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THE RADIUS OF STARLIKENESS OF NORMALIZED BESSEL FUNCTIONS OF THE FIRST KIND

ÁRPÁD BARICZ, PÁL AUREL KUPÁN, AND RÓBERT SZÁSZ

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Abstract. In this note our aim is to determine the radius of starlikeness of the normalized Bessel functions of the first kind for three different kinds of normalization. The key tool in the proof of our main result is the Mittag-Leffler expansion for Bessel functions of the first kind and the fact that, according to Ismail and Muldoon, the smallest positive zeros of some Dini functions are less than the first positive zero of the Bessel function of the first kind.

1. Introduction

Let $\mathbf{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ be the open disk with center $z_0 \in \mathbb{C}$ and radius $r > 0$ and let us denote the particular disk $\mathbf{D}(0, 1)$ by $\mathbf{D}$. Moreover, let $\mathcal{A}$ be the class of analytic functions defined in the unit disk $\mathbf{D}$, which can be normalized as $f(z) = z + a_2z^2 + \ldots$, that is, $f(0) = f'(0) - 1 = 0$. The class of starlike functions, denoted by $\mathcal{S}^*$, is the subclass of $\mathcal{A}$ which consists of functions f for which the domain $f(\mathbf{D})$ is starlike with respect to 0. An analytic description of $\mathcal{S}^*$ is

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} \left| \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0 \text{ for all } z \in \mathbf{D} \right. \right\}.$$

Moreover, consider the class of starlike functions of order $\beta \in [0, 1)$, that is,

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} \left| \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \beta \text{ for all } z \in \mathbf{D} \right. \right\}.$$

The real numbers

$$r^*(f) = \sup\left\{ r > 0 \left| \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0 \text{ for all } z \in \mathbf{D}(0, r) \right. \right\}$$

and

$$r^*_\beta(f) = \sup\left\{ r > 0 \left| \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \beta \text{ for all } z \in \mathbf{D}(0, r) \right. \right\}$$

are called the radius of starlikeness and the radius of starlikeness of order β of the function f, respectively. We note that in fact $r^*(f)$ is the largest radius such that $f(\mathbf{D}(0, r^*(f)))$ is a starlike domain with respect to 0.

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Now, consider the Bessel function of the first kind [19], which is a particular solution of the second-order linear homogeneous Bessel differential equation. This function has the infinite series representation

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

where $z \in \mathbb{C}$ and $\nu \in \mathbb{C}$ such that $\nu \neq -1, -2, \ldots$. Observe that the Bessel function J_ν does not belong to class $\mathcal{A}$. Thus, it is natural to consider the following three kinds of normalization of the Bessel function of the first kind:

$$f_\nu(z) = \left[2^\nu \Gamma(\nu+1) J_\nu(z)\right]^{1/\nu}, \nu \neq 0, \tag{1.1}$$

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z) \tag{1.2}$$

and

$$h_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu/2} J_\nu(\sqrt{z}). \tag{1.3}$$

Clearly the functions f_ν, g_ν and h_ν belong to the class $\mathcal{A}$. We note here that in fact

$$f_\nu(z) = \exp\left[\frac{1}{\nu} \operatorname{Log}\left(2^\nu \Gamma(\nu+1) J_\nu(z)\right)\right],$$

where Log represents the principal branch of the logarithm function and every many-valued function considered in this paper is taken with the principal branch.

Now, let us recall some results on the geometric behavior of the functions f_ν, g_ν and h_ν. Brown [6] determined the radius of starlikeness for f_ν in the case when $\nu > 0$. Namely, in [6, Theorem 2] it was shown that the radius $r^*(f_\nu)$ is the smallest positive zero of the function $z \mapsto J'_\nu(z)$. Moreover, in [6, Theorem 3] Brown proved that if $\nu > 0$, then the radius of starlikeness of the function g_ν is the smallest positive zero of the function $z \mapsto zJ'_\nu(z) + (1-\nu)J_\nu(z)$. Kreyszig and Todd [13, Theorem 3] proved that when $\nu > -1$ the function g_ν is univalent in the circle $|z| \leq \rho_\nu$ but not in any concentric circle with larger radius, where ρ_ν is the first maximum of the function g_ν on the positive real axis. Brown [6, p. 282] pointed out that when $\nu > 0$ the radius of starlikeness of the function g_ν, that is, $r^*(g_\nu)$, is exactly the radius of univalence ρ_ν obtained by Kreyszig and Todd [13]. Furthermore, Brown [7, Theorem 5.1] showed that the radius of starlikeness of the function g_ν is also ρ_ν when $\nu \in (-1/2, 0)$. On the other hand, Hayden and Merkes [9, Theorem C] deduced that when $\mu = \operatorname{Re}\nu > -1$ the radius of starlikeness of g_ν is not less than the smallest positive zero of g'_μ. It is worth mentioning that Brown used the methods of Nehari [15] and Robertson [16], and an important tool in the proofs was the fact that the Bessel function of the first kind is a particular solution of the Bessel differential equation. For related (more general) results the interested reader is referred to [8, 14, 16, 20] and to the references therein. Finally, let us mention that other geometric properties of the functions g_ν and h_ν were obtained in [2, 4, 5, 17, 18]. See also the references therein.

Motivated by the above results in this paper we make a contribution to the subject and we determine the radius of starlikeness of order β for the functions f_ν, g_ν and h_ν. We note that our approach is much simpler than the methods used in [6, 7, 9, 13] and is based only on the Mittag-Leffler expansion for Bessel functions of the first kind and on the fact that the smallest positive zeros of certain Dini functions are less than the first positive zero of the Bessel function of the first kind, according to Ismail and Muldoon [10, 11].

2. Starlikeness of order β of normalized Bessel functions

Our main result is the following theorem. Here I_ν denotes the modified Bessel function of the first kind, which in view of the relation $I_\nu(z) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}z)$ is also sometimes called the Bessel function of the first kind with imaginary argument.

Theorem 1. *Let $1 > \beta \geq 0$. Then the following assertions are true:*

- **a.** *If $\nu \in (-1, 0)$, then $r^*_\beta(f_\nu) = x_{\nu,\beta}$, where $x_{\nu,\beta}$ denotes the unique positive root of the equation $zI'_\nu(z) - \beta\nu I_\nu(z) = 0$. Moreover, if $\nu > 0$, then we have $r^*_\beta(f_\nu) = x_{\nu,\beta,1}$, where $x_{\nu,\beta,1}$ is the smallest positive root of the equation $zJ'_\nu(z) - \beta\nu J_\nu(z) = 0$.*
- **b.** *If $\nu > -1$, then $r^*_\beta(g_\nu) = y_{\nu,\beta,1}$, where $y_{\nu,\beta,1}$ is the smallest positive root of the equation $zJ'_\nu(z) + (1 - \beta - \nu)J_\nu(z) = 0$.*
- **c.** *If $\nu > -1$, then $r^*_\beta(h_\nu) = z_{\nu,\beta,1}$, where $z_{\nu,\beta,1}$ is the smallest positive root of the equation $zJ'_\nu(z) + (2 - 2\beta - \nu)J_\nu(z) = 0$.*

In particular, when $\beta = 0$, we get the following result.

Corollary 1. *The following assertions are true:*

- **a.** *If $\nu \in (-1, 0)$, then the radius of starlikeness of f_ν is $x_{\nu,0}$, where $x_{\nu,0}$ is the unique positive root of the equation $I'_\nu(z) = 0$. If $\nu > 0$, then the radius of starlikeness of the function f_ν is $x_{\nu,0,1}$, which denotes the smallest positive root of the equation $J'_\nu(z) = 0$.*
- **b.** *If $\nu > -1$, then the radius of starlikeness of the function g_ν is $y_{\nu,0,1}$, which denotes the smallest positive root of the equation $zJ'_\nu(z) + (1-\nu)J_\nu(z) = 0$.*
- **c.** *If $\nu > -1$, then the radius of starlikeness of the function h_ν is $z_{\nu,0,1}$, which denotes the smallest positive root of the equation $zJ'_\nu(z) + (2-\nu)J_\nu(z) = 0$.*

Observe that parts **a** and **b** of Corollary 1 complement the results of [6, Theorem 2], [6, Theorem 3] and [7, Theorem 5.1], mentioned above. Part **c** complements the results from [2, 5, 17, 18]. It is of interest to note here that very recently Szász [17] proved that the normalized Bessel function h_ν is starlike if and only if $\nu \geq \nu_0$, where $\nu_0 = -0.5623\ldots$ is the root of the equation $h'_\nu(1) = 0$, that is, $J'_\nu(1) + (2 - \nu)J_\nu(1) = 0$. Finally, we mention that if we consider the function $z \mapsto \lambda_\nu(z) = h_\nu(z)/z$, then parts **c** of Theorem 1 and Corollary 1 can be rewritten in terms of convex functions. The idea is to use the differentiation formula

$$\lambda'_\nu(z) = -\frac{1}{4(\nu+1)}\lambda_{\nu+1}(z)$$

together with the well-known duality theorems of Alexander [1] and Jack [12]. See also [4, p. 25] for the results of Alexander and Jack, and also [4, Ch. 2] for similar results on convex Bessel functions.

Proof of Theorem 1. First we prove part **a** for $\nu > 0$ and parts **b** and **c** for $\nu > -1$. We need to show that the inequalities

$$\mathrm{Re}\left[\frac{zf'_\nu(z)}{f_\nu(z)}\right] > \beta, \quad \mathrm{Re}\left[\frac{zg'_\nu(z)}{g_\nu(z)}\right] > \beta \quad \text{and} \quad \mathrm{Re}\left[\frac{zh'_\nu(z)}{h_\nu(z)}\right] > \beta \tag{2.1}$$

are valid for all $\nu > 0$ and $z \in \mathbf{D}(0, x_{\nu,\beta,1})$, $\nu > -1$ and $z \in \mathbf{D}(0, y_{\nu,\beta,1})$, and $\nu > -1$ and $z \in \mathbf{D}(0, z_{\nu,\beta,1})$, respectively, and each of the above inequalities does not hold in any larger disk.

Lommel's well-known result states that if $\nu > -1$, then the zeros of the Bessel function J_ν are all real. Thus, if $j_{\nu,n}$ denotes the n-th positive zero of the Bessel function J_ν, then the Bessel function admits the Weierstrassian decomposition of the form [19, p. 498]

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right), \tag{2.2}$$

and this infinite product is uniformly convergent on each compact subset of $\mathbb{C}$. Logarithmic differentiation of (2.2) yields

$$\frac{zJ_\nu'(z)}{J_\nu(z)} = \nu - \sum_{n\geq 1} \frac{2z^2}{j_{\nu,n}^2 - z^2}, \tag{2.3}$$

which in view of the recurrence relation [19, p. 45] $zJ_\nu'(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z)$ is equivalent to the Mittag-Leffler expansion [19, p. 498]

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n\geq 1} \frac{2z}{j_{\nu,n}^2 - z^2}.$$

Consequently, in view of (1.1), (1.2), (1.3) and (2.3) we obtain

$$\frac{zf_\nu'(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zJ_\nu'(z)}{J_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1} \frac{2z^2}{j_{\nu,n}^2 - z^2},$$

$$\frac{zg_\nu'(z)}{g_\nu(z)} = 1 - \nu + \frac{zJ_\nu'(z)}{J_\nu(z)} = 1 - \sum_{n\geq 1} \frac{2z^2}{j_{\nu,n}^2 - z^2}$$

and

$$\frac{zh_\nu'(z)}{h_\nu(z)} = 1 - \frac{\nu}{2} + \frac{1}{2}\frac{\sqrt{z}J_\nu'(\sqrt{z})}{J_\nu(\sqrt{z})} = 1 - \sum_{n\geq 1} \frac{z}{j_{\nu,n}^2 - z}.$$

It is known [19, p. 597] that in cases $\alpha + \nu > 0$ and $\nu > -1$ the so-called Dini function $z \mapsto zJ_\nu'(z) + \alpha J_\nu(z)$ has only real zeros, and according to Ismail and Muldoon [11, p. 11] we know that the smallest positive zero of the above function is less than $j_{\nu,1}$. This in turn implies that $x_{\nu,\beta,1} < j_{\nu,1}$ for all $\nu > 0$, $y_{\nu,\beta,1} < j_{\nu,1}$ for all $\nu > -1$, and $z_{\nu,\beta,1} < j_{\nu,1}$ for all $\nu > -1$. In other words, for all $0 \leq \beta < 1$ and $n \in \{2, 3, \ldots\}$ we have $\mathbf{D}(0, x_{\nu,\beta,1}) \subset \mathbf{D}(0, j_{\nu,1}) \subset \mathbf{D}(0, j_{\nu,n})$ when $\nu > 0$, $\mathbf{D}(0, y_{\nu,\beta,1}) \subset \mathbf{D}(0, j_{\nu,1}) \subset \mathbf{D}(0, j_{\nu,n})$ when $\nu > -1$, and $\mathbf{D}(0, z_{\nu,\beta,1}) \subset \mathbf{D}(0, j_{\nu,1}) \subset \mathbf{D}(0, j_{\nu,n})$ when $\nu > -1$. On the other hand, it is known [17] that if $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ such that $\alpha > |z|$, then

$$\frac{|z|}{\alpha - |z|} \geq \operatorname{Re}\left(\frac{z}{\alpha - z}\right). \tag{2.4}$$

By using (2.4), we obtain for all $\nu > -1$, $n \in \{1, 2, \ldots\}$ and $z \in \mathbf{D}(0, j_{\nu,1})$ the inequality

$$\frac{|z|^2}{j_{\nu,n}^2 - |z|^2} \geq \operatorname{Re}\left(\frac{z^2}{j_{\nu,n}^2 - z^2}\right), \tag{2.5}$$

which in turn implies that

$$\operatorname{Re}\left[\frac{zf_\nu'(z)}{f_\nu(z)}\right] = 1 - \frac{1}{\nu}\operatorname{Re}\left[\sum_{n\geq 1} \frac{2z^2}{j_{\nu,n}^2 - z^2}\right] \geq 1 - \frac{1}{\nu}\sum_{n\geq 1} \frac{2|z|^2}{j_{\nu,n}^2 - |z|^2} = \frac{|z|f_\nu'(|z|)}{f_\nu(|z|)},$$

$$\operatorname{Re}\left[\frac{zg_\nu'(z)}{g_\nu(z)}\right]=1-\operatorname{Re}\left[\sum_{n\geq1}\frac{2z^2}{j_{\nu,n}^2-z^2}\right]\geq1-\sum_{n\geq1}\frac{2|z|^2}{j_{\nu,n}^2-|z|^2}=\frac{|z|g_\nu'(|z|)}{g_\nu(|z|)}$$

and

$$\operatorname{Re}\left[\frac{zh_\nu'(z)}{h_\nu(z)}\right]=1-\operatorname{Re}\left[\sum_{n\geq1}\frac{z}{j_{\nu,n}^2-z}\right]\geq1-\sum_{n\geq1}\frac{|z|}{j_{\nu,n}^2-|z|}=\frac{|z|h_\nu'(|z|)}{h_\nu(|z|)},$$

with equality when $z=|z|=r$. The minimum principle for harmonic functions and the previous inequalities imply that the corresponding inequalities in (2.1) are valid if and only if we have $|z|<x_{\nu,\beta,1}$, $|z|<y_{\nu,\beta,1}$, and $|z|<z_{\nu,\beta,1}$, respectively, where $x_{\nu,\beta,1}$, $y_{\nu,\beta,1}$ and $z_{\nu,\beta,1}$ are the smallest positive roots of the equations

$$rf_\nu'(r)/f_\nu(r)=\beta,\quad rg_\nu'(r)/g_\nu(r)=\beta$$

and

$$rh_\nu'(r)/h_\nu(r)=\beta,$$

which are equivalent to

$$rJ_\nu'(r)-\beta\nu J_\nu(r)=0,\quad rJ_\nu'(r)+(1-\beta-\nu)J_\nu(r)=0$$

and

$$rJ_\nu'(r)+(2-2\beta-\nu)J_\nu(r)=0,$$

respectively.

Now, we prove the statement of part **a** when $\nu\in(-1,0)$. First observe that the counterpart of (2.4), that is,

$$\operatorname{Re}\left(\frac{z}{\alpha-z}\right)\geq\frac{-|z|}{\alpha+|z|},\tag{2.6}$$

is valid for all $\alpha\in\mathbb{R}$ and $z\in\mathbb{C}$ such that $\alpha>|z|$. Indeed, if we have $z=x+\mathrm{i}y$ and $m=|z|=\sqrt{x^2+y^2}$, then (2.6) is equivalent to $\alpha(\alpha-m)(m+x)\geq0$, which is clearly true. By using (2.6), we obtain for all $\nu>-1$, $n\in\{1,2,\dots\}$ and $z\in\mathbf{D}(0,j_{\nu,1})$ the inequality

$$\operatorname{Re}\left(\frac{z^2}{j_{\nu,n}^2-z^2}\right)\geq\frac{-|z|^2}{j_{\nu,n}^2+|z|^2},\tag{2.7}$$

which in turn implies that

$$\operatorname{Re}\left[\frac{zf_\nu'(z)}{f_\nu(z)}\right]=1-\frac{1}{\nu}\operatorname{Re}\left[\sum_{n\geq1}\frac{2z^2}{j_{\nu,n}^2-z^2}\right]\geq1+\frac{1}{\nu}\sum_{n\geq1}\frac{2|z|^2}{j_{\nu,n}^2+|z|^2}=\frac{\mathrm{i}|z|f_\nu'(\mathrm{i}|z|)}{f_\nu(\mathrm{i}|z|)}.$$

This time we have equality if $z=\mathrm{i}|z|=\mathrm{i}r$, and from the above inequality we conclude that the first inequality in (2.1) holds if and only if $|z|<x_{\nu,\beta}$, where $x_{\nu,\beta}$ denotes the positive root of the equation $\mathrm{i}rf_\nu'(\mathrm{i}r)/f_\nu(\mathrm{i}r)=\beta$, which is equivalent to $rI_\nu'(r)-\beta\nu I_\nu(r)=0$. All we need to prove is that $x_{\nu,\beta}$ is unique and $x_{\nu,\beta}<j_{\nu,1}$ for all $\beta\in[0,1)$ and $\nu\in(-1,0)$, since in order to use (2.7) we tacitly assumed that for all $\beta\in[0,1)$ and $n\in\{2,3,\dots\}$ we have $\mathbf{D}(0,x_{\nu,\beta})\subset\mathbf{D}(0,j_{\nu,1})\subset\mathbf{D}(0,j_{\nu,n})$ when $\nu\in(-1,0)$. For this recall that in case $-1<\nu<-\alpha$ the Dini function $z\mapsto zJ_\nu'(z)+\alpha J_\nu(z)$ has all its zeros real and a single pair of conjugate purely imaginary zeros [19, p. 597]. Moreover, due to Ismail and Muldoon [10, eq. (3.2)] we know that if $\pm i\xi$ (ξ real) denotes the purely imaginary zeros of the Dini function $z \mapsto zJ'_\nu(z) + \alpha J_\nu(z)$, then

$$\xi^2 < -\frac{\alpha+\nu}{2+\alpha+\nu} j_{\nu,1}^2.$$

This in turn implies that

$$x_{\nu,\beta}^2 < -\frac{\nu(1-\beta)}{2+\nu(1-\beta)} j_{\nu,1}^2 < j_{\nu,1}^2,$$

as we required. Finally, consider the function $q_\nu : (0,\infty) \to \mathbb{R}$, defined by $q_\nu(r) = rI'_\nu(r)/I_\nu(r) - \beta\nu$. By using the asymptotic relations for small and large values of r for the function $r \mapsto I_\nu(r)$, it can be verified that $rI'_\nu(r)/I_\nu(r)$ tends to ν as $r \to 0$, and tends to infinity as $r \to \infty$. Moreover, it is known (see for example [3]) that the function $r \mapsto rI'_\nu(r)/I_\nu(r)$ is increasing on $(0,\infty)$ for all $\nu > -1$. Thus the function q_ν is increasing, $q_\nu(r)$ tends to $\nu(1-\beta) < 0$ as $r \to 0$, and tends to infinity as $r \to \infty$. Consequently, the graph of q_ν intersects the r-axis only once, and thus the equation $rI'_\nu(r) - \beta\nu I_\nu(r) = 0$ has only one solution. This completes the proof. □

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References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. (2) **17** (1915), no. 1, 12–22, DOI 10.2307/2007212. MR1503516

[2] Árpád Baricz, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73** (2008), no. 1-2, 155–178. MR2429033 (2009e:33012)

[3] Árpád Baricz, *Turán type inequalities for modified Bessel functions*, Bull. Aust. Math. Soc. **82** (2010), no. 2, 254–264, DOI 10.1017/S000497271000002X. MR2685149 (2011k:33009)

[4] Árpád Baricz, *Generalized Bessel functions of the first kind*, Lecture Notes in Mathematics, vol. 1994, Springer-Verlag, Berlin, 2010. MR2656410 (2011f:33007)

[5] Árpád Baricz and Saminathan Ponnusamy, *Starlikeness and convexity of generalized Bessel functions*, Integral Transforms Spec. Funct. **21** (2010), no. 9-10, 641–653, DOI 10.1080/10652460903516736. MR2743533 (2012c:33016)

[6] R. K. Brown, *Univalence of Bessel functions*, Proc. Amer. Math. Soc. **11** (1960), 278–283. MR0111846 (22 #2706)

[7] R. K. Brown, *Univalent solutions of $W'' + pW = 0$*, Canad. J. Math. **14** (1962), 69–78. MR0131009 (24 #A863)

[8] R. K. Brown, *Univalence of normalized solutions of $W''(z) + p(z)W(z) = 0$*, Internat. J. Math. Math. Sci. **5** (1982), no. 3, 459–483, DOI 10.1155/S0161171282000441. MR669101 (84g:30014)

[9] T. L. Hayden and E. P. Merkes, *Chain sequences and univalence*, Illinois J. Math. **8** (1964), 523–528. MR0166344 (29 #3621)

[10] M. E. H. Ismail and M. E. Muldoon, *Zeros of combinations of Bessel functions and their derivatives*, Appl. Anal. **31** (1988), no. 1-2, 73–90, DOI 10.1080/00036818808839817. MR1017504 (90k:33022)

[11] Mourad E. H. Ismail and Martin E. Muldoon, *Bounds for the small real and purely imaginary zeros of Bessel and related functions*, Methods Appl. Anal. **2** (1995), no. 1, 1–21. MR1337450 (96e:33007)

[12] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2) **3** (1971), 469–474. MR0281897 (43 #7611)

[13] Erwin Kreyszig and John Todd, *The radius of univalence of Bessel functions. I*, Illinois J. Math. **4** (1960), 143–149. MR0110827 (22 #1695)

[14] E. P. Merkes, M. S. Robertson, and W. T. Scott, *On products of starlike functions*, Proc. Amer. Math. Soc. **13** (1962), 960–964. MR0142747 (26 #316)

[15] Zeev Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551. MR0029999 (10,696e)

[16] M. S. Robertson, *Schlicht solutions of $W'' + pW = 0$*, Trans. Amer. Math. Soc. **76** (1954), 254–274. MR0061171 (15,786c)

[17] Róbert Szász, *On starlikeness of Bessel functions of the first kind*, 8th Joint Conference on Mathematics and Computer Science—MaCS 2010, NOVADAT Ltd., Győr, 2011, pp. 63–70. MR2882512 (2012k:30058)

[18] Róbert Szász and Pál Aurel Kupán, *About the univalence of the Bessel functions*, Stud. Univ. Babeş-Bolyai Math. **54** (2009), no. 1, 127–132. MR2486953 (2010e:30015)

[19] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, England, 1944. MR0010746 (6,64a)

[20] Herbert S. Wilf, *The radius of univalence of certain entire functions*, Illinois J. Math. **6** (1962), 242–244. MR0138734 (25 #2177)

Department of Mathematics and Informatics, Sapientia Hungarian University of Transylvania, Târgu Mureş 540485, Romania

Current address: Department of Economics, Babeş-Bolyai University, Cluj-Napoca 400591, Romania

E-mail address: bariczocsi@yahoo.com

Department of Mathematics and Informatics, Sapientia Hungarian University of Transylvania, Târgu Mureş 540485, Romania

E-mail address: kupanp@ms.sapientia.ro

Department of Mathematics and Informatics, Sapientia Hungarian University of Transylvania, Târgu Mureş 540485, Romania

E-mail address: rszasz@ms.sapientia.ro