THE RADIUS OF UNIVALENCE OF THE ERROR FUNCTION

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Communicated by Richard Bellman, July 29, 1958

We shall determine the radius of univalence of the error function

$$\operatorname{erf} z = \int_0^z e^{-t^2} dt,$$

that is, the radius of the largest open circular disk, $|z| < \rho$, in which erf z is schlicht. Some lower bounds for ρ have been obtained previously, namely:

$$\left\{\frac{1}{2}\left[(\pi^2+1)^{1/2}-1\right]\right\}^{1/2} = 1.07 \cdots, \qquad \text{[Nehari, 1]},$$

$$(\pi/2)^{1/2} = 1.25 \cdots, \qquad \text{[Rogozin, 2]},$$

the largest positive root R, of $x - \arctan x = \pi$, where $x = (4R^4-1)^{1/2}$; $R = 1.51 \cdots$, [Reade, **3**]. These bounds were obtained by different, rather general methods. Our methods are based on special properties of erf z, and were suggested by a detailed study of actual numerical values of erf z, which were computed on the IBM 704 at the National Bureau of Standards by E. Brauer and J. C. Gager.

THEOREM. *The radius of univalence of* erf z *is the minimum distance from the origin of points, not on the x-axis, for which* erf z *is real.*

Two proofs of this are given, one depending on the properties of the maps of $|z| = r$, and the other on the properties of the curves in the z-plane on which arg erf z is constant.

Our proofs have a constructive character and can be used to obtain bounds for ρ. With a small amount of hand calculation we find

$$1.5666 < \rho < 1.5858.$$

If we make use of the results of the elaborate calculations already referred to, we find that a plausible, seven decimal value of ρ is 1.5748376.

Added in proof, October 10, 1958. We have now shown that the situation is quite different if we use another normalization: the radius of univalence of $E(z) = \exp z^2 \operatorname{erf} z$ is 0.92413887 . . . .

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FUNCTIONS WHOSE PARTIAL DERIVATIVES ARE MEASURES

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Communicated by W. S. Massey, July 16, 1958

Let x denote a generic point of euclidean N-space $R^N (N \geqq 2)$. We consider the space $\mathfrak{F}$ of all summable functions $f(x)$ such that the gradient grad f (in the distribution theory sense) is a totally finite measure. $I(f)$ denotes the total variation of the vector measure grad f. In case grad f is a function F we have

$$I(f) = \int_{R^N} | F(x) | \, dx.$$

We write H_k for Hausdorff k-measure; and fr E for the frontier of a set E. Fr E is *rectifiable* if it is the Lipschitzian image of a compact subset of R^{N-1}.

One ought to be able to determine the primitive f with greater precision than grad f, at least in certain cases. Our main result is that indeed f can be determined up to H_{N-1}-measure 0 in two (quite opposed) cases: (1) grad f is a function; (2) the range of f is a discrete set, which we may take to be the integers. More precisely, let $\mathfrak{F}_1, \mathfrak{F}_2$ be the sets of those $f \in \mathfrak{F}$ satisfying (1) and (2) respectively. Let $\mathfrak{F}_{01}$ be the set of all Lipschitzian functions f with compact support. Let $\mathfrak{F}_{02}$ be the set of all functions f with the following property: there exist a closed oriented $(N-1)$-polyhedron A and a Lipschitzian mapping $g(w)$ from A into R^N such that, for every $x \notin g(A)$, $f(x)$ is the degree of the mapping g at x, and $f(x) = 0$ for $x \in g(A)$. Write $J(w)$ for the Jacobian vector of $g(w)$, wherever it exists. Let Q denote the set of points $x \in g(A)$ at which there is a nonunique tangent; more precisely, we say that $x \in Q$ if there exist $w, w' \in A$ such that: (1) g is