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THE RADON-NIKODYM THEOREM FOR BANACH SPACE VALUED MEASURES

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The present notes are an updated version of a brief history of the Radon-Nikodym theorem for Banach space valued measures written by the first author in January, 1973. Since that time much progress has been made in this subject, and these notes are aimed at conveying some of the flavor of this progress and hopefully interesting readers in some of the problems that remain.

Our style will be informal. A few proofs are included. Complete details for most of what appears herein will be found in the finished version of [0].

There appears to be roughly three aspects to the theory of differentiation of vector-valued measures: analytic, operator theoretic and geometric. While these aspects necessarily are intimately interrelated, we shall try to discuss the Radon-Nikodym theorem for these three viewpoints separately. Our presentation will be, to a large extent, along historical lines, though occasionally we stray from this path.

In writing these notes, we have benefitted from conversations with many mathematicians. They have shown us examples and counterexamples, and have been kind enough to send us preprints of their related work (oftentimes they even sent handwritten copies of their work!). A nonexhaustive list includes: J. Batt, W. J. Davis, B. Faires, T. Figiel, A. Gleit, W. B. Johnson, P. Kranz, E. Leonard, D. R. Lewis, J. Lindenstrauss, R. H. Lohman, H. Maynard, P. Morris, R. R. Phelps, H. P. Rosenthal, C. Stegall and K. Sundaresan. To each we extend our gratitude. We were especially fortunate to have a number of long conversations with Bob Huff which were extremely beneficial and had a definite effect upon this version of these notes.

I. **Analytic Aspects of the Radon-Nikodym Theorem.** The start of the theory of vector-valued Radon-Nikodym theorems coincides (not too surprisingly) with the introduction of the first vector-valued integration theory by S. Bochner [1]. In this first paper on integration of vector-valued functions, Bochner notes that if every X-valued function of bounded variation defined on $[0, 1]$ is differentiable almost everywhere then each X-valued absolutely continuous function on $[0, 1]$ can be recovered from its derivative via the "Bochner" integral. It

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was left open, however, whether any infinite dimensional Banach space had the afore-mentioned property (called by some the Gelfand-Frechet property). The next year in a short note [2], Bochner demonstrated that $L_\infty[0, 1]$ did not possess the Gelfand-Frechet property. The existence of infinite dimensional Banach spaces with the Gelfand-Frechet property remained open until 1935 when G. Birkhoff established in [3] that Hilbert spaces possess the Gelfand-Frechet property. His proof was rather direct though tedious. An easier proof of a (formally) stronger statement is worth outlining: if $F : \Sigma \to H$ is countably additive and possesses finite variation $|F|$ then in a natural way F defines a continuous linear functional on the Hilbert space (natural inner product) $L_2(|F|; H)$. Applying the Riesz Representation Theorem gets the desired derivative. The reader most likely recognizes this as the classical von-Neumann argument for the Radon-Nikodym Theorem slightly generalized.

More difficult results soon followed. In a now-famous paper [4] introducing the class of uniformly convex Banach spaces, J. Clarkson showed that every uniformly convex Banach space is a Gelfand-Frechet space. He also observes that ℓ_1 is a Gelfand-Frechet space but c_0 and $L_1[0, 1]$ were *not* Gelfand-Frechet spaces; for example, the function $f: [0, 1] \to L_1[0, 1]$ given by $f(t) = c_{[0,t)}$ is nowhere differentiable but has bounded variation a fact that is easily established (that c_0 is not a Gelfand-Frechet space will be established momentarily).

In the same issue of the Transactions that Clarkson's paper appeared—the very next paper [5]—N. Dunford and M. Morse extended Clarkson's observation about ℓ_1 to the class of Banach spaces having "boundedly complete" Schauder bases. A sequence (x_n) of members of a Banach space X is called a *Schauder basis* whenever each $x \in X$ has a unique representation in the form $x = \sum_n a_n x_n$; the linear functionals $x \to a_n$ are always continuous and we denote them by f_n and call them coefficient functionals. The Schauder basis (x_n) is said to be *boundedly complete* whenever given any sequence (a_n) of scalars, if $\sup_n \|\sum_{k=1}^n a_k x_k\| < \infty$ then the series $\sum_n a_n x_n$ converges. Dunford and Morse showed that Banach spaces with boundedly complete bases are Gelfand-Frechet spaces; more precisely they proved the,

THEOREM. *Let (Ω, Σ) be a measurable space and let $F : \Sigma \to X$ be a countably additive measure possessing finite variation $|F|$. Suppose that X possesses a boundedly complete basis (x_n) (coefficient functionals (f_n)). Then F is differentiable with respect to $|F|$; moreover, the "natural" derivative works.*

The Dunford-Morse theorem was soon to be generalized by the Dunford-Pettis results. However, their proof is so elegant that it bears (frequent) repeating:

PROOF. We start by making the basis "monotone," i.e., if $x \in X$ so $x = \sum_n f_n(x)x_n$ — then define a new norm $|||\cdot|||$ on L as follows:

$$|||\cdot||| = \sup_n \left\| \sum_{k=1}^{n} f_k(x)x_k \right\|.$$

$|||\cdot|||$ is equivalent to $\|\cdot\|$ so (x_n, f_n) is still a boundedly complete Schauder basis for $(X, |||\cdot|||)$. Moreover, for any $n, j \geqq 1$ we have

$$\left|\left|\left| \sum_{k=1}^{n} a_k x_k \right|\right|\right| \leqq \left|\left|\left| \sum_{k=1}^{n+j} a_k x_k \right|\right|\right| .$$

Since F's differentiability is clearly invariant under isomorphism, we will show F is differentiable into X with $|||\cdot|||$.

For each n, let $F_n(A) = \sum_{k+1}^{n} f_k(F(A))x_k$. Then for each n, $f_n \circ F$ is a countably additive scalar-valued, $|F|$-continuous measure defined on Σ. Thus, $df_n \circ F/d|F|$ exists for each n.

Let $g_n : \Omega \to X$ be given by

$$g_n(w) = \sum_{k=1}^{n} \frac{df_k \circ F}{d|F|} x_k.$$

Clearly $g_n \in L_1(|F|; X)$ and for $n, j \geqq 1$, $|||g_n(w)||| \leqq |||g_{n+j}(w)|||$. Furthermore, $F_n = \int g_n \, d|F|$ so that

$$\int |||g_n(w)|||d|F|(w) = |F_n|(\Omega) \leqq |F|(\Omega).$$

Thus $d(w) = \lim_n |||g_n(w)|||$ exists $|F|$-almost everywhere and $d \in L_1(|F|)$, by the Bounded Convergence Theorem. By the form of the g_n's and the fact that $\sup_n |||g_n(w)||| = \lim_n |||g_n(w)||| < \infty$ for $|F|$-almost all $w \in \Omega$, we have by (x_n)'s boundedly complete nature that $g \equiv \sum_n (df_n \circ F/d|F|)x_n$ is well-defined $|F|$-almost everywhere and, because $|||g(\cdot)||| \leqq d(\cdot)$ $|F|$-almost everywhere, satisfies $g \in L_1(|F|, X)$. It is easily seen that g is the derivative of F with respect to $|F|$.

The late 30's experienced a number of basic papers in vector measures concerned with the Radon-Nikodym theorem in some variation or another. The most prominent of these was probably the classic of I. M. Gelfand [6] which showed that a function of bounded variation on [0, 1] and having values in a separable dual was weakly differentiable. As $L_1[0, 1]$ did not possess this property (the same function as defined by Clarkson works) Gelfand was able to conclude that $L_1[0, 1]$ was not isomorphic to a dual space. Also noteworthy (especially for the purposes of these notes) was a paper of Bochner and Taylor [7] in which it was essentially (modulo some abstract measure theory) shown that the Gelfand-Frechet spaces were the same as those Banach spaces X with the property (henceforth and for obvious reasons called the *Radon-Nikodym property*) that given a countably additive X-valued map F defined on a sigma-algebra possessing finite variation $|F|$ then there exists a Bochner $|F|$-integrable function f such that $F(A) = \int_A f\,d|F|$ for each A in F's domain. (So Dunford and Morse did indeed establish that spaces with boundedly complete bases were Gelfand-Frechet spaces).

The papers of the late 30's were largely preparatory for the major work of this time period and indeed of the analytic aspect of the Radon-Nikodym theorem: the *Linear Operations on Summable Functions* of N. Dunford and B. J. Pettis [13]. The results of this paper evolved over a period of several years as seen particularly in the work of Gelfand ([6]), Dunford ([8], [9], [10]) and Pettis ([11], [12]) and contains some of the most beautiful theorems in functional analysis of the pre-war period. We cite only those related to the Radon-Nikodym theorem:

(*DP* 1) *If X is a separable dual space, then X possesses the Radon-Nikodym property.*

(*DP* 2) *If $(\Omega, \Sigma, \lambda)$ is a (finite) measure space and $T : L_1(\lambda) \to X$ is a weakly compact linear operator, then there exists a λ-essentially bounded, strongly measurable function $f : \Omega \to X$ such that*

$$(*) \quad Tg = \int fg\,d\lambda$$

holds for all $g \in L_1(\lambda)$. Moreover $\|T\| = \|f\|_{\text{ess.sup.}}$.

(*DP* 3) *If $(\Omega, \Sigma, \lambda)$ is a finite measure space and $T : L_1(\lambda) \to X$ is a compact linear operator, then the f of (DP 2) has λ-essentially precompact range, hence, is approximable in essential supremum norm by simple functions.*

(*DP* 4) *If $(\Omega, \Sigma, \lambda)$ is a finite measure space and $T : L_1(\lambda) \to X$ is any linear operator which is representable in the form $(*)$ of (DP 2), then T maps weakly convergent sequences into norm convergent sequences.*

(To be accurate it must be remarked that Dunford and Pettis did not prove (DP 2) in the form it is stated; they established the conclusion under the additional hypothesis that the weakly compact operator's range was separable. However, R. S. Phillips in his classic paper *On linear transformations* [14] showed that separability of the range was a consequence of the operator's weak compactness).

EXAMPLE. If a Banach space X possesses the Radon-Nikodym property and if $(\Omega, \Sigma, \lambda)$ is a finite measure space then every continuous linear operator $T: L_1(\lambda) \to X$ has a representation as in (DP 2) so, by (DP 4), maps weak null sequences into norm null sequences. The operator $T: L_1[0, 2\pi] \to c_0$ defined by $(Tf)_k = \int_0^{2\pi} f(t) \sin(kt)\, dt$ is continuous, linear and maps the weak null sequence $(\sin(kt))$ in $L_1[0, 2\pi]$ into a sequence bounded away from zero in c_0. Thus c_0 does not possess the Radon-Nikodym property.

A particular consequence of (DP 2) worth special mention is the fact that all *reflexive Banach spaces possess the Radon-Nikodym property*. Actually, this fact can also be derived from (DP 1) if one allows the following fact (we give a proof in the discussion of the geometric aspects of the Radon-Nikodym theorem) about the stability of the Radon-Nikodym property: a Banach space X possesses the Radon-Nikodym property if and only if every *separable* closed subspace of X possesses it. Thus, as separable closed subspaces of reflexive spaces are separable duals, we obtain the asserted fact.

Something must be said here. Though it might seem like the above proof of reflexive Banach spaces having the Radon-Nikodym property is a bit round-about, the only proofs we know of spaces having the Radon-Nikodym property depend ultimately upon establishing that separable subspaces are isomorphic to subspaces of separable duals. One might conjecture an affirmative answer to the following:

PROBLEM 1. If X is a separable Banach space possessing the Radon-Nikodym property then need X imbed in a separable conjugate?

The stability result referred to above gives rise to the possibility that one need not even look at all the separable subspaces (such is the case in testing for reflexivity, weak sequential completeness or quasi-reflexivity, for example; see [15], [16]). Thus the

PROBLEM 2. If every closed subspace of X possessing a Schauder basis possesses the Radon-Nikodym property, then need X also possess it?

Related to this is

PROBLEM 3. Is there an intrinsic characterization of Schauder bases that span spaces with the Radon-Nikodym property?

The Dunford-Pettis results suggest other problems. For example, from (DP 3) we get the following:

THEOREM. *The compact operators on an L_1-space that achieve their norm are dense in the space of compact operators.*

PROOF. Maharam's theorem [17] tells us that the problem lies in the case of finite λ's. By (DP 3), we need only show that if $s: \Omega \to X$ is a simple function then the operator $S: L_1(\lambda) \to X$ given by $Sf = \int fs\,d\lambda$ achieves its norm. If $s = \sum_{i=1}^{n} c_{A_i} x_i$ where $A_1, \cdots, A_n$ are pairwise disjoint members of $\sum$ then choose k between 1 and n such that $\|x_k\| = \max\{\|x_1\|, \cdots, \|x_n\|\}$.

$$\|S\| = \|s\|_\infty = \|x_k\| = \frac{\lambda(A_k)}{\lambda(A_k)}\|x_k\| = \left\| \int \lambda(A_k)^{-1} c_{A_k} s\, d\lambda \right\|$$

and $\lambda(A_k)^{-1} c_{Ak}$ has norm 1.

It is easily verified that every compact operator on a reflexive Banach space achieves its norm and that if Ω is compact, Hausdorff and dispersed then the compact operators achieving their norm are dense in the space of compact operators. Unanswered however is the

PROBLEM 4. Does every Banach space X have the property that the compact operators on X (to any other Banach space) that achieve their norm are dense in the space of compact operators? What about $C(\Omega)$'s?

(DP 4) suggests a number of problems. It is easy to construct operators on $L_1(0, 1)$ which map weakly convergent sequences into norm convergent sequences but haven't a kernel. For example, the operator $T: L_1(0, 1) \to C(0, 1]$ given by $(Tf)(t) = \int_0^t f(w)\,dw$ is such a linear operator. However, the following is not yet resolved.

PROBLEM 5. If a Banach space X possesses the property that every continuous linear operator $T: L_1[0, 1] \to X$ maps weakly convergent sequences into norm convergent sequences then need X possess the Radon-Nikodym property?

An affirmative answer to Problem 5 will similarly answer

PROBLEM 6. If weak and norm convergence of sequences in X coincide, then need X possess the Radon-Nikodym property?

As we mentioned above the only way new classes of spaces have been shown to possess the Radon-Nikodym property is by showing that separable subspaces are subspaces of separable conjugates. We illustrate this with an example (other such examples will be mentioned in later sections).

A Banach space X is said to be *weakly compactly generated* whenever there exists a weakly compact set in X whose linear span is dense. Separable Banach spaces are precisely the compactly generated spaces. Of course, all reflexive Banach spaces are weakly compactly generated. In a sense, the prototypes of the class of weakly compactly generated spaces are spaces of the form $c_0(\Gamma)$ (Γ any set). An $L_1(\mu)$-space is weakly compactly generated if and only if μ is σ-finite. This class of spaces has been studied by a number of authors; the basic information regarding weakly compactly generated spaces is contained in [18].

The next result was remarked to us by Bill Johnson and Charles Stegall [19]; the proof here is that of Bob Huff [20].

THEOREM. *Weakly compactly generated dual spaces possess the Radon-Nikodym property.*

PROOF. If Y is a separable subspace of the weakly compactly generated space X^*, then there is a separable subspace S of X such that Y is a subspace of S^*. We will show S^* is separable. Clearly S^* is a quotient of X^*; thus, S^* is also weakly compactly generated. Let K be a weakly compact convex subset of S^* that generates S^*. Then K is a weak-star compact. But S is separable so K is weak-star metrizable. Hence, K is weak-star separable. But K's compactness in both Hausdorff topologies (weak and weak-star) says these topologies coincide on K, i.e., K is weakly separable. By Mazur's theorem, K is norm separable and thus S^* is separable-being the closed linear span of K.

COROLLARY. *$L_1(\mu)$ is not isomorphic to a dual for μ σ-finite unless μ is purely atomic.*

None of the consequences thus far mentioned of the Dunford-Pettis results was really out of the reach of Dunford and Pettis; the reason for their not proving them lies primarily in the fact that certain concepts (weakly compactly generated, operators achieving norm) had not yet come of age. It is curious though that the problem of necessary and sufficient conditions for a given operator $T: L_1(\lambda) \to X$ (λ a finite measure) to have a derivative was not given any attention especially since the tools for the solution of this problem were already developed. This problem waited for a satisfactory solution until the late '60's when M. Metivier [21] and M. A. Rieffel [22] established necessary and sufficient conditions for a vector measure to have a Bochner derivative with respect to some finite positive measure. Their presentation, while close in spirit to several proofs of Dunford, Pettis and Phillips (especially [23]), was not exactly direct; however, in [24], S. Moedomo and J. J. Uhl derived the Metivier-Rieffel results and more, using nothing but the Dunford-Pettis-Phillips theorem (DP 2) and clever observation. The final result is the

THEOREM. *Let* $(\Omega, \Sigma, \lambda)$ *be a finite positive measure space. Let* $F : \Sigma \to X$ *be a countably additive* λ*-continuous vector measure. Then TFAE:*

(1) *There exists a strongly* λ*-measurable function* $f : \Omega \to X$ *such that for each* $A \in \Sigma$

$$F(A) = (\text{Pettis}) - \int_A f(w)\, d\lambda(w)$$

(2) *given* $\epsilon > 0$ *there exists* $\Omega_\epsilon \in \Sigma$ *such that* $\lambda(\Omega/\Omega_\epsilon) < \epsilon$ *and* $\{F(A)/\lambda(A) : A \in \Sigma, A \subset \Omega_\epsilon\}$ *is relatively norm compact;*

(3) *given* $A \in \Sigma, \lambda(A) > 0$ *there exists* $B \in \Sigma$, $B \subset A$, $\lambda(B) > 0$ *such that* $\{F(C)/\lambda(C) : C \in \Sigma, C \subset B\}$ *is relatively norm compact;*

(4) *same as* (2) *with "norm compact" replaced by "weakly compact";*

(5) *same as* (3) *with "norm compact" replaced by "weakly compact".*

For f *to be Bochner* λ*-integrable, it is necessary and sufficient that* F *possess finite variation.*

The Moedomo-Uhl proof of this theorem is, it seems, the most natural proof and certainly was accessible to the mathematicians of the early 1940's.

We first show (1) implies (2).

Let $\epsilon > 0$ be given. Let (f_n) be a sequence of simple functions converging λ almost everywhere to the Pettis λ-integrable function f. Egoroff's theorem ensures that there exists a set $\Omega_\epsilon \in \Sigma$ such that Ω/Ω_ϵ has λ-measure $< \epsilon$ and f_n converges uniformly on Ω_ϵ to f. It is now an easy matter to show that $U : L_1(\lambda) \to X$ defined by $Ug = \int_{\Omega_\epsilon} gf\, d\lambda$ is the operator limit of the finite rank operators $U_n : L_1(\lambda) \to X$ given by $U_n g = \int_{\Omega_\epsilon} gf_n\, d\lambda$ and hence U is a compact linear operator. Note that if $A \in \Sigma$, $A \subset \Omega_\epsilon$ then $\|c_A \lambda(A)^{-1}\|_1 \leqq 1$ so $F(A)/\lambda(A) = U(c_A/\lambda(A))$ and hence $\{F(A)/\lambda(A) : A \in \Sigma,\ A \subset \Omega_\epsilon\}$ is contained in the relatively norm compact image under U of $L_1(\lambda)$'s unit ball.

That (2) implies (3) is obvious. We will next show that (3) implies (1) and stop there; the equivalence of (4) and (5) with the other conditions is a standard exhaustion argument.

Suppose (3) holds. Let $\epsilon > 0$ be given. Choose Ω_ϵ in accordance with (3). Let $s = \sum_{i=1}^{n} a_i c_{A_i}$ be a simple function in $L_1(\lambda)$. Define $U_\epsilon(s)$ by $U_\epsilon(s) = \sum_{i=1}^{n} a_i F(\Omega_\epsilon \cap A_i)$. U_ϵ is continuous and linear and for s having L_1-norm $\leqq 1$, we have $U_\epsilon(s) \in$ absolutely closed convex hull of $\{F(E)/\lambda(E) : E \in \Sigma, E \in \Omega_\epsilon\}$ which by (3) and the Krein-Smulian theorem is a weakly compact set in X. Thus U_ϵ extends to a weakly compact linear operator $U_\epsilon : L_1(\lambda) \to X$. The Dunford-Pettis-Phillips theorem (*DP* 2) yields a Bochner derivative for U_ϵ. Now let $\epsilon \to 0$ and piece together the ϵ-derivatives to get the Pettis derivative of F.

II. **Operator Theoretic Aspects of the Radon-Nikodym Theorem.** The results of the early analytic era were, to a large extent, unused and unappreciated until the late 60's. There is one notable exception: the work of A. Grothendieck.

In his Memoir [25] and his Resumé [26], Grothendieck introduced several new classes of operators and studied the problems of approximation by finite rank operators and the structure of Banach spaces in terms of these classes of operators. Some of Grothendieck's most striking results are ultimately dependent upon Radon-Nikodym considerations.

The role that vector measures play in the theory of tensor products derives largely (entirely?) from the introduction by Grothendieck of the notion of an integral bilinear functional. R. Schatten and J. von Neumann had started, in the early 1940's, a systematic analysis of tensor products of Banach spaces ([27]). Their basic hangups were twofold: first, they could not get a handle on the dual of the space of compact operators (or more generally the dual of the λ tensor product of two Banach spaces) and second after the appearance of J. Dixmier's study of tensor products of Hilbert spaces [29], they tried to mimic the Hilbert space situation. What was needed was a new idea. This was provided by Grothendieck who showed how the entire apparatus of measure theory entered *naturally* via the notion of integral operators (and integral bilinear functionals) into the study of the general structure of Banach spaces.

Recall the basic definitions (or equivalences thereof) of the classes of absolutely summing, 2-summing, integral and nuclear operators. A continuous linear operator $T : X \to Y$ is said to be:

absolutely summing whenever given an unconditionally convergent series $\sum_n x_n$ in X the series $\Sigma_n Tx_n$ is absolutely convergent in Y;

2-*summing* whenever given a formal series $\sum_n x_n$ in X such that $\sum_n |fx_n|^2 < \infty$ for each $f \in X^*$ it follows that $\sum_n \|Tx_n\|^2 < \infty$;

integral whenever there exists a regular Borel measure μ defined on the compact Hausdorff space $\Omega(X^*) \times \Omega(Y^{**})$ formed by taking the Cartesian product of the unit balls of X^* and Y^{**} in their respective weak-star topologies with

$$gTx = \int_{\Omega(X^*)\times\Omega(Y^{**})} f(x)G(T^*g)\, d\mu(f, G)$$

holding for all $x \in X$ and $g \in Y^*$;

nuclear whenever there exist sequences $(f_n) \subset X^*$ and $(y_n) \subset Y$ such that $\sum_n \|f_n\| \, \|y_n\| < \infty$ and $Tx = \sum_n f_n(x) y_n$ holds for all $x \in X$. We denote by $\Pi_1(X; Y)$, $\Pi_2(X; Y)$, $I(X; Y)$ and $N(X; Y)$ the classes of absolutely summing, 2-summing, integral and nuclear operators $T : X \to Y$. The basic containments between these classes may be summarized as follows:

$$N(X; Y) \subset I(X; Y) \subset \Pi_1(X; Y) \subset \Pi_2(X; Y).$$

To present some of the deeper results that follow from [25], [26] and [30] we introduce the notation: by C we denote any space of all continuous functions (perhaps on a locally compact Hausdorff space, in which case we ask vanishing at infinity), by L we denote a space of all absolutely integrable functions with respect to some measure and by H we denote any Hilbert space. Then Grothendieck basically showed

(G 1) $\mathcal{L}(L; H) = \Pi_1(L; H)$;

(G 2) $\mathcal{L}(C; H) = \Pi_2(C; H)$ *and* $\mathcal{L}(C; L) = \Pi_2(C; L)$;

(G 3) $\Pi_1(C; Y) = I(C; Y)$ *and* $\Pi_1(X; C) = I(X; C)$;

(G 4) *Integral operators to or from H are nuclear*;

(G 5) *The composition of 2-summing operators is nuclear.*

(G 4) depends in an essential manner upon the Radon-Nikodym property and (G 5) is also a consequence of the Radon-Nikodym theorem for Hilbert spaces though it can be established independently of this result.

A fascinating consequence of (G 1) through (G 5) is the *Six Theorem*. It is a fact that if $T : X \to Y$ is a nuclear linear operator then T can be factored in the form

$$\begin{array}{ccccccccccc} X & \multicolumn{9}{c}{\longrightarrow} & Y \\ \downarrow & & & & & & & & & & \uparrow \\ Z_1 & \to & Z_2 & \to & Z_3 & \to & Z_4 & \to & Z_5 & \to & Z_6 \end{array}$$

where each z_i is either a C, L or H and we can do this so that each class appears twice and no C, L or H succeeds itself. Using (G 1) through (G 5) the converse can be shown also to hold, that is, we have the

SIX THEOREM. *If $T : X \to Y$ can be factored as above, then T is nuclear.*

In addition to the above type of results, Grothendieck contributed in other basic ways to the theory and applications of the differentiation of vector measures. Among the results at least implicitly found in either [25] or [26] and closely related to the theory of the Radon-Nikodym theorem are:

(G6) *A continuous linear operator $T : X \to L_1(\lambda)$* (λ any measure) *is integral if and only if T maps X's unit ball into a lattice bounded subset of $L_1(\lambda)$;*

(G7) *A continuous linear operator $T : X \to L_1(\lambda)$ is nuclear if and only if T maps X's unit ball into a lattice bounded, equimeasurable subset of $L_1(\lambda)$, where $K \subset L_1(\lambda)$ is equimeasurable means that given $\epsilon > 0$ and a set P of positive, finite λ-measure, then there exists a subset P_ϵ of P such that $\lambda(P/P_\epsilon) \leqq \epsilon$ and such that K/P_ϵ is relatively norm compact in $L_\infty(\lambda/P_\epsilon)$.*

(G7) can be restated as a Radon-Nikodym Theorem as: a measure $F : \Sigma$ (sigma-algebra) $\to L_1(\lambda)$ possesses finite variation $|F|$ and possesses a Bochner derivative with respect to $|F|$ if and only if the range of F is lattice bounded and equimeasurable. To our knowledge this is the only result which characterizes Bochner differentiability in terms of only the range of a measure. It should be remarked that the lattice boundedness of F's range and the fact that F has its values in an $L_1(\lambda)$ space insure F's finiteness of variation.

Grothendieck not only proved *new* Radon-Nikodym theorems but gave new directions for the application of Radon-Nikodym results. This is largely due to his characterizations of integral operators in terms of factorization. For example, he showed that for a continuous linear operator $T : X \to Y$ to be integral it is both necessary and sufficient that there exist a compact Hausdorff space Ω and a regular Borel measure μ defined on Ω such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \longrightarrow & Y^{**} \\ S\big\downarrow & & & & \big\uparrow R \\ C(\Omega) & & \longrightarrow & & L_1(\mu) \end{array}$$

commutes for some bounded linear operators $R : L_1(\mu) \to Y^{**}$ and $S : X \to C(\Omega)$. Thus the study of integral operators is reduced to a large extent to the study of operators on $C(\Omega)$ spaces. On such spaces it can be shown (and is implicit in [25], explicit in [31], [97], [99]) that absolutely summing and integral operators correspond to vector measures possessing finite variation while nuclear operators correspond to vector measures possessing finite variation and which are differentiable with respect to their variation. Using these ideas, Grothendieck derived the somewhat startling,

(G8) *Let $T : X \to Y$ and $S : Y \to Z$ be continuous linear operators. If T is integral and S is weakly compact, then ST is nuclear; while, if T is weakly compact and S is integral, then ST is nuclear into Z^{**}.*

(G9) *If $T : X \to Y^*$ is integral and either X^* or Y^* possesses the Radon-Nikodym property, then T is nuclear into Y^*.*

Based upon (G8) and G9) and his duality theory for topological tensor products, Grothendieck was then able to (essentially) show,

(G10) *If X possesses the Radon-Nikodym property, the approximation property and is complemented in X^{**} via a norm one projection (in particular if X is a dual space), then X possesses the metric approximation property.*

The recent results of T. Figiel and W. B. Johnson [32] indicate the special nature of the Radon-Nikodym theorem in (G10).

The work of Grothendieck was virtually ignored until the mid-1960's when renewed efforts in the structure theory of Banach space theory brought his work out of storage so-to-say. Motivated largely by the deep (and mysterious) results of [26], J. Lindenstrauss and A. Pelczynski introduced in [30] the class of $\mathcal{L}_p$-spaces. These spaces to a large extent are the isomorphic versions of complemented subspaces of the $L_p(\mu)$ space. Utilizing the $\mathcal{L}_p$ space theory developed by Lindenstrauss, Pelczynski, Rosenthal, Stegall and Retherford in [30], [33] and [34] one of the more spectacular applications of notions related to the Radon-Nikodym theorem was obtained by D. R. Lewis and C. Stegall [35]: Let $\Pi : L_1[0,1] \to L_1[0,1]$ be a continuous linear projection having infinite dimensional range. Suppose the measure P : Borel sets in $[0,1] \to L_1[0,1]$ is given by $P(A) = \Pi(c_A)$. Then P is a countably additive vector measure possessing finite variation. For P to be differentiable with respect to its variation, it is necessary and sufficient that $\Pi L_1[0,1]$ be isomorphic to ℓ_1.

This gives rise to several questions regarding L_1-valued measures. The first was suggested to us by W. B. Johnson.

PROBLEM 6. Suppose $\Pi : L_1[0,1] \to L_1[0,1]$ is a continuous linear projection and let P : Borel sets $\to L_1[0,1]$ be given by $P(A) = \Pi(c_A)$. If P is nowhere differentiable with respect to P's variation, need $\Pi L_1[0,1]$ be isomorphic to $L_1[0,1]$?

PROBLEM 7. Suppose X is a subspace of $L_1[0,1]$ which does not possess the Radon-Nikodym property. Need X contain an isomorph of $L_1[0,1]$?

Before leaving the discussion of the operator theoretic aspects of the Radon-Nikodym theorem, it is worth mentioning several results related to projections in L_1-spaces, L_1-valued measures and differentiability of these measures.

Particularly noteworthy is the as-yet-unpublished result on splittings of $L_1[0,1]$ of Per Enflo: if $L_1]0,1] = X \oplus Y$ then either X or Y is isomorphic to $L_1[0,1]$.

With regards to non-differentiability, we note the example of A. Costé [36] of an L_1 [0 1]-valued measure of finite variation having relatively compact range, but without a derivative with respect to its variation. A consequence of this example is the existence (a) of a compact, integral operator $T : C[0,1] \to L_1[0,1]$ which is not nuclear and (b) the existence of a Dunford-Pettis operator $S : L_1[0,1] \to L_1[0,1]$ with no derivative. Costé's construction (actually carried out over the 2-dimensional torus) is ultimately dependent upon a deep fact from harmonic analysis (due originally to Menchoff) to the effect that on $[0, 2\pi]$ there exists a regular Borel measure μ singular with respect to Lebesgue measure for which the Fourier coefficients $\hat{\mu}(n)$ tend to zero as $n \to \pm\infty$.

III. **Geometric Aspects of the Radon-Nikodym Theorem.** Perhaps the greatest break through in the theory of Radon-Nikodym is due to M. A. Rieffel who in [37] tried to recover a classical differentiation theorem of Phillips by introducing the geometric notion of dentability. While Rieffel's efforts to obtain Phillip's result were unsuccessful, something more important came out of them — the establishment of a close interrelationship between the Radon-Nikodym theorem (and consequently the Radon-Nikodym property) and the geometry of a Banach space. It is this aspect of the Radon-Nikodym theorem that has seen the most spectacular advances in the theory in recent years.

Recall the notion of dentability: a bounded subset B of a Banach space X is *dentable* whenever given $\epsilon > 0$ one can find a point $x_\epsilon \in B$ such that if one sweeps out the open ϵ-ball about x_ϵ (see Figure 1) then fills up the remainder (i.e., takes the closed convex hull of $B/S_\epsilon(x_\epsilon)$), (see Figure 2) one doesn't get x_ϵ back again. If the same x works for each $\epsilon > 0$ then x is called a *denting point* of B.

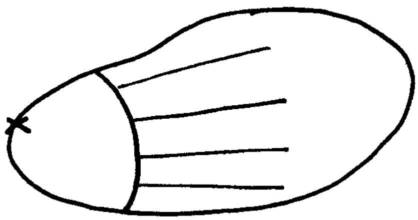

Figure 1

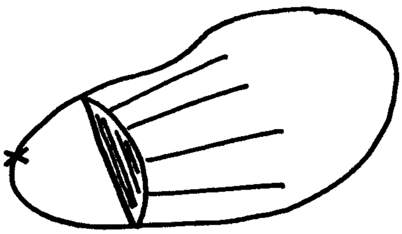

Figure 2

The basic facts regarding dentable sets are the following:

(1) if the closed convex hull of B is dentable so is B (M. Rieffel [37]);

(2) if B is a compact convex set then the extreme points of B are all denting points (and, of course, conversely) (M. Rieffel [37]);

(3) strongly exposed points of B are denting points (a point $x \in B$ is *strongly exposed* whenever there is $f \in X^*$, $\|f\| = 1$ such that $f(x) > f(y)$ for $y \in B$, $y \neq x$ and such that if $(y_n) \subset B$ and $f(y_n) \rightarrow f(x)$ then $\|x - y_n\| \rightarrow 0$);

(4) weakly compact sets are dentable (J. Lindenstrauss [38] showed that weakly compact convex sets in locally uniformly convex Banach spaces are the closed convex hull of their strongly exposed points, hence are dentable; S. Troyanski [34] showed that weakly compact sets always live in locally uniformly convex spaces);

(5) if every countable subset of B is dentable, then B is dentable (H. Maynard [40]).

(We ought to remark that important related work of I. Namioka ([41], [42]) contains a number of conditions for dentability of sets as well as an easy proof of (4) above.

The basic result of Rieffel that uncovered the fundamental relationship of the geometry of a Banach space to the Radon-Nikodym theorem goes as follows:

RIEFFEL'S DENTABILITY THEOREM: *Let* (Ω, Σ, μ) *be a finite positive measure space,* X *be a Banach space and* $F : \Sigma \to X$ *be a countably additive vector measure possessing finite variation* $|F|$ *with* $|F| \ll \mu$. *Then TFAE:*

(1) *F is Bochner differentiable with respect to* μ;

(2) *given* $\epsilon > 0$ *there exists* $\Omega_\epsilon \in \Sigma$ *such that* $\mu(\Omega/\Omega_\epsilon) < \epsilon$ *and such that*

$$\left\{ \frac{F(A)}{\mu(A)} : A \in \Sigma,\ \mu(A) > 0,\ A \subset \Omega_\epsilon \right\}$$

is dentable;

(3) *given* $A \in \Sigma$, $\mu(A) > 0$ *there exists* $B \subset A$, $B \in \Sigma$, $\mu(B) > 0$ *such that*

$$\left\{ \frac{F(C)}{\mu(C)} : C \in \Sigma, C \subset B, \mu(C) > 0 \right\}$$

is dentable.

Since the paper [37] of Rieffel is a bit inaccessible we indicate briefly the proof of (3) implies (1).

First we make a general remark concerning the existence of Radon-Nikodym derivatives for a vector measure F with respect to a scalar measure μ: *to show the existence of* $dF/d\mu$ *it suffices to show that given* $\epsilon > 0$ *there exist sequences* $(x_n^\epsilon) \subset X$ *and* $E_n^\epsilon \in \Sigma$ *where* $E_n^\epsilon \cap E_m^\epsilon = \emptyset$ *for* $n \neq m$, $\mu(E_n^\epsilon) > 0$ *and* $\Omega = \bigcup_n E_n^\epsilon$ *(*μ *almost) and*

$$\left\{ \frac{F(E)}{\mu(E)} : E \in \Sigma,\ \mu(E) > 0,\ E \subset E_n^\epsilon \right\} \subset \text{Ball}\,(x_n^\epsilon, \epsilon).$$

Why? Basically because in such a situation if one chooses for fixed $\epsilon > 0$ an n_ϵ such that $\mu(\bigcup_{n>n_\epsilon} E_n^\epsilon) < \epsilon$ and consider

$$x_\epsilon = \sum_{k=1}^{n} x_k^\epsilon C_{E_{k\epsilon}},$$

then the sequence $(x_{1/n})$ is mean-Cauchy in $L_1(\mu; X)$, converging to the derivative of F with respect to μ. Verification of this is somewhat tedious and can be found in [0].

Now to indicate how the dentability condition of (3) leads us to the above situation, we will show that *given* $E \in \Sigma$, *of positive* μ*-measure and* $\epsilon > 0$, *there exists* $D \in \Sigma$ *contained in* E *of positive* μ*-measure and there exists an* $x \in X$ *such that the average range of* F *over* D, $\mathcal{A}_D(F)$, *given by*

$$\mathcal{A}_D(F) = \left\{ \frac{F(C)}{\mu(C)} : C \in \Sigma, C \subset D, \mu(C) > 0 \right\}$$

is contained in Ball (x, ϵ).

From this an easy exhaustion argument yields the desired situation.

By dentability we know that there exists $E_d \in \Sigma$, $E_d \subset E$ with $\mu(E_d) > 0$ and $\mathcal{A}_{E_d}(F)$ dentable. Let $x \in \mathcal{A}_{E_d}(F)$ be such that

$$x \notin \overline{\text{conv}}(\mathcal{A}_{E_d}(F) \backslash \text{Ball}\,(x, \epsilon)).$$

Suppose $x = F(D_0)/\mu(D_0)$ where $D_0 \in \Sigma$, $D_0 \subset E_d$, and $\mu(D_0) > 0$. If D_0 is the desired D, good; otherwise $\mathcal{A}_{D_0}(F) \not\subset$ Ball (x, ϵ). Thus there is $E_1 \subset D_0$ such that $\mu(E_1) > 0$ and $\|(F(E_1)/\mu(E_1)) - x\| \geqq \epsilon$. Note that $F(E_1)/\mu(E_1) \in \mathcal{A}_{D_0}(F)$ yet $F(E_1)/\mu(E_1) \notin$ Ball (x, ϵ). Thus

$$\frac{F(E_1)}{\mu(E_1)} \in \overline{\text{conv}}\,(\mathcal{A}_{E_d}(F)/\text{Ball}\,(x, \epsilon)).$$

Let k_1 be the smallest positive integer $\geqq 2$ for which there exists $E_1 \subset D_0$ such that $\mu(E_1) \geqq k_1^{-1}$, and

$$\frac{F(E_1)}{\mu(E_i)} \in \overline{\text{conv}}\,(\mathcal{A}_{E_d}(F)/\text{Ball}\,(x, \epsilon)).$$

Take any such E_1. Let $D_1 = D_0 \backslash E_1$.

We claim $\mu(D_1) > 0$. Otherwise, $\mu(D_1) = 0$ so that $\mu(D_0) - \mu(E_1) = \mu(D_0/E_1) = \mu(D_1) = 0$ so that $\mu(D_0) = \mu(E_1)$. As F is μ-continuous, $F(D_0 \backslash E_1) = 0$ so $F(D_0) = F(E_1)$ as well. Hence $F(D_0)/\mu(D_0) = F(E_1)/\mu(E_1)$ an impossibility in light of the inclusion of the right side and non-inclusion of the left side in the set $\overline{\text{conv}}\,(\mathcal{A}_{E_d}(F)/\text{Ball}\,(x, \epsilon))$.

If D_1 fits the bill — good; otherwise, $\cdots$.

We generate a disjoint sequence (E_n) of members of Σ, a non-decreasing sequence (k_n) of positive integers such that $\mu(E_n) \geqq (k_n)^{-1}$ and

$$\frac{F(E_n)}{\mu(E_n)} \in \overline{\text{conv}}\,(\mathcal{A}_{E_d}(F)/\text{Ball}\,(x, \epsilon)).$$

As $\mu(E_n) \to 0$ we have $k_n \to \infty$. Letting $E_0 = \bigcup_n E_n$ and $D = D_0/E_0$ we claim that $\mu(D) > 0$ and $\mathcal{A}_D(F) \subset$ Ball (x, ϵ).

That $\mathcal{A}_D(F) \subset \text{Ball}(x, \epsilon)$ is proved as follows: Let $D' \subset D$, $D' \in \Sigma$, $|\mu(D') > 0$. Then D' is contained in $D_0 \backslash \bigcup_{k=1}^n E_k$ for all n so that if $F(D')/\mu(D') \notin \text{Ball}(x, \epsilon)$ then $F(D')/\mu(D') \in \overline{\text{conv}}(\mathcal{A}_{E_d}(F) \backslash \text{Ball}(x, \epsilon))$. But then by choice of the (k_n)'s, $\mu(D') \leqq (k_n - 1)^{-1} \rightarrow 0$ contradicting $\mu(D') > 0$.

To see that $\mu(D) > 0$ suppose $\mu(D) = 0$. Then as before $\mu(D_0) = \mu(E_0)$ and $F(D_0) = F(E_0)$ so that

$$\begin{aligned} \frac{F(D_0)}{\mu(D_0)} &= \frac{F(E_0)}{\mu(E_0)} \\ &= \frac{\sum_n F(E_n)}{\mu(E_0)} \\ &= \sum_n \frac{F(E_n)}{\mu(E_n)} \cdot \quad \mu_n = \frac{\mu(E_n)}{\mu(E_0)} \\ &= \sum_n \mu_n \frac{F(E_n)}{\mu(E_n)} \in \overline{\text{conv}}(\mathcal{A}_{E_d}(F) \backslash \text{Ball}(x, \epsilon)) \end{aligned}$$

since $\sum_n \mu_n = 1$ where $\mu_n \geqq 0$.

An immediate consequence of the Rieffel Dentability theorem is the

THEOREM. *If every (closed convex) bounded subset of the Banach space X is dentable, then X possesses the Radon-Nikodym property.*

After the Rieffel result appeared there was a period of absorption of the notion of dentability. Then in 1972, Hugh Maynard provided a major breakthrough; using a notion closely related to that of dentability he gave the first internal characterization of Banach spaces possessing the Radon-Nikodym property. Though Maynard's result did not itself provide the converse to Rieffel's sufficient condition it was the basis for the eventual proof of the converse. Within several days during the summer of 1973, W. J. Davis and R. R. Phelps and, independently, R. E. Huff proved the following:

THEOREM. *If the Banach space X possesses the Radon-Nikodym property, then every bounded subset of X is dentable.*

The Davis-Phelps proof depended upon Maynard's result itself while Huff's proof was achieved by a suitably clever modification of Maynard's construction. As Maynard's construction is possible of interest in other connections we present the original Huff proof (in [43], Huff's proof has been considerably streamlined):

PROOF OF DAVIS-HUFF-MAYNARD-PHELPS THEOREM. Let K be a bounded, non-dentable subset of the Banach space X. Suppose X possesses the Radon-Nikodym property. We assume that K is contained in the unit ball of X. Let $0 < \epsilon < 1$ be chosen so that

$$(*)\ x \in K \text{ implies } x \in \overline{\text{co}}\,(K \backslash S_\epsilon(x)).$$

Let $\Omega = [0, 1)$ and let λ denote Lebesgue measure on $[0, 1)$.

We will define inductively an increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_n \subset \cdots$ of finite algebras or subsets of Ω and additive maps $F_n : \Sigma_n \to X$ such that

(i) the atoms of Σ_n partition Ω into half-open intervals $\{I_1^n, I_2^n, \cdots, I_{p(n)}^n\}$;

(ii) $F_n(I_i^n)/\lambda(I_i^n) \in K$ for all n and all $1 \leqq i \leqq p(n)$;

(iii) $I_j^{n+1} \subset I_i^n$ implies

$$\left\| \frac{F_{n+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{F_{n+1}(I_i^n)}{\lambda(I_i^n)} \right\| \geqq \frac{2^n - 1}{2^n}\,\epsilon.$$

and

(iv) $\|F_n(E) - F_{n+1}(E)\| \leqq \epsilon\lambda(E)/2^n$ for each $E \in \Sigma_n$.

To start the construction we let $\Sigma_0 = \{\emptyset, \Omega\}$ and choose any $x_0 \in K$. Define $F_0(\emptyset) = 0$ and $F_0(\Omega) = x_0$. Since $x_0 \in K$ and K is non-dentable there exists $\alpha_1^1, \cdots, \alpha_{p(1)}^1 > 0$, $\sum_{j=1}^{p(1)} \alpha_j^1 = 1$ and $x_1^1, \cdots, x_{p(1)}^1 \in K$ such that

$$\left\| \frac{F_0(\Omega)}{\lambda(\Omega)} - x_j^1 \right\| = \|x_0 - x_j^1\| \geqq \epsilon$$

and

$$\left\| \frac{F_0(\Omega)}{\lambda(\Omega)} - \sum_{j=1}^{p(1)} \alpha_j^1 x_j^1 \right\| = \left\| x_0 - \sum_{j=1}^{p(1)} \alpha_j^1 x_j^1 \right\| < \frac{\epsilon}{2^n}.$$

Partition Ω into finitely many disjoint half-open intervals $I_1^1, \cdots, I_{p(1)}^1$ such that $\lambda(I_j^1) = \alpha_j^1$. Let Σ_1 be the algebra of subsets of Ω generated by $\{I_1^1, \cdots, I_{p(1)}^1\}$. Define F_1 on Σ_1 by defining $F_1(I_j^1) = \alpha_j^1 x_j^1$ for $1 \leqq j \leqq p(1)$.

Generally, if Σ_n and F_n have been defined and Σ_n has the (pairwise disjoint) half-open intervals $I_1^n, \cdots, I_{p(n)}^n$ as atoms and $F_n(I_j^n)/\lambda(I_j^n) \in K$ for each $1 \leqq j \leqq p(n)$, we proceed to define Σ_{n+1} and F_{n+1} as follows:

By the non-dentability of K we have for each $1 \leqq j \leqq p(n)$ that there exists $\alpha_1(j), \cdots, \alpha_{q(j)}(j) > 0$, $\sum_{\ell=1}^{q(j)} \alpha_\ell(j) = 1$ and $x_1(j), \cdots, x_{q(j)}(j) \in K$ for which

$$\left\| x_\ell(j) - \frac{F_n(I_j^n)}{\lambda(I_j^n)} \right\| \geqq \epsilon$$

and

$$\left\| \frac{F_n(I_j^n)}{\lambda(I_j^n)} - \sum_{\ell=1}^{q(j)} \alpha_\ell(j)x_\ell(j) \right\| < \frac{\epsilon}{2^n}.$$

Partition I_j^n into pairwise disjoint half-open intervals $J_1(j), \cdots, J_{q(j)}(j)$ such that if $1 \leqq \ell \leqq q(j)$ then $\lambda(J_\ell(j)) = \lambda(I_j^n)\alpha_\ell(j)$. Let Σ_{n+1} be the algebra generated by the collection $\{J_\ell(j) : 1 \leqq j \leqq p(n), 1 \leqq \ell \leqq q(j)\}$ and define F_{n+1} on Σ_{n+1} by $F_{n+1}(J_\ell(j)) = \lambda(J_\ell(j))x_\ell(j)$.

This completes the basic construction of the proof. We now establish (i) through (iv) for the sequence of pairs (Σ_n, F_n). The construction itself contains (i) and (ii). Note that

$$\begin{aligned}
\left\| \frac{F_{n+1}(J_\ell(j))}{\lambda(J_\ell(j))} - \frac{F_{n+1}(I_j^n)}{\lambda(I_j^n)} \right\| &= \left\| x_\ell(j) - \sum_{\ell=1}^{q(j)} \frac{F_{n+1}(J_\ell(j))}{\lambda(I_j^n)} \right\| \\
&= \left\| x_\ell(j) - \sum_{\ell=1}^{q(j)} \alpha_\ell(j)x_\ell(j) \right\| \\
&\geqq \left\| x_\ell(j) - \frac{F_n(I_j^n)}{\lambda(I_j^n)} \right\| \\
&\quad - \left\| \sum_{\ell=1}^{q(j)} \alpha_\ell(j)x_\ell(j) - \frac{F_n(I_j^n)}{\lambda(I_j^n)} \right\| \\
&\geqq \epsilon - \frac{\epsilon}{2^n} = \frac{2^n - 1}{2^n}\epsilon.
\end{aligned}$$

which establishes (iii). Next observe that

$$\begin{aligned}
\|F_n(I_j^n) - F_{n+1}(I_j^n)\| &= \left\| F_n(I_j^n) - \sum_{\ell=1}^{q(j)} F_{n+1}(J_\ell(j)) \right\| \\
&= \left\| F_n(I_j^n) - \sum_{\ell=1}^{q(j)} \lambda(I_j^n)\alpha_\ell(j)x_\ell(j) \right\| \\
&= \lambda(I_j^n) \left\| \frac{F_n(I_j^n)}{\lambda(I_j^n)} - \sum_{\ell=1}^{q(j)} \alpha_\ell(j)x_\ell(j) \right\| \\
&\leqq \lambda(I_j^n) \frac{\epsilon}{2^n}
\end{aligned}$$

which by the fact that each $E \in \Sigma_n$ is the disjoint union of some I_j^n's yields

$$\|F_n(E) - F_{n+1}(E)\| \leqq \frac{\epsilon}{2n}\lambda(E)$$

for each $E \in \Sigma_n$, i.e., (iv) is established.

Along with (i) and (iv) we have

(v) $\|F_n(E)\| \leqq \lambda(E)$, for each $E \in \Sigma_n$,

and (as an easy consequence of (iv)),

(vi) $\|F_n(E)/\lambda(E) - F_{n+1}(E)/\lambda(E)\| \leqq \epsilon/2^n$ for all $E \in \Sigma_n$, with $\lambda(E) > 0$.

Let $\mathcal{A} = \bigcup_n \Sigma_n$. Then $\mathcal{A}$ is an algebra of subsets of Ω. By (iv), if we let $E \in \mathcal{A}$ then $E \in \Sigma_n$ for all sufficiently large n and $\lim_n F_n(E) \equiv F(E)$ exists. Clearly $F: \mathcal{A} \to X$ is additive and by (v) satisfies $\|F(E)\| \leqq \lambda(E)$ for each $E \in \mathcal{A}$. Thus $F: \mathcal{A} \to X$ is countable additive and strongly additive on the algebra $\mathcal{A}$. By the Kluvanek extension theorem [45], F extends to a countable additive $\tilde{F}$ defined on the sigma-algebra Σ of subsets of Ω generated by $\mathcal{A}$. If $f \in X^*$ is given and $\|f\| \leqq 1$ then for each $E \in \Sigma$ we have by (v)

$$\begin{aligned} |f(\tilde{F}(E))| &= |\lim_\pi \textstyle\sum fF(E_i)| = \lim_\pi |\textstyle\sum fF(E_i)| \\ &\leqq \lim_\pi \textstyle\sum |fF(E_i)| \leqq \lim_\pi \textstyle\sum \lambda(E_i) \\ &= \lambda(E), \end{aligned}$$

(where π ranges over all countable disjoint coverings of E by members $E_1, \cdots, E_k, \cdots$ of $\mathcal{A}$ and $\pi_1 \leqq \pi_2$ means each member of π_2 is a union of members of π_1). Thus,

(vii) $\|\tilde{F}(E)\| \leqq \lambda(E)$

holds for each $E \in \Sigma$. Thus $\tilde{F}$ is of bounded variation and is dominated by λ.

Since X has the Radon-Nikodym property, $\tilde{F}$ is differentiable with respect to λ, say $\tilde{F}(E) = \int_E f(w)\, d\lambda(w)$ holds for each $E \in \Sigma$.

By the necessity of Rieffel's dentability theorem (Theorem 2.1) and the result of STEP ONE of the sufficiency part of the same theorem we have that given $B' \in \Sigma$ with $\lambda(B') > 0$ there exists $B \in \Sigma$, $B \subset B'$, $\lambda(B) > 0$ with diameter $\mathcal{A}_B(\tilde{F}) < \epsilon/10$. We shall show however that $\tilde{F}$ as constructed satisfies diameter $\mathcal{A}_B(\tilde{F}) \geqq \epsilon/4$ for all $B \in \Sigma$ with $\lambda(B) > 0$. The resulting contradiction will finish the proof.

First we observe that for $n \geqq 4$ if I_j^{n+1} is contained in I_i^n then

(viii) $$\left\| \frac{\tilde{F}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{\tilde{F}(I_i^n)}{\lambda(I_i^n)} \right\| \geqq \frac{\epsilon}{2}.$$

For if $n \geqq 4$ and I_j^{n+1} is contained in I_i^n then for all $m \geqq n+1$

$$\left\| \frac{F_m(1_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{F_m(I_i^n)}{\lambda(I_i^n)} \right\| \geqq \left\| \frac{F_{n+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{F_{n+1}(I_i^n)}{\lambda(I_i^n)} \right\|$$

$$- \left[\left\| \frac{F_n(I_i^n)}{\lambda(I_i^n)} - \frac{F_{n+1}(I_i^n)}{\lambda(I_i^n)} \right\| + \left\| \frac{F_{n+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{F_m(I_n^{n+1})}{\lambda(I_j^{n+1})} \right\| \right]$$

$$\geqq \frac{2^n - 1}{2^n}\epsilon - \frac{\epsilon}{2^n} - \sum_{k=n+1}^{m-1} \left\| \frac{F_k(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{F_{k+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} \right\|$$

by (iii) and (vi)

$$\geqq \quad \frac{2^n - 2}{2n}\epsilon - \sum_{k=n+1}^{m-1} \frac{\epsilon}{2^k} \geqq \frac{2^n-2}{2^n}\epsilon - \frac{\epsilon}{2^n} = \frac{2^n-3}{2^n}\epsilon > \frac{\epsilon}{2}.$$

Now (viii) follows by letting $m \to \infty$.

Let $B \in \Sigma$ be given with $\lambda(B) > 0$.

Since Σ is generated by $\mathcal{A}$ there exists $E \in \mathcal{A}$ such that $\lambda(E \backslash B) + \lambda(B \backslash E) < (\epsilon/16)\lambda(B)$, so that

$$\begin{aligned} \lambda(E \backslash B) &< \frac{\epsilon}{16}\lambda(B) - \lambda(B \backslash E) \\ &\leqq \frac{\epsilon}{16}[\lambda(B) - \lambda(B \backslash E)] \\ &= \frac{\epsilon}{16}\lambda(B \cap E). \end{aligned}$$

E must be a Σ_n for some $n \geqq 4$. Thus E is the union of some I_i^n's, say $E = \bigcup_{j \in K} I_j^n$. Then clearly some $j_0 \in K$ has

$$0 < \lambda(I^n_{j_0} \backslash B) < \epsilon/16\, \lambda(I^n_{j_0} \cap B)$$

since E is the disjoint union of the I_i^n's as i ranges through K and $\lambda(E \backslash B) < (\epsilon/16)\lambda\,(E \cap B)$.

Look at $I^n_{j_0} \cap B = C$. Clearly

(ix) $\lambda(I^n_{j_0} \backslash C) < \epsilon/16\, \lambda(C)$

so that $\lambda(C) > 0$. So we are in the same position with C that we started with B. We can conclude that there exists $I^{n+1}_{j_1} \subset I^n_{j_0}$ such that if $D = I^{n+1}_{j_1} \cap C$ then $\lambda(I^{n+1}_{j_1} \backslash D) < (\epsilon/16)\lambda(D)$. Again $\lambda(D) > 0$.

By (viii) and the inclusion $I^{n+1}_{j_1} \subset I^n_{j_0}$ we have

$$(*) \qquad \left\| \frac{\tilde{F}(I^{n+1}_{j_1})}{\lambda(I^{n+1}_{j_1})} - \frac{\tilde{F}(I^n_{j_0})}{\lambda(I^n_{j_0})} \right\| \geqq \frac{\epsilon}{2}.$$

It is readily checked that

$$(**)\quad \left\| \frac{\tilde F(I^n_{j_0})}{\lambda(I^n_{j_0})} - \frac{\tilde F(C)}{\lambda(C)} \right\| = \frac{\lambda(I^n_{j_0}\backslash C)}{\lambda(C)} \left\| \frac{\tilde F(I^n_{j_0}\backslash C)}{\lambda(I^n_{j_0}\backslash C)} - \frac{\tilde F(I^n_{j_0})}{\lambda(I^n_{j_0})} \right\|$$

which by (ix), $\leqq (\epsilon/16)\cdot 2 = \epsilon/8$. Likewise,

$$(***)\quad \left\| \frac{\tilde F(I^{n+1}_{j_1})}{\lambda(I^{n+1}_{j_1})} - \frac{\tilde F(D)}{\lambda(D)} \right\| \leqq \frac{\epsilon}{8}.$$

Thus, from $(*)$, $(**)$ and $(***)$, $\|(\tilde F(C)/\lambda(C)) - (\tilde F(D)/\lambda(D))\| \geqq \epsilon/4$ and we have shown that diameter $\mathcal{A}_B(F) \geqq \epsilon/4$, as we wanted.

The combined effect, of course, of the above results is that *a Banach space X possesses the Radon-Nikodym property if and only if every (closed) bounded (convex) subset of X is dentable.* Immediate from this and Maynard's observation (5) above (J. Uhl has given a prior analytic proof of this result in [46]) is the

THEOREM. *Let X be a Banach space. Then TFAE:*

(1) *X possesses the Radon-Nikodym property;*

(2) *every closed linear subspace of X possesses the Radon-Nikodym property;*

(3) *every separable closed linear subspace of X possesses the Radon-Nikodym property.*

Consequently if every separable subspace of X has a separable dual then X^* posseses the Radon-Nikodym property.

The fact that the Radon-Nikodym property is separably determined is an immediate consequence of Maynard's observation that a set is dentable whenever each of its countable subsets is dentable. That this is so is most easily seen by considering a non-dentable set B in the Banach space X: pick any point $b \in B$. There is an $\epsilon > 0$ such that $b \in \overline{\text{conv}}\,(B\backslash \text{Ball}\ (b,\epsilon))$. Thus there exist $b_1^{(1)}, \cdots, b^{(1)}_{n(1)}$ in B at distance more than ϵ away from b with $\sum_{i=1}^{n(1)} \lambda_i^{(1)} b_i^{(1)}$ close to b, for some $\lambda_1^{(1)}, \cdots, \lambda^1_{n(1)} \geqq 0$, $\sum_{i=1}^{n(1)} \lambda_i^{(1)} = 1$. Similarly with each $b_i^{(1)}$ we can find $b_j^{(2)}(i)$'s in $B\backslash\text{Ball}\,(b_i^{(1)},\epsilon)$ such that an appropriate convex combination of the $b_j^{(2)}(i)$ yields a good estimate to $b^{(1)}$. Repeat this procedure on $b_j^{(2)}(i)$'s, etc., etc.: the resulting countable collection is non-dentable.

Another immediate consequence of the Davis-Huff-Maynard-Phelps Theorem along with Rieffel's Theorem is the following result of Davis and Phelps [44]:

COROLLARY. *A Banach space X possesses the Radon-Nikodym property if and only if every equivalent renorming of X produces a dentable closed unit ball.*

Before continuing with our discussion of the geometric aspects of the Radon-Nikodym Theorem it is perhaps wise to recall historically the first applications of the theorem to the structure theory of Banach spaces. Probably the best known (though not always known to be an application of a Radon-Nikodym Theorem) is the application mentioned in Part I: neither c_0 nor L_1 (0, 1) are isomorphic to dual spaces. Why? Again, both are separable and do not possess the Radon-Nikodym property. Now everyone recalls from a first course in functional analysis the usual proof that c_0 and L_1 (0, 1) are not (isometric to) dual spaces. Their unit balls haven't nearly enough extreme points, so by the Krein-Milman Theorem they can't be duals. Of course, using *just* the classical Krein-Milman Theorem the isometric result is all that one can conclude. An isomorphic version is also available.

We say that a Banach space X possesses the *Krein-Milman property* ([47]) whenever every closed bounded convex subset of X possesses an extreme point. Clearly the Krein-Milman property is an isomorphic invariant. If X possesses the Krein-Milman property, then every closed bounded convex subset of X is the closed convex hull of its extreme points ([47]). The classical Krein-Milman Theorem yields that reflexive Banach spaces possess the Krein-Milman property. J. Lindenstrauss ([47]) showed that $\ell_1(\Gamma)$ possesses the Krein-Milman property and C. Bessaga and A. Pelczynski [48] showed that all separable conjugate spaces possess the Krein-Milman property. As c_0 and $L_1(0, 1)$ do not possess the Krein-Milman property, again one can see that neither is isomorphic to a dual.

The close relationship between the Krein-Milman and Radon-Nikodym properties both historically (insofar as applications are concerned) and conceptually (especially in light of the Rieffel-Davis-Huff-Maynard-Phelps Theorem) leads us to ask

PROBLEM 8. Are the Krein-Milman and Radon-Nikodym properties equivalent?

We first posed this in the first write-up of these notes and there has been considerable progress already. First, J. Lindenstrauss showed that *the Radon-Nikodym property implies the Krein-Milman property* (his proof appears in [49]). His proof is so elegant and simple it requires repetition: let B be a non-empty closed bounded convex subset of the Banach space X. Let $\epsilon \geqq 0$ be given. Pick $x \in B$ so that x is not a member of $\overline{\text{conv}}\,(B)\backslash\text{Ball}\,(x, \epsilon/2))$. Pictorially, see Figure 3. Now separate x from $\overline{\text{conv}}\,(B\backslash\text{Ball}\,(x, \epsilon/2))$ by a hyperplane; pictorially see Figure 4. By the Bishop-Phelps Theorem we can push the hyperplane to miss $\overline{\text{conv}}\,(B\backslash\text{Ball}\,(x, \epsilon/2))$ yet support B. (See Figure 5). Of course the face (the face of a hyperplane is the collection of points of B at which the corresponding functional achieves its max) of this new hyperplane in B — call it B_1 — has diameter $\leqq \epsilon$.

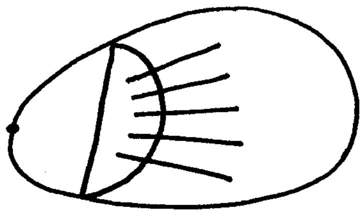

Figure 3

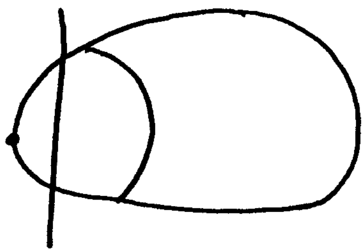

Figure 4

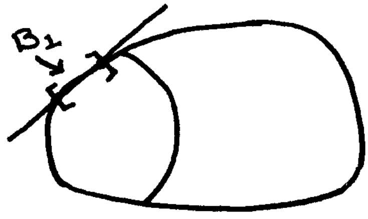

Figure 5

Now apply the same procedure to B_1 yielding a face of B_1 of diameter $\leqq \epsilon/2$. Continuing in this fashion we get a sequence B_n of closed bounded convex sets, diameter $B_n \leqq \epsilon/n$, each B_n a face of B_{n-1}. Thus $\bigcap_n B_n$ consists of a single point which by the facial nature of the B_n's is necessarily extreme.

The converse to Problem 8 remains open (there has been a significant contribution which we discuss later). In fact the following is not yet known concerning the Krein-Milman property:

PROBLEM 9. If every separable subspace of X possesses the Krein-Milman property need X possess the Krein-Milman property?

After Lindenstrauss' result, a number of improvements of the basic implications regarding the extremal structure of closed bounded convex sets in Banach spaces with the Radon-Nikodym property were discovered by R. R. Phelps [49]. We summarize them in the following:

THEOREM. *Let X be a Banach space. Then TFAE:*
(1) *X possesses the Radon-Nikodym property;*
(2) *every bounded subset of X is dentable;*
(3) *every closed, bounded, convex subset of X is dentable;*
(4) *every closed, bounded, convex subset of X has a denting point;*
(5) *every closed, bounded, convex subset of X is the closed convex hull of its denting points;*
(6) *every closed, bounded, convex subset of X has a strongly exposed point;*
(7) *every closed, bounded, convex subset of X is the closed convex hull of its strongly exposed points.*

The proofs of many of the implications in [49] are reminiscent of those found in the proof of the original Bishop-Phelps Theorem ([50]). This similarity is probably not accidental.

We say that a Banach space X possesses the *Bishop-Phelps property* whenever given a closed, bounded, absolutely convex set $B \subset X$ and a Banach space Y the collection of continuous linear operators from X to Y which achieve the maximum norm on B is uniformly dense in the space of all continuous linear operators from X to Y. Clearly, the Bishop-Phelps property is an isomorphic invariant. This property was studied by J. Lindenstrauss in [38] who showed that if X is a reflexive Banach space then X possesses the Bishop-Phelps property. In light of [38] and the aforementioned results of R. Phelps we have that *any Banach space with an equivalent locally uniformly convex norm* (a Banach space is said to be *locally uniformly convex* whenever given $x_n, x_0 \in X$ $\|x_n\| = 1 = \|x_0\|$ and $\|x_n + x_0\| \to 2$ implies $\|x_n - x_0\| \to 0$) *possessing the Bishop-Phelps property possesses the Radon-Nikodym property.* Based partially upon Phelps results and partially upon the knowledge of existing examples we ask

PROBLEM 10. If a Banach space X possesses the Bishop-Phelps property (respectively the Radon-Nikodym property) then need X be locally uniformly convexifiable?

Similarly,

PROBLEM 11. Are the Bishop-Phelps and Radon-Nikodym properties equivalent?

Another indication of the relationship of the Bishop-Phelps theorem and the Radon-Nikodym property can be found in the study of various types of smoothness and convexity properties of a norm in Banach spaces. Recall that a Banach space X is called *smooth* whenever each non-zero point $x \in X$ has a *unique* support functional in X^*; in this case one obtains a natural norm-to-weak-star continuous map from the sphere of X to that of X^*. If this spherical image map is norm-to-weak continuous X is said to be *very smooth*; if it is norm-to-norm continuous X's norm is said to be *Frechet differentiable.*

An easy consequence of Mazur's theorem and the Bishop-Phelps theorem is the

THEOREM. If X *possesses an equivalent very smooth norm then* X^* *possesses the Radon-Nikodym property.*

This result was first proved somewhat indirectly by E. Leonard and K. Sundaresan in [51] where it was shown that if E's norm was Frechet differentiable then for $1 < p < \infty$ so too was $L_E{}^p(0, 1)$'s norm in which case $L_E{}^p(0, 1)^* = L_{E^*}{}^{p'}(0, 1)$; an appeal to the results of N. Gretsky and J. J. Uhl Jr. [52] finishes the proof. A direct proof of the above theorem was given in [19]; a consequence of the above results found in [19] concerns renorming of $C(\Omega)$-spaces. Its proof indicates how Radon-Nikodym considerations can play an effective role in Banach space theory.

THEOREM. *If* $C(\Omega)$ *is a Grothendieck space* (Ω *an infinite compact Hausdorff space*) *then* $C(\Omega)$ *cannot be renormed smoothly.*

PROOF. (Recall a Grothendieck space is a Banach space X for which weak-star and weak sequential convergence in X^* are the same).

Suppose $C(\Omega)$ is a smoothable Grothendieck space. In its smooth norm, $C(\Omega)$ is still a Grothendieck space. Thus, in this new norm $C(\Omega)$ is very smooth. Thus, $C(\Omega)^*$ possesses the Radon-Nikodym property.

As the Radon-Nikodym property is an isomorphic invariant we can assume that $C(\Omega)$ — in its usual norm — has a dual with the Radon-Nikodym property. But $C(\Omega)$'s dual is an $L_1(\mu)$ and $L_1(\mu)$'s have the Radon-Nikodym property if and only if μ is purely atomic, i.e., $C(\Omega)^*$ is an $\ell_1(\Gamma)$ space. But then Ω must be a dispersed compact Hausdorff space ([53]), i.e., contains no perfect subsets. If we now take any sequence of distinct points of Ω it must have a convergent subsequence, otherwise, its set of limit points would be perfect; thus Ω is sequentially compact. Let (w_n) be a sequence of distinct points of Ω converging in Ω to w_0. Then denoting by δ_w the point-mass concentrated at $w \in \Omega$ we have $\delta_{w_n} \to \delta_{w_0}$ weak-star in $C(\Omega)^*$. But $C(\Omega)$ is a Grothendieck space so $\delta_{w_n} \to \delta_{w_0}$ weakly. As $C(\Omega)^* = \ell_1(\Gamma)$, $\delta_{w_n} \to \delta_{w_0}$ in norm. This is absurd since for $n \neq m$ $\|\delta_{w_n} - \delta_{w_m}\| = 2$!!

The above proof avoids the usual calculations of M. M. Day [54] and also extends his result to some extent. Also it suggests a possible affirmative answer to the

Problem 12. If X is a Grothendieck space and X^* possesses the Radon-Nikodym property then need X be reflexive? In particular, if X is a Grothendieck space with an equivalent (very) smooth norm need X be reflexive?

The smoothness conditions on a Banach space are invariably somewhat dual to convexity conditions on the dual space.

We say that the Banach space X is *strictly convex* whenever every norm one element of X is an extreme point of the closed unit ball of X; we say X is *weakly locally uniformly convex* whenever given $x_n, x_0 \in X$, $\|x_n\| = 1 = \|x_0\|$ and $\|x_n + x_0\| \to 2$ we have $x_n \to x_0$ weakly. Of course, locally uniformly convex spaces are weakly locally uniformly convex; in turn, weakly locally uniformly convex spaces are strictly convex.

If X^* is weakly locally uniformly convex then X^* is strictly convex so X is smooth. Thus there is a (unique) spherical image map of X's unit sphere to X^*'s which is $\|\cdot\|$-to-weak-star continuous. As is easily seen, weak-star and weak sequential convergence on the unit sphere of X^* coincide so X is very smooth. Thus *(weakly) locally uniformly convex dual spaces possess the Radon-Nikodym property.* We know of no dual space which possesses the Radon-Nikodym property which is not "in some way" dually (weakly) locally uniformly convexifiable. Thus

Problem 13. If X^* possesses the Radon-Nikodym property then need there exist a Banach space Y such that Y^* is isometric (isomorphic) to X^* and Y^* is dually locally uniformly convex?

As one quickly sees most of the affirmative information that has been gathered has concerned dual spaces. In fact in the case of dual spaces several of our above questions (Problems 1 and 8) have been answered in the affirmative. The most spectacular result is that of C. Stegall who showed in [55]:

THEOREM. *For any Banach space X, TFAE:*
(1) *X^* possesses the Radon-Nikodym property;*
(2) *every separable subspace of X has a separable dual;*
(3) *every separable subspace of X^* imbeds in some separable dual.*
Consequently, the dual of a separable Banach space possesses the Radon-Nikodym property if and only if it is separable.

The proof of Stegall is found in somewhat modified form in [0] (most of the modifications in [0] are due to R. E. Huff); perhaps a few applications of Stegall's theorem are in order; first, however we remark that using the Stegall construction, R. E. Huff and P. Morris ([56]) have proved that *for dual spaces the Krein-Milman property is equivalent to the Radon-Nikodym property.* So for example *Banach spaces with boundedly complete Markusevich bases* ([57]) *possess the Radon-Nikodym property* (this result was noted by P. Kranz and the authors but follows more easily from the results of S. Troyanski [57] and the Huff-Morris result). Similarly, *if X is a strong differentiability space in the sense of E. Asplund* ([58]) *then X^* possesses the Radon-Nikodym property* (this follows from results of [59] for example). Thus for dual spaces there are a number of ways in which one can test for the Radon-Nikodym property; of some interest would be an affirmative answer to the following,

PROBLEM 14. If X^* possesses the Radon-Nikodym property then need X^* possess a boundedly complete Markusevich basis (we refer the interested reader to [57], [60], and [61] for various discussions of Markusevich bases)?

Returning to Stegall's theorem we note first the following:

COROLLARY. *Let X^* possesses the Radon-Nikodym property. Let Y be any quotient of any subspace of X. Then Y^* possesses the Radon-Nikodym property.*

On to the applications of Stegall's theorem:

APPLICATION 1. If X^* possesses the Radon-Nikodym property then bounded sequences in X have weak Cauchy subsequences.

Indeed, if $(x_n) \subset X$ is a bounded sequence then the closed linear span $[x_n]$ of (x_n) is a separable subspace of X. By Stegall's theorem and its corollary, $[x_n]^*$ possesses the Radon-Nikodym property hence is separable; thus (x_n) has a $\sigma([x_n], [x_n]^*)$-Cauchy subsequence. Clearly this subsequence is also weak Cauchy in X.

Thus *a weakly sequentially complete Banach space X with X^* possessing the Radon-Nikodym property is reflexive.*

(If one wanted to, one could prove the above avoiding the Stegall result quoted above; however some hammers must be used—for example the results of [62], [63] and the fact that $L_1[0,1]$ does not possess the Radon-Nikodym property will suffice).

Another application is to topological tensor products:

APPLICATION 2. *Suppose X, Y are Banach spaces with X^* and Y^* possessing the Radon-Nikodym property and either X^* or Y^* possessing the approximation property. Then the projective tensor product $X^* \hat{\otimes} Y^*$ possesses the Radon-Nikodym property.*

In fact, by (G 10) of Part II, whichever of X^* or Y^* possesses the approximation property also possesses the metric approximation property. It follows therefore that $X^* \hat{\otimes} Y^*$ coincides (isometrically) with the class of nuclear operators from X to Y^* which by (G 9) is isometric to the class of integral operators from X to Y^*. But the integral operators from X to Y^* is the dual of the injective tensor product $X \check{\otimes} Y$ of X and Y. Thus $X^* \hat{\otimes} Y^*$ is a dual space. Let S be a separable closed linear subspace of $X \check{\otimes} Y$. Then there exists separable closed linear subspaces $\tilde{X}_0$ and $\tilde{Y}_0$ of X and Y such that S is a subspace of $\tilde{X}_0 \check{\otimes} \tilde{Y}_0$. But $(\tilde{X}_0 \check{\otimes} \tilde{Y}_0)^*$ is isometric to the space of integral operators from $\tilde{X}_0$ to $\tilde{Y}_0^*$. By the corollary to Stegall's theorem, $\tilde{Y}_0^*$ possesses the Radon-Nikodym property so by (G 9) the space of integral operators from $\tilde{X}_0$ to $\tilde{Y}_0^*$ is identical to the space of nuclear operators from $\tilde{X}_0$ to $\tilde{Y}_0^*$. By Stegall's theorem both $\tilde{X}_0^*$ and $\tilde{Y}_0^*$ are separable, so the space of nuclear operators from $\tilde{X}_0$ to $\tilde{Y}_0^*$ is separable. Thus $(\tilde{X}_0 \check{\otimes} \tilde{Y}_0)^*$ is separable and so, by the Dunford-Pettis theorem, possesses the Radon-Nikodym property. By the corollary to Stegall's theorem S^* is separable. Thus every separable subspace of the predual of $X^* \hat{\otimes} Y^*$ has separable dual. It follows that $X^* \hat{\otimes} Y^*$ possesses the Radon-Nikodym property.

The use of the fact that the spaces involved were dual spaces in the above proof is clear; it is not clear that the dual nature of the spaces is necessary. Thus the

PROBLEM 15. If X and Y possess the Radon-Nikodym property then does $X \hat{\otimes} Y$?

Epilogue. In the preceding, we've developed a number of formulations of the Radon-Nikodym property. It might be of some use to collect these. So we state the,

THEOREM. *Let X be a Banach space. Then TFAE:*

(1) *X possesses the Radon-Nikodym property;*

(2) *every closed linear subspace of X possesses the Radon-Nikodym property;*

(3) *every separable closed linear subspace of X possesses the Radon-Nikodym property;*

(4) *every function* $f : [0, 1] \to X$ *of bounded variation is differentiable almost everywhere;*

(5) *every absolutely continuous function* $f : [0, 1] \to X$ *is differentiable almost everywhere with*

$$f(b) - f(a) = \int_a^b f'(t)\,dt$$

for any $a, b \in [0, 1]$;

(6) *every continuous linear operator* $T : L_1[0, 1] \to X$ *factors through* $\mathcal{L}_1$, i.e., *given T there exist continuous linear operators* $R : \mathcal{L}_1 \to X$ *and* $S : L_1[0, 1] \to \mathcal{L}_1$ *such that the diagram*

$$\begin{array}{ccc} L_1[0,1] & \xrightarrow{\;T\;} & X \\ & {\scriptstyle S}\searrow \quad \nearrow{\scriptstyle R} & \\ & \mathcal{L}_1 & \end{array}$$

commutes;

(7) *for every compact Hausdorff topological space* Ω *the absolutely summing, integral and nuclear operators from* $C(\Omega)$ *to X are isometrically identical;*

(8) *for every Banach space Y the Pietsch integral operators* ([64]) *from Y into X coincide (isometrically) with the nuclear linear operators from Y into X;*

(9) *every (closed) bounded (convex) subset of X is dentable;*

(10) *every closed bounded convex subset of X has denting points and is the closed convex hull of such;*

(11) *every closed bounded convex subset of X has strongly exposed points and is the closed convex hull of such.*

If there is a Banach space Y such that X is isomorphic to Y^* *then* (1) *through* (11) *are equivalent to*

(12) *every separable subspace of Y has a separable dual;*

(13) *every separable subspace of X is isomorphic to a subspace of a separable dual*;

(14) *every closed bounded convex subset of X has extreme points and is the closed convex hull of such*;

(15) *For* $1 \leqq p < \infty$, $L_p(\mu, X)^* = L_{p'}(\mu, X^*)$ *where* $p^{-1} + (p')^{-1} = 1$.

Concerning spaces with or without the Radon-Nikodym property we have

Radon-Nikodym Property

Yes	No
separable duals	C_0
reflective spaces	$L_1[0, 1]$
weakly compactly generated duals	$C(\Omega)$, Ω infinite compact, Hausdorff
duals of Frechet differentiable spaces	$L_1(\mu)$, μ not purely atomic
duals of strong differentiable spaces [49]	$L_\infty(\mu)$, μ non-trivial
locally uniformly convex duals,	
$L_p(\mu)$ $1 < p < \infty$	
$\ell_1(\Gamma)$, Γ any set	
nuclear operators between L_p, L_q	
quasi-reflexive spaces	
X^{**}, X^* when X^{**}/X is separable [65]	
X^*, if $\{G \in X^{**} \|G\| \leqq 1\}$ is Eberlein compact weak star topology [65]	
X^* when contained in a weakly compactly generated space [65]	
unconditionally convergent series in reflexive space (or any dual space with Radon-Nikodym	
absolutely convergent series in a reflexive space (or any space with Radon-Nikodym)	

Largely because it never seemed to fit in the previous sections we have not remarked upon a fundamental method of W. J. Davis, T.

Figiel, W. B. Johnson and A. Pelczynski [66] concerning factorization of operators. One particular variation of this method yields that weakly compact linear operators factor through reflexive spaces. A moment's reflection leads to a quick derivation of the Dunford-Pettis-Phillips theorem as an immediate consequence of the Dunford-Pettis theorem. Another variation of the Davis et al. construction is the fact that separable duals are subspaces of spaces with boundedly complete bases (this is actually dependent also upon the deep results of [67]) so the Dunford-Pettis theorem follows from the Dunford-Morse result. As all of classical vector measures is, in a sense, the study of weakly compact operators the paper of Davis, Figiel, Johnson and Pelczynski is must reading for workers in the area.

Finally, we have to admit to concentrating our attentions on a rather special topic: the Radon-Nikodym theorem as it pertains to Banach space theory. Applications to the study of non-linear operators (see [68], [69] for example) or probability theory (see [69], [70] or [71]) were not discussed. The role of the Radon-Nikodym theorem in the topological classification of Frechet spaces (see [72]) was never mentioned. Perhaps our greatest oversight has been neglecting the basic work of E. Thomas [73] on the Radon-Nikodym theorem in general spaces.

Added in Proof. Since the submission of the original manuscript, there has been a great deal of progress in the study of the Radon-Nikodym property. It seems like no time is better than the present to report on this progress. As is usual, attendant to this progress are new problems — we mention these as well.

Once one knows of the existence of many extreme points for closed bounded convex subsets of a Banach space with the Radon-Nikodym property, it is natural to ask if Choquet-type theorems might not be possible. Such is the case. G. A. Edgar [A1] demonstrated the first such result showing the following

THEOREM. *Let K be a separable closed bounded convex subset of a Banach space X that has the Radon-Nikodym property. Let $x \in K$. Then there is a Borel probability measure μ defined on K such that x is the resolvent of μ and $\mu(\text{ext } K) = 1$.*

It should be mentioned that in the case of separable Banach spaces (in fact, more generally, separable Fréchet spaces) the extreme points of closed bounded convex sets are universally measurable, a fact due to R. Bourgin [A2]; thus the statement $\mu(\text{ext } K) = 1$ makes sense.

The possibility of *neat* extensions of Edgar's result to nonseparable closed bounded convex sets is discussed in [A3]; it seems plausible at this time that a Choquet theorem of the sort that each point of a closed bounded convex set K in a Radon-Nikodym space is representable by a probability measure μ defined on the Borel subsets of K which satisfies $\mu(B) = 1$ for any Borel set $B \supseteq \text{ext } K$ can be proved. Such a result would be a very exciting complement to Edgar's result.

A few words on Edgar's method of proof are in order, particularly since the proof is so beautiful. The idea is as follows: take a point x in K. If x is extreme, then the obvious point mass has x as a resolvent. If not, then x is the midpoint of some nontrivial line segment in K; if the endpoints of this segment are extreme points, fine: average the point masses. Otherwise, one-or-both-of these endpoints is the midpoint of some nontrivial line segment in K; continue this process. The hope is that the endpoints will fan out to the edges of the set K; of course, this isn't going to happen but one can hope. What is happening though is that we are constructing a K-valued martingale (look at the Huff construction). Now a fact of life in Banach spaces (proved by S. D. Chatterji [A4]) is that the Radon-Nikodym property for X is equivalent to the convergence of bounded equi-integrable martingales in $L_1(\mu, X)$ for all μ's. Thus the martingale constructed above converges to some bounded measurable K-valued function f defined on K which, it is hoped, lives further out towards the edges of K than x is. We wish to continue pushing the range of f out towards the edges — it is unlikely that f's range already lies in the extreme points of K. To keep spreading f out, we use the Kuratowski-Ryll-Nardzewski selection theorem [A5] to find universally measurable functions f_1, f_2 on K to K which satisfy $(f_1 + f_2)/2 = f$ with $f_1(k) = f(k)$ if and only if $f(k)$ is extreme. Repetition of the first part of the argument, only now dealing with measurable functions rather than simple functions again produces a convergent martingale; now a transfinite induction allows one to continue the process through all the countable ordinals. The process is easily seen at each point of K to be a continuous process from the ordinals less than the first uncountable ordinal to K. Thus, there is a *countable* stopping time at each point of K. But at the stopping time, the procedure of construction insures us that the value is an extreme point. Of course, along the way, we've been carrying the probability measure μ' defined on the Borel sets of the two-point space $\{0, 1\}$ raised to the power Ω, where Ω denotes the first uncountable ordinal. The fact that the limit function g given by the stopping time's value has its values in the extreme points of K allows one to define $\mu(B) = \int_B g \, d\mu'$. It is a routine calculation to see that μ is the representing measure for x.

Since Edgar's result, there has been considerable effort on proving uniqueness theorems for the representing measures; the paper of R. Bourgin and G. A. Edgar [**A6**] extends the classical simplex characterization in a totally natural *but* highly nontrivial manner to this noncompact situation.

Though the equivalence of the Krein-Milman property with the Radon-Nikodym property remains open, a variation of the Krein-Milman property has been shown to be equivalent to the Radon-Nikodym property by Bob Huff and Peter Morris. We say that a Banach space X has the *strong Krein-Milman property* whenever each closed bounded subset of X contains an extreme point of its closed convex hull. Huff and Morris [**A7**] proved the following

THEOREM. *Each of the following statements imply all of the others*:

(1) *X has the Radon-Nikodym property*;

(2) *each closed bounded subset of X contains an extreme point of its convex hull*;

(3) *X has the strong Krein-Milman property*;

(4) *for each closed bounded convex subset K of X, the set of elements in X^* which strongly expose some point of X is dense in X^*.*

(5) *X does not have the following property*: *There exists a closed subset K contained in the interior of a closed bounded convex set C in X with $\overline{\text{co}}(K) = C$.*

A key step in the proof of the above theorem is a lemma which is a modification of the key lemma of the Davis-Phelps proof of the equivalence of dentability of arbitrary bounded subsets of X with X having the Radon-Nikodym property. This lemma, which promises to be of considerable use in future constructions highlights dentability as a compactness type property. It bears mention.

LEMMA (DAVIS-HUFF-MORRIS-PHELPS). *Suppose K is a closed convex nondentable set with nonempty interior K^0. Then there exists $r > 0$ such that*

$$K^0 = \text{co}(K^0 \setminus \bigcup_{i=1}^{n} \{y \in X : \|y - x_i\| < r\})$$

for any finite subset $\{x_1, \cdots, x_n\} \subseteq K$.

In the fall of 1974, Isaac Namioka and Bob Phelps turned their attentions to a class of spaces introduced by E. Asplund. Called by Asplund strong differentiability spaces; these spaces have come to be known as Asplund spaces: a Banach space X is said to be an *Asplund space* whenever every continuous convex function defined on an open convex domain in X has a dense domain of Fréchet differentiability.

Asplund [**A8**] essentially showed that *if X admits an equivalent norm whose dual norm is locally uniformly convex, then X is an Asplund space; in particular, if X* is reflexive or separable then X is an Asplund space* (this requires a number of deep renorming theorems which one may find in [**A9**] or [**A10**]). Asplund also established the

THEOREM. *If X is an Asplund space, then each weak* compact convex subset of X* is the weak* closed convex hull of its weak* strongly exposed points* (A weak* strongly exposed point of $K \subseteq X^*$ is a strongly exposed point whose exposing functional is back in X).

Namioka and Phelps established the converse, i.e., *X is an Asplund space if and only if weak* compact convex subsets of X* are the weak* closed convex hulls of their weak* strongly exposed points.*

Using this result, they were able to establish some striking stability properties for the class of Asplund spaces; previously Asplund had shown that the property of being an Asplund space was preserved by quotient maps; using the Namioka-Phelps characterization, one actually has the

THEOREM. *If X is any subspace of a quotient of an Asplund space, then X is also an Asplund space.*

Moreover, if $(X_n)_{n=1}^{\infty}$ are Asplund spaces, then $(\Sigma_n \oplus X_n)_{\ell_p}$, $(\Sigma_n \oplus X_n)_{c_0}$ are Asplund spaces for any $1 < p < \infty$.

It is easily established that if X is an Asplund space, X^* has the Radon-Nikodym property. This suggests the

PROBLEM 16. If X^* has the Radon-Nikodym property, need X be an Asplund space?

This problem was originally raised by Bob Phelps and already there has been significant progress on it. Collier [**A11**] and Namioka and Phelps [**A12**] have shown that *if X* is weakly compactly generated, then X is an Asplund space.* Also, *if $C(\Omega)^*$ has the Radon-Nikodym property* (which happens if and only if Ω is dispersed), *then $C(\Omega)$ is an Asplund space.* Peter Morris showed that *if X is a subspace of some weakly compactly generated Banach space and X* has the Radon-Nikodym property, then X is an Asplund space.*

It seems likely that an affirmative solution is in the offing to

PROBLEM 17. If X^* has the Radon-Nikodym property and X embeds in a weakly compactly generated space, need X be weakly compactly generated?

This would constitute a simultaneous extension of the Friedland-John-Zizler theorem ([**A13**], [**A14**]): *if X has a Frechet differentiable norm and X embeds in a weakly compactly generated space, then X is weakly compactly generated,* and the recent result of John and Zizler [**A16**]: *if both X and X^* embed in weakly compactly generated spaces, then X is weakly compactly generated.* That the latter result would be extended is a consequence of Kuo's theorem: *if X^* is a subspace of a weakly compactly generated space, then X^* has the Radon-Nikodym property.* When the original manuscript of this paper was submitted only Kuo's original proof of this fact was available; since that time, Peter Morris has derived a truly beautiful and elegant proof which bears repeating: we want to show that every separable subspace of X has a separable dual; take a separable subspace S of X. By a classical result of Banach and Mazur, there is a quotient map Q: $\ell_1 \overset{\text{onto}}{\to} S$. $Q^*: S^* \to \ell_\infty$ is an imbedding. Since S is a subspace of X, S^* is a quotient of X^*. So we have the diagram

$$\begin{array}{ccccc} X^* & \overset{R}{\to} & S^* & \overset{Q^*}{\to} & \ell_\infty, \\ \downarrow & & & & \\ Y & & & & \end{array}$$

where R is the quotient map of X^* onto S^* and Y is the weakly compactly generated space that contains X^*. The map Q^*R takes its values in the *injective space* ℓ_∞([**A18**]) so can be extended to a bounded linear operator $M: Y \to \ell_\infty$. Of course, $\overline{MY}$ is a weakly compactly generated linear subspace of ℓ_∞; but weakly compact subsets of ℓ_∞ are separable, so the range of M (and consequently, Q^*S^*) is separable. Since S^* is isomorphic to Q^*S^*, S^* is separable. FINIS

It should be remarked here that the above results are sharp; in fact, H. P. Rosenthal's example [**A19**] of a nonweakly compactly generated subspace of a weakly compactly generated space is a dual space. Hence, it has the Radon-Nikodym property though it is not weakly compactly generated.

Before leaving the subject of Asplund spaces, we mention a few more problems:

PROBLEM 18. If X is an Asplund space, need X admit an equivalent Fréchet differentiable norm?

PROBLEM 19. If X admits an equivalent Fréchet differentiable norm then need X be an Asplund space?

PROBLEM 20. If X^* has the Radon-Nikodym property, then need X have an equivalent Fréchet differentiable norm?

PROBLEM 21. If X and Y are Asplund spaces, need $X \check{\otimes} Y$, the injective tensor product of X, Y, be an Asplund space?

When studying elementary functional analysis, the first example of a separable Banach space with nonseparable dual is ℓ_1. For many years, it was open as to whether or not this might not be the fundamental culprit, i.e., if X is separable and X^* not, then maybe ℓ_1 lies somewhere in X. Such a possibility was put to rest by R. C. James [**A20**] who constructed a Banach space called "James tree space" denoted by JT that had the following startling properties (many of these properties were established by J. Lindenstrauss and C. Stegall [**A21**]):

1. JT is a separable dual space which does not contain ℓ_1;
2. JT^* is nonseparable and does not contain ℓ_1 or c_0;
3. $JT^{**}, JT^{****}, \cdots$, all the even duals of JT are weakly compactly generated;
4. $JT^*, JT^{***}, \cdots$, all the odd duals of JT fail the Radon-Nikodym property; in particular, none are weakly compactly generated.

Notwithstanding the space JT, the containment or noncontainment of ℓ_1 is closely related to geometric problems with Radon-Nikodym flavor. Most of these relations are based on a spectacular result of Haskell Rosenthal [**A22**]: *a Banach space X contains ℓ_1 if and only if X admits a bounded sequence (x_n) which has no weak Cauchy subsequences.* Based in part on the Rosenthal constructions, Rosenthal and Ted Odell [**A23**] showed the results for *separable spaces*:

THEOREM. *X contains no subspace isomorphic to ℓ_1 if and only if weak* compact convex subsets of X^* are the norm-closed convex hull of their extreme points.*

Thus the James tree space shows that the full power of Krein-Milman is needed in dual spaces to deduce the Radon-Nikodym property; Stegall's construction of [**A24**] is sharp!

Using the tools developed by Rosenthal in [**A22**] and calling upon his own rather impressive collection of mathematical weapons, Bill Johnson [**A25**] showed the following:

THEOREM. *Suppose X^* contains ℓ_1; however, whenever ℓ_1 imbeds in X^*, the unit vector basis of ℓ_1 never has a weak* convergent subsequence. Then ℓ_1 imbeds in X.*

COROLLARY. *If X is a Grothendieck space and X^* has the Radon-Nikodym property, then X is reflexive.*

COROLLARY. *If X is a Grothendieck space admitting an equivalent smooth norm, then X is reflexive.*

The attentive reader will note that these last two corollaries also resolve problem 12 in the affirmative.

It ought to be mentioned at this juncture that Bill Johnson [**A26**] has also given the following improvement on Stegall's theorem: *Let X^* have the Radon-Nikodym property and $S \subseteq X^*$ be a separable closed linear subspace, then there exists a separable dual Y contained in X^* containing S with Y complemented in X^*.* This gives further evidence of an affirmative response to the

PROBLEM 22. If X has the Radon-Nikodym property and S is a separable subspace of X, is S contained in a complemented separable subspace of X?

As we saw in our discussion of the Dunford-Pettis results and later in our discussion of the Bishop-Phelps property, there appears to be a close relationship between the Radon-Nikodym property and density of norm-attaining operators. This belief in the relationship between these formally different notions is further fortified by the Huff-Morris results alluded to above. Recently, the second author of this paper has proved the following: *Suppose X is a strictly convex Banach space. Then in order that the operators from $L_1[0, 1]$ to X that attain their norm be dense in the space of operators, it is necessary and sufficient that X have the Radon-Nikodym property.*

The Phelps results on the plenitude of strongly exposed points in arbitrary closed bounded convex subsets of spaces with the Radon-Nikodym property establishes this property as a central notion in convexity theory of Banach spaces; as is usual with the notions related to convexity it is fruitful to search for dual notions of smoothness. Such a program has been initiated by M. Edelstein [**A27**] and, modulo a few oversights on the part of Edelstein, continued by D. C. Kemp [**A28**]. The work of Edelstein and Kemp is still in its infancy; however, it raises the following general

PROBLEM 23. Find (smoothness) conditions on X^* that insure that X has the Radon-Nikodym property. Hopefully, the conditions will be necessary, sufficient *and* applicable!

In this connection, we must mention some recent very elegant work of Fran Sullivan [**A29**]. It is a basic fact of nonreflexive Banach spaces that the convexity and smoothness of the norm deteriorates badly as one passes to higher duals. So if X is a nonreflexive Banach space, then (a) X^{****} is not strictly convex, (b) X^{***} is not smooth,

(c) X^{**} is not weakly locally uniformly convex, *and* (d) X^* is not very smooth (see [A10]). Similarly, for nonRadon-Nikodym dual spaces, we have a deterioration of convexity and smoothness properties; stated positively, we have for X^* to have Radon-Nikodym it is sufficient that (a) X be very smooth, (b) X^* be weakly locally uniformly convex, (c) X^{**} be smooth, or (d) X^{***} be strictly convex. The last two conditions are to be found in Sullivan's paper along with a number of other results related to the problem of finding conditions on X that insure the strict convexity or smoothness of X^*.

As we saw in the section dealing with Grothendieck's contributions to the Radon-Nikodym theorem, the Radon-Nikodym property is a useful tool in studying spaces of operators. When do various spaces of operators have the Radon-Nikodym property? It is not surprising that for the space of bounded linear operators, the conditions are rather restrictive. In fact, *if X^* and Y have the Radon-Nikodym property and if every continuous linear operator from X to Y is compact, then $\mathcal{L}(X; Y)$ — the space of bounded linear operators from X to Y has the Radon-Nikodym property.* This result is due in the separable case to the first author; W. B. Johnson showed how to reduce the general case to the separable case. A noteworthy corollary of this result is the following: *the space $\ell_1\langle X\rangle$ of unconditionally convergent series in a Banach space X with the Radon-Nikodym property has the Radon-Nikodym property.* This in turn suggests the

Problem 24. Let α be an accessible $\otimes$ norm [A30] and X be a Banach space with the Radon-Nikodym property. Does $\ell_1 \otimes_\alpha X$ have the Radon-Nikodym property?

For $\alpha = \lambda$ or γ (least and greatest reasonable norms), the response is yes.

In connection with the aforementioned result on $\mathcal{L}(X; Y)$ having the Radon-Nikodym property, we mention that Terry Morrison has given an extremely elegant proof of the following partial converse: *if $\mathcal{L}(X; Y)$ has the Radon-Nikodym property and Y has a complemented subspace with an unconditional basis, then every operator from X to Y is compact.* In fact, otherwise Morrison shows the existence of a Boolean algebra $\mathcal{A}$ and a bounded additive set function $F : \mathcal{A} \to \mathcal{L}(X; Y)$ which is not strongly additive (see [A31]), an implication of which condemns $\mathcal{L}(X; Y)$ to lack the Radon-Nikodym property.

Finally, we mention some results of Heinrich Lotz [A32] pertaining to the Radon-Nikodym property in dual Banach lattices. The most striking result demonstrating the special nature of Banach lattice goes as follows:

THEOREM. *Let X be a dual Banach lattice and suppose X does not contain an order isomorphic copy of c_0 or $L_1[0, 1]$. Then X has the Radon-Nikodym property.*

Similar stronger statements in the presence of additional conditions on the order structure of X are contained in [**A32**]. In passing, we mention the following intriguing problem posed by Lotz:

PROBLEM 25. If X is a Banach lattice with the Radon-Nikodym property, is X a dual space?

Some final comments are in order. Problems 5 and 6 have been answered in the negative by Jim Hagler who has exhibited a Banach space JH with the following properties:

(1) JH is separable and hereditarily c_0;

(2) JH^* is nonseparable; hence, by Stegall's theorem, lacks the Radon-Nikodym property;

(3) weak Cauchy sequences in JH^* are norm convergent.

The space JH is modeled after the James tree space (with c_0 as a building block rather than ℓ_2) and possesses a number of fascinating properties in addition to (1), (2) and (3).

The result of Peter Morris to which we alluded in the discussion of Asplund spaces has also been obtained by K. John and V. Zizler. With regards to Asplund spaces, a recent advance has been made by I. Ekeland and G. Lebourg who have given an affirmative response to Problem 16; we recommend the reader to the lecture notes of John Rainwater regarding the Akeland-Lebourg result — these notes are particularly clear, self-contained (and, we are told, self typed).

Finally, Gerry Edgar has demonstrated the following stability result: if $Y \subseteq X$ and both Y and X/Y have the Radon-Nikodym property, then X has the Radon-Nikodym property.

We regret the omission of any reference to the work of J. Gil de la Madrid on the relationship of the theory of vector measures and tensor products of Banach spaces; Gil de la Madrid's papers appear in Trans. AMS **114** (1965), 98–121 and Canad. J. Math. **18** (1966), 762–793. The reader interested in the topics of Part II will be well advised to refer to these papers.

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