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# THE RADON-NIKODYM PROPERTY AND DENTABLE SETS IN BANACH SPACES

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ABSTRACT. In order to prove a Radon-Nikodym theorem for the Bochner integral, Rieffel [5] introduced the class of "dentable" subsets of Banach spaces. Maynard [3] later introduced the strictly larger class of "$s$-dentable" sets, and extended Rieffel's result to show that *a Banach space has the Radon-Nikodym property if and only if every bounded nonempty subset of E is s-dentable*. He left open, however, the question as to whether, in a space with the Radon-Nikodym property, every bounded nonempty set is dentable. In the present note we give an elementary construction which shows this question has an affirmative answer.

**Definitions.** A Banach space $E$ has the *Radon-Nikodym property* if for each totally finite positive measure space $(X, \Sigma, \mu)$ and each $E$-valued, $\mu$-continuous measure $m$ on $\Sigma$ with $|m|(X) < \infty$, there is a Bochner integrable $f$ from $X$ to $E$ such that $m(E) = \int_E f\, d\mu$ $(E \in \Sigma)$.

A subset $A$ of $E$ is *dentable* if for every $\epsilon > 0$, there exists $x \in A$ such that $x \notin \operatorname{cl}\operatorname{co}(A \setminus S_\epsilon(x))$. [Here $\operatorname{co} B$ denotes the convex hull of $B$, $\operatorname{cl}\operatorname{co} B$ is its closure and $S_\epsilon(x) = \{y \in E: \|x - y\| \leq \epsilon\}$.] A bounded set $A$ is called *$s$-dentable* if for each $\epsilon > 0$ there exists $x \in A$ such that $x \notin s(A \setminus S_\epsilon(x))$. [Here $s(B) = \{\sum_{i=1}^{\infty} \lambda_i x_i: \lambda_i \geq 0, \sum \lambda_i = 1, \{x_i\} \subset B\}$.] A point $x \in A$ is a *denting* [*$s$-denting*] *point* if for all $\epsilon > 0$, $x \notin \operatorname{cl}\operatorname{co}(A \setminus S_\epsilon(x))$ [$x \notin s(A \setminus S_\epsilon(x))$].

Dentable sets are $s$-dentable, and Maynard has given an example of a bounded set which is $s$-dentable but not dentable. Rieffel has shown that if $A$ is not dentable, then neither is $\operatorname{clco} A$. The analogous assertion fails for $s$-dentability: The closed unit ball of $C([0, 1])$ is not dentable [5], but the constantly 1 function is an $s$-denting point. By Lemma 2 below, its interior is not $s$-dentable.

**Lemma 1.** *A subset $A$ of $E$ is not dentable if and only if there exists*

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*$\epsilon > 0$ such that $A \subset \operatorname{cl}\operatorname{co}(A \setminus S_\epsilon(x))$ for each $x \in A$. If $A$ is closed and convex, this is equivalent to $A = \operatorname{cl}\operatorname{co}(A \setminus S_\epsilon(x))$ for each $x \in A$.*

**Proof.** One implication is trivial. For the other, suppose $A$ is not dentable. Then there exists $2\epsilon > 0$ such that for each $y \in A$, $y \in \operatorname{cl}\operatorname{co}(A \setminus S_{2\epsilon}(y))$. Suppose that $x, y \in A$ and $\|x - y\| > \epsilon$. Then $y \in \operatorname{cl}\operatorname{co}(A \setminus S_\epsilon(x))$. On the other hand, if $\|x - y\| \leq \epsilon$, then $S_\epsilon(x) \subset S_{2\epsilon}(y)$ so that $y \in \operatorname{cl}\operatorname{co}(A \setminus S_{2\epsilon}(y)) \subset \operatorname{cl}\operatorname{co}(A \setminus S_\epsilon(x))$, completing the proof.

**Lemma 2.** *Suppose $C$ is a closed convex set in $E$ with nonempty interior (denoted by $\operatorname{int} C$) and suppose $C$ is not dentable. Then there exists $\epsilon > 0$ such that for each $x \in C$, $\operatorname{int} C \subset \operatorname{co}[\operatorname{int} C \setminus S_\epsilon(x)]$. In particular, $\operatorname{int} C$ is not s-dentable.*

**Proof.** By Lemma 1, there exists $\epsilon > 0$ such that $C = \operatorname{cl}\operatorname{co}(C \setminus S_\epsilon(x))$ for each $x \in C$. Let $J_x = C \setminus S_\epsilon(x)$ so that $\operatorname{int} J_x = \operatorname{int} C \setminus S_\epsilon(x)$. For $\epsilon$ sufficiently small, $\operatorname{int} J_x \neq \emptyset$ for each $x \in C$. Fix $x$ and let $J = J_x$. Then $C = \operatorname{cl}\operatorname{co} J$ and we want to show that $\operatorname{int}(\operatorname{cl}\operatorname{co} J) \subset \operatorname{co}(\operatorname{int} J)$. Note that $J \subset \operatorname{cl}(\operatorname{int} J)$: If $y \in J$, then $y \in C$ and $\|y - x\| > \epsilon$. Let $z \in \operatorname{int} C$, so that $[z, x) \subset \operatorname{int} C$ and there exists $u \in [z, x) \cap S_\epsilon(x)$. Therefore, $[u, y) \subset \operatorname{int} C$ so for some $v \in [u, y)$ we have $[v, y) \subset \operatorname{int} C \setminus S_\epsilon(x)$. Thus $y \in \operatorname{cl}(\operatorname{int} J)$. It now follows that $\operatorname{co} J \subset \operatorname{cl}\operatorname{co}(\operatorname{int} J)$. Taking the interior of each side, we conclude that

$$\operatorname{int}(\operatorname{cl}\operatorname{co} J) = \operatorname{int}(\operatorname{co} J) \subset \operatorname{int}(\operatorname{cl}\operatorname{co}(\operatorname{int} J)) = \operatorname{co}(\operatorname{int} J).$$

The equalities follow from the fact that the interior of a convex set coincides with the interior of its closure.

**Proposition.** *If $E$ contains a bounded nonempty set which is not dentable, then it contains a bounded closed convex and symmetric set $C$ which is not dentable and which has nonempty interior. In particular, $E$ can be renormed so that the new unit ball is not dentable and the interior of the new unit ball is not s-dentable.*

**Proof.** If $A$ is a bounded nonempty set which is not dentable, then the same is true of the sets $A_1 = A \cup (-A)$ (definition), $A_2 = \operatorname{cl}\operatorname{co} A_1$ (Rieffel [5, Proposition 2]) and $A_3 = S + A_2$, where $S$ is the closed unit ball of $E$ (easy computation). Let $C$ be the closure of $A_3$. Again, by Rieffel's proposition, $C$ is not dentable. By Lemma 2, $\operatorname{int} C$ is not $s$-dentable.

What we have shown is that every bounded subset of $E$ is dentable if and only if every bounded subset of $E$ is $s$-dentable. This yields the following corollary.

**Corollary:** *A Banach space $E$ has the Radon-Nikodym property if and only if every bounded nonempty set in $E$ is dentable.*

J. Diestel has raised the question of the relationship between the Radon-Nikodym property and the Krein-Milman property [every closed bounded convex set is the closed convex hull of its extreme points]. It is known that a space has both properties if it is reflexive or is a separable conjugate space [1], [4]. The Proposition shows that every bounded nonempty subset of $E$ is dentable if and only if the unit ball of every Banach space isomorphic to $E$ is dentable. Is the analogous assertion true for the Krein-Milman property? That is, if $E$ contains a closed, bounded nonempty convex set which is not the closed convex hull of its extreme points, can $E$ be renormed so that its new unit ball is not the closed convex hull of its extreme points? The answer is affirmative for the spaces $(m)$ and $L_1([0, 1])$ [2]. In spaces with the Radon-Nikodym property does every closed bounded convex set have a denting point ($s$-denting point, extreme point)?

**Added in proof.** Since this paper was submitted, a number of related results have appeared. Huff [6] has given an independent proof of the above corollary, by modifying the proof of Maynard's main theorem [3]. Lindenstrauss has used the corollary to show (cf. [7]) that the Radon-Nikodym property implies the Krein-Milman property, and the question concerning denting points has also received an affirmative answer in [7]. Huff and Morris [8] have shown that the Krein-Milman property implies the Radon-Nikodym property in conjugate spaces, but this implication remains open for arbitrary spaces. Finally, Edgar [9] has generalized Lindenstrauss' result by obtaining a Choquet-type representation theorem for bounded closed convex subsets of a Banach space with the Radon-Nikodym property.

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