto a union of components of strata of Q.

Examples. The map $s_\lambda\colon I^k \to I^{n+k}$ of λ given by

$$s_\lambda(x_1, x_2, \dots, x_k) = (\tfrac{1}{2}, \dots, \tfrac{1}{2}, x_1, \tfrac{1}{2}, \dots, \tfrac{1}{2}, x_2, \tfrac{1}{2}, \dots, \tfrac{1}{2}, x_k, \tfrac{1}{2}, \dots)$$

(see the end of Section 1) is a stratified embedding.

If M is a manifold with corners and Q is a smooth manifold without boundary, then the projection $M \times Q \to M$ is a stratified embedding.

The inclusions $I^n \subset \mathbb{E}^n \subset \mathbb{R}^n$ are embeddings of manifolds with corners.

A face map (in the sense of Section 1) $\lambda\colon I^k \to I^n$ is a face map of manifolds with corners.

Definition (Smooth CW complex)**.** We refer to [25, pages 13–14] for the definition and basic properties of convex linear cells in $\mathbb{R}^n$ (which we shall abbreviate to convex cells). Convex cells have well-defined faces and if e' is a face of e we write $e' < e$. Convex cells are smooth manifolds with corners. A smooth face map between convex cells (in the sense of manifolds with corners) is the same as a diffeomorphism onto a face.

A *smooth CW complex* is a collection of convex cells glued by smooth face maps. More precisely, it comprises a CW complex W and for each cell $c \in W$ a preferred characteristic map $\chi_c\colon e_c \to W$ where e_c is a convex cell such that for each face $e' < e_c$ there is a cell $d \in W$ with preferred characteristic map $\chi_d\colon e_d \to W$ and a diffeomorphism $\mu\colon e_d \to e'$ such that the following diagram commutes:

$$\begin{tikzcd} e_d \arrow[r, "\chi_d"] \arrow[d, "\mu"] & W \\ e' \arrow[r, "\text{inc.}"] & e_c \arrow[u, "\chi_c"'] \end{tikzcd}$$

Examples. A (realised) $\square$-set or Δ-set, a simplicial complex or a convex linear cell complex are all examples of smooth CW complexes.

We say that a smooth CW complex W gives rise to a *smooth decomposition* of a smooth manifold M (possibly with corners) if there is a homeomorphism $h\colon W \to M$ such that $h \circ \chi_c\colon e_c \to M$ is an embedding of manifolds with corners for each cell $c \in W$. Usually we identify W and M via h in this situation and say that W is a smooth decomposition of M.

Definition 2.1 (Smooth mock bundle)**.** Let W be a smooth CW complex. A *mock bundle ξ over W* of codimension q (denoted ξ^q/W) comprises a *total space* E_ξ and a *projection* $p_\xi\colon E_\xi \to W$ with the following property.[1]

Let c be an n-cell of W with characteristic map $\chi_c\colon e_c \to W$, then there is a smooth manifold with corners B_c of dimension $n-q$ called the *block* over c and a stratified map $p_c\colon B_c \to e_c$ and a map $b_c\colon B_c \to E_\xi$ such that the following diagram is a pull-back:

$$\begin{tikzcd} B_c \arrow[r, "b_c"] \arrow[d, "p_c"] & E_\xi \arrow[d, "p_\xi"] \\ e_c \arrow[r, "\chi_x"] & W \end{tikzcd}$$

Amalgamation lemma 2.2. *Suppose that ξ^q/W is a smooth mock bundle and that W is a smooth decomposition of a smooth manifold with corners M^m. Then E_ξ can be given the structure of a smooth manifold with corners of dimension $m-q$ such that $p_\xi\colon E_\xi \to M$ is ε-homotopic through mock bundle projections to a stratified map.*

Proof. By standard embedding theorems we may assume that each block B_c of ξ is a stratified submanifold of $e_c \times \mathbb{R}^N$ for some N and that these embeddings fit together to give an embedding of E_ξ in $M \times \mathbb{R}^N$ such that p_ξ is the restriction of projection on the first coordinate. We shall isotope this embedding so that E_ξ becomes a smooth submanifold of $M \times \mathbb{R}^N$. We work inductively over the skeleta of W. Assume inductively that this isotopy has already been carried out over a neighbourhood of the $(i-1)$-skeleton of W.

Now fix attention on the interior of a particular block B_c° which is the subset of E_ξ lying over the interior c° of an i-cell of W. The standard tubular neighbourhood theorem applied to the submanifold c° of $|W| = M$ yields a tubular neighbourhood λ of c° in $|W| = M$ formed by tubular neighbourhoods in each of the incident cells. Using the tubular neighbourhood theorem for manifolds with corners we can construct a (non-smooth) tubular neighbourhood μ of B_c° in E_ξ formed by (smooth) tubular neighbourhoods in each of the incident blocks, which extends to a tubular neighbourhood μ^+ on $c^\circ \times \mathbb{R}^N$ in $M \times \mathbb{R}^N$. By inductively applying uniqueness we can deform μ^+ near B_c° to $\lambda \times \mathbb{R}^N$ by an ε-isotopy. This carries E_ξ to a smooth submanifold of $M \times \mathbb{R}^N$ so that projection on M is a stratified map and, moreover, the isotopy determines a homotopy through mock bundle projections of the projection on $|W|$. By induction E_ξ is already smooth near ∂B and we can keep a neighbourhood of ∂B fixed through the isotopy. Do this for each i-cell of W to complete the induction step. □

Definition (Maps of smooth CW complexes)**.** A *linear projection* of convex cells is a surjective map $f\colon e_1 \to e_2$, where e_1, e_2 are convex cells, which is the restriction of an affine map, and such that $f(e') < e_2$ for each face $e' < e_1$. Examples include simplicial maps of one simplex onto another, projections $I^{n+k} \to I^n$ and projections of the form $d \times e \to e$ where d, e are convex cells.

[1]Note that the notation used here for dimension of a mock bundle, namely that q is *codimension*, is the negative of that used in [1] where ξ^q/W meant a mock bundle of *fibre dimension q*, i.e., codimension $-q$. The notation used here is consistent with the usual convention for cohomology.

A *smooth projection* of convex cells is a linear projection composed with a diffeomorphism of e_1.

A *smooth map* $f\colon W \to Z$ *of smooth CW complexes* is a map such that for each cell $c \in W$ there is a cell $d \in Z$ and a smooth projection $\phi\colon e_c \to e_d$ such that the following diagram commutes:

$$\begin{tikzcd} e_c \arrow[r, "\phi"] \arrow[d, "\chi_c"] & e_d \arrow[d, "\chi_d"] \\ W \arrow[r, "f"] & Z \end{tikzcd}$$

Examples include $\Box$-maps, Δ-maps, simplicial maps and projections $W \times Z \to W$, where W, Z are any two smooth CW complexes.

The following lemma follows from the definitions.

Pull-back lemma 2.3. *Let ξ^q/Z be a smooth mock bundle and $f\colon W \to Z$ a smooth map of smooth CW complexes, then the pull-back $f^*(E_\xi) \to W$ is a mock bundle of the same codimension (denoted $f^*(\xi^q)/W$).*

Properties of mock bundles. We shall summarise properties of mock bundles. The details of all the results stated here are analogous to the similar results for PL mock bundles over cell complexes proved in [1].

The set of cobordism classes of mock bundles with base a smooth CW complex forms an abelian group (under disjoint union of total spaces) and there is a relative group (the total space is empty over the subcomplex). This all fits together with the pull-back construction to define a cohomology theory which can be identified with smooth cobordism (classified by the Thom spectrum MO).

If the base is a manifold, then the amalgamation lemma defines a map from q-cobordism to $(n-q)$-bordism. This map is the Poincaré duality isomorphism.

Given two mock bundles ξ^q/W and η^r/W we can define the mock bundle $(\xi \cup \eta)^{q+r}/W$ in various equivalent ways analogous to the Whitney sum of bundles. We can pull one bundle back over the total space of the other and then compose. This is equivalent to making the projection of the first bundle transverse to the projection of the section and then pulling back. We can take the external product $\xi \times \eta/W \times W$ and restrict to the diagonal. These equivalent constructions define the cup product in cobordism. If the base is a manifold, then pull-back (or transversality) defines the cap product which coincides under Poincaré duality with the cup product.

Mock bundles can be generalised and extended in a number of ways. The simplest is to use orientation. If each block (and cell) is oriented in a compatible way (cf. [1, page 82]), then the resulting theory of oriented mock bundles defines oriented cobordism (classified by MSO). More generally, we can consider restrictions on the stable normal bundle of blocks and this yields the corresponding cobordism theory. A particular example of relevance here is the case when blocks are stably framed manifolds; in this case the resulting theory is stable cobordism classified by the sphere spectrum $\mathbb{S}$. By considering manifolds with singularities, the resulting theory can be further generalised and such a mock bundle theory can represent the cohomology theory corresponding to an arbitrary spectrum [1, Chapter 7]. The corresponding homology theory is represented by the bordism theory given by using manifolds with the same allowed singularities. Coefficients and sheaves of coefficients can also be defined geometrically (see [1, Chapters 3 and 6]).

Key example (James bundles)**.** At the end of the last section we defined a projection

$$p_n\colon |J^n(C)| \to |C|$$

where $J^n(C)$ is the nth associated James complex of the $\square$-set C.

Now if we choose a particular $(n+k)$-cell σ of C, then the pull-back of p_n over I^{n+k} (by the characteristic map for σ) is a k-manifold (in fact, it is the $\binom{n+k}{k}$ copies of I^k corresponding to the elements of P_k^{n+k}). Therefore, p_n is the projection of a mock bundle of codimension n, which we shall call the *nth James bundle* of C denoted $\zeta^n(C)$.

Definition 2.4 (Embedded mock bundle)**.** Let W be a smooth CW complex and Q a smooth manifold without boundary. An *embedded mock bundle* in $W \times Q$ is defined to be a mock bundle ξ/W with an embedding $E_\xi \subset W \times Q$ such that p_ξ is the restriction of the projection $W \times Q \to W$ and such that for each cell $c \in W$ the induced embedding $B_c \subset e_c \times Q$ is a stratified embedding. The proof of the amalgamation lemma (with $\mathbb{R}^n$ replaced by Q) implies that if W is a smooth decomposition of a manifold, then E_ξ can be ε-isotoped to a smooth submanifold of $W \times Q$ so that projection on W is still a mock bundle projection.

We shall be particularly concerned with the case when Q is $\mathbb{R}^t$ for some t and each block is framed in $e_c \times \mathbb{R}^t$. The theory defined by mock bundles of this type is unstable cohomotopy. See, in particular, [12, 3.6 and 5.1].

Key example (Embedding the James bundle in $|C| \times \mathbb{R}$)**.** Let C be a $\square$-set. The James bundles can be embedded in $|C| \times \mathbb{R}$. This is done by ordering the cubes of $J^n(C)$ over a particular cube of C and lifting in that order. Recall that the k-cubes of $J^n(C)$ lying over a $(k+n)$-cube are indexed by projections $\lambda = (i_1, \ldots, i_n) \in P_n^{n+k}$. These may be ordered lexicographically. The lexicographic order is compatible with face maps and can be used to define the required embedding by induction on dimension of cells of C as follows.

Suppose inductively that the embedding has been defined over cells of C of dimension $\leq k+n-1$.

Consider a $(k+n)$-cube $c \in C$ with characteristic map $\chi_c\colon I^{k+n} \to |C|$. Pulling the embedding back (where it is already defined) over χ_c gives an embedding of $\zeta^n(\partial I^{k+n})$ in $\partial I^{k+n} \times \mathbb{R}$. Now embed the centres of the k-cubes of $J^n(I^{k+n})$ at $(\frac{1}{2}, \ldots, \frac{1}{2}) \times r_\lambda$ where r_λ are real numbers for $\lambda \in P_n^{n+k}$ chosen to increase strictly corresponding to the lexicographic order on P_n^{n+k}.

Now embed each k-cube of $J^n(I^{k+n})$ as the cone on its (already embedded) boundary. Finally, smooth the resulting embedding and push it forward to $|C| \times \mathbb{R}$ using $\chi_c \times \mathrm{id}$.

Precise smooth formulae for this embedding can be found in [12, Section 3] using a bump function.

The embedding is in fact framed. This can be seen as follows. Each k-cube $p_n(c, \lambda)$, where $\lambda = (i_1, \ldots, i_n)$, of $J^n(I^{k+n})$ is framed in I^{k+n} by the n vectors parallel to directions $i_1, \ldots, i_n$. These lift to parallel vectors in $I^{n+k} \times \mathbb{R}$ and the framing is completed by the vector parallel to the positive $\mathbb{R}$ direction (vertically up). This framing is compatible with faces and defines a framing of $\zeta^n(C)$ in $|C| \times \mathbb{R}$. The formulae in [12, Section 3] also give formulae for the framing.

For the special case $n = 1$ the map of $\zeta^n(C)$ to $\mathbb{R}$ can be simply described: the centre of $c_{(c_k)}$ is mapped to k. This determines a map, linear on simplexes of

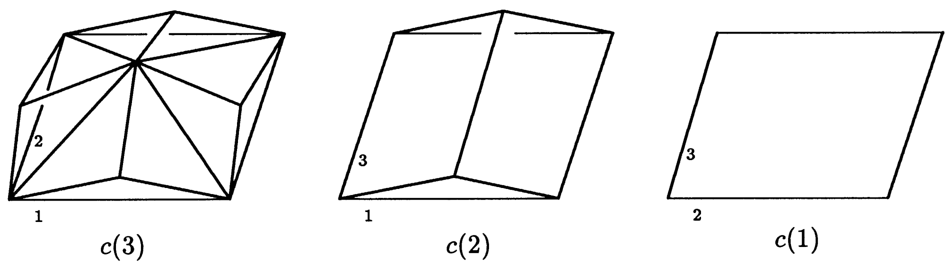


FIGURE 3.

$Sd_\Delta\zeta^1(C)$, to $\mathbb{R}$. It follows that the centre of $\partial_i^\epsilon c_{(k)}$ is mapped to k if $i \geq k$ and to $k-1$ if $i < k$.

In Figure 2 we illustrated $J^1(C)$ for a 3-cube $c \in C$. The embeddings in $|C| \times \mathbb{R}$ (before smoothing) above each of the three 2-cubes are illustrated in Figure 3.

We finish this section with a discussion of transversality with respect to a $\square$-set, which will be important for the main classification theorems of Section 4. A similar treatment can be given for any smooth CW complex, but we shall not need this in this paper.

Transversality.

Definition (Transverse map to a $\square$-set)**.** Let M be a smooth manifold (possibly with boundary) of dimension m and C a $\square$-set. Let c be an n-cell of C with characteristic map $\chi_c\colon I^n \to |C|$ denote $\chi_c(I^n - \partial I^n)$ by c° (the interior of c) and $\chi_c(\frac{1}{2}, \ldots, \frac{1}{2})$ by $\hat{c}$ (the centre of c).

Let $f\colon M \to |C|$ be a map. Let M_c denote the closure of $f^{-1}(c^\circ)$ and N_c the closure of $f^{-1}(\hat{c})$.

We say f is *transverse* to c if M_c is a smooth m-manifold with corners embedded (as a manifold with corners) in M and equipped with a diffeomorphism $\iota_c\colon N_c \times I^n \to M_c$ such that the following diagram commutes:

$$\begin{tikzcd} N_c \times I^n \arrow[r, "\iota_c"] \arrow[d, "p_2"] & M_c \arrow[d, "f"] \\ I^n \arrow[r, "\chi_c"] & {|C|} \end{tikzcd}$$

where p_2 denotes projection on the second coordinate. Thus, N_c is a framed submanifold (with corners) of codimension n framed by copies of I^n on each of which f is the characteristic map for c.

A map $f\colon M \to |C|$ is *transverse* if it is transverse to each cell $c \in C$ and the framings are compatible with face maps in the following sense. Let $\lambda\colon I^q \to I^n$ be a face map and $d = \lambda^*(c)$. Then there is a face map (of manifolds with corners) $\lambda_c^*\colon N_c \to N_d$ and the following diagram commutes:

$$\begin{tikzcd} N_c \times I^q \arrow[r, "\mathrm{id} \times \lambda"] \arrow[d, "\lambda_c^* \times \mathrm{id}"] & N_c \times I^n \arrow[d, "i_c"] \\ N_d \times I^q \arrow[r, "i_d"] & M \end{tikzcd}$$

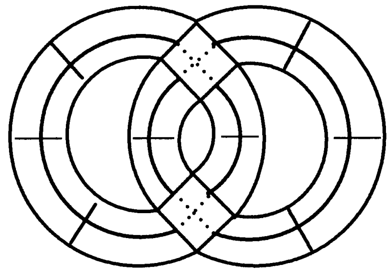


FIGURE 4.

where $i_c = \text{inc.} \circ \iota_c$, using the notation established above. The last condition can be summarised by saying that M is the realisation of a $\square$-space and f is the realisation of a $\square$-map.

To see this, define a $\square$-space by $X_n = \coprod_{c \in C_n} N_c$ and $\lambda^* = \coprod \lambda_c^*$. Then the diffeomorphisms ι_c define a homeomorphism $\iota\colon |X| \to M$ and if we identify M with $|X|$ via ι, then the commuting diagrams above imply that f is the realisation of a $\square$-map $|X| \to |C|$.

Remark. The framing compatibility condition in the definition of a transverse map is unnecessary. If $f\colon M \to |C|$ is transverse to each cell of C, then the framings can, in fact, be changed to become compatible (without altering f). However, the full definition is the one that we shall need in practice.

Ellucidation. To help the reader understand the (somewhat complicated) concept of a transverse map to a $\square$-set we shall describe transversality for maps of closed surfaces and 3-manifolds (possibly with boundary) into C.

A transverse map of a closed surface Σ into a $\square$-set C meets only the 2-skeleton of C. The pull-backs of the squares of C are a number of disjoint little squares in Σ (each of which can be identified with the standard square I^2 and maps by a characteristic map to a 2-cell of C). The pull-backs of the 1-cells are a number of bicollared 1-manifolds which are either bicollared closed curves or are attached to edges of the little squares (and each edge of each square is used in this way). Each bicollar line can be identified with I^1 and is mapped by f to a 1-cell of C (by the characteristic map for that cell). Finally, each component of $\Sigma - \{\text{little squares and collared 1-manifolds}\}$ is mapped to a vertex of C.

Thus, we can think of the transverse map as defining a thickened diagram of self-transverse curves (with the squares at double points). This is illustrated in Figure 4. We shall explore the connection between transverse maps and diagrams in the next section.

A transverse map of a closed 3-manifold into a $\square$-set C meets only the 3-skeleton of C. The pull-back of the 3-cubes are a number of little cubes each of which can be identified with I^3 and which map onto 3-cubes of C by characteristic maps. The pull-back of squares are framed 1-manifolds (framed by a copy of the standard square) and such that each such square is mapped onto a square of C by a characteristic map. These framed 1-manifolds are attached to the square faces of the little cubes at their boundaries. The pull-back of edges are framed sheets (framed by copies of I^1 and mapped by characteristic maps to edges of C). The edges of the sheets are attached to edges of the framed 1-manifold (i.e., along 1-manifold

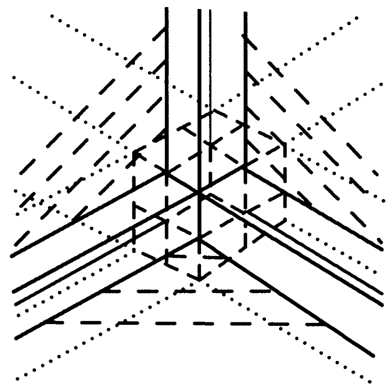


FIGURE 5.

$\times$ edge of square) and the two framings are required to be the same here (this is the framing compatibility condition in this case). The remainder of M is then a 3-manifold (with corners) each component of which is mapped to a vertex of C. A general view near one of the little cubes is illustrated in Figure 5.

For a 3-manifold M with boundary, transversality has a similar description. In this case the little cubes are all in the interior of M, $f|\partial M$ is a transverse map of a surface in C and the framed 1-manifolds can terminate at little squares in ∂M as well as faces of little cubes.

Theorem 2.5 (Transversality for $\square$-sets)**.** *Let M be a smooth manifold* (*possibly with boundary*) *and C a $\square$-set. Let $f\colon M \to |C|$ be a map. Then f is homotopic to a transverse map.*

If $f|\partial M$ is already transverse, then the homotopy can be assumed to keep $f|\partial M$ fixed.

Proof. We shall first prove the theorem in the case that M is a closed surface Σ, as this case contains all the ideas for the general case.

We start by using standard cellular approximation techniques to homotope f to meet only the 2-skeleton. Next, we make f transverse to the centres of the squares of C. The result is that f maps a number of small squares in Σ diffeomorphically onto neighbourhoods of centres of squares in C. By radial homotopies, we can assume that each of these small squares in fact maps onto the whole of a square in C and that the rest of Σ now maps to the 1-skeleton. It is clear how to identify each little square with I^2 so that f maps each by a characteristic map. Now let Σ' denote the closure of $\Sigma - \{\text{small squares}\}$. Then Σ' is a surface with boundary (with corners) and $f|\partial\Sigma'$ is transverse to the centres of the edges of C. By relative transversality and further radial homotopies, we can homotope f rel $\partial\Sigma'$ so that the preimages of the centres of the edges of C are framed 1-manifolds in Σ' with framing lines mapped onto the relevant edges of C. Moreover, we can identify each framing line with I^1 so that it is mapped by a characteristic map. If Σ_0 now denotes the closure of $\Sigma - \{\text{small squares and framed 1-manifolds}\}$, then Σ_0 is mapped to the 0-skeleton, i.e., each component of Σ_0 is mapped to a vertex of C. The map f is now transverse.

For the general case of an n-manifold (perhaps with boundary), we can assume inductively that $f|\partial M$ is already transverse and uses cellular approximation rel ∂M to ensure that f meets only the n-skeleton. We next make f transverse to the centres of n-cells. By radial homotopies as in the 2-dimensional case we can assume that the closure of the preimage of the interiors of the n-cells denoted M_n is a collection of disjoint little n-cubes in $M-\partial M$ each of which can be identified with I^n and is mapped by a characteristic map, i.e., f is now transverse to the n-cells of C. Now let M' be the closure of $M-M_n$. Then $f|\partial M'$ is transverse to the $(n-1)$-cells of C. By relative transversality and radial homotopies we can homotope f rel $\partial M'$ to be transverse to the $(n-1)$-cells. We then proceed by downward induction to complete the construction of a transverse map homotopic to f rel ∂M. Notice that the process produces compatible framings automatically. $\square$

Pulling back mock bundles by transverse maps. Now suppose that $f\colon M^m \to |C|$ is a transverse map and ξ^q/C is a mock bundle. Then by choosing smooth CW decompositions of each manifold N_c so that the face maps λ_c^* are inclusions of subcomplexes (notation from the definition of a transverse map, above), then the product structure on M_c for each $c \in C$ defines a smooth decomposition of M so that f is a smooth map (of smooth CW complexes). It follows from 2.2 and 2.3 that $f^*(\xi)$ is a mock bundle of codimension q over M and that $E(f^*(\xi))$ is a smooth manifold of dimension $m-q$ representing the Poincaré dual to $f^*(\xi)$. Moreover, if ξ is embedded in $|C| \times Q$, then $E(f^*(\xi))$ is ε-isotopic to a smooth submanifold of $M \times Q$.

The case when ξ is a James bundle ζ^i of C will be particularly important for the rest of the paper. Let P_i denote $E(f^*(\zeta^i))$. Then P_i is a smooth manifold of dimension $m-i$ which can be assumed to be embedded smoothly in $M \times \mathbb{R}$. Moreover, we can see from construction that the image of P_1 in M is a framed immersed self-transverse submanifold V of M of codimension 1. (This is illustrated for the case $m=2$ in Figure 4 above.) Moreover, the images of P_i are the i-tuple points of V and this is illustrated in Figure 5. In this figure the image of P_2 is the immersed 1-manifold defined by the double lines and P_3 is the 0-manifold of triple points.

The choice of terminology is explained in [12, Section 3] where James bundles are related to classical James-Hopf invariants. There is then a connection with the results of [20] which also relate generalised James-Hopf invariants to multiple points of immersions; see [12, Remark 3.7]. In the next section we shall establish the connection with links and diagrams suggested by Figures 4 and 5.

3. Links and diagrams

In this section we use the transversality theorem proved above to deduce the main classification theorem stated at the start of the paper, together with several related classification results.

We start by defining diagrams in arbitrary dimensions. First we need the concept of a self-transverse immersion. Let $\boxplus^p$ be the p-cube I^p together with the p hyperplanes $x_i = \frac{1}{2}$, $i = 1, \dots, p$, and let T^p denote the union of the p hyperplanes. Then $\boxplus^p \times I^{n-p}$ consists of an n-cube with p central hyperplanes meeting in an $n-p$-dimensional subspace called the *core*. An immersed smooth manifold M of dimension $n-1$ in a manifold Q of dimension n is called *self-transverse* if each point x of M has a neighbourhood in Q like $\boxplus^p \times I^{n-p}$ in which the point x corresponds to an interior point of the core and in which $T^p \times I^{n-p}$ corresponds to the image of M. The integer p is called the *index* of x and the set of points of index p is called the *stratum of index p*. The stratum of index one is locally embedded and the closures of components of index one points will be called the *sheets* of the immersed manifold M. Sheets can be locally continued through points of higher index. At a point of index p there are locally p such extended sheets which meet in a manifold of codimension p.

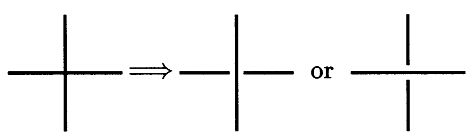
Notation. We write $M \ltimes Q$ if M is a self-transverse immersed submanifold of codimension 1 of Q.

Remark. Any immersed manifold of codimension 1 can be regularly homotoped so that it is self-transverse (see Lashof and Smale [22]).

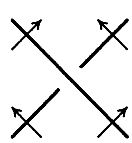
Definition ((Framed) diagram)**.** A *diagram D* in an n-manifold Q is a framed immersed self-transverse submanifold M such that the sheets are locally totally ordered and this ordering is preserved in a neighbourhood of points of higher index. The ordering is thought of as "vertical" and we speak of a sheet being "above" another if it follows in the order.

We call the components of $Q - M$ the *regions* of the diagram and also refer to these as the stratum of index 0 of the diagram. In general, the stratum of index p of D is the stratum of index p of M as described above.

Example 1 (Diagram on a surface)**.** A diagram D on a closed surface Σ is a familiar concept. It comprises a collection of framed immersed circles in general position Σ such that at each crossing one of the components is locally regarded as the overcrossing curve and the other as the undercrossing curve.

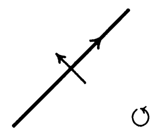


The framing can be pictured as a transverse arrow for each arc of the diagram, the direction of which is preserved through crossings.



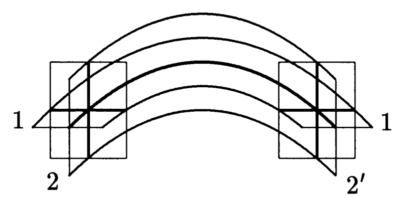
The component of $\Sigma - D$ are the *regions* of the diagram and we call the components of $D - \{\text{double points}\}$ the *arcs* of D. These are what we called sheets above.

If the surface is oriented, then the framing determines an orientation on the arcs of the diagram (and conversely) by the left-hand rule illustrated:



This framing is usually called the "blackboard framing".

Example 2 (Diagram in a 3-manifold)**.** A diagram in a closed 3-manifold is a self-transverse immersed surface (i.e., transverse double curves and triple points) equipped with a compatible ordering of sheets at double curves and triple points. Compatible means that the ordering of sheets at the triple points restricts to give the ordering at adjacent double arcs. It follows that the ordering of sheets at double arcs is preserved in a neighbourhood of a triple point. So, for example, if in the following diagram sheet 2 is above sheet 1, then sheet $2'$ is above sheet $1'$.



Remarks. (1) If $M \ltimes Q$ and both M and Q are oriented, then the orientations determine a canonical framing of M in Q analogous to the blackboard framing for a diagram on a surface.

(2) In the case that M or Q have boundary, we assume that diagrams are *proper*, i.e., that M meets ∂Q in its boundary (which thus defines a diagram in ∂Q).

Framed embeddings and diagrams. A diagram $D\colon M \ltimes Q$ determines a framed embedding of M in $Q\times\mathbb{R}$ (i.e., a *link*) by lifting the sheets at multiple points in the $\mathbb{R}$ direction in the order given. In other words, a sheet "above" another sheet is lifted higher. The framing is given by using the given framing of the diagram and taking vertically upwards as the last framing vector. It is clear that the resulting link, called the *lift of* D is well-defined up to an isotopy which moves points vertically.

There is a converse to this process. A framed link determines a diagram. This is a consequence of the compression theorem. Let M be a framed embedding in $Q\times\mathbb{R}$. We call M *horizontal* if the last framing vector is always vertically up. Note that a horizontal embedding covers an immersion in Q.

Compression Theorem. *Let M be a framed embedding in $Q \times \mathbb{R}$. Then M is isotopic* (*by a small isotopy*) *to a horizontal embedding; moreover, if M is already horizontal in the neighbourhood of some compact set, then the isotopy can be assumed fixed on that compact set.*

The theorem follows from a deep result of Gromov [16]. Direct proofs are given in [26].

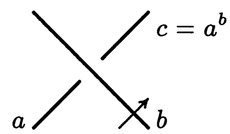
Corollary 3.1. *Any framed link of codimension* 2 *is isotopic to the lift of some diagram.*

Proof. This follows at once from the Compression Theorem and Lashof-Smale [22]. □

There are relative versions of both results used in the corollary so that, for example, if M and Q have boundary and M is embedded properly in $Q \times \mathbb{R}$ such that the embedding of the boundary is the lift of a diagram, then the diagram determined by M can be taken to extend the given diagram of the boundary.

The fundamental rack of a diagram and a link. For details here see [8, pages 369–375]. A diagram in $\mathbb{R}^2$ determines a *fundamental rack* by labelling the arcs by generators and reading a relation at each crossing:

(3.2)



More generally, any diagram determines a rack in a similar way. Components of index 1 of D are labelled by generators and a relation is read from each component of index 2 by the same rule as in the 2-dimensional case (think of a perpendicular slice); points of higher index are not used (cf. [8, Remark (2), page 375]).

(Note that there are really two versions of diagram (3.2) because the framing of the understring is immaterial.)

Notation. If D is a diagram, then we denote the fundamental rack of D defined as above by $\Gamma(D)$.

A framed codimension 2 embedding L also determines a *fundamental rack* denoted $\Gamma(L)$ (see the next definition for details); moreover, if D is a diagram in $\mathbb{R}^n$ and L is a lift in $\mathbb{R}^{n+1}$, then $\Gamma(D)$ can be naturally identified with $\Gamma(L)$ [8, Theorem 4.7 and Remark (2), page 375].

We need to interpret the rack $\Gamma(D)$ in the case that the diagram is not in $\mathbb{R}^n$. This case was not covered in [8].

Definition (The reduced fundamental rack)**.** Let $L\colon M \subset Q \times \mathbb{R}$ be a framed codimension two embedding (i.e., a link) and choose a basepoint $* \in Q \times \mathbb{R} - M$. Consider paths α from the parallel (framing) manifold of M to $*$ in $Q \times \mathbb{R} - M$. Recall that the fundamental rack $\Gamma(L)$ comprises the set of homotopy classes of these paths with the rack operation a^b, where $a = [\alpha]$, $b = [\beta]$, given by the class of the composition of α with the "frying pan" loop determined by β, namely $\overline{\beta} \circ \mu \circ \beta$ where μ is the meridian at the start of β [8, page 359]. To define the reduced fundamental rack $\overline{\Gamma}(L)$ we kill the action of $\pi_1(Q)$. More precisely, two paths α, β starting from the same point are equivalent if $\overline{\beta} \circ \alpha$ is a product of conjugates of elements of $\pi_1(Q)$ in $\pi_1(Q \times \mathbb{R} - M)$, where Q is identified with $Q \times 1$ and we assume that the link lies below level 1. This is extended to homotopy classes in the obvious way.

It can be checked that this is an equivalence relation and that the rack operation is defined on equivalence classes. The resulting rack is the *reduced fundamental rack* $\overline{\Gamma}(L)$.

There is a simple interpretation of the reduced rack. Replace $\mathbb{R}$ by $[-1, 1]$ (assume that the link lies between levels -1 and 1). Now define $\overline{Q}$ to be $Q \times [-1, 1]/Q \times 1$ (i.e., a copy of the cone on Q) based at the image of $Q \times 1$. Note that the definition of the fundamental rack does not use that the bigger manifold is actually a manifold. Thus, we can define the fundamental rack $\Gamma(M \subset \overline{Q})$. It is not hard to check that $\overline{\Gamma}(L) = \Gamma(M \subset \overline{Q})$.

Note that if Q is simply connected, then the reduced fundamental rack coincides with the usual fundamental rack.

Lemma 3.3. *Let $D\colon M \ltimes Q$ be a diagram and let L be the link given by lifting D to $Q \times \mathbb{R}$. Then $\Gamma(D)$ can be naturally identified with $\overline{\Gamma}(L)$.*

Proof. Using the interpretation of $\overline{\Gamma}(L)$ as $\Gamma(M \subset \overline{Q})$, the proof given in [8, pages 372–375] adapts with obvious changes. $\square$

Remark. A detailed proof for the analogous quandle case can be found in [18, Proposition 5.1].

Labelling diagrams by racks. Let D be a diagram. We say that D is *labelled* by a rack X if each component of the stratum of index 1 of D is labelled by an element of X with compatibility at strata of index 2 given by rule (3.2) on a perpendicular slice, where a, b, c now denote elements of X (rather than generators of $\Gamma(D)$) and $c = a^b$ in X.

There is a natural labelling of any diagram by its fundamental rack, which we call the *identity labelling*. More generally, we have the following result which follows immediately from definitions:

Lemma 3.4. *A labelling of a diagram D by a rack X is equivalent to a rack homomorphism $\Gamma(D) \to X$.* $\square$

The lemma implies that labelling is functorial in the sense that a rack homomorphism $X \to Y$ induces a labelling by Y, it also implies that labelling is really a property of the lift, not the diagram:

Corollary 3.5. *A labelling of a diagram by a rack X is equivalent to a rack homomorphism to X of the reduced fundamental rack of the lifted framed embedding.* $\square$

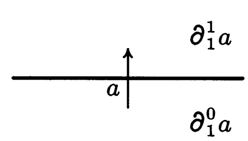
We often speak of a framed link having a *representation in X* to mean that the fundamental rack has a homomorphism to X.

Labelling diagrams by $\square$-sets. A diagram is *labelled by a $\square$-set C* if, for each p, each component of the stratum of index p is labelled by a p-cube of C with compatibility conditions. We shall explain these in detail for diagrams in surfaces and 3-manifolds. The general case is a straightforward extension.

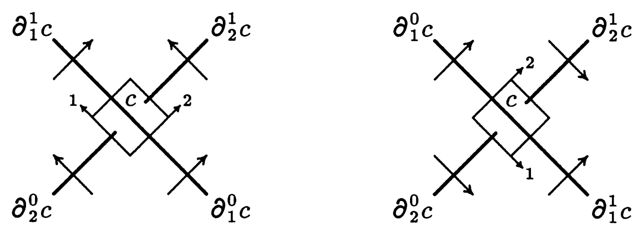
A diagram D on a surface is *labelled* by a $\square$-set C if:

(1) The regions are labelled by vertices of C.

(2) The arcs are labelled by edges of C compatibly with adjacent regions. This means that the edge labelling an arc α is attached (in C) to the two vertices labelling the adjacent regions. To decide which vertex labels which side, identify the edge with the transverse framing arrow:



(3) The double points are labelled by squares of C compatibility with adjacent arcs. This means that the four adjacent arcs are labelled by the four 1-faces of the square. The rule for determining which face labels which arc is this: position a copy of the standard square (i.e., I^2) at the double point with faces oriented correctly by the framing arrows and axis 1 parallel to the overcrossing. Now the four faces intersect the appropriate adjacent arcs, see the diagram below, where we have drawn both possible orientations for the square:



Let D be a diagram in a 3-manifold M. Call the components of the double curves minus triple points *double arcs*, the components of the surface minus the double curves *sheets*, and the components of M minus the surface *regions*.

A labelling of D by a $\square$-set C is a labelling of regions (resp. sheets, double arcs, triple points) by vertices (resp. edges, squares, 3-cubes) of C subject to compatibility conditions at sheets, double curves and triple points.

The compatibility conditions at sheets and double curves are the same as for a 2-dimensional diagram (imagine working in a transverse slice) whilst the compatibility condition at a triple point can be described as follows. Say the *positive* side of a sheet coincides with the head of its framing arrow. Suppose a triple point is labelled by $x \in C_3$. Suppose a nearby double curve is labelled by $y \in C_2$ and the missing sheet is number $i \in \{1, 2, 3\}$. Then $\partial_i^\epsilon x = y$ where $\epsilon = 1$ if the double curve is on the positive side of the missing sheet and $\epsilon = 0$ otherwise. (Notice that a small copy of the standard cube I^3 can be placed at a triple point by orienting axes according to framing of sheets and ordering axes so that axis i is perpendicular to the ith sheet. If this is done, then the small cube meets an adjacent double arc in the face given by the above labelling rules.)

These compatibility conditions extend to a general diagram in the obvious way. If a component of the stratum of index p is labelled by $c \in C_p$, then the neighbouring components of the index $(p-1)$ stratum are labelled by $\partial_i^\epsilon(c)$ with the rule for determining i and ϵ being analogous to the 3-dimensional case.

Remarks. (1) If $C = BX$, where X is a rack, then labelling in C is precisely the same as labelling in X. This is the same as labelling index 1 components by elements of X with the usual compatibility requirement at index 2 points (diagram (3.2)). Points of higher index play no part because a 3-cube of $C = BX$ is determined by its faces (see the discussion in section 1, the key example). Also, there is no need to label index 1 components explicitly since a square in BX is determined by its edges. The compatibility condition ensures that the required square exists.

(2) Every diagram has the *trivial* labelling, namely labelling by T, the trivial $\square$-set (with one cell of each dimension). This can also be regarded as labelling by the trivial rack (with one element).

(3) Labelling is functorial: Given a diagram labelled in C and a $\square$-map $f\colon C \to D$, then f transforms the labels to a labelling in D.

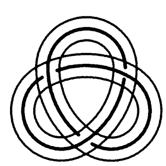
(4) There is also the concept of labelling by a *trunk* T; in other words, labelling by the nerve $\mathcal{N}T$ (see [10]). In this case regions are labelled by vertices and index 1 components by edges (between the vertices labelling adjacent regions) such that at index 2 components the adjacent index 1 components form a preferred square. As for racks, points of higher index play no part in the labelling.

(5) The case of labelling by the *action rack space* $B_Y X$ [10, Example 3.1.2] is worth describing in detail. This is the same as labelling the diagram by the rack X together with a *regional labelling* by Y. Here Y is a set on which X acts (see [10, above 1.4]). In other words, regions are labelled in Y compatibly with the labelling on sheets; if a region is labelled $a \in Y$ and a sheet labelled $b \in X$ is crossed (in the framing direction), then the region on the other side is labelled a^b (the action of b on a). An important special case is when $X = Y$ and the action is the rack operation and all labels lie in X. This type of labelling was used in [27], with X the three colour rack, to distinguish left and right trefoils. The space $B_X X$ is called the *extended rack space.*

Labelling and transversality. In section 2 we defined a transverse map of a manifold M in a $\square$-set C and at the end of the section we observed that such a map defines a framed self-transverse immersed submanifold V of M (see Figures 4, 5), which can be seen as the image of the pull-back of the first James bundle $\zeta^1(C)$. We also observed that $\zeta^1(C)$ embeds in $|C|\times\mathbb{R}$ and hence this immersed submanifold is covered by an embedding in $M \times \mathbb{R}$; in other words, it is a *diagram.* Moreover, this diagram is labelled by C in a natural way. Recall that each component of index p is surrounded by a p-cube bundle, the fibres of which are mapped to a p-cube of C. Label the component by this cube. It can readily be checked that this labelling is compatible; indeed, the definition of compatibility (above) was set up precisely in this way.

Thus, a transverse map *to* C determines a diagram labelled *by* C. We now describe the converse process which is given by the construction of a *neighbourhood system* for the diagram. To help understand the somewhat complicated construction, we shall first deal with the case $n = 2, 3$ in detail.

2-dimensional case. Suppose that we are given a diagram in a surface Σ labelled by C. We construct a neighbourhood system for the diagram by drawing little squares around the double points and continue to construct bi-collars around the arcs.



This determines a transverse map into C by mapping the regions outside the squares and bi-collars to the labelling vertex, collapsing the bi-collars onto fibres and mapping to the labelling edge and, finally, mapping the little squares to the labelling squares for the double points, using the orientations for edges and squares described in the definition of labelling by a $\square$-set (above).

It is clear that this construction is unique up to minor choices which only affect the neighbourhood system up to an ambient isotopy fixing the diagram setwise. To be precise, define two transverse maps of Σ in C to be *diagram isotopic* if they differ by an ambient isotopy fixing the pull-back diagram setwise, then we have a well-defined process for turning a labelled diagram into a diagram isotopy class of transverse maps.

3-dimensional case. Given a diagram in a 3-manifold M we construct a neighbourhood system as follows. We choose little 3-cubes around the triple points which meet neighbouring sheets in the three central 2-cubes. Each of these can be identified with I^3 in a canonical way using the ordering of sheets as described above. We call the portions of the double arcs outside these cubes, *reduced double arcs*. Next construct a trivial bundle with fibre a square around each reduced double arc such that each square meets neighbouring sheets in the central cross and which fits onto a face of the relevant 3-cube at boundary points (precisely how to construct these trivial bundles will be explained in Lemma 3.6 below). Finally, construct bi-collars around the reduced sheets (outside the square bundles) which fit onto edges of the squares. (See Figure 5 for a view of part of this construction.)

Now map the 3-cubes to the labelling 3-cube of C, collapse the square bundles onto a single square and map to the labelling square, and likewise collapse the bi-collars and map to labelling edges and finally map the reduced regions to labelling vertices. The result is the required transverse map to C.

The only element of choice in the construction was the choice of the neighbourhood system. In the next lemma we show how to choose this system using collaring arguments. Using uniqueness of collars we can then see that the system is unique up to ambient isotopy fixing the diagram setwise.

Lemma 3.6 (Constructing neighbourhood systems by collars)**.** *Let D be a diagram in a 3-manifold M. Then D has a neighbourhood system and any two such systems differ by an ambient isotopy of M fixing D setwise.*

Proof. Start by choosing a collar for each triple point in each double arc (i.e., an interval). Now concentrate on a particular triple point p and label the extended sheets near p $1, 2, 3$ (in the order given by the diagram) and use the label 12 for example for the extended double arc $1 \cap 2$, etc. The collar on p in 12 is a double interval J, say. Choose a bi-collar on J in 1 extending the chosen collar in 13. This defines a square S, say, in 1. Do the same for 12 in 2 and 13 in 3. We now have three mutually perpendicular squares. Complete the little cube at p by choosing a bi-collar on S in M extending the chosen (partially defined) collars on 12 in 2 and 13 in 3.

To define the square bundle on a reduced double arc α, say, choose bi-collars in both intersecting sheets (extending collars given by the little cubes on ∂J) and then extend one of these to a bi-collar on the total space of the other collar (extending the collars given by the squares over ∂J). Finally, construct the bi-collars over the reduced sheets extending the already constructed collars (given by the square bundles) over the boundary.

It is clear that a neighbourhood system defines all the above collars and the uniqueness part of the lemma now follows from uniqueness of collars. □

The general case. The extension of the case $n = 3$ to the general case is straightforward: a neighbourhood system for a diagram D labelled in C is constructed by choosing little n-cubes around each point of index n meeting nearby sheets in central $(n-1)$-cubes. Then the faces are extended to trivial $(n-1)$-cubes bundles around the reduced index $n-1$ strata and the process is completed by downward induction on index. Uniqueness is proved in the same way as the case $n = 3$. The neighbourhood system and the labelling in C determines a transverse map to C unique up to *diagram isotopy*, that is, up to an isotopy fixing the diagram setwise.

It is clear that the two processes: transverse map to labelled diagram and labelled diagram to transverse map (via neighbourhood system) are inverse and we can summarise this in the following result.

Proposition 3.7. *There is a bijection between labelled diagrams in M labelled by a $\square$-set C and diagram isotopy classes of transverse maps of M in C.*

The bijection is given by pulling back the first James bundle $\zeta^1(C)$. $\square$

Remarks. (1) Note that the lift of the diagram in $M \times \mathbb{R}$ is also obtained by pulling back the embedded first James bundle $\zeta^1(C) \subset C \times \mathbb{R}$.

(2) There is a relative version of the proposition for the case M has boundary: the restriction of the bijection to the boundary coincides with the bijection for the boundary.

In order to interpret π_n of a cubical set, we need a based version of the proposition. choose basepoint $* \in S^n$ and base vertex $* \in C_0$ and identify $S^n - \{*\}$ with $\mathbb{R}^n$.

Proposition 3.8. *There is a bijection between labelled diagrams in $\mathbb{R}^n$ labelled by a $\square$-set C such that the non-compact region is labelled by the vertex $* \in C_0$ and diagram isotopy classes of based transverse maps of S^n in C.*

The bijection is given by pulling back the first James bundle $\zeta^1(C)$. $\square$

The classification theorems. We now interpret homotopy classes of maps into C and in particular $\pi_n(C)$. To do this we shall need the following definition.

Definition (Cobordism of diagrams)**.** Diagrams D_0, D_1 in M are *cobordant* if there is a diagram D in $M \times I$ which meets $M \times \{0, 1\}$ in D_0, D_1, respectively. We call D the *cobordism* between D_0 and D_1. It can readily be checked that diagram cobordism is an equivalence relation.

There is a similar notion of cobordism for labelled diagrams and we denote the set of cobordism classes of diagrams in M labelled in a $\square$-set C by $\mathcal{D}(M, C)$.

Now let C be a $\square$-set with a basepoint $* \in C_0$. Define the set $\mathcal{D}(n, C)$ to be the set of labelled cobordism classes of diagrams in $\mathbb{R}^n$ (labelled by C) such that the non-compact region is labelled by the vertex $*$.

The set of diagrams in $\mathbb{R}^n$ has an addition given by disjoint union. To be precise, given diagrams D_1, D_2 choose copies in disjoint half spaces, then define $D_1 + D_2$ to be $D_1 \coprod D_2$. This addition is well-defined up to cobordism, is compatible with cobordism, and makes $\mathcal{D}(n, C)$ into an abelian group.

In the case when $C = BX$, a rack space, we abbreviate $\mathcal{D}(M, BX)$ and $\mathcal{D}(n, BX)$ to $\mathcal{D}(M, X)$ and $\mathcal{D}(n, X)$, respectively.

Theorem 3.9 (Classification of labelled diagrams)**.** *Let C be a $\square$-set. There is a natural bijection between the set of homotopy classes of maps $[M, |C|]$ and $\mathcal{D}(M, C)$. If C has basepoint $* \in C_0$, there is a natural isomorphism between $\pi_n(C)$ and $\mathcal{D}(n, C)$.*

The isomorphisms can be described as given by pulling back the first James bundle $\xi^1(C)$ which thus plays the rôle of classifying bundle for labelled diagrams.

Proof. By Proposition 3.7, a labelled diagram in M determines a (transverse) map of M in C. Similarly by a cobordism determines a homotopy. Thus, we have a function $\Phi\colon \mathcal{D}(M,C) \to [M,|C|]$. By transversality (Theorem 2.5) Φ is a bijection. For the second bijection we use Proposition 3.8 instead of 3.7. The rest of the theorem follows from definitions. □

We now specialise to the case when C is the rack space BX and labelling in C is the same as labelling in X and is a property of the link which covers the diagram. Note that BX is based at the unique 0-cell.

We need to define cobordism of links:

Definition (Cobordism of links)**.** We say that framed links L_0, L_1 in W are *cobordant* if there is a framed link L properly embedded in $W \times I$ which meets $W \times \{0,1\}$ in L_0, L_1, respectively.

Let X be a rack, then there is an analogous notion of cobordism of links with representation in X, namely a cobordism with a representation in X (i.e., a homomorphism of the fundamental rack in X) extending the given representations on the ends.

It can readily be checked that cobordism is an equivalence relation and we denote the set of cobordism classes of framed links in W with representation in X by $\mathcal{F}(W,X)$.

If $W = M \times \mathbb{R}$, then we can consider representations using *reduced* fundamental rack as defined earlier; we use the notation $\overline{\mathcal{F}}(M \times \mathbb{R}, X)$ for the set of framed links in $M \times \mathbb{R}$ with homomorphism of reduced fundamental rack in X up to cobordism also with homomorphism of reduced fundamental rack in X.

If $W = \mathbb{R}^{n+1}$, we abbreviate the notation $\mathcal{F}(\mathbb{R}^{n+1}, X)$ to $\mathcal{F}(n+1, X)$.

There is addition on $\mathcal{F}(n+1,X)$ given by disjoint union. To be precise, given links L_1, L_2, choose copies in disjoint half spaces, then define $L_1 + L_2$ to be $L_1 \coprod L_2$. We need to explain how to represent $L_1 \coprod L_2$ in X. The simplest way to do this is to use diagrams. If L_i is given by the diagram D_i, then $L_1 \coprod L_2$ is given by $D_1 \coprod D_2$ and representations correspond to labellings of D_1, D_2 which define a labelling of $D_1 \coprod D_2$ in the obvious way. We can also define the required representation using the fact that the fundamental rack of $L_1 \coprod L_2$ is the free product of the fundamental racks of L_1, L_2 (see [8, page 357]). The homomorphisms of the two factors determine a homomorphism on the entire rack.

The addition on $\mathcal{F}(n+1,X)$ is well defined up to cobordism, is compatible with cobordism, and makes $\mathcal{F}(n+1,X)$ into an abelian group.

Proposition 3.10. *There is a bijection $\mathcal{D}(M,X) \to \overline{\mathcal{F}}(M \times \mathbb{R}, X)$ and an isomorphism $\mathcal{D}(n,X) \to \mathcal{F}(n+1,X)$ both induced by lifting diagrams.*

Proof. This follows from definitions and Corollaries 3.1 and 3.5. □

Combining the last two results, we deduce our main classification theorem:

Theorem 3.11 (Classification of links)**.** *Let X be a rack. There is a natural bijection between $[M,|BX|]$ and $\overline{\mathcal{F}}(M \times \mathbb{R}, X)$ and there is a natural isomorphism between $\pi_n(BX)$ and $\mathcal{F}(n+1,X)$. The embedded first James bundle plays the rôle of classifying bundle in both cases.*

Double cobordism. Diagrams are *doubly cobordant* if they are cobordant by a simultaneous cobordism of diagram and the containing manifold. More precisely, suppose that $D_i\colon V_i \ltimes M_i$ is a diagram for $i = 1, 2$. Then D_1 is doubly cobordant to D_2 if there is a diagram $U \ltimes W$ with boundary, such that the boundary is the disjoint union $D_1 \coprod D_2$.

There is a similar notion of double cobordism of diagrams with labelling in a $\square$-set or a rack and there is an analogous notion of double cobordism of links possibly with representation in a rack. We shall consider the special case of *product links* by which we mean links in $L\colon V \subset M \times \mathbb{R}$ up to double cobordism of V and M with representation of reduced fundamental rack in a given rack X.

These sets all form abelian groups under disjoint union.

The proofs of Theorems 3.11 and 3.9 extend to prove the following theorem.

Theorem 3.12 (Classification up to double cobordism). *The set of double cobordism classes of diagrams in n-manifolds labelled in the $\square$-set C is in natural bijection with $\mathfrak{N}_n(C)$ (the unoriented bordism group). If the containing manifolds are oriented and we use oriented bordism, then the set is in bijection with $\Omega_n(C)$ (the oriented bordism group).*

The set of double cobordism classes of product links in n-manifolds cross $\mathbb{R}$ with representation of reduced fundamental rack in a given rack X is in natural bijection with $\mathfrak{N}_n(BX)$. If all manifolds are oriented, then the set is in bijection with $\Omega_n(BX)$.

Calculations. Later in the paper (section 5) we shall report on calculations of homotopy and homology of rack spaces. Any such calculation gives an immediate calculation of a link group using the appropriate classification theorem above. We will not spell out all such corollaries, but here are a couple of sample results:

(1) Let K be the trefoil knot with any framing. Any link in $\mathbb{R}^3$ with representation in $\Gamma(K)$ is cobordant (with representation) to the disjoint union of n copies of K with identity representation.

(2) The group of double cobordism classes of three coloured product links in oriented 3-manifolds is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$. Hence, there is a particular three coloured link which is non-trivial under double cobordism, but for which the disjoint union of three copies is trivial.

Here *three colouring* means representation in the *three colour rack* $D_3 = \{0, 1, 2 \mid a^b = c \text{ iff } a, b, c \text{ are all the same or all different}\}$.

The first result follows from Theorems 3.11 and 5.4. Theorem 5.4 implies that π_2 of the rack space of the fundamental rack of any irreducible link in $\mathbb{R}^3$ is $\mathbb{Z}$. (Result (1) is therefore true with the trefoil replaced by any irreducible link.)

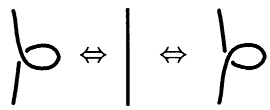
The second result follows from Theorem 3.12, the fact that $\Omega_3(X) \cong H_3(X)$ for any space X, and the calculation of $H_3(BD_3)$ as $\mathbb{Z} \oplus \mathbb{Z}_3$ given in [27].

4. The classical case

We now turn to the lowest non-trivial dimension ($n = 2$). In this case the cobordism classes can readily be described as equivalence classes under simple moves and this gives a combinatorial description of $\pi_2(C)$ which can be used for calculations.

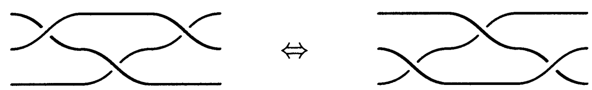
Framed embeddings and diagrams. Up to isotopy any framed embedding in $\mathbb{R}^3$ is the lift of a diagram in $\mathbb{R}^2$. This is seen by choosing any diagram to represent the (unframed) link and then correcting the framing by introducing twists

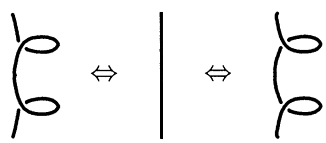
(Reidemeister 1-move):

(R1) 

(This argument is the proof of the Compression Theorem in this easy case.) The resulting diagram is unique up to regular homotopy (or equivalently Reidemeister 2 and 3 moves) together with the double Reidemeister 1-move illustrated below. For a proof see [8, pages 369–370]. We shall refer to these moves as R2, R3, and R11 (pronounced r-one-one), respectively:

(R2) 

(R3) 

(R11) 

There is a similar result with the same proof for any surface.

Classification by cobordism. Recall that $\mathcal{D}(2, C)$ is the (group) of labelled cobordism classes of diagrams in $\mathbb{R}^2$ labelled in the based $\square$-set C (such that the infinite region is labelled by $*$). We recall Theorem 3.9 in this case:

Theorem 4.1 (Classification of labelled diagrams)**.** *Let C be a $\square$-set with basepoint $* \in C_0$. There is a natural isomorphism between $\pi_2(C)$ and $\mathcal{D}(2, C)$.*

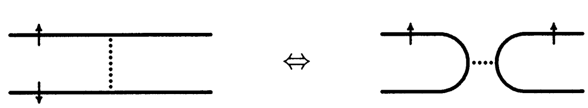
The theorem cuts both ways. It classifies diagrams up to cobordism and also provides an interpretation of $\pi_2(C)$ which can be used for calculations. For this purpose we need to break a cobordism into a sequence of combinatorial moves which we now describe.

Cobordism by moves. Suppose that we are given a cobordism (a diagram D in $\mathbb{R}^2 \times I$) between diagrams D_0, D_1. Think of $\mathbb{R}^2 \times I$ as a sequence of copies of $\mathbb{R}^2$. This breaks D into a sequence of slices. By general position this is an isotopy apart from a finite number of critical slices which are maxima, minima and saddles of the sheets, maxima and minima of the double curves and triple points. Corresponding to these are the diagram moves listed below:

Diagram moves.

BD Births and deaths of little circles: $D \Leftrightarrow D \cup O$.

Br Bridge between arcs with compatible framing:

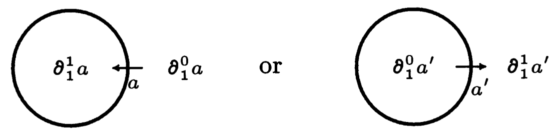


R2 move

R3 move

If the cobordism is labelled, then the labelling before and after a move satisfies the following compatibility conditions:

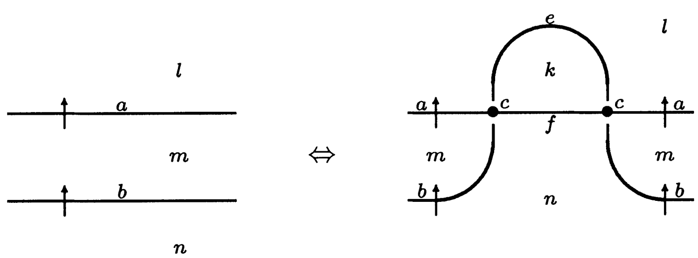
BD Births must correspond to an edge of C:



There is no condition for deaths.

Br Bridges must be between arcs with the same label.

An **R2** move must involve two double points labelled by the same square (with opposite orientations):



In the figure the following face equalities hold: $n = \partial_1^0 b = \partial_1^0 f$, $m = \partial_1^1 b = \partial_1^0 a$, $l = \partial_1^1 a$, $a = \partial_1^1 c$, $f = \partial_1^0 c$, $e = \partial_2^1 c$, $b = \partial_2^0 c$, $k = \partial_1^1 f = \partial_1^0 e$.

An **R3** move must correspond to a 3-cube of C.

Proposition 4.2. *Diagrams labelled in* C *are cobordant iff they differ by compatible moves* BD, Br, R2, R3 (*above*). $\square$

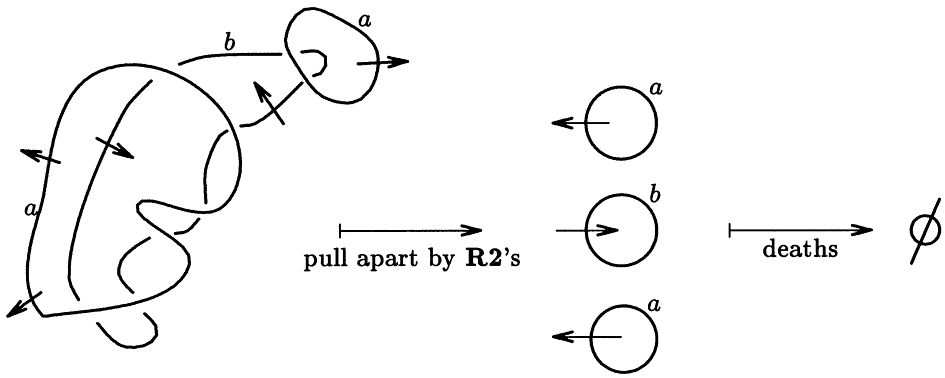
Note. If the labelling is by a rack, then R2 and R3 moves are always compatible (this is essentially what the rack laws are designed for [8, section 4]), and births can have arbitrary labels.

Digression on π_2.

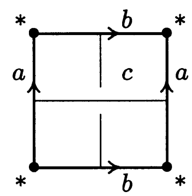
Remark. Theorem 4.1 and the descriptions of cobordisms in terms of moves allows us to make calculations of π_2. See for example Theorem 5.15 for a report of calculations made by this method. Note at once that the writhe of a diagram is invariant under moves and hence we always have a map to $\mathbb{Z}$. the *writhe* is defined (as usual) to be the number of double points counted with a sign—a right-hand crossing counting $+1$ and a left-hand one -1.

This map to $\mathbb{Z}$ is not always onto as the illustrative example below shows. The writhe has the following interpretation: Let $t_C\colon C \to T$ be the constant map, where T denotes the trivial $\square$-set as usual. Then writhe is the same as $\pi_2(C) \xrightarrow{(t_C)_*} \pi_2(T) = \pi_2(\Omega(S^2)) = \pi_3(S^2) = \mathbb{Z}$.

Example (Calculation of π_2 (torus))**.** As an illustration of the use of diagrams to calculate π_2 of a $\square$-set we calculate π_2(2-torus).



where $T^2 =$



Notice that, in the initial diagram, arcs labelled a always cross over arcs labelled b and there are no other crossings. Hence, the link is an unlink and can be pulled apart as suggested.

The Whitehead conjecture. To illustrate the applicability of this method in general, we now give a translation of the Whitehead conjecture [30] into a conjecture about coloured diagrams.

Recall that the Whitehead conjecture states that if K is a subcomplex of the 2-dimensional complex L, then $\pi_2(L) = 0$ implies $\pi_2(K) = 0$. By [12, Proposition 1.2] it is sufficient to establish the conjecture for 2-dimensional $\square$-sets and subsets.

We consider plane diagrams up to moves Br, BD and R2. No R3's are allowed because the dimension is at most 2.

A *colouring* of a diagram is a colouring of arcs, regions and crossings. The outside (infinite) region is always coloured white, say. A *colour scheme* is a list of allowable colours (three lists, one each for regions, arcs and crossings) and rules about neighbouring colours. The rules prescribe the two neighbouring colours for a given colour on an edge and all the neighbouring colours for a given colour on a crossing.

Moves are *allowable* if Br and BD moves respect the colour scheme and R2 moves involve two crossings with the same colour (but opposite orientation).

A diagram is *reducible* if it can be changed to the empty (all white) diagram by allowable moves.

Using Proposition 4.2, the Whitehead conjecture now has the following equivalent statement.

Conjecture 4.3. *Suppose that for a given colour scheme all diagrams are reducible. Then any diagram can be reduced without using any colours not already used in the diagram.*

Classification of links using moves. Let X be a rack; recall that $\mathcal{F}(3, X)$ is the group of cobordism classes of framed links in $\mathbb{R}^3$ with representation in X; recall also that $\mathcal{F}(3, X) \cong \mathcal{D}(2, X) \cong \pi_2(BX)$ as a special case of Theorem 3.9 and Proposition 3.10. It is possible to analyse cobordisms combinatorially in this dimension and this provides an alternative proof of the first of these two isomorphisms.

Theorem 4.4 (Special case of Proposition 3.10)**.** $\mathcal{F}(3, X) \cong \mathcal{D}(2, X)$.

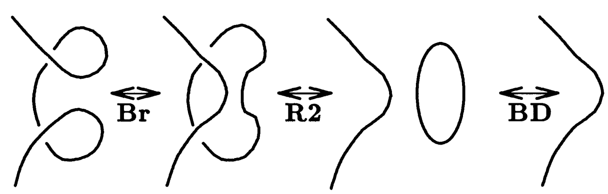
Proof. Any diagram labelled in X lifts to a framed link with representation in X and a cobordism of diagrams to a cobordism of links. Thus, we have a homomorphism

$$\Psi \colon \mathcal{D}(2, X) \to \mathcal{F}(3, X),$$

where Ψ is surjective since any framed link is the lift of a diagram (unique up to moves R11, R2, R3). To see that Ψ is injective we have to show that labelled diagrams whose lifts are cobordant are themselves cobordant (as diagrams) or equivalently (using Proposition 4.2) that they differ by moves BD, Br, R2 and R3. We will show this by analysing the cobordism. Since the representation (or labelling) in X plays no real rôle in the proof, it will henceforth be suppressed.

Now the cobordism is a framed 3-manifold in $\mathbb{R}^3 \times I$ and by slicing by parallel $\mathbb{R}^3$'s we obtain (using general position) a sequence of framed links with the following critical stages (where a slice contains a critical point of the projection to I): Bridge moves, births, and deaths of small circles. Moreover, by rotating small neighbourhoods of these critical points (if necessary) we can assume that near the critical point nearby slices are lifts of diagrams; i.e., that the R1 moves needed to correct the framing are away from the critical points.

Thus we can choose diagrams near these critical stages whose lifts are isotopic to the nearby slices and which differ by diagram moves BD or Br. Now away from critical stages each link in the sequence corresponds to a diagram unique up to moves R2, R3, R11 and combining the two sets of moves, we see that the cobordism corresponds to diagram moves BD, Br, R2, R3, R11. But the following sequence of pictures shows how to achieve an R11 as a combination of a bridge move, an R2 and a death. The rest of the theorem is clear. □



Virtual links. Virtual links have been introduced by Kauffman [19] and studied by several authors including Carter-Saito-Kamada [3], Fenn-Jordan-Kauffman [7], Kamada-Kamada [18] and Kuperberg [21]. Here we shall show that the second homology group of the rack space BX classifies framed virtual links with representation in a rack X, up to cobordism.

Oriented Gauss codes and oriented crossing graphs. There are several equivalent definitions of a virtual link. Kauffman [19] introduced the subject and defined a virtual link as an equivalence class of oriented Gauss codes up to Reidemeister moves (R1, R2 and R3). An oriented Gauss code is the same as a 4-valent graph such that each vertex can be identified with a standard crossing in the plane up to rotation through π. In other words, we know which of the edges at the vertex are the overcrossing arcs and which are the undercrossing arcs and we also have a cyclic ordering of the arcs at the vertex—over, under, over, under. We call such a 4-valent graph an *oriented crossing graph*. So a virtual link is an equivalence class of oriented crossing graphs under Reidemeister moves.

Virtual link diagrams. Now an oriented crossing graph can be immersed in the plane with the vertices forming crossings with the correct orientation. Such an immersion is well defined up to changing the immersion on edges and this leads to the more usual definition of virtual links in terms of diagrams. The crossings which come from the vertices of the graph are the real crossings and the ones which come from crossings of the immersed edges are the virtual crossings. The result is a *virtual link diagram*. The equivalence relation on virtual link diagrams is generated by Reidemeister moves on real crossings together with the ability to move an arc containing only virtual crossings to any other position with the same endpoints. This latter move can be replaced by a set of *extended Reidemeister moves*—Reidemeister moves R1, R2 and R3 for virtual crossings and one mixed move, namely an R3 move with two virtual and one real crossing. See Kauffman [19, Figures 2 and 3].

Framed virtual links. We need to extend these definitions to *framed virtual links*. To frame a virtual link, we orient the components and use the blackboard framing convention. Moreover, in the above definitions we replace the R1 move for real crossings with the double Reidemeister 1-move (the R11 move). Thus a framed virtual link is an equivalence class of oriented crossing graphs, with edges oriented compatibly with crossings, under R2, R3 and R11. Equivalently, it is an equivalence class of oriented virtual link diagrams under moves R2, R3 and R11 for real crossings plus the extended moves, *including R1 for virtual crossings*.

Stable equivalence. We need to interpret (framed or unframed) virtual links as equivalence classes of genuine (framed or unframed) links in oriented surfaces cross I.

Definitions. Let Σ be an oriented surface and let L be a link in $\Sigma \times I$. Suppose that $D_1, D_2 \subset \Sigma$ are discs disjoint from the projection of L. If we add an oriented handle to Σ with feet at D_1, D_2 to form Σ', then we say that $L \subset \Sigma' \times I$ is a *stabilization* of $L \subset \Sigma \times I$.

Links $L_1 \subset \Sigma_1 \times I$ and $L_2 \subset \Sigma_2 \times I$ are *stably equivalent* if they differ by a sequence of stabilizations and their inverses.

The following result is "well known". It was suggested by Kauffman in [19] and can be deduced from [3, Proposition 3.4]. However, the proof of this last result spreads over three papers [3, 18, 19], so it seems worthwhile to include here a short direct proof.

Theorem 4.5. *Virtual links (respectively framed virtual links) are in bijective correspondence with stable equivalence classes of links (respectively framed links) in oriented surfaces cross I.*

Proof. Let $\mathcal{VL}$ denote the set of virtual links and let $\mathcal{SL}$ denote stable equivalence classes of links in oriented closed surfaces cross I.

There is a function $\phi\colon \mathcal{SL} \to \mathcal{VL}$ given as follows: Let $L \subset \Sigma \times I$ be a link and project it to form a diagram in Σ. The diagram determines an oriented crossing graph and hence a virtual link. It is clear that the result is unaltered by stabilization. An isotopy can be replaced by Reidemeister moves which correspond to Reidemeister moves on the oriented crossing graph. Hence ϕ is well defined.

There is another function $\psi\colon \mathcal{VL} \to \mathcal{SL}$ given as follows. Use the definition of a virtual link as an equivalence class of oriented crossing graphs. Let Γ be such a graph; consider an immersion of Γ in the plane with the vertices forming crossings with the correct orientation. Let N be an induced regular neighbourhood. It is easy to see that N depends only on the original graph. From an oriented surface Σ by capping the boundary circles of N with discs. We obtain a link diagram in Σ and hence a link in $\Sigma \times I$. The result is unchanged by an R1 or R3 move on the original graph whilst an R2 may add (or delete) a handle disjoint from the diagram. Thus ψ is also well defined.

It is clear that $\phi \circ \psi$ is the identity on $\mathcal{VL}$. To see that $\psi \circ \phi$ is the identity on $\mathcal{SL}$ observe that the result of applying $\psi \circ \phi$ to a link in $\Sigma \times I$ with diagram D is the same as surgering along the circles which form the boundary of a regular neighbourhood of D. The surgery is a stable equivalence.

For the framed case, the proof is exactly the same with R1 moves replaced by R11 moves throughout. □

Remark. Kuperberg [21] has proved a far stronger result: A virtual link has a unique *irreducible* representation as a link in an oriented surface cross I, where irreducible means that no destabilizations are possible.

The fundamental rack of a virtual link. There is a notion of a fundamental rack of a framed virtual link obtained from any diagram by labelling arcs with generators (ignoring all virtual crossings) and reading a relation at real crossings by the usual rule (3.2). Since this is the same as reading the fundamental rack of any corresponding diagram in an oriented surface, it follows from Lemma 3.3 that this rack coincides with the reduced fundamental rack of any corresponding link in an oriented surface cross I.

Thus we have a good notion of a representation in a rack X for a framed virtual link, namely a homomorphism of fundamental rack in X.

Cobordism of virtual links. Virtual links are *cobordant* if they differ by the allowed moves plus the two cobordism moves BD (birth-death) and Br (bridge) introduced earlier. Note that R11 moves can be obtained from R2, Br and BD moves and do not need to be included here. Combining the proofs of Theorems 4.4 and 4.5 we see that this corresponds to double cobordism of any corresponding link in surface cross I. (Notice that cobordism of surfaces is generated by stabilization.)

There are similar definitions of cobordism of framed links and of links with representation in a rack X.

Cobordism classes form an abelian group under disjoint union and we denote the group of cobordism classes of framed virtual links with representation in a rack X by $\mathcal{VL}(X)$.

Theorem 4.6 (Classification of virtual links)**.** *There is a natural isomorphism between $\mathcal{VL}(X)$ and the second homology group of the rack space $H_2(BX)$.*

Proof. Theorem 3.12 shows that there is a natural bijection between the set of double cobordism classes of links in oriented surfaces cross I with representation in X and the oriented bordism group $\Omega_2(BX)$. But $H_2(BX) = \Omega_2(BX)$ and the result follows. $\square$

Calculations. In section 5 we report on calculations of H_2 of rack spaces, in particular, Greene's results in Theorem 5.16. Using the last result, any of these calculations implies results about virtual links.

Below are some samples. By *n-colouring* we mean a representation in the dihedral rack $D_n := \{0, 1, \dots, n-1\}$ with $i^j = 2j - i \bmod n$ for all i, j. The *writhe* of a framed virtual link is the number of crossings counted algebraically (the framing gives signs to crossings in the usual way). Writhe is a cobordism invariant and can be interpreted as the element of $H_2(T) \cong \mathbb{Z}$ determined by the link. Here T is the trivial $\square$-set and is the rack space of the trivial rack (i.e., no labelling).

(1) *Any odd coloured or uncoloured virtual link is cobordant to any other with the same writhe.*

(2) $\mathcal{VL}(D_{2n})$ *for* $n > 0$ *is at least* $\mathbb{Z}^4$*; thus, there are at least three infinite families of even coloured cobordism classes with any given writhe. Furthermore,* $\mathcal{VL}(D_4)$ *has* 2*-torsion as well.*

5. The algebraic topology of rack spaces

In this section we turn to invariants of rack spaces. These are important because any invariant of rack spaces automatically becomes a knot or link invariant by calculating it for the rack space of the fundamental rack of the knot or link. Moreover, further invariants can be derived by considering representations of the fundamental rack in other racks, for example, finite racks.

All the invariants considered in this section are homotopy type invariants; however, the homotopy type of the rack space is not a complete link invariant for links in S^3 even when the rack itself is (see the remarks on Theorem 5.4 below). In this context it is worth reiterating that the combinatorial structure of the rack space is equivalent to the rack itself and therefore it is valuable to construct combinatorial invariants of a $\square$-set which are not homotopy type invariants. The main new invariants introduced in [12] (the James-Hopf invariants), and also the associated generalised cohomology theories, are combinatorial invariants of this type.

Here we start by identifying the fundamental group of rack spaces and proving that they are simple. We then turn to calculations of homotopy groups. We calculate all the homotopy groups of the rack space of an irreducible link in a 3-manifold and the second homotopy group of the rack space of any link in S^3. We determine the homotopy type of the rack space of an irreducible link in an irreducible 3-manifold with infinite fundamental group. We also determine the homotopy type of the rack space of a free rack and of the trivial rack with n elements. (Note that Wiest [31] has also determined the homotopy type of the rack space of an irreducible link in a general 3-manifold.)

We conclude with some results on homology groups and a review of results of Flower [14] and Greene [15].

Fundamental group. The fundamental groupoid of a $\square$-set is discussed in [10]. We repeat the computation of the fundamental group of the rack space.

Recall from [8] that the *associated group* $As(X)$ to a rack X is the group generated by the elements of X subject to the relations $a^b = b^{-1}ab$.

Proposition 5.1. *The fundamental group $\pi_1(BX)$ of the rack space of a rack X is isomorphic to the associated group $As(X)$ of X.*

Proof. Recall that the rack space BX has a single vertex and edges in bijection with the elements of X which therefore generate $\pi_1(BX)$. Moreover, the relations given by the squares of BX are $a^b = b^{-1}ab$ for $a, b \in X$. The result now follows from the definition of $As(X)$. $\square$

Simplicity of the rack space. We next prove that BX is a simple space for any rack X.

Proposition 5.2. *Let X be any rack. Then the action of the fundamental group $\pi_1(BX)$ on $\pi_n(BX)$ is trivial.*

Proof. Let $\beta \in \pi_n(BX)$, and let $\alpha \in \pi_1(BX)$ be a generator corresponding to $x \in X$ as above. We may represent β by a labelled diagram D in $\mathbb{R}^n$. Then $\alpha \cdot \beta$ is represented by the diagram which comprises a framed standard sphere in $\mathbb{R}^n$ labelled by x and containing D in its interior. But the following diagram cobordism shows that the two diagrams are equivalent. Pull the sphere under D to one side (without changing any labels on D) and then eliminate it. $\square$

Notation. Let X be a rack. Recall from section 3 that $\mathcal{D}(n, X)$, $\mathcal{F}(n+1, X)$ denote the group of cobordism classes of diagrams in $\mathbb{R}^n$ labelled by X and the group of framed cobordism classes of framed codimension 2 links in $\mathbb{R}^{n+1}$ with representation in X, respectively.

We shall use the notation $[D, \lambda]$ for an element of $\mathcal{D}(n, X)$, where D denotes a diagram and λ a labelling and we shall use the notation $[L, F, \rho]$ for an element of $\mathcal{F}(n+1, X)$ where L is a codimension 2 link (an $(n-1)$-manifold) in $\mathbb{R}^{n+1}$, F is a framing of L and ρ a representation in X, i.e., a homomorphism of the fundamental rack of (L, F) to X.

If $x \in X$ is an element of X, then we denote by λ^x the labelling obtained from λ by operating on all labels by x. That this is also a labelling follows from the rack law. Similarly, we denote by ρ^x the representation obtained by composing ρ with the automorphism $a \mapsto a^x$ of X.

Corollary 5.3. *Let X be a rack. Let $[D, \lambda] \in \mathcal{D}(n, X)$ and let $[L, F, \rho] \in \mathcal{F}(n+1, X)$ and $x \in X$. Then $[D, \lambda] = [D, \lambda^x]$, and $[L, F, \rho] = [L, F, \rho^x]$.*

Proof. To see that $[D, \lambda] = [D, \lambda^x]$ simply pull the sphere over instead of under in the proof of Proposition 5.2. The other result now follows from the isomorphism of the two groups (Proposition 3.10). $\square$

Remark. Notice that if X is the fundamental rack of a link L in S^n, then the action of x is given by $a^x = a \cdot \partial x$ where ∂ is the augmentation to $\pi_1(L) := \pi_1(S^3 - L)$, hence the corollary also implies invariance of $\mathcal{D}(n, X)$ and $\mathcal{F}(n+1, X)$ under the action of $\pi_1(L)$ in this case.

Homotopy groups of the rack space of an irreducible link. Let L be a framed link in a 3-manifold M^3. We say that L is *irreducible* if each embedded 2-sphere in $M^3 - L$ bounds a 3-ball (i.e., $M - L$ is an irreducible 3-manifold). We say that L is *trivial* if $M = S^3$ and L is equivalent to the unknot in S^3 with zero framing.

Let Λ be a framed submanifold of $\mathbb{R}^{n+1}$ with framing $F\colon \Lambda \times D^2 \to \mathbb{R}^{n+1}$. Write $N(\Lambda) = \mathrm{im}(F)$ for the tubular neighbourhood of Λ and $\Lambda^c = \overline{\mathbb{R}^{n+1} - N(\Lambda)}$ for the closure of the complement. Also write $\Lambda^+ = F(\Lambda \times \{(1,0)\})$ for the *parallel manifold* to Λ.

Theorem 5.4. *Let M^3 be a 3-manifold and let L be a framed non-trivial irreducible link in M^3. Let $\Gamma(L)$ denote the fundamental rack of L. Then for $n > 1$,*

$$\pi_n(B\Gamma(L)) \cong \pi_{n+1}(M^3).$$

Proof. Recall from Theorem 3.11 that $\pi_n(B\Gamma(L)) \cong \mathcal{F}(n+1, \Gamma(L))$,

Define the homomorphism

$$\phi\colon \pi_{n+1}(M^3) \to \mathcal{F}(n+1, \Gamma(L))$$

as follows. Let $\alpha \in \pi_{n+1}(M^3)$ be represented by a map $f\colon \mathbb{R}^{n+1} \to M^3$ which is constant outside a compact set. Homotope f to be transverse to L. Then $\Lambda = f^{-1}L$ is a framed submanifold with framing F such that f is compatible with the framings. Let $\rho\colon \Gamma(\Lambda) \to \Gamma(L)$ be the induced homomorphism of racks. We define $\phi(\alpha) = [\Lambda, F, \rho]$. If f and g are both transverse representatives of α, then a homotopy between f and g can also be made transverse producing a bordism and we see that ϕ is well defined.

ϕ is surjective. For each component Λ_i^+ and Λ^+ let L_i^+ be the corresponding component of L which is in the image of Λ_i^+. Notice that the L_i^+'s may not be distinct. For each i choose an embedded path p_i from Λ_i^+ to the basepoint representing an element of $\Gamma(\Lambda)$ so that the chosen paths only meet at the basepoint. Similarly, choose a path q_i for L_i^+, where $\rho[p_i] = [q_i]$. Corresponding to our choices we have longitudinal and meridinal subgroups $\Lambda_i(l)$, $\Lambda_i(m)$ of $\pi_1(\Lambda^c)$ and subgroups $L_i(l)$, $L_i(m)$ of $\pi_1(L^c)$. Now $As(\Gamma(\Lambda)) \cong \pi_1(\Lambda^c)$ and $As(\Gamma(L)) \cong \pi_2(M, L^c)$. Then $\rho\colon \Gamma(\Lambda) \to \Gamma(L)$ induces a homomorphism $\rho_1\colon \pi_1(\Lambda^c) \to \pi_1(L^c)$ by composing with the boundary map in the homotopy exact sequence of the pair (M, L^c). See [8, page 360].[2] The homomorphism has restrictions $\rho_i(l)\colon \Lambda_i(l) \to L_i(l)$ and $\rho_i(m)\colon \Lambda_i(m) \to L_i(m)$.

We claim that the hypotheses on L imply that the group $L_i(l)$ is infinite cyclic. To see this, suppose not. Then some multiple of the longitude is null homotopic in L^c. By the loop theorem there is a closed essential curve in the neighbourhood of the longitude L_i^+ which bounds a disc in L^c. But this neighbourhood is an annulus and the only possibility is that L_i^+ itself bounds a disc. By irreducibility, we then see that $L = L_i$ and it is trivial, contradicting our hypotheses.

The space L_i^+ together with its embedded "tail" q_i is an Eilenberg-Mac Lane space so there is a unique map, up to homotopy, $f_0\colon \Lambda_i^+ \cup p_i(I) \to L_i^+ \cup q_i(I)$ inducing

[2] It has been pointed out by Wiest that the proof of the result used here [8, Proposition 3.2] contains a misleading statement. To be precise, the statement made at the top of page 361 in [8] is open to misinterpretation. The paths can also be varied by an isotopy which moves one little disc around another—essentially a pure braid automorphism. This can be realised by two interchanges and is implicit in the subsequent text.

the homomorphism $\rho_i(l)$. Furthermore, we can assume, for later convenience, that $f_0\Lambda_i^+ \subset L_i^+$ and $f_0 p_i = q_i$. Since ρ_1 is induced by sending meridians to meridians, there is a unique extension $f_1 \colon N(\Lambda) \to N(L)$ which preserves framing. We can finally extend f over Λ^c since L^c is also an Eilenberg-Mac Lane space.

ϕ **is injective.** First we observe that the map f constructed above is unique up to based homotopy. To see this suppose alternative choices p_i' and q_i' are made in place of p_i and q_i respectively, and assume end points agree. Consider $f_0' \colon \Lambda_i^+ \cup p_i'(I) \to L_i^+ \cup q_i'(L)$. We can assume the set of p_i's together with the p_i''s do not meet on interiors. Now suppose $f_0 \circ (\overline{p_i} \cdot r \cdot p_i) \simeq \overline{q_i} \cdot l^{n_i} \cdot q_i$ where r is some loop in Λ_i^+ and l is "once round" L_i^+. Notice $p_i' \simeq p_i \circ (\overline{p_i} \cdot p_i')$. , Then from the fact that a homomorphism of racks preserves the action of the associated groups we have, after an easy calculation, $\rho_1[\overline{p'}_i \cdot r \cdot p_i'] = [\overline{q'}_i \cdot l^{n_i} \cdot q_i']$. It follows that f_0 and f_0' can be taken to agree on Λ_i^+ and f_1 and f_1' agree on $N(\Lambda)$.

Now in completing the constructing of f consider the construction over the 1-skeleton. If we have $fp_i = q_i$, then essentially the same argument shows we can assume $fp_i' = q_i'$. We are now ready to prove ϕ is injective. Suppose (W, F, ρ) is a bordism between (W_0, F_0, ρ_0) and (W_1, F_1, ρ_1) and the latter are associated with maps g_0 and g_1, respectively. Now repeat the proof that ϕ is onto but this time using $S^{n+1} \times I$ and W and ρ in place of S^{n+1} and Λ and ρ. The resulting bordism then must give g_0 and g_1 on the two ends by the above observations. □

Remarks on and consequences of Theorem 5.4. It is worth pointing out that irreducibility of the *link* L in M does not imply irreducibility of the 3-*manifold* M. Indeed, any connected 3-manifold contains an irreducible link (see [8, page 380, Remark (2)]). The higher homotopy groups of a general (non-irreducible) 3-manifold can be very complicated. Each separating 2-sphere potentially determines a copy of $\pi_*(S^2)$ which is then subject to action by π_1 of the manifold.

The theorem therefore implies that the higher homotopy groups of rack spaces of irreducible links can in general be complicated. In the case that the 3-manifold *is* irreducible, its higher homotopy groups are either all zero (the case when the fundamental group of M is infinite) or coincide with the homotopy groups of the 3-sphere. Turning first to the infinite fundamental group case, we deduce:

Corollary 5.5. *Let L be a framed non-trivial irreducible link in an irreducible 3-manifold with infinite fundamental group and let $\Gamma(L)$ denote the fundamental rack of L. Then $B\Gamma(L)$ is a $K(\pi, 1)$ where π is the kernel of $\pi_1(M - L) \to \pi_1(M)$.*

Proof. By the theorem and the remarks above, $B\Gamma(L)$ is a $K(\pi, 1)$ where π is the associated group of $\Gamma(L)$ by Proposition 5.1. But since $\pi_2(M) = 0$, [8, Proposition 3.2] implies that the associated group is the kernel of $\pi_1(M - L) \to \pi_1(M)$. □

In the case when M is irreducible and has finite fundamental group (e.g., when $M = S^3$), then (as remarked earlier) the theorem implies that the higher homotopy groups of $B\Gamma(L)$ coincide with the higher homotopy groups of S^3 (with an index shift of 1). Thus the theorem gives a plentiful supply of new geometric interpretations for these groups. We now describe one such interpretation which does not need the concept of fundamental rack for its statement.

We need to define the writhe of a framed link in a higher dimensional sphere. Let M^{n-1} be a framed submanifold of S^{n+1}. Let M^+ denote the parallel manifold to M as usual and let $\phi \colon M^c \to S^1$ be the map defined by any Seifert bounding manifold for the co-dimensional 2 submanifold M. Then the composition $M \to M^+ \xrightarrow{\phi|M^+} S^1$ defines an element of $H^1(M)$ called the *writhe*. There is a similar notion for the writhe of a cobordism and a cobordism which preserves writhe.

Corollary 5.6. *Fix an integer $w > 0$. Then $\pi_{n+1}(S^3)$ is isomorphic to the set of equivalence classes of framed $(n-1)$-manifolds embedded in S^{n+1} with writhe divisible by w, under framed cobordism also with writhe divisible by w.*

Proof. Let U_w denote the unknot in S^3 with framing w. Then U_w is irreducible, non-trivial and the theorem applies. But writhe divisible by w is the same as having a representation in the cyclic rack of order w which is $\Gamma(U_w)$. □

Since we now know that $\pi_{n+1}(S^3)$ is isomorphic to the set of equivalence classes of framed $(n-1)$-manifolds embedded in S^{n+1} with writhe divisible by w, we can consider the forgetful map which ignores the writhe condition on framings.

Corollary 5.7. *The forgetful map is multiplication by w.*

Proof. This follows from observing that the map from $\pi_3(S^3)$ to $\pi_3(S^2)$ which is given by applying the Thom-Ponrjagin construction to U_w is w times the Hopf map. But the forgetful map is effectively composition with this map. □

By contrast in the case $w = 0$, the cobordism groups are all zero. This is essentially what Theorem 5.13 below says.

Wiest [31] has extended these results in a number of ways. He has shown that the augmented rack space $B_G X$ (where G is $\pi_1(M - L)$) has the same homotopy type as $\Omega(M^3)$. This implies that BX has the homotopy type of $\Omega(M^3)$ factored by an action of G. Further, he has shown that, for irreducible links in homotopy 3-spheres, the homotopy type of the rack space is determined by the fundamental group of the link. Thus for example if R is the reef knot in S^3 (square knot in American English) and G is the granny knot (both taken with framing zero for definiteness), then $B\Gamma(R)$ and $B\Gamma(G)$ have the same homotopy type. In this case the fundamental rack is a classifying invariant and the two racks differ. Thus, the homotopy type of the rack space (as distinct from the combinatorial structure) contains strictly less information than the rack itself.

The second homotopy group for links in S^3. Now let L be a framed link in S^3. We say that L is *non-split* or *irreducible* if no embedded 2-sphere in $S^3 - L$ divides the components of L into two non-empty subsets. (This is consistent with the usage of irreducible for links in M^3 given earlier.)

In general, a link can be written as a union $L = L_1 \cup \cdots \cup L_k$ where each L_i is a maximal irreducible sublink. We call the sublinks L_i the *blocks* of L. A block is said to be *trivial* if it is equivalent to the unknot with zero framing.

Theorem 5.8. *Let L be a framed link in S^3. Then $\pi_2(B(\Gamma(L))) \cong \mathbb{Z}^p$ where p is the number of non-trivial blocks of L. Furthermore, a basis of $\pi_2(B(\Gamma(L)))$ is given by diagrams representing these blocks.*

Paraphrased, the theorem says that any link in S^3 with representation in $\Gamma(L)$ is cobordant respecting the representation to a unique *standard* link comprising a number of separate copies of the blocks of L.

Proof. Case 1: L is irreducible and non-trivial.

This is a special case of Theorem 5.4.

Case 2: L is irreducible and trivial.

In this case the homomorphism $\phi\colon \pi_3(S^3) \to \mathcal{F}(n+1, \Gamma(L))$ defined in Case 1 is surjective by exactly the same argument. But $\phi[\mathrm{id}]$ is represented by the trivial link with identity representation which is null cobordant.

This completes Case 2 and we now turn to the general case $L = L_1 \cup \cdots \cup L_k$ where each L_i is maximal irreducible. We need the following lemmas.

Lemma 5.9. *Let M be an irreducible 3-manifold. Let M_0 be M with the interior of k balls $B_1, \ldots, B_k$ removed. Then $\pi_2(M_0)$ is generated as a $\mathbb{Z}[\pi_1(M_0)]$ module by the spheres ∂B_i.*

Proof. Let $\widetilde{M}$ and $\widetilde{M}_0$ be the universal covers of M and M_0, respectively. Then $\widetilde{M}$ can be obtained from $\widetilde{M}_0$ by filling in holes with copies gB_i of B_i one for each element g in $\pi_1(M_0)$, $i = 1, \ldots, n$. Since M is irreducible $\pi_2(M) \cong H_2(\widetilde{M}) \cong 0$. Then using a Mayer-Vietoris sequence we see that as an abelian group $\pi_2(M_0) \cong H_2(\widetilde{M}_0)$ has one generator for each pair (g, B_i) where $g \in \pi_1(M_0)$. □

Lemma 5.10. *Let M be the connected sum $M = M_1 \sharp \cdots \sharp M_k$ of k irreducible 3-manifolds each with non-trivial fundamental group. Then as a $\mathbb{Z}[\pi_1(M)]$ module $\pi_2(M)$ is generated by the separating spheres $S_1, \ldots, S_{k-1}$.*

Proof. Let an element of $\pi_2(M)$ be represented by a map $f\colon S^2 \to M$ of the 2-sphere into M which we may assume is transverse to the separating spheres. Consider an innermost disc D in S^2 which has boundary in the intersection of $f(S^2)$ and S_i, say. Let D' be a (singular) disc in S_i which bounds ∂D. Then by the previous lemma the homotopy class of the sphere $D \cup D'$ is in the subgroup generated as a $\pi_1(M)$ module by the separating spheres $S_1, \ldots, S_{k-1}$. By subtracting this element, we may perform a homotopy to remove this intersection curve. We can now argue by induction on the number of intersections. □

Returning now to the proof of the main theorem we look at π_3 and attempt to construct as before a map $f\colon \mathbb{R}^3 \to S^3$. This time the complement of L may not be an Eilenberg-Mac Lane space. The construction of the mapping on the tubular neighbourhood of Λ runs as before and there is no obstruction to extending to the 2-skeleton of the complement, but obstructions to mapping in the 3-cells may be non-zero. By the above lemma it will be sufficient to consider the case of a single 3-cell, which may be taken to be far away from Λ and where the map on the bounding 2-sphere is as follows. The map is constant on the equator and the upper hemi-disc is wrapped around a separating sphere with degree ± 1. The separating sphere contains just one component L_i of L. Maps on great arcs running from the south pole to the equator give the same path in S^3. We can now map the lower hemi 3-ball to the image of this path so that the map is constant on the equatorial 2-disc. At this point, by considering the upper hemisphere together with the equatorial 2-disc, we see that the problem is reduced to the case where the bounding 2-sphere is mapped to the separating 2-sphere. But now if we add to Λ a copy of L_i in the 3-ball, the map extends in the obvious way. Notice that the representation on the copy of $\Gamma(L_i)$ which this determines, is conjugated by a fixed element of $\pi_1(L^c)$. Call such a link *conjugated.* This means that the map f can be defined for the extended link. Indeed, we may further extend the link by adding copies of L so that f has degree zero. If we now make the resulting null homotopy transverse to L, we construct a bordism between $[\Lambda, F, \rho]$ and a number of disjoint copies of conjugated links L_i. By Proposition 5.2 and the result in Case 1 we can assume that these added links are labelled by identities and not conjugated. Thus, the L_i with identity labelling form a generating set for π_2. Note that any trivial sublinks can be eliminated by a bordism as in Case 2.

Now suppose some non-trivial linear combination of the $[L_i, F_i, id_i]$ is bordant to zero. We need the following lemma.

Lemma 5.11. *Let X be the free product of racks X_i and let Y_j be the union of the orbits in X determined by X_j. Then there is a retraction of Y_j onto X_j.*

Proof. This follows from the definition of the free product of racks [8, page 357]. This retraction is defined by setting the action of other orbits equal to the trivial action. □

The rack homomorphism commutes with the action of the associated groups and components correspond to orbits of the section. Thus the null bordism cannot mix components. Now observe that $\Gamma(L)$ is the free product of the $\Gamma(L_i)$ and therefore there is a retraction of the orbit determined by L_i onto $\Gamma(L_i)$ by the lemma. Applying this retraction to the appropriate pieces of the null bordism we see that $[L_i, F_i, id_i] = 0$. This contradicts Case 1.

It follows that the $[L_i, F_i, id_i]$ form a basis for π_2 and Theorem 5.9 is proved. □

Homotopy type of the space of a trivial rack. Let $X(n) = \{1, \ldots, n\}$ be the trivial rack with n elements; so that $x^y = x$ for all x and y.

Theorem 5.12. *The classifying space $BX(n)$ has the homotopy type of $\Omega(S^2 \vee \cdots \vee S^2)$, the loop space on the wedge of n copies of S^2.*

Proof. First observe the simple form of the faces in $BX(n)$:

$$\partial_i^\epsilon(x_1, \ldots, x_n) = (x_1, \ldots, \hat{x}_i, \ldots, x_n) \quad \text{for } 1 \le i \le n,\ \epsilon \in \{0, 1\}.$$

Now $\Omega(S^2 \vee \cdots \vee S^2) \simeq \Omega S(S^1 \vee \cdots \vee S^1)$, but $\Omega S(S^1 \vee \cdots \vee S^1)$ has the homotopy type of $(S^1 \vee \cdots \vee S^1)_\infty$, the free monoid on $(S^1 \vee \cdots \vee S^1) \setminus \{*\}$, by the James construction. Identify S^1 with $I/\partial I$ and let (k, t) denote the point t in the kth copy of S^1 in $S^1 \vee \cdots \vee S^1$. Then there is a homeomorphism $BX(n) \to (S^1 \vee \cdots \vee S^1)_\infty$ given by

$$[(i_1, \ldots, i_n), (t_1, \ldots, t_n)] \to (i_1, t_1) \cdot \cdots \cdot (i_n, t_n).$$

It follows that $\pi_k(B(X(n)))$ can be viewed geometrically as bordism classes of framed codimension two manifolds in S^{k+1} with components labelled in $\{1, \ldots, n\}$. □

Remark. For $k = 2$, by the Hilton-Milnor theorem [17], $\pi_2(BX(n)) \cong \mathbb{Z}^{n+\binom{n}{2}}$ where the second set of generators are Whitehead products. Geometrically, $\pi_2(BX(n))$ is interpreted as cobordism classes of links with components labelled by n distinct labels, or equivalently as a link divided into n disjoint sublinks, and the first n integer invariants are total writhes of the sublinks and the remaining $\binom{n}{2}$ are the mutual linking numbers. For more details, see [29].

Homotopy type of the space of a free rack.

Theorem 5.13. *Let FR_n denote the free rack on n elements. Then BFR_n has the homotopy type of $S^1 \vee \cdots \vee S^1$, the wedge of n copies of S^1.*

Proof. Recall from Theorem 3.11 that $\pi_n(BFR_n) \cong \mathcal{F}(n+1, FR_n)$.

We observe that FR_n is the fundamental rack of D_n, which is the link comprising n framed points in the 2-disc D^2. The proof of Theorem 5.4 shows that if (Λ, FR_n, ρ) represents an element of $\pi_n(BFR_n)$ (for $n \geq 2$), then (Λ, F, ρ) pulls back from a transverse map of $\mathbb{R}^{n+1}$ to D_n. But since D^2 is contractible, this map is null homotopic and applying relative transversality we obtain a null cobordism of (Λ, FR_n, ρ). By Proposition 5.1 $\pi_1(BFR_n) \cong *_n\mathbb{Z}$ (the free product of n copies of $\mathbb{Z}$). The result follows. $\square$

Remark. Recall from [8, page 376] that there is the concept of an *extended free rack*, which has free operator generators in addition to the usual free rack generators. This can be identified with the fundamental rack of a number of framed points in an orientable surface. A similar proof (using the fact that the higher homotopy groups of a surface vanish) then shows that if F is an extended free rack, then BF has the homotopy type of a wedge of circles. Moreover, both proofs extend with a little care to arbitrary free (or extended free) racks (in other words, there is no need for the generating sets to be finite). Thus, BF has the homotopy type of a 1-complex where F is any free (or extended free) rack.

A remark on homology groups. Let X be any rack. Recall that there is the concept of the *extended* rack space $B_X X$ [10, Example 3.1.1] which is a covering space of BX [10, Theorem 3.7]. Now there is a chain equivalence between $C^*(BX, *)$ and $C^{*-1}(B_X X)$ where $* \in BX_0$ is the unique vertex. There are corresponding isomorphisms of homology and cohomology groups (with a shift of one dimension). This is defined by mapping $(x, x_1, \dots, x_n) \in B_X X^{(n-1)}$ to $(x, x_1, \dots, x_n) \in BX^{(n)}$ using the description given in [10, Examples 3.4.1 and 2]. Note that $\partial^{\epsilon}_{i-1}$ in $B_X X$ coincides with ∂^{ϵ}_i in BX for $i = 2, 3, \dots, n$ whilst $\partial^1_1 = \partial^0_1$ in BX (both are given by $(x_1, x_2, \dots, x_n) \to (x_2, \dots, x_n)$) and that these two cancel out as a pair of the boundary formula.

Thus we have:

Theorem 5.14. *The rack space BX of any rack admits a covering space with the same homology groups but in dimensions all shifted one lower.* $\square$

The chain equivalence can be realised using a map $|B_X X| \times S^1 \to |BX|$ defined as follows. Embed the $(n-1)$-cube $(x, x_1, \dots, x_n)$ of $B_X X$ as the central $(n-1)$-cube perpendicular to the first direction in the n-cube $(x, x_1, \dots, x_n)$ of BX. Then use the remaining coordinate to extend to $B_X X \times [-1, 1]$. Since $\partial^1_1 = \partial^0_1$ in BX, this map factors via $B_X X \times [-1, 1]/-1 \sim 1$, that is, $B_X X \times S^1$. Then the above chain equivalence is given by crossing with the fundamental class of S^1 and using this map.

It is an interesting question to characterise spaces which have the property of admitting a covering space with the same homology groups shifted one dimension.

Permutation and dihedral racks. Let $\rho\colon P \to P$ be a fixed permutation of the set P. Then the *permutation rack* P_ρ is P with $i^j = \rho(i)$ for all i, j.

Combining results of Flower [14] and Greene [15], which were proved using the cobordism by moves technique of section 4, we have:

Theorem 5.15. *For a permutation rack P_ρ:*

(1) $\pi_2(BP_\rho)$ *is freely generated by one twisted unknot for each finite orbit, with the number of twists equal to the length of the orbit, together with a pair of linked unknots (each unknot having a single twist) for each unordered pair of unequal orbits.*

(2) $H_2(BP\rho)$ *is as in* (1) *except that there is a generator for each ordered pair of unequal orbits.* □

Let D_n denote the *dihedral rack* on n elements: $D_n = \{0, 1, \ldots, n-1\}$, and $i^j = 2j - i \bmod n$ for all i, j.

Theorem 5.16 ([15])**.**

$$H_2(BD_n) = \begin{cases} \mathbb{Z} & \text{for } n \text{ odd}, \\ H_2(BD_n) = \mathbb{Z}^4 & \text{for } n = 2 \bmod 4, \\ H_2(BD_n) \geq \mathbb{Z}^4 & \text{otherwise.} \qquad \square \end{cases}$$

Remark. A calculation using Maple shows $H_2(BD_4) = \mathbb{Z}^4 + \mathbb{Z}_2^2$, so the inequality of the theorem can be strict. For further details see [14, 15].

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