THE RADON-NIKODYM THEOREM FOR VON NEUMANN ALGEBRAS

BY

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Abstract

Let φ be a faithful normal semi-finite weight on a von Neumann algebra $\mathfrak{M}$. For each normal semi-finite weight ψ on $\mathfrak{M}$, invariant under the modular automorphism group Σ of φ, there is a unique self-adjoint positive operator h, affiliated with the sub-algebra of fixed-points for Σ, such that $\psi = \varphi(h\,\cdot\,)$. Conversely, each such h determines a Σ-invariant normal semi-finite weight. An easy application of this non-commutative Radon-Nikodym theorem yields the result that $\mathfrak{M}$ is semi-finite if and only if Σ consists of inner automorphisms.

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1. Introduction

The classical Radon-Nikodym theorem asserts that if a measure ν is absolutely continuous with respect to a measure μ, then there is a measurable function h such that $\nu = \mu(h\cdot)$. It is easy to see that the absolute continuity of ν with respect to μ is equivalent to the condition that ν can be regarded as a normal functional on L^∞_μ. Since L^∞_μ is the proto-

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type of an abelian von Neumann algebra, the question therefore naturally arises whether one can find generalizations of the Radon-Nikodym theorem in the non-commutative case. The most successful results in this direction are due to H. Dye [7] and I. Segal [23], who show that if φ is a faithful normal semi-finite trace on a von Neumann algebra $\mathfrak{M}$ then for each normal positive functional ψ on $\mathfrak{M}$ there is a unique (unbounded) self-adjoint positive operator h affiliated with $\mathfrak{M}$ such that $\psi = \varphi(h\cdot)$.

Simple counterexamples show that one cannot hope for a Radon-Nikodym theorem with arbitrary functionals φ and ψ. Even the weaker result that $\psi = \varphi(h\cdot h)$ does not hold in general if one requires h to be a closed operator (the BT-theorem is the best possible result in that direction). We shall accordingly ask the Radon-Nikodym theorem to hold only for pairs φ, ψ of functionals which "commute" in a sense to be made precise below.

In [28, 29] M. Tomita, in his fundamental study of the relation between a von Neumann algebra and its commutant, discovered that with each full left Hilbert algebra is associated a one-parameter group of automorphisms of the left von Neumann algebra. In particular, each faithful normal positive functional ψ on a von Neumann algebra $\mathfrak{M}$ gives rise to a one-parameter group Σ of automorphisms of $\mathfrak{M}$, since φ induces a left Hilbert algebra structure on $\mathfrak{M}$. Using the theory of weights (semi-finite positive functionals) F. Combes showed in [2] that, conversely, each full left Hilbert algebra arises from a faithful normal semi-finite weight on the left von Neumann algebra. This fact was also discovered by M. Tomita in [29]. We shall adopt this last point of view since it allows us to regard the triple (φ, $\mathfrak{M}$, Σ) as an analogue of a measure and its L^∞-algebra. The modular automorphism group Σ, which is trivial in the commutative case (and when φ is a trace), serves as the extra information that compensate the lack of invariance (trace-structure) in φ. The weights ψ which are invariant under Σ, i.e. $\psi(\sigma_t(x)) = \psi(x)$ for all σ_t in Σ and x in $\mathfrak{M}_+$, are the ones which "commute" with φ; and these are the functionals for which the Radon-Nikodym theorem holds.

After a brief summary in § 2 of the theory of normal semi-finite weights, their representations and their modular automorphism groups we describe in § 3 the important class of analytic elements for Σ in $\mathfrak{M}$. In particular we characterize the elements in the von Neumann algebra $\mathfrak{M}^\Sigma$ of fixed-points for Σ in $\mathfrak{M}$ as those elements h in $\mathfrak{M}$ for which the functionals $\varphi(h\cdot)$ and $\varphi(\cdot h)$ are semi-finite and equal.

We can therefore in § 4 introduce an affine map: $h \mapsto \varphi(h\cdot)$ from $\mathfrak{M}^\Sigma_+$ to the set of Σ-invariant normal semi-finite weights on $\mathfrak{M}$. A little extra work shows that the map can be extended to all self-adjoint (unbounded) positive operators affiliated with $\mathfrak{M}^\Sigma$. The main problem in this paragraph is to show that the elements in the modular automorphism group of a weight $\psi = \varphi(h\cdot)$ have the form $\sigma_t^\psi(\cdot) = h^{it}\sigma_t(\cdot)h^{-it}$.

In § 5 we prove the Radon-Nikodym theorem which says that given a triple $(\varphi, \mathcal{M}, \Sigma)$, each Σ-invariant normal semi-finite weight ψ on $\mathcal{M}$ is of the form $\varphi(h\cdot)$, where h is a unique self-adjoint, positive operator affiliated with $\mathcal{M}^{\Sigma}$. As a prerequisite for the theorem we establish the result that if ψ satisfies the Kubo-Martin-Schwinger condition with respect to Σ then $\psi=\varphi(h\cdot)$ where h is affiliated with the center of $\mathcal{M}$. This is used to show that if ψ is a normal semi-finite weight on $\mathcal{M}$ with modular automorphism group Σ^{ψ} then the Σ-invariance of ψ implies that the groups Σ and Σ^{ψ} commute. Conversely, if Σ and Σ^{ψ} commute then ψ is Σ-invariant under fairly mild extra conditions, but not in general. Only as a corollary of the Radon-Nikodym theorem do we learn that Σ-invariance of ψ implies Σ^{ψ}-invariance of φ (so that "commutativity" is a symmetric relation). The main reason for solving the problems of commuting automorphism groups first (instead of collecting the results as corollaries of the main theorem) is that we need this material to show that two "commuting" weights which agree on a dense subalgebra are equal. For arbitrary weights this is not always true.

Section 6 consists of applications of the Radon-Nikodym theorem to automorphism groups of von Neumann algebras. We consider a triple $(\varphi, \mathcal{M}, \Sigma)$ and a group G of automorphisms of $\mathcal{M}$ which commutes with Σ and leaves the center of $\mathcal{M}$ pointwise fixed. Then $\varphi\circ g=\varphi(h_g\cdot)$ for each g in G, where h_g is a self-adjoint positive non-singular operator affiliated with the center of $\mathcal{M}$; and the map $g\mapsto h_g$ is a homomorphism. In some cases this implies that φ is G-invariant (i.e. if φ is finite or if G is compact). We show that if there is some G-invariant weight ψ on $\mathcal{M}$ and if ψ is also Σ-invariant (so that $\psi=\varphi(h\cdot)$) then φ is G-invariant under certain integrability conditions on h, though in general it need not be. The results are inspired by works of N. Hugenholtz and E. Størmer on the corresponding problems for normal states on $\mathcal{M}$ (see [12] and [24]). We finally give a construction for the implementation of G as a group of unitaries on the Hilbert space of φ.

In § 7 we give four more applications of the Radon-Nikodym theorem. We first show that each normal weight on a von Neumann algebra is the sum of normal positive functionals. As a corollary each lower semi-continuous weight on a C*-algebra is the sum of positive functionals. We next present a short proof of the result from [25] that a von Neumann algebra $\mathcal{M}$ is semi-finite if and only if there is a faithful normal semi-finite weight on $\mathcal{M}$ whose modular automorphism group is implemented by a one-parameter unitary group in $\mathcal{M}$. Specializing to normal semi-finite weights on a semi-finite von Neumann algebra $\mathcal{M}$, we consider the problem whether $\psi=\varphi(t\cdot t)$ for some t in $\mathcal{M}_+$, whenever $\psi\leqslant\varphi$. We show that this is true, with a unique t, and that $\|t\|\leqslant 1$, provided that φ is finite or majorized by a normal semi-finite trace on $\mathcal{M}$; but that it need not hold in general. Thus S. Sakai's non-commutative Radon-Nikodym theorem for finite weights ([22]) cannot be extended to semi-finite weights. Finally we use the bijective correspondence between self-adjoint positive operators in $\mathfrak{H}$ and normal semi-finite weights on $\mathcal{B}(\mathfrak{H})$, obtained by applying the Radon-Nikodym theorem to the triple (Tr, $\mathcal{B}(\mathfrak{H})$, $\{\iota\}$), to define a strong sum between certain pairs of self-adjoint positive operators in $\mathfrak{H}$.

2. Weights and modular automorphism groups

For the convenience of the reader we summarize in this paragraph the basic facts about weights and their associated automorphism groups. Detailed information can be found in [1], [2] and [25]. At the same time we develop a set of notations which will be used throughout the paper.

A *weight* on a von Neumann algebra $\mathcal{M}$ is a map φ from $\mathcal{M}_+$ (the positive part of $\mathcal{M}$) to $[0, \infty]$ satisfying

$$\varphi(\alpha x) = \alpha\varphi(x) \quad \text{for all } \alpha \text{ in } \mathbf{R}_+ \text{ and all } x \text{ in } \mathcal{M}_+,$$

$$\varphi(x+y) = \varphi(x)+\varphi(y) \quad \text{for all } x \text{ and } y \text{ in } \mathcal{M}_+.$$

It is called *normal* if there is a set $\{\omega_\iota\}$ of (bounded) normal positive functionals on $\mathcal{M}$ such that

$$\varphi(x) = \sup \omega_\iota(x) \quad \text{for each } x \text{ in } \mathcal{M}_+.$$

F. Combes has shown in [1; Lemma 1.9] that the set of positive functionals ω which are completely majorized by a weight φ (i.e. there is an $\varepsilon > 0$ such that $(1+\varepsilon)\omega \leqslant \varphi$) form an increasing net such that $\lim \omega(x) = \varphi(x)$ for each x in $\mathcal{M}_+$ of lower semi-continuity (in norm) for φ. It follows from [1; Lemma 4.3] that if φ is σ-weakly lower semi-continuous on $\mathcal{M}_+$ then there is a largest normal weight $\varphi_0 \leqslant \varphi$ (the *normalization* of φ); and $\varphi_0(x) = \varphi(x)$ for each x with $\varphi(x) < \infty$. It is not known whether one may have $\varphi_0 \neq \varphi$.

The (complex) linear span $\mathfrak{m}$ of the set

$$\mathfrak{m}_+ = \{x \in \mathcal{M}_+ \mid \varphi(x) < \infty\}$$

is a *-algebra of $\mathcal{M}$ and there is a natural extension of φ to a positive linear functional on $\mathfrak{m}$ (again denoted by φ). We say that φ is *semi-finite* if $\mathfrak{m}$ is σ-weakly dense in $\mathcal{M}$. The weight φ is *faithful* if $\varphi(x) = 0$ implies $x = 0$ for each x in $\mathcal{M}_+$. We shall work almost exclusively with faithful normal semi-finite weights. However, if φ is normal then there are projections p and q in $\mathcal{M}$ with $p \leqslant q$ such that φ is semi-finite on $q\mathcal{M}q$ and faithful on $(1-p)\mathcal{M}(1-p)$. Thus the restriction to faithful semi-finite weights is for most considerations only a matter of convenience.

Let φ be a faithful normal semi-finite weight on $\mathcal{M}$. Then φ determines an inner product on the left ideal

$$\mathfrak{n} = \{x \in \mathcal{M} \mid \varphi(x^*x) < \infty\}.$$

We denote by η the linear injection of $\mathfrak{n}$ into the completed Hilbert space $\mathfrak{H}$ so that $(\eta(x) \mid \eta(y)) = \varphi(y^*x)$, for all x and y in $\mathfrak{n}$. Put $\mathfrak{A} = \mathfrak{n}^* \cap \mathfrak{n}$. Then $\eta(\mathfrak{A})$ is a *full* (achevée) *left Hilbert algebra* in $\mathfrak{H}$, and each full left Hilbert algebra arises in this manner. A repetition of the usual Gelfand-Naimark-Segal construction gives a faithful normal representation π of $\mathcal{M}$ on $\mathfrak{H}$ such that $\pi(x)\eta(y) = \eta(xy)$ for each x in $\mathcal{M}$ and y in $\mathfrak{n}$. The conjugate linear map in $\mathfrak{H}$ that sends $\eta(x)$ to $\eta(x^*)$ extends to a closed operator S with polar decomposition $J\Delta^{\frac{1}{2}}$. Here J is a conjugate linear isometry with period two, and Δ, the *modular operator* associated with φ, is a self-adjoint, positive, non-singular operator in $\mathfrak{H}$. For each complex α one has $J\Delta^{\alpha}J = \Delta^{-\bar{\alpha}}$. The map: $x \mapsto JxJ$ in $\mathcal{B}(\mathfrak{H})$ is a conjugate linear isomorphism of $\pi(\mathcal{M})$ onto $\pi(\mathcal{M})'$, so that π is a *standard representation* of $\mathcal{M}$. Since φ will be fixed throughout the paper we shall identify $\mathcal{M}$ with its image $\pi(\mathcal{M})$.

The one-parameter group of unitary transformations of $\mathcal{B}(\mathfrak{H})$ given by $x \mapsto \Delta^{it}x\Delta^{-it}$ leaves $\mathcal{M}$ invariant (as a set) and can therefore be regarded as a group Σ of automorphisms of $\mathcal{M}$. This *modular automorphism group* will play a central rôle in the sequel. Note that for each σ_t in Σ we have $\eta(\sigma_t(x)) = \Delta^{it}\eta(x)$ when $x \in \mathfrak{n}$, so that $\mathfrak{n}$, $\mathfrak{A}$ and $\mathfrak{m}$ are invariant under Σ. It follows that $\varphi(\sigma_t(x)) = \varphi(x)$ for each x in $\mathcal{M}_+$ so that φ is Σ-*invariant.*

For each x and y in $\mathfrak{A}$ there is a function f continuous and bounded on the strip

$$\{\alpha \in \mathbf{C} \mid 0 \leqslant \operatorname{Im} \alpha \leqslant 1\}$$

and holomorphic in the interior, such that for each real t

$$f(t) = \varphi(\sigma_t(x)y) \quad \text{and} \quad f(t+i) = \varphi(y\sigma_t(x)).$$

We say that φ satisfies the Kubo-Martin-Schwinger (KMS) boundary condition with respect to Σ. If, conversely, there is a strongly continuous one-parameter group $\{g_t\}$ of automorphism of $\mathcal{M}$ leaving φ invariant, and such that φ satisfies the KMS boundary condition for each pair x and y in $\mathfrak{A}$, then $g_t = \sigma_t$ for all t. The weight φ is a trace if and only if the modular operator is the identity. On the other hand a knowledge about Σ or the various KMS boundary conditions will usually compensate the non-tracelike behavior of φ.

3. Analytic vectors in von Neumann algebras

A function f from a complex domain Ω into a Banach space B is called *holomorphic* if the complex function: $\alpha \mapsto \Phi(f(\alpha))$ is holomorphic for each Φ in B^*; or, equivalently, if f is (complex) differentiable in norm on Ω [11; Chap. II, § 2]. If F is a closed region in $\mathbf{C}$ we denote by $\mathcal{A}(F, B)$ the Banach space of continuous and bounded functions from F to B that are holomorphic in the interior of F.

For a von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite weight φ and modular automorphism group Σ let $\mathcal{M}_0$ denote the set of *analytic elements* of $\mathcal{M}$, i.e. those elements h for which the function $t \mapsto \sigma_t(h)$ has an extension (necessarily unique) to an analytic (entire) function $\sigma_\alpha(h)$ from $\mathbf{C}$ to $\mathcal{M}$. It is easy to verify that for each x in $\mathcal{M}$ and $\gamma > 0$ the element

$$h_\gamma = \gamma^{\frac{1}{2}} \pi^{-\frac{1}{2}} \int \exp(-\gamma t^2) \sigma_t(x)\, dt \tag{*}$$

is analytic; with

$$\sigma_\alpha(h_\gamma) = \gamma^{\frac{1}{2}} \pi^{-\frac{1}{2}} \int \exp(-\gamma (t-\alpha)^2) \sigma_t(x)\, dt.$$

With $\gamma \to \infty$ the elements $\{h_\gamma\}$ tend σ-weakly to x, which shows that $\mathcal{M}_0$ is σ-weakly dense in $\mathcal{M}$. It is not hard to see that $\mathcal{M}_0$ is a *-subalgebra of $\mathcal{M}$ such that

$$\sigma_\alpha(hk) = \sigma_\alpha(h)\sigma_\alpha(k) \quad \text{for } h \text{ and } k \text{ in } \mathcal{M}_0;$$

$$\sigma_\alpha(h)^* = \sigma_{\bar{\alpha}}(h^*) \quad \text{for } h \text{ in } \mathcal{M}_0;$$

$$\sigma_{\alpha+\beta}(h) = \sigma_\alpha(\sigma_\beta(h)) \quad \text{for } h \text{ in } \mathcal{M}_0.$$

LEMMA 3.1. *Let $t \mapsto x(t)$ be a strongly continuous function from $\mathbf{R}$ to $\mathcal{M}_+$. If $x(t)$ is integrable and $x = \int x(t)\, dt$, then $\varphi(x) = \int \varphi(x(t))\, dt$.*

Proof. For each normal functional ω we have $\omega(x) = \int \omega(x(t)) dt$. Let $\{\omega_i\}$ be an increasing net of normal positive functionals with limit φ. Then $\{\omega_i(x(t))\}$ is an increasing net of continuous functions with limit $\varphi(x(t))$. Consequently

$$\int \varphi(x(t))\, dt = \lim \int \omega_i(x(t))\, dt = \lim \omega_i(x) = \varphi(x). \quad \text{Q.E.D.}$$

Applying Lemma 3.1 to the equation (*) and using the Σ-invariance of φ one has $\varphi(h_\gamma) = \varphi(x)$. This shows that the *Tomita algebra* $\mathfrak{A}_0 = \mathcal{M}_0 \cap \mathfrak{A}$ is σ-weakly dense in $\mathcal{M}$. Using the σ-weak lower semi-continuity of φ one can prove that $\eta(\mathfrak{A}_0)$ is also dense in $\mathfrak{H}$.

The following lemma is basic for our investigation.

LEMMA 3.2. *Let H be a self-adjoint positive non-singular operator on $\mathfrak{H}$. For a vector ξ in $\mathfrak{H}$ and $\delta > 0$ the following two conditions are equivalent:*

(i) *ξ belongs to the definition domain $\mathcal{D}(H^\delta)$ of H^δ.*

(ii) *The function $t \mapsto H^{it}\xi = \xi(t)$ can be extended from $\mathbf{R}$ to a function $\xi(\alpha)$ in $\mathcal{A}(-\delta \leq \operatorname{Im} \alpha \leq 0, \mathfrak{H})$.*

Proof. (i) $\Rightarrow$ (ii): If $\alpha = t + is$ then $\mathcal{D}(H^{i\alpha}) = \mathcal{D}(H^{-s})$. Assuming that $\xi \in \mathcal{D}(H^{\delta})$ we have $\xi \in \mathcal{D}(H^{i\alpha})$ for $-\delta \leqslant \operatorname{Im} \alpha \leqslant 0$, and

$$\|H^{i\alpha}\xi\| = \|H^{-s}\xi\| \leqslant \|(1+H)^{-s}\xi\| \leqslant \|(1+H)^{-\delta}\xi\|.$$

It follows that the function $\xi(\alpha) = H^{i\alpha}\xi$ is bounded and continuous for $-\delta \leqslant \operatorname{Im} \alpha \leqslant 0$. In the interior of this strip we have

$$\mathcal{D}(i\alpha(\log H)H^{i\alpha}) = \mathcal{D}((\log H)H^{-s}) \supseteq \mathcal{D}(H^{\delta}),$$

which shows that $\xi(\alpha)$ is holomorphic for $-\delta < \operatorname{Im} \alpha < 0$.

(ii) $\Rightarrow$ (i): Suppose $\xi(t) = H^{it}\xi$ has an extension $\xi(\alpha)$ in $\mathcal{A}(-\delta \leqslant \operatorname{Im} \alpha \leqslant 0, \mathfrak{H})$. For each vector ζ in $\mathcal{D}(H^{\delta})$ the function $\zeta(\alpha) = H^{i\alpha}\zeta$ belongs to $\mathcal{A}(-\delta \leqslant \operatorname{Im} \alpha \leqslant 0, \mathfrak{H})$ as we saw above. We therefore have two holomorphic functions

$$\alpha \mapsto (\xi(\alpha) \mid \zeta) \quad \text{and} \quad \alpha \mapsto (\xi \mid H^{-i\bar{\alpha}}\zeta)$$

which are equal when $\operatorname{Im} \alpha = 0$. They then coincide in the whole strip, which shows that the functional

$$\zeta \mapsto (\xi \mid H^{-i\bar{\alpha}}\zeta)$$

is bounded (by $\|\xi(\alpha)\|$) so that $\xi \in \mathcal{D}(H^{i\alpha})$. In particular $\xi \in \mathcal{D}(H^{\delta})$. Q.E.D.

PROPOSITION 3.3. (i) $\mathfrak{A}$ *is a two-sided module over* $\mathcal{M}_0$.

(ii) $\mathfrak{m}$ *is a two-sided module over* $\mathcal{M}_0$.

(iii) $\mathfrak{A}_0$ *is a two-sided ideal of* $\mathcal{M}_0$.

Proof. (i) $\mathfrak{A}$ and $\mathcal{M}_0$ are both *-algebras, so it suffices to prove that $\mathcal{M}_0\mathfrak{A} \subset \mathfrak{A}$. Since moreover $\mathfrak{A} = \mathfrak{n}^* \cap \mathfrak{n}$ and $\mathfrak{n}$ is a left ideal, it is enough to show that if $h \in \mathcal{M}_0$ and $x \in \mathfrak{A}$ then $hx \in \mathfrak{n}^*$. Now $\eta(\mathfrak{A})$ is a full left Hilbert algebra and therefore $x \in \mathfrak{n}^*$ if and only if $\eta(x) \in \mathcal{D}(\Delta^{\frac{1}{2}})$, and

$$\|\Delta^{\frac{1}{2}}\eta(x)\|^2 = \|S\eta(x)\|^2 = \|\eta(x^*)\|^2 = \varphi(xx^*).$$

By Lemma 3.2 the function $\alpha \mapsto \Delta^{i\alpha}\eta(x)$ belongs to $\mathcal{A}(-\frac{1}{2} \leqslant \operatorname{Im} \alpha \leqslant 0, \mathfrak{H})$; and since $h \in \mathcal{M}_0$ there is an analytic extension $\sigma_\alpha(h)$ of the function $t \mapsto \sigma_t(h)$. With $\alpha = t + is$ we have

$$\|\sigma_\alpha(h)\| = \|\sigma_t \circ \sigma_{is}(h)\| = \|\sigma_{is}(h)\|$$

which shows that $\sigma_\alpha(h)$ is bounded on the strip $\{-\frac{1}{2} \leqslant \operatorname{Im} \alpha \leqslant 0\}$. Consider now the function

$$\alpha \mapsto \sigma_\alpha(h)\Delta^{i\alpha}\eta(x).$$

Since the map: $(h, \xi)\mapsto h\xi$ from $\mathcal{M}\times\mathfrak{H}$ to $\mathfrak{H}$ is norm continuous the above function belongs to $\mathcal{A}(-\frac{1}{2}\leq \operatorname{Im}\alpha\leq 0, \mathfrak{H})$. On the real line its values are

$$\sigma_t(h)\Delta^{it}\eta(x)=\Delta^{it}h\eta(x)=\Delta^{it}\eta(hx).$$

By Lemma 3.2 this implies that $\eta(hx)\in\mathcal{D}(\Delta^{\frac{1}{2}})$ and therefore $hx\in\mathfrak{n}^*$.

(ii) If $h\in\mathcal{M}_0$ and $x\in\mathfrak{m}$ then assuming, as we may, that $x\geq 0$ we have $x^{\frac{1}{2}}\in\mathfrak{A}$ and thus from (i)

$$hx=(hx^{\frac{1}{2}})x^{\frac{1}{2}}\in\mathfrak{A}^2=\mathfrak{m}.$$

(iii) Follows immediately from (i). Q.E.D.

Lemma 3.4. *If $h\in\mathcal{M}$ such that $h\mathfrak{m}\subset\mathfrak{m}$ and $\mathfrak{m}h\subset\mathfrak{m}$ then for each pair x, y in $\mathfrak{A}_0$ there is an analytic function f which is bounded on each horizontal strip and such that*

$$f(t)=\varphi(\sigma_t(h)xy^*)\quad \textit{and}\quad f(t+i)=\varphi(xy^*\sigma_t(h)).$$

Proof. Define an analytic function f by

$$f(\alpha)=(h\Delta^{-i\alpha}\eta(x)\,|\,\Delta^{-i\bar{\alpha}+1}\eta(y)). \tag{**}$$

We then calculate

$$\begin{aligned}f(t)&=(h\Delta^{-it}\eta(x)\,|\,\Delta^{-it+1}\eta(y))=(\Delta^{-it}\sigma_t(h)\eta(x)\,|\,\Delta^{-it+1}\eta(y))\\&=(\Delta^{\frac{1}{2}}\eta(\sigma_t(h)x)\,|\,\Delta^{\frac{1}{2}}\eta(y))=(S\eta(y)\,|\,S\eta(\sigma_t(h)x))=\varphi(\sigma_t(h)xy^*);\\ f(t+i)&=(h\Delta^{-it+1}\eta(x)\,|\,\Delta^{-it}\eta(y))=(\Delta^{\frac{1}{2}}\eta(x)\,|\,\Delta^{\frac{1}{2}}\sigma_t(h^*)\eta(y))\\&=(S\eta(\sigma_t(h^*)y)\,|\,S\eta(x))=\varphi(xy^*\sigma_t(h)).\quad \text{Q.E.D.}\end{aligned}$$

Lemma 3.5. *If $h\in\mathcal{M}_0$ and $z\in\mathfrak{m}$ then the function f: $\alpha\mapsto\varphi(\sigma_\alpha(h)z)$ is analytic, and it is bounded on each horizontal strip. Moreover,*

$$f(t)=\varphi(\sigma_t(h)z)\quad \textit{and}\quad f(t+i)=\varphi(z\sigma_t(h)).$$

Proof. We may assume that $z=xy^*$ with x and y in $\mathfrak{A}$, since $\mathfrak{m}$ is linearly spanned by such elements. The argument in Proposition 3.3 showed that the function $\alpha\mapsto\sigma_\alpha(h)\Delta^{i\alpha}\eta(x)$ belongs to $\mathcal{A}(-\frac{1}{2}\leq\operatorname{Im}\alpha\leq 0, \mathfrak{H})$. This function coincides with the function $\alpha\mapsto\Delta^{i\alpha}\eta(hx)$ on the real line, and therefore on the whole strip. It follows that

$$\Delta^{\frac{1}{2}}\eta(hx)=\sigma_{-i/2}(h)\Delta^{\frac{1}{2}}\eta(x).$$

Using this with $\sigma_\alpha(h)$ instead of h we get

$$\Delta^{\frac{1}{2}}\sigma_\alpha(h)\eta(x)=\sigma_{\alpha-i/2}(h)\Delta^{\frac{1}{2}}\eta(x).$$

Now we can calculate

$$f(\alpha) = \varphi(\sigma_\alpha(h)xy^*) = (S\eta(y) \mid S\sigma_\alpha(h)\eta(x)) = (\Delta^{\frac{1}{2}}\sigma_\alpha(h)\eta(x) \mid \Delta^{\frac{1}{2}}\eta(y)) = (\sigma_{\alpha-i/2}(h)\Delta^{\frac{1}{2}}\eta(x) \mid \Delta^{\frac{1}{2}}\eta(y)).$$

It is clear from this expression that f is analytic and that it is bounded on horizontal strips. Moreover,

$$\begin{aligned} f(t+i) &= (\sigma_{t+i/2}(h)\Delta^{\frac{1}{2}}\eta(x) \mid \Delta^{\frac{1}{2}}\eta(y)) = (\Delta^{\frac{1}{2}}\eta(x) \mid \sigma_{t-i/2}(h^*)\Delta^{\frac{1}{2}}\eta(y)) \\ &= (\Delta^{\frac{1}{2}}\eta(x) \mid \Delta^{\frac{1}{2}}\sigma_t(h^*)\eta(y)) = (S\eta(\sigma_t(h^*)y) \mid S\eta(x)) = \varphi(xy^*\sigma_t(h)). \quad \text{Q.E.D.} \end{aligned}$$

THEOREM 3.6. *For h in $\mathcal{M}$ the following conditions are equivalent:*

(i) $\sigma_t(h) = h$ *for all real t;*

(ii) $h\mathfrak{m} \subset \mathfrak{m}$, $\mathfrak{m}h \subset \mathfrak{m}$ *and* $\varphi(hz) = \varphi(zh)$ *for all z in* $\mathfrak{m}$.

Proof. (i) $\Rightarrow$ (ii): The constant function $\sigma_\alpha(h) = h$ is analytic so that $h \in \mathcal{M}_0$. Therefore h is a two-sided multiplier of $\mathfrak{m}$ by Proposition 3.3, and for each z in $\mathfrak{m}$ there is by Lemma 3.5 an analytic function f such that

$$f(t) = \varphi(\sigma_t(h)z) = \varphi(hz) \quad \text{and} \quad f(t+i) = \varphi(z\sigma_t(h)) = \varphi(zh).$$

Since f is constant on the real line, it is constant everywhere. Thus $\varphi(hz) = \varphi(zh)$.

(ii) $\Rightarrow$ (i): Take x and y in $\mathfrak{A}_0$. Then by Lemma 3.4 there is an analytic function f such that

$$f(t) = \varphi(\sigma_t(h)xy^*) = \varphi(xy^*\sigma_t(h)) = f(t+i).$$

Since f is bounded on horizontal strips and periodic in vertical direction it must be bounded. By Liouville's theorem f is constant. Thus $\varphi(\sigma_t(h)xy^*) = \varphi(hxy^*)$ for all x and y in $\mathfrak{A}_0$. Put $k = \sigma_t(h) - h$. Then k satisfies the same assumptions and $\varphi(kxy^*) = 0$ for all x and y in $\mathfrak{A}_0$. Using formula (**) from the proof of Lemma 3.4 with $\alpha = 0$ we get $(k\eta(x) \mid \Delta\eta(y)) = 0$ for all x and y in $\mathfrak{A}_0$. But $\eta(\mathfrak{A}_0)$ and $\Delta\eta(\mathfrak{A}_0)$ are both dense in $\mathfrak{H}$ and therefore $k = 0$. Q.E.D.

The elements satisfying Theorem 3.6 constitute the fixed-point algebra of Σ, denoted by $\mathcal{M}^\Sigma$. It is clearly a von Neumann subalgebra of $\mathcal{M}$, and can also be defined as the set of elements in $\mathcal{M}$ that commute with Δ. In particular, the center $\mathcal{Z}$ of $\mathcal{M}$ is contained in $\mathcal{M}^\Sigma$.

The restriction of φ to $\mathcal{M}^\Sigma$ is a trace but need not be semi-finite. In fact the restriction is semi-finite if and only if there is a normal projection of $\mathcal{M}$ onto $\mathcal{M}^\Sigma$ leaving φ invariant [3; Proposition 4.3]. For such *strictly semi-finite* weights the Radon-Nikodym problem reduces to the corresponding problem for traces by projecting $\mathfrak{m}$ onto $\mathfrak{m} \cap \mathcal{M}^\Sigma$. In order to treat the more general case we shall instead make use of the fact that $\mathcal{M}^\Sigma$ consists of two-sided multipliers of $\mathfrak{m}$.

4. Derived weights

Let as before φ be a faithful, normal, semi-finite weight on $\mathcal{M}$. For each h in $\mathcal{M}_+^\Sigma$ define $\varphi(h\cdot)$ on $\mathcal{M}_+$ by $\varphi(hx)=\varphi(h^{\frac{1}{2}}xh^{\frac{1}{2}})$.

PROPOSITION 4.1. *The map $h\mapsto\varphi(h\cdot)$ is affine and order-preserving from $\mathcal{M}_+^\Sigma$ into the set of Σ-invariant normal semi-finite weights on $\mathcal{M}$.*

Proof. It is clear that each $\varphi(h\cdot)$ is a Σ-invariant normal weight on $\mathcal{M}$. By Proposition 3.3 the weight $\varphi(h\cdot)$ is finite on $\mathfrak{m}$, and therefore semi-finite.

If h and k are in $\mathcal{M}_+^\Sigma$ then there are elements u and v in the unit ball of $\mathcal{M}^\Sigma$ such that

$$h^{\frac{1}{2}}=u(h+k)^{\frac{1}{2}} \quad \text{and} \quad k^{\frac{1}{2}}=v(h+k)^{\frac{1}{2}}$$

(in fact $u=\lim h^{\frac{1}{2}}(h+k+\varepsilon)^{-\frac{1}{2}}$). If $\varphi((h+k)x)<\infty$ for some x in $\mathcal{M}_+$ then $y=(h+k)^{\frac{1}{2}}x(h+k)^{\frac{1}{2}}\in\mathfrak{m}$. It follows from Proposition 3.3, that uyu^* and vyv^* belong to $\mathfrak{m}$; which shows that $\varphi(hx)<\infty$ and $\varphi(kx)<\infty$. Moreover,

$$\varphi(hx)+\varphi(kx)=\varphi(uyu^*+vyv^*)=\varphi((u^*u+v^*v)y)=\varphi(y)=\varphi((h+k)x)$$

since u^*u+v^*v is the range projection of $h+k$. Now

$$\begin{aligned}(h+k)^{\frac{1}{2}}x(h+k)^{\frac{1}{2}}&=\lim_{\varepsilon}\,(h+k+\varepsilon)^{-\frac{1}{2}}(h+k)x(h+k)(h+k+\varepsilon)^{-\frac{1}{2}}\\&\leqslant\lim_{\varepsilon}\,(h+k+\varepsilon)^{-\frac{1}{2}}2(hxh+kxk)(h+k+\varepsilon)^{-\frac{1}{2}}\\&=2(u^*h^{\frac{1}{2}}xh^{\frac{1}{2}}u+v^*k^{\frac{1}{2}}xk^{\frac{1}{2}}v).\end{aligned}$$

This shows that if $\varphi(hx)<\infty$ and $\varphi(kx)<\infty$ then $\varphi((h+k)x)<\infty$. Thus $\varphi((h+k)\cdot)=\varphi(h\cdot)+\varphi(k\cdot)$.

If $h\leqslant k$ then $k=h+h'$ with h' in $\mathcal{M}_+^\Sigma$; hence $\varphi(h\cdot)\leqslant\varphi(k\cdot)$ from the preceding. Q.E.D.

For our purposes we shall also need weights which are derived from unbounded operators. We first introduce some terminology. If h and k are self-adjoint positive operators on $\mathfrak{H}$ and $\varepsilon>0$ we put $h_\varepsilon=h(1+\varepsilon h)^{-1}$. We write $h\leqslant k$ if $h_\varepsilon\leqslant k_\varepsilon$ for some (and hence any) $\varepsilon>0$. This is equivalent to the two conditions

$$\mathcal{D}(h^{\frac{1}{2}})\supset\mathcal{D}(k^{\frac{1}{2}}) \quad \text{and} \quad \|h^{\frac{1}{2}}\xi\|^2\leqslant\|k^{\frac{1}{2}}\xi\|^2$$

for each ξ in $\mathcal{D}(k^{\frac{1}{2}})$. We say that a net $\{h_i\}$ of self-adjoint positive operators increases to the self-adjoint operator h, and write $h_i\nearrow h$ if $h_{i\varepsilon}\nearrow h_\varepsilon$. Thus $h_\varepsilon\nearrow h$ when $\varepsilon\searrow 0$.

Not let h be a self-adjoint, positive operator affiliated with $\mathcal{M}^\Sigma$. Then $h_\varepsilon\in\mathcal{M}^\Sigma$ for each $\varepsilon>0$. We define $\varphi(h\cdot)$ as the limit of the increasing set of normal semi-finite weights $\{\varphi(h_\varepsilon\cdot)\}$.

PROPOSITION 4.2. *The map $h \mapsto \varphi(h\cdot)$ is order-preserving and normal from the set of self-adjoint positive operators affiliated with $\mathcal{M}^\Sigma$ into the set of Σ-invariant normal semi-finite weights on $\mathcal{M}$.*

Proof. Since $\varphi(h\cdot)$ is the supremum of Σ-invariant normal weights it is itself Σ-invariant and normal. Let e_n be the spectral projection of h corresponding to $[0, n]$. The set $\cup\, e_n \mathfrak{m} e_n$ is contained in $\mathfrak{m}$ by Proposition 3.3 and dense in $\mathcal{M}$; and $\varphi(h\cdot)$ is finite on this set by Proposition 4.1 since $he_n \in \mathcal{M}^\Sigma$. Thus $\varphi(h\cdot)$ is semi-finite.

If $h \leqslant k$ with h and k affiliated with $\mathcal{M}^\Sigma$ then $h_\varepsilon \leqslant k_\varepsilon$ and it follows from Proposition 4.1 and the definition of $\varphi(h\cdot)$ that $\varphi(h\cdot) \leqslant \varphi(k\cdot)$.

Suppose now that $h_\iota \nearrow h$ with h_ι and h affiliated with $\mathcal{M}^\Sigma$. Then from the above the net $\{\varphi(h_\iota\cdot)\}$ increases to a normal weight $\psi \leqslant \varphi(h\cdot)$. However, φ is σ-weakly lower semi-continuous and $h_{\iota\varepsilon} \nearrow h_\varepsilon$ so that

$$\varphi(hx) = \lim_\varepsilon \varphi(h_\varepsilon x) \leqslant \lim_\varepsilon \lim_\iota \varphi(h_{\iota\varepsilon} x) \leqslant \lim_\iota \varphi(h_\iota x) = \psi(x).$$

Thus $\varphi(h_\iota) \nearrow \varphi(h\cdot)$. Q.E.D.

PROPOSITION 4.3. (The chain rule.) *Suppose h and k are commuting self-adjoint positive operators affiliated with $\mathcal{M}^\Sigma$. Put $\psi = \varphi(h\cdot)$. Then $\psi(k\cdot) = \varphi(h\cdot k\cdot)$, where $h\cdot k$ denotes the closed operator obtained from hk.*

Proof. If $x \in \mathfrak{n}_\psi^*$ and $\delta > 0$ then

$$\psi(k_\delta^{\frac{1}{2}} xx^* k_\delta^{\frac{1}{2}}) = \lim_\varepsilon \varphi(h_\varepsilon^{\frac{1}{2}} k_\delta^{\frac{1}{2}} xx^* k_\delta^{\frac{1}{2}} h_\varepsilon^{\frac{1}{2}}) \leqslant \delta^{-1} \lim_\varepsilon \varphi(h_\varepsilon^{\frac{1}{2}} xx^* h_\varepsilon^{\frac{1}{2}}) = \delta^{-1}\psi(xx^*) < \infty,$$

using the fact that $h_\varepsilon k_\delta \leqslant \delta^{-1} h_\varepsilon$. Thus $k_\delta^{\frac{1}{2}} x \in \mathfrak{n}_\psi^*$. It follows that $k_\delta \mathfrak{m}_\psi \subset \mathfrak{m}_\psi$ and $\mathfrak{m}_\psi k_\delta \subset \mathfrak{m}_\psi$. If $z \in \mathfrak{m}_\psi$ then

$$\psi(k_\delta z) = \lim_\varepsilon \varphi(h_\varepsilon^{\frac{1}{2}} k_\delta z h_\varepsilon^{\frac{1}{2}}) = \lim_\varepsilon \varphi(h_\varepsilon^{\frac{1}{2}} z k_\delta h_\varepsilon^{\frac{1}{2}}) = \psi(z k_\delta).$$

From Theorem 3.6 we conclude that k is affiliated with the fixed-point algebra of the modular automorphism group of ψ. Thus $\psi(k\cdot)$ is well defined.

The net $\{h_\varepsilon k_\delta\}$ increases to $h\cdot k$. Therefore

$$\varphi(h\cdot kx) = \lim_{\varepsilon,\delta} \varphi(h_\varepsilon k_\delta x) \leqslant \lim_\delta \psi(k_\delta x) = \psi(kx)$$

by Proposition 4.2. On the other hand

$$\psi(kx) = \lim_\delta \psi(k_\delta x) = \lim_\delta \lim_\varepsilon \varphi(h_\varepsilon k_\delta x) \leqslant \varphi(h\cdot kx).$$

Thus $\psi(k\cdot) = \varphi(h\cdot k\cdot)$. Q.E.D.

LEMMA 4.4. *Let h be an invertible positive element in $\mathfrak{M}^{\Sigma}$ and take x in $\mathfrak{A}_0$ and y in $\mathfrak{A}$. Then the analytic function f given by*

$$f(\alpha) = (h^{i\alpha+1}\Delta^{i\alpha+1}\eta(x) \mid Sh^{-i\alpha}\eta(y))$$

satisfies the boundary conditions

$$f(t) = \varphi(hh^{it}\sigma_t(x)h^{-it}y) \quad \text{and} \quad f(t+i) = \varphi(hyh^{it}\sigma_t(x)h^{-it}).$$

Proof. By straightforward computations we get

$$\begin{aligned} f(t) = (h^{it+1}\Delta^{it+1}\eta(x) \mid Sh^{-it}\eta(y)) &= (\Delta h^{it+1}\Delta^{it}\eta(x) \mid \Delta^{-\frac{1}{2}}Jh^{-it}\eta(y)) \\ &= (h^{-it}\eta(y) \mid Sh^{it+1}\Delta^{it}\eta(x)) = \varphi(hh^{it}\sigma_t(x)h^{-it}y); \end{aligned}$$

$$f(t+i) = (h^{it}\Delta^{it}\eta(x) \mid Sh^{-it+1}\eta(y)) = \varphi(h^{-it}hyh^{it}\sigma_t(x)) = \varphi(hyh^{it}\sigma_t(x)h^{-it}). \quad \text{Q.E.D.}$$

LEMMA 4.5. *Let h be an invertible positive element in $\mathfrak{M}^{\Sigma}$ and put $\psi = \varphi(h\,\cdot)$. Then the modular automorphism group $\{\sigma_t^{\psi}\}$ of ψ is given by $\sigma_t^{\psi}(x) = h^{it}\sigma_t(x)h^{-it}$.*

Proof. By Proposition 3.3 we have

$$\mathfrak{m}_{\psi} = h^{-\frac{1}{2}}\mathfrak{m}h^{-\frac{1}{2}} \subset \mathfrak{m} \quad \text{and} \quad \mathfrak{m} = h^{-\frac{1}{2}}h^{\frac{1}{2}}\mathfrak{m}h^{\frac{1}{2}}h^{-\frac{1}{2}} \subset \mathfrak{m}_{\psi}.$$

Thus $\mathfrak{m}_{\psi} = \mathfrak{m}$ and $\mathfrak{A}_{\psi} = \mathfrak{A}$. If x and y are elements of $\mathfrak{A}$ let $\{x_n\}$ be a sequence in $\mathfrak{A}_0$ such that $\eta(x_n) \to \eta(x)$ and $S\eta(x_n) \to S\eta(x)$, i.e. $\eta(x_n^*) \to \eta(x^*)$. Then from Lemma 4.4 we have a sequence $\{f_n\}$ of analytic functions such that

$$f_n(t) = \varphi(hh^{it}\sigma_t(x_n)h^{-it}y) = (\eta(h^{-it}yh^{it}h) \mid \eta(\sigma_t(x_n^*))),$$

$$f_n(t+i) = \varphi(hyh^{it}\sigma_t(x_n)h^{-it}) = (\eta(\sigma_t(x_n)) \mid \eta(h^{-it}y^*h^{it}h)).$$

From these expressions it follows that $\{f_n\}$ converges uniformly on the lines $\{\mathrm{Im}\,\alpha = 0\}$ and $\{\mathrm{Im}\,\alpha = 1\}$, since $\eta(z) \mapsto \eta(\sigma_t(z))$ and $\eta(z) \mapsto \eta(h^{-it}zh^{it})$ are both unitary transformations in $\mathfrak{H}$. By the Phragmen-Lindelöf theorem the functions converge uniformly on the strip to a function f in $\mathcal{A}(0 \leqslant \mathrm{Im}\,\alpha \leqslant 1)$ such that

$$f(t) = \varphi(hh^{it}\sigma_t(x)h^{-it}y) \quad \text{and} \quad f(t+i) = \varphi(hyh^{it}\sigma_t(x)h^{-it}).$$

Thus ψ satisfies the KMS condition with respect to the group $\{h^{it}\sigma_t(\cdot)h^{-it}\}$. From the unicity of the modular automorphism group it follows that $\sigma_t^{\psi}(\cdot) = h^{it}\sigma_t(\cdot)h^{-it}$. Q.E.D.

THEOREM 4.6. *Let h be a self-adjoint positive operator affiliated with $\mathfrak{M}^{\Sigma}$ and put $\psi = \varphi(h\,\cdot)$. Then the modular automorphism group $\{\sigma_t^{\psi}\}$ of ψ is given by $\sigma_t^{\psi}(x) = h^{it}\sigma_t(x)h^{-it}$.*

Proof. Let e_n denote the spectral projection of h corresponding to $[n^{-1}, n]$. Then $\psi(e_n\cdot)=\varphi(he_n\cdot)$ by Proposition 4.3. Restricting to the von Neumann algebra $e_n\mathcal{M}e_n$ we see that $\varphi(e_n\cdot)$ is faithful normal and semi-finite with modular automorphism group Σ (since $\sigma_t(e_n)=e_n$). By Lemma 4.5 we get $\sigma_t^\psi(x)=(he_n)^{it}\sigma_t(x)(he_n)^{-it}=h^{it}\sigma_t(x)h^{-it}$ for x in $e_n\mathcal{M}e_n$. Now ψ is faithful only on $[h]\mathcal{M}[h]$, where $[h]$ means the range projection of h; hence its modular group is only well-defined on this von Neumann algebra. But $\sigma_t^\psi(\cdot)=h^{it}\sigma_t(\cdot)h^{-it}$ on $\bigcup e_n\mathcal{M}e_n$, which is σ-weakly dense in $[h]\mathcal{M}[h]$. Therefore $\sigma_t^\psi(\cdot)=h^{it}\sigma_t(\cdot)h^{-it}$ on $[h]\mathcal{M}[h]$. Q.E.D.

COROLLARY 4.7. *If h is a self-adjoint positive operator affiliated with the center $\mathcal{Z}$ of $\mathcal{M}$ then the weight $\varphi(h\cdot)$ satisfies the* KMS *condition with respect to Σ.*

5. Radon-Nikodym derivatives

Let as before φ be a faithful normal semi-finite weight on $\mathcal{M}$. In this paragraph we characterize the weights which can be written in the form $\varphi(h\cdot)$.

LEMMA 5.1. *If ψ is a Σ-invariant normal semi-finite weight then its support is contained in $\mathcal{M}^\Sigma$. If, moreover, ψ satisfies the* KMS *condition with respect to Σ then its support is in $\mathcal{Z}$.*

Proof. The orthogonal complement of the support of ψ is the largest projection p such that $\psi(p)=0$. If ψ is Σ-invariant then $\psi(\sigma_t(p))=0$; hence $\sigma_t(p)\leqslant p$. It follows that $p\in\mathcal{M}^\Sigma$. If, moreover, ψ satisfies the KMS condition with respect to Σ then for x in $\mathfrak{A}_\psi$ there is an analytic function f such that

$$f(t)=\psi(\sigma_t(xp)px^*)=\psi(\sigma_t(x)px^*),$$

$$f(t+i)=\psi(px^*\sigma_t(xp))=\psi(px^*\sigma_t(x)p)=0;$$

since xp and px^* belong to $\mathfrak{A}_\psi$ by Proposition 3.3. It follows that $f=0$; hence $xpx^*\in p\mathcal{M}p$ for all x in $\mathfrak{A}_\psi$ and $p\in\mathcal{Z}$. Q.E.D.

LEMMA 5.2. *Let ψ be a Σ-invariant normal semi-finite weight on $\mathcal{M}$. If there exists a weakly dense *-subalgebra $\mathcal{B}$ in $\mathfrak{m}$, invariant under Σ, such that $\varphi=\psi$ on $\mathcal{B}$ then $\psi\leqslant\varphi$, and ψ is faithful. If, moreover, ψ satisfies the* KMS *condition with respect to Σ then $\varphi=\psi$.*

Proof. If x and y are in $\mathcal{B}$ then $\varphi(x\cdot y)$ and $\psi(x\cdot y)$ are two normal functionals on $\mathcal{M}$ which agree on $\mathcal{B}$, since $\mathcal{B}$ is an algebra. Therefore $\varphi(x\cdot y)=\psi(x\cdot y)$.

Since $\mathcal{B}$ is a dense *-algebra there is an increasing net $\{u_\lambda\}$ in $\mathcal{B}_+$ such that $u_\lambda \nearrow 1$. Put

$$h_\lambda = \pi^{-\frac{1}{2}} \int \exp(-t^2)\, \sigma_t(u_\lambda)\, dt.$$

Since $\mathcal{B}$ is invariant under Σ we have

$$\varphi(\sigma_t(u_\lambda) x \sigma_s(u_\lambda)) = \psi(\sigma_t(u_\lambda) x \sigma_s(u_\lambda))$$

for all s and t and each x in $\mathcal{M}$. It follows from Lemma 3.1, by the polarization identity, that $\varphi(h_\lambda x h_\lambda) = \psi(h_\lambda x h_\lambda)$.

Each h_λ is an analytic element with

$$\sigma_\alpha(h_\lambda) = \pi^{-\frac{1}{2}} \int \exp(-(t-\alpha)^2)\, \sigma_t(u_\lambda)\, dt.$$

For each vector ξ in $\mathfrak{H}$ the function f_λ: $t \mapsto \|\sigma_t(1-u_\lambda)\xi\|$ is continuous, and the net $\{f_\lambda\}$ decreases pointwise to 0. By Dini's theorem $f_\lambda \searrow 0$ uniformly on compact sets. It follows that

$$\begin{aligned}
\|(1-\sigma_\alpha(h_\lambda))\,\xi\| &= \left\| \left(1 - \pi^{-\frac{1}{2}} \int \exp(-(t-\alpha)^2)\, \sigma_t(u_\lambda)\, dt\right) \xi \right\| \\
&= \left\| \pi^{-\frac{1}{2}} \int \exp(-(t-\alpha)^2)\, \sigma_t(1-u_\lambda)\, \xi dt \right\| \\
&\leqslant \pi^{-\frac{1}{2}} \int |\exp(-(t-\alpha)^2)|\, \|\sigma_t(1-u_\lambda)\,\xi\|\, dt \\
&= \pi^{-\frac{1}{2}} \exp(\operatorname{Im}\alpha)^2 \int \exp\, -(t - \operatorname{Re}\alpha)^2\, \|\sigma_t(1-u_\lambda)\,\xi\|\, dt \searrow 0,
\end{aligned}$$

so that $\{\sigma_\alpha(h_\lambda)\}$ converges strongly (and bounded) to 1 for every α. In particular $h_\lambda \nearrow 1$.

Take now x in $\mathfrak{m}_+$. Using the σ-weak lower semi-continuity of ψ and the analyticity of h_λ we get

$$\begin{aligned}
\psi(x) \leqslant \lim \psi(h_\lambda x h_\lambda) &= \lim \varphi(h_\lambda x h_\lambda) = \lim \|\eta(x^{\frac{1}{2}} h_\lambda)\|^2 = \lim \|S h_\lambda S \eta(x^{\frac{1}{2}})\|^2 \\
&= \lim \|J \Delta^{\frac{1}{2}} h_\lambda \Delta^{-\frac{1}{2}} J \eta(x^{\frac{1}{2}})\|^2 = \lim \|\sigma_{-i/2}(h_\lambda) J \eta(x^{\frac{1}{2}})\|^2 \\
&= \|J\eta(x^{\frac{1}{2}})\|^2 = \varphi(x).
\end{aligned}$$

Thus $\psi \leqslant \varphi$.

Let $1-q$ denote the support of ψ and take x in $\mathfrak{m}_+$ with $x \leqslant q$. Then, using the formula $hxh \leqslant 2((1-h)x(1-h)+x)$ we have

$$\begin{aligned}\varphi(x) &\leq \lim \varphi(h_\lambda x h_\lambda) = \lim \psi(h_\lambda x h_\lambda) \leq 2 \lim \psi((1-h_\lambda)x(1-h_\lambda)) \\ &\leq 2 \lim \varphi((1-h_\lambda)x(1-h_\lambda)) = 2 \lim \|\eta(x^{\frac{1}{2}}(1-h_\lambda))\|^2 \\ &= 2 \lim \|S(1-h_\lambda)S\eta(x^{\frac{1}{2}})\|^2 = 2 \lim \|(1-\sigma_{-i/2}(h_\lambda))\eta(x^{\frac{1}{2}})\|^2 = 0.\end{aligned}$$

Since $q \in \mathcal{M}^\Sigma$ we have $qu_\lambda q = 0$ from the above. It follows that $q = 0$ so that ψ is faithful.

Assume now that ψ also satisfies the KMS condition with respect to Σ. We can then interchange the rôle of φ and ψ in the preceding argument to obtain $\varphi \leq \psi$. Thus $\varphi = \psi$. Q.E.D.

LEMMA 5.3. *If ψ is a Σ-invariant normal semi-finite weight on $\mathcal{M}$ that satisfies the* KMS *condition with respect to Σ then $\varphi+\psi$ is semi-finite and satisfies the* KMS *condition with respect to Σ.*

Proof. The sets $\mathcal{M}_0 \cap \mathfrak{m}$ and $\mathcal{M}_0 \cap \mathfrak{m}_\psi$ of analytic elements for Σ in $\mathfrak{m}$ and $\mathfrak{m}_\psi$, respectively, are both σ-weakly dense in $\mathcal{M}$ by the remarks in the beginning of § 3. Therefore the set of products $(\mathcal{M}_0 \cap \mathfrak{m})(\mathcal{M}_0 \cap \mathfrak{m}_\psi)$ is σ-weakly dense in $\mathcal{M}$. However, this set is contained in both $\mathfrak{m}$ and $\mathfrak{m}_\psi$ by Proposition 3.3 (ii). Thus $\mathfrak{m} \cap \mathfrak{m}_\psi$ is σ-weakly dense in $\mathcal{M}$ and $\varphi+\psi$ is semi-finite. Since $\sigma_t^\varphi = \sigma_t^{\varphi+\psi} = \sigma_t$ it is immediate that $\varphi+\psi$ satisfies the KMS condition with respect to Σ. Q.E.D.

THEOREM 5.4. *If ψ is a Σ-invariant normal semi-finite weight that satisfies the* KMS *condition with respect to Σ then there is a unique self-adjoint positive operator h affiliated with $\mathcal{Z}$ such that $\psi = \varphi(h\,\cdot)$.*

Proof. Since the support q of ψ belongs to $\mathcal{Z}$ and since the modular automorphism group of $\varphi(q\,\cdot)$ is the restriction of Σ to $q\mathcal{M}$ we may assume that ψ is faithful.

Consider first the case $\psi \leq \varphi$. Then there is a unique element h in $\mathcal{M}'$ with $0 \leq h \leq 1$ such that $\psi(y^*x) = (h\eta(x) \mid \eta(y))$ for all x and y in $\mathfrak{A}$. Since ψ is Σ-invariant we have

$$(h\eta(x) \mid \eta(y)) = \psi(y^*x) = \psi(\sigma_t(y^*x)) = (h\eta(\sigma_t(x)) \mid \eta(\sigma_t(y))) = (h\Delta^{it}\eta(x) \mid \Delta^{it}\eta(y)).$$

Thus h commutes with Δ^{it} for all t, and $h\Delta \subset \Delta h$. Now take x and y in $\mathfrak{A}_0$. Since $\mathfrak{A}_0 \subset \mathfrak{A} \subset \mathfrak{A}_\psi$ there is a function f in $\mathcal{A}(0 \leq \operatorname{Im} \alpha \leq 1)$ such that

$$f(t) = \psi(\sigma_t(x)y) = (h\eta(y) \mid S\Delta^{it}\eta(x)) = (\eta(y) \mid hS\Delta^{it}\eta(x));$$
$$f(t+i) = \psi(y\sigma_t(x)) = (h\Delta^{it}\eta(x) \mid S\eta(y)).$$

But $\eta(x)$ is analytic and therefore $f(\alpha+i) = (h\Delta^{i\alpha}\eta(x) \mid S\eta(y))$ for all α. Using this formula we get

$$f(t) = (h\Delta^{i(t-i)}\eta(x)\,|\,\Delta^{-\frac{1}{2}}J\eta(y)) = (\Delta^{-\frac{1}{2}}h\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}+it}\eta(x)\,|\,J\eta(y))$$
$$= (h\Delta^{\frac{1}{2}+it}\eta(x)\,|\,J\eta(y)) = (\eta(y)\,|\,JhJS\Delta^{it}\eta(x)).$$

Since $\eta(\mathfrak{A}_0)$ is dense in $\mathfrak{H}$ we have $JhJ=h$. As $JhJ\in\mathcal{M}$ we conclude that $h\in\mathcal{Z}$. The two functionals ψ and $\varphi(h\cdot)$ coincide on a dense subalgebra $\mathfrak{A}$ and both satisfy the KMS condition with respect to Σ. Therefore $\psi=\varphi(h\cdot)$ by Lemma 5.2.

In the general case put $\tau=\varphi+\psi$. By Lemma 5.3 τ is semi-finite and satisfies the KMS condition with respect to Σ. From the first part of the proof we have non-singular operators h and k in $\mathcal{Z}_+$ such that $\varphi=\tau(h\cdot)$ and $\psi=\tau(k\cdot)$. Since $\varphi+\psi=\tau$ we have $h+k=1$. From Proposition 4.3 we conclude that

$$\varphi(h^{-1}k\cdot)=\tau(k\cdot)=\psi;$$

where $h^{-1}k$ is a self-adjoint positive operator affiliated with $\mathcal{Z}$. Q.E.D.

THEOREM 5.5. *Let G be a group of automorphisms of $\mathcal{M}$ that leaves $\mathcal{Z}$ fixed (pointwise). Then G and Σ commute if and only if there is a homomorphism $g\mapsto h_g$ of G into the multiplicative group of non-singular self-adjoint positive operators affiliated with $\mathcal{Z}$ such that $\varphi(g(\cdot))=\varphi(h_g\cdot)$.*

Proof. For each g in G the weight $\varphi(g(\cdot))$ is faithful normal and semi-finite; and the elements of its modular automorphism group are of the form $g^{-1}\circ\sigma_t\circ g$. If therefore G and Σ commute then $\varphi(g(\cdot))$ satisfies the KMS condition with respect to Σ. By Theorem 5.4 there is a unique non-singular self-adjoint positive operator h_g affiliated with $\mathcal{Z}$ such that $\varphi(g(\cdot))=\varphi(h_g\cdot)$. Since G leaves $\mathcal{Z}$ fixed we have

$$\varphi(h_{gg'}\cdot)=\varphi(gg'(\cdot))=\varphi(h_g g'(\cdot))=\varphi(g'(h_g\cdot))=\varphi(h_{g'}\cdot h_g\cdot),$$

so that $h_{gg'}=h_g\cdot h_{g'}$.

Conversely, assume that a homomorphism $g\mapsto h_g$ exists, and for fixed g in G consider the group with elements $\sigma'_t=g^{-1}\circ\sigma_t\circ g$. Then

$$\varphi(\sigma'_t(x))=\varphi(h_g^{-1}\sigma_t(g(x)))=\varphi(h_g^{-1}g(x))=\varphi(x)$$

so that φ is invariant under $\{\sigma'_t\}$. Let e_n be the spectral projection of h_g corresponding to $[n^{-1}, n]$ and take x and y in $\mathfrak{A}$. Then $g(e_nx)\in\mathfrak{A}$, $g(e_ny)\in\mathfrak{A}$ and $h_g^{-1}g(e_nx)\in\mathfrak{A}$ since h_ge_n and $h_g^{-1}e_n$ are both bounded. There is therefore a function f_n in $\mathcal{A}(0\leqslant\operatorname{Im}\alpha\leqslant 1)$ such that

$$f_n(t)=\varphi(\sigma_t(h_g^{-1}e_ng(x))\,g(e_ny))=\varphi(h_g^{-1}e_n\sigma_t(g(x))\,g(y))$$
$$=\varphi(g^{-1}(e_n\sigma_t(g(x))\,g(y)))=\varphi(e_n\sigma'_t(x)\,y);$$

$$f_n(t+i) = \varphi(g(e_n y)\,\sigma_t(h_g^{-1} e_n g(x))) = \varphi(h_g^{-1} e_n g(y)\,\sigma_t(g(x)))$$
$$= \varphi(g^{-1}(e_n g(y)\,\sigma_t(g(x)))) = \varphi(e_n y \sigma'_t(x)).$$

When $n \to \infty$ the functions f_n converge uniformly on $\{\operatorname{Im} \alpha = 0\}$ and $\{\operatorname{Im} \alpha = 1\}$. By the Phragmen-Lindelöf theorem there is therefore a function f in $\mathcal{A}(0 \leqslant \operatorname{Im} \alpha \leqslant 1)$ such that

$$f(t) = \varphi(\sigma'_t(x)\,y) \quad \text{and} \quad f(t+i) = \varphi(y\sigma'_t(x)).$$

Thus φ satisfies the KMS condition with respect to $\{\sigma'_t\}$, and therefore $\sigma'_t = \sigma_t$ for all t. Q.E.D.

COROLLARY 5.6. *If ψ is a Σ-invariant faithful normal semi-finite weight on $\mathfrak{M}$ then the modular groups Σ^ψ and Σ commute.*

PROPOSITION 5.7. *Let G be a strongly continuous one-parameter group of automorphisms of $\mathfrak{M}$ that leaves $\mathfrak{Z}$ fixed and commutes with Σ. There is then a unique non-singular self-adjoint positive operator h affiliated with $\mathfrak{Z}$ such that $\varphi(g_s(\cdot)) = \varphi(h^s \cdot)$ for all s.*

Proof. By Theorem 5.5 $\varphi(g_s(\cdot)) = \varphi(h_s \cdot)$. Put $h = h_1$. Then $h_s = h^s$ for every dyadic rational number s. Let e_n be the spectral projection of h corresponding to $[n^{-1}, n]$. Then for each x in $\mathfrak{A}$ the function

$$s \mapsto \varphi(h^s e_n x^* x) = (h^s e_n \eta(x) \,|\, \eta(x))$$

is continuous and bounded; whereas the function

$$s \mapsto \varphi(h_s e_n x^* x) = \varphi(g_s(e_n x^* x)) = (h_s e_n \eta(x) \,|\, \eta(x))$$

is lower semi-continuous, since φ is σ-weakly lower semi-continuous. We have $((h_s e_n - h^s e_n)\,\eta(x) \,|\, \eta(x)) \leqslant 0$ on a dense set and therefore for all s. Since this holds for each x in $\mathfrak{A}$ we conclude that $h_s e_n \leqslant h^s e_n$. But then $s \mapsto h_s e_n (h^s e_n)^{-1} e_n$ is a one-parameter group of positive operators with norm less than or equal to one. The only such group is the constant one and therefore $h_s e_n = h^s e_n$ for all s. Since $e_n \nearrow 1$ we get $h_s = h^s$. Q.E.D.

LEMMA 5.8. *Let ψ be a faithful normal semi-finite weight on $\mathfrak{M}$ with modular automorphism group Σ^ψ. If Σ and Σ^ψ commute then ψ is Σ-invariant provided that $\psi \leqslant \varphi$ or $\varphi \leqslant \psi$.*

This is a special case of Theorem 6.6.

Proof. By Proposition 5.7 we have $\psi(\sigma_t(\cdot)) = \psi(h^t \cdot)$ where h is affiliated with $\mathfrak{Z}$. If e_ε is the spectral projection of h corresponding to $[1+\varepsilon, \infty[$ then for $x \geqslant 0$ we have

$$\psi(e_\varepsilon \sigma_t(x)) = \psi(h^t e_\varepsilon x) \geqslant (1+\varepsilon)^t \psi(e_\varepsilon x) \tag{*}$$

for $t \geqslant 0$. Suppose that $\psi \leqslant \varphi$ and $x \in \mathfrak{m}$. Then from (*) we have $\varphi(e_\varepsilon x) \geqslant (1+\varepsilon)^t \psi(e_\varepsilon x)$ for all $t > 0$. Therefore $\psi(e_\varepsilon x) = 0$, so that $e_\varepsilon x = 0$ for all x in $\mathfrak{m}$. Hence $e_\varepsilon = 0$ for all $\varepsilon > 0$ which implies that $h \leqslant 1$.

Suppose instead that $\varphi \leqslant \psi$ and take x in $\mathfrak{m}_\psi$. Exchanging x with $\sigma_{-t}(x)$ in (*) yields $\psi(e_\varepsilon x) \geqslant (1+\varepsilon)^t \varphi(e_\varepsilon x)$ for all $t > 0$. As before this implies that $h \leqslant 1$.

Similar arguments apply to show that the spectral projection of h corresponding to $[0, 1-\varepsilon[$ is zero. The conclusion is that in both cases $\varphi \leqslant \psi$ and $\psi \leqslant \varphi$ we have $h = 1$. Q.E.D.

PROPOSITION 5.9. *If ψ is a Σ-invariant normal semi-finite weight on $\mathcal{M}$ which is equal to φ on a σ-weakly dense Σ-invariant *-subalgebra of $\mathfrak{m}$ then $\psi = \varphi$.*

Proof. From Lemma 5.2 we know that $\psi \leqslant \varphi$ and that ψ is faithful. Therefore φ is Σ^ψ-invariant by Lemma 5.8. The elements x in $\mathcal{M}_+$ such that $\varphi(x) = \psi(x) < \infty$ form the positive part of a hereditary *-subalgebra of $\mathcal{M}$ (since $\psi \leqslant \varphi$) which is Σ^ψ-invariant and σ-weakly dense. We can therefore use Lemma 5.2 again, interchanging φ and ψ, to obtain $\varphi \leqslant \psi$. Q.E.D.

PROPOSITION 5.10. *Let ψ be a faithful normal semi-finite weight on $\mathcal{M}$ with modular automorphism group Σ^ψ. If Σ and Σ^ψ commute then $\varphi + \psi$ is semi-finite.*

Proof. If k is an analytic element for Σ then $\sigma_s^\psi(k)$ is analytic for Σ and $\sigma_\alpha(\sigma_s^\psi(k)) = \sigma_s^\psi(\sigma_\alpha(k))$. It follows that for each x in $\mathcal{M}$ and $\gamma > 0$ the element

$$h_\gamma = \gamma\pi^{-1} \iint \exp(-\gamma(t^2+s^2))\, \sigma_t \circ \sigma_s^\psi(x)\, dt\, ds$$

is analytic for both Σ and Σ^ψ. Moreover, the elements $\{h_\gamma\}$ tend σ-weakly to x when $\gamma \to \infty$.

By Proposition 5.7 we have $\psi(\sigma_t(\cdot)) = \psi(h^t \cdot)$ where h is affiliated with $\mathcal{Z}$. Let e_n be the spectral projection of h corresponding to $[0, n]$. With x in $\mathfrak{m}_\psi$, $x \geqslant 0$, and h_γ defined as above we get

$$\psi(h_\gamma e_n) = \gamma\pi^{-1} \iint \exp(-\gamma t^2) \exp(-\gamma s^2)\, \psi(h^t e_n x)\, dt\, dt$$

$$= \gamma^{\frac{1}{2}}\pi^{-\frac{1}{2}} \iint \exp(-\gamma t^2)\, \psi(h^t e_n x)\, dt$$

$$\leqslant \gamma^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int \exp(-\gamma t^2)\, n^t \psi(x)\, dt < \infty.$$

It follows that the set $\mathcal{M}_0^\psi \cap \mathcal{M}_0 \cap \mathfrak{m}_\psi$ is σ-weakly dense in $\mathcal{M}$, since $h_\gamma e_n \to x$ when

$(\gamma, n)\to\infty$. By a symmetric argument $\mathcal{M}_0 \cap \mathcal{M}_0^\psi \cap \mathfrak{m}$ is σ-weakly dense in $\mathcal{M}$. Therefore the product of these sets form a dense set in $\mathcal{M}$. But

$$(\mathcal{M}_0^\psi \cap \mathcal{M}_0 \cap \mathfrak{m})\,(\mathcal{M}_0 \cap \mathcal{M}_0^\psi \cap \mathfrak{m}) \subset \mathfrak{m}_\psi \cap \mathfrak{m}$$

by Proposition 3.3. Thus $\varphi+\psi$ is semi-finite. Q.E.D.

We see from Theorem 5.5 that if ψ is a normal semi-finite weight which is Σ-invariant then the modular automorphism group Σ^ψ of ψ commutes with Σ. If, conversely, Σ^ψ and Σ commute then ψ is Σ-invariant under fairly mild extra conditions (see Proposition 6.1, Corollary 6.4 and Theorem 6.6). However, the fact that Σ^ψ and Σ commute does not in general imply that ψ is Σ-invariant, even though it is sufficient to ensure that $\varphi+\psi$ is semi-finite (Proposition 5.10). The point is that the modular automorphism group of $\varphi+\psi$ need not commute with Σ.

PROPOSITION 5.11. *There exists a pair φ and ψ of faithful normal semi-finite weights with modular automorphism groups Σ and Σ^ψ, such that Σ and Σ^ψ commute but ψ is not Σ-invariant.*

Proof. Let $\mathcal{M}=\mathcal{B}(L^2(\mathbf{R}))$ and take P and Q as the canonical pair in the commutation relations; i.e.

$$P\xi(\gamma)=\gamma\xi(\gamma) \quad \text{and} \quad Q\xi(\gamma)=-i\xi'(\gamma).$$

With $h=\exp P$ and $k=\exp Q$ we define two non-singular self-adjoint positive operators (affiliated with $\mathcal{M}$) such that $h^{it}\xi(\gamma)=e^{it\gamma}\xi(\gamma)$ and $k^{is}\xi(\gamma)=\xi(\gamma+s)$. Thus $h^{it}k^{is}=e^{-ist}k^{is}h^{it}$. Since the trace Tr is a faithful normal semi-finite weight with trivial modular automorphism group we get two faithful normal semi-finite weights by defining $\varphi=\operatorname{Tr}(h\cdot)$ and $\psi=\operatorname{Tr}(k\cdot)$. (Proposition 4.2.) The modular automorphism groups of φ and ψ are given by Theorem 4.6 and we have

$$\sigma_s^\psi\circ\sigma_t(x)=k^{is}h^{it}xh^{-it}k^{-is}=e^{ist}h^{it}k^{is}xk^{-is}h^{-it}e^{-ist}=\sigma_t\circ\sigma_s^\psi(x).$$

Thus Σ and Σ^ψ commute; but

$$\psi(\sigma_t(x))=\operatorname{Tr}(kh^{it}xh^{-it})=e^t\operatorname{Tr}(kx)=e^t\psi(x),$$

so that ψ is not Σ-invariant. Q.E.D.

We shall now prove our main result.

THEOREM 5.12. *If ψ is a Σ-invariant normal semi-finite weight on $\mathcal{M}$ then there is a unique self-adjoint positive operator h affiliated with $\mathcal{M}^\Sigma$ such that $\psi=\varphi(h\cdot)$.*

Proof. Assume first that $\psi \leqslant \varphi$. There is then a unique operator h' in $\mathcal{M}'$ with $0 \leqslant h' \leqslant 1$ such that $\psi(x^*x) = \|h'^{\frac{1}{2}}\eta(x)\|^2$ for each x in $\mathfrak{A}$. Since ψ is Σ-invariant we have $\Delta^{-it}h'\Delta^{it} = h'$ for all t, so that $h'\Delta \subset \Delta h'$. Put $h = Jh'J$. Then $h \in \mathcal{M}^{\Sigma}$ and

$$\varphi(hx^*x) = \|\eta(xh^{\frac{1}{2}})\|^2 = \|Sh^{\frac{1}{2}}S\eta(x)\|^2$$
$$= \|\Delta^{-\frac{1}{2}}Jh^{\frac{1}{2}}J\Delta^{\frac{1}{2}}\eta(x)\|^2 = \|\Delta^{-\frac{1}{2}}h'^{\frac{1}{2}}\Delta^{\frac{1}{2}}\eta(x)\|^2 = \|h'^{\frac{1}{2}}\eta(x)\|^2 = \psi(x^*x).$$

Thus $\varphi(h\cdot)$ and ψ coincide on $\mathfrak{m}$, and therefore $\varphi(h\cdot) = \psi$ by Proposition 5.9.

In the general case put $\tau = \varphi + \psi$. By Proposition 5.10 τ is semi-finite; and clearly τ is Σ-invariant. By Theorem 5.5 and Lemma 5.8 we conclude that φ is Σ^{τ}-invariant. Therefore ψ is also Σ^{τ}-invariant. From the first part of the proof we get h and k in $\mathcal{M}^{\Sigma^{\tau}}$ such that $\varphi = \tau(h\cdot)$ and $\psi = \tau(k\cdot)$. Since φ is faithful, h is non-singular, and since $\varphi + \psi = \tau$ we have $h + k = 1$. By Theorem 4.6 we have $\sigma_t(x) = h^{it}\sigma_t^{\tau}(x)h^{-it}$ which shows that $h \in \mathcal{M}^{\Sigma}$. Thus $k \in \mathcal{M}^{\Sigma}$ as well. Using the chain rule (Proposition 4.3) we finally get

$$\varphi(h^{-1}k\cdot) = \tau(k\cdot) = \psi,$$

where $h^{-1}k$ is a self-adjoint positive operator affiliated with $\mathcal{M}^{\Sigma}$. Q.E.D.

COROLLARY 5.13. *If ψ is a Σ-invariant faithful normal semi-finite weight on $\mathcal{M}$ with modular automorphism group Σ^{ψ} then φ is Σ^{ψ}-invariant.*

6. Applications to automorphism groups

Let G be a group of automorphisms of a von Neumann algebra $\mathcal{M}$ that leaves the center $\mathcal{Z}$ fixed; and let φ be a faithful normal semi-finite weight on $\mathcal{M}$. In this section we apply the Radon-Nikodym theorem to the problem, already touched in Lemma 5.8, of finding conditions under which φ is G-invariant.

From Theorem 5.5 we see that a necessary condition for G-invariance of φ is that G and Σ commute. If φ is finite this condition is also sufficient by [10; Theorem 1.1]. For completeness we include here a short proof of this result.

PROPOSITION 6.1. *If φ is a faithful normal finite weight on $\mathcal{M}$ and G is a group of automorphisms of $\mathcal{M}$ that leaves $\mathcal{Z}$ fixed and commutes with Σ then φ is G-invariant.*

Proof. By Theorem 5.5 we have $\varphi(g(\cdot)) = \varphi(h_g\cdot)$ where h_g is affiliated with $\mathcal{Z}$. Let e_ε be the spectral projection of h_g corresponding to $[1+\varepsilon, \infty[$. Then

$$\|\varphi\| \geqslant \varphi(g^n(e_\varepsilon)) = \varphi(h_g^n e_\varepsilon) \geqslant (1+\varepsilon)^n \varphi(e_\varepsilon).$$

This implies that $e_\varepsilon = 0$ for all $\varepsilon > 0$ so that $h_g \leqslant 1$. Since the map $g \rightarrow h_g$ is a homomorphism we have $h_\varepsilon = 1$ for all g in G. Q.E.D.

An automorphism g of $\mathcal{M}$ is said to be σ-weakly *recurrent* if for each x in $\mathcal{M}$ and $\varepsilon > 0$ and each finite set $\{\omega_k\}$ of normal states there are infinitely many n such that for all k

$$|\omega_k(g^n(x) - x)| < \varepsilon \quad \text{and} \quad |\omega_k(g^{-n}(x) - x)| < \varepsilon.$$

THEOREM 6.2. *Let G be a group of automorphisms of $\mathcal{M}$ that leaves $\mathcal{Z}$ fixed and commutes with Σ. The set G_0 of elements in G under which φ is invariant is a normal subgroup of G such that the quotient group G/G_0 is abelian. Moreover, G_0 contains all σ-weakly recurrent elements from G.*

Proof. If $g \mapsto h_g$ is the homomorphism of G into the multiplicative group (abelian) of non-singular self-adjoint positive operators affiliated with $\mathcal{Z}$, given by Theorem 5.5, then G_0 consists of those elements g for which $h_g = 1$. Therefore G_0 is a normal subgroup of G and G/G_0 is abelian.

Let g be a σ-weakly recurrent element of G and let e_ε be the spectral projection of h_g corresponding to $[1+\varepsilon, \infty[$. For x in $\mathfrak{m}_+$ there is then a net $\{n_i\}$ of positive integers such that $n_i \rightarrow \infty$ and $\{g^{-n_i}(e_\varepsilon x)\}$ tend σ-weakly to $e_\varepsilon x$. Since φ is σ-weakly lower semi-continuous this implies that

$$\varphi(e_\varepsilon x) \leqslant \liminf \varphi(g^{-n_i}(e_\varepsilon x)) = \liminf \varphi(h_g^{-n_i} e_\varepsilon x) \leqslant \lim\, (1+\varepsilon)^{-n_i} \varphi(e_\varepsilon x) = 0.$$

Therefore $e_\varepsilon = 0$ for all $\varepsilon > 0$ so that $h_g \leqslant 1$. Since g^{-1} is also σ-weakly recurrent we have $h_g^{-1} \leqslant 1$; hence $g \in G_0$. Q.E.D.

PROPOSITION 6.3. *Let G be a σ-weakly continuous topological group of automorphisms of $\mathcal{M}$ that leaves $\mathcal{Z}$ fixed and commutes with Σ. Then with G_0 as in Theorem 6.2 the group G/G_0 contains no non-zero compact subgroups.*

Proof. If G/G_0 has a non-zero compact subgroup then passing if necessary to a subgroup we may assume that G is generated (topologically) by an element g such that G/G_0 is compact. Then either there is a sequence $\{n_k\}$ of numbers tending to infinity such that $g^{n_k} \rightarrow \iota$ (the identity automorphism) or there is a neighborhood U of ι in G such that $g^n \notin U$ for any $n \geqslant 1$. Choosing U symmetric we may also assume that no negative powers of g belongs to U. Since g generates G this implies that G is discrete. Then G/G_0 is discrete and compact; hence finite. But G/G_0 is isomorphic to $\{h_g^n\}$ which implies that $h_g = 1$. Therefore the first possibility must occur. But in that case g is σ-weakly recurrent since

$$\omega(g^{n_k}(x)) \rightarrow \omega(x) \quad \text{and} \quad \omega(g^{-n_k}(x)) \rightarrow \omega(x)$$

for all ω and x. Therefore $h_g = 1$ by Theorem 6.2; a contradiction. Q.E.D.

COROLLARY 6.4. *If G is a σ-weakly continuous compact group of automorphisms of $\mathcal{M}$ that leaves $\mathcal{Z}$ fixed and commutes with Σ then φ is G-invariant.*

If $\mathcal{M}$ is a semi-finite factor with trace τ and if ψ is a normal state of $\mathcal{M}$ which is invariant under a group G of automorphisms of $\mathcal{M}$ then τ is G-invariant (see [12] and [24]). The optimal generalization of this result would be that if G is a group of automorphisms of $\mathcal{M}$ which leaves $\mathcal{Z}$ fixed and commutes with Σ and if there exists some Σ-invariant faithful normal semi-finite weight ψ which is G-invariant then the weight φ is also G-invariant. This generalization is valid if one imposes certain integrability conditions on the Radon-Nikodym derivative of ψ with respect to φ. Otherwise it may be false, as we shall see.

The appropriate counterexample (Proposition 6.9) and a weaker version of Theorem 6.6 have been obtained independently by N. H. Petersen in [20].

PROPOSITION 6.5. *Suppose that ψ is a Σ-invariant normal semi-finite weight on $\mathcal{M}$ and put $\psi=\varphi(h\cdot)$ with h affiliated with $\mathcal{M}^{\Sigma}$. Then the following conditions are equivalent:*

(i) *$\varphi(x)<\infty$ implies $\psi(x)<\infty$ for all x in $\mathcal{M}^+$;*

(ii) *If e_m is the spectral projection of h corresponding to the interval $[m, \infty[$ then $\varphi(he_m)<\infty$ for large m;*

(iii) *$\psi=\psi_1+\psi_\infty$ with ψ_1 and ψ_∞ normal weights on $\mathcal{M}$ (Σ-invariant if desired) such that ψ_1 is finite and ψ_∞ is majorized by a multiple of φ;*

(iv) *For each sequence $\{x_n\}$ in $\mathcal{M}$ with $0\leqslant x_n\leqslant 1$ such that $\varphi(x_n)\to 0$ we have $\psi(x_n)\to 0$.*

When the above conditions are satisfied we say that ψ *belongs to $O(\varphi)$.*

Proof. (i) $\Rightarrow$ (ii): If (ii) does not hold then $\varphi(e_m)=\infty$ for all m. Otherwise we would have $\psi(e_m)=\varphi(he_m)<\infty$ for some m by (i). Therefore with $m=2^n$ we can find x_n in $e_m\mathcal{M}e_m$ such that $0\leqslant x_n\leqslant 1$ and $\varphi(x_n)\geqslant 1$. Put $x=\Sigma\, 2^{-n}\varphi(x_n)^{-1}x_n$. Then $x\in\mathcal{M}_+$ and $\varphi(x)<\infty$; but

$$\psi(x_n)=\varphi(hx_n)\geqslant 2^n\varphi(x_n),$$

so that $\psi(x)=\infty$, a contradiction.

(ii) $\Rightarrow$ (iii): Put $\psi_1(x)=\varphi(he_m x)$ and $\psi_\infty(x)=\varphi(h(1-e_m)x)$. Since $\varphi(he_m)<\infty$ the functional ψ_1 is finite; and since $h(1-e_m)\leqslant m$ we have $\psi_\infty\leqslant m\varphi$.

(iii) $\Rightarrow$ (iv): If $\varphi(x_n)\to 0$ then $\omega_i(x_n)\to 0$ for each normal functional ω_i majorized by φ. Then with e_i the support of ω_i, the sequence $\{e_i x_n e_i\}$ tends strongly to zero by [4; Chap. I, § 4, Proposition 4]. But $e_i\nearrow 1$ as $\omega_i\nearrow\varphi$; hence $\{x_n\}$ tends strongly to zero. If now $\psi=\psi_1+\psi_\infty$ then from the above $\psi_1(x_n)\to 0$. But also $\psi_\infty(x_n)\to 0$ since ψ_∞ is majorized by a multiple of φ. Thus $\psi(x_n)\to 0$.

(iv) $\Rightarrow$ (i): If $\varphi(x)<\infty$ then $\varphi(n^{-1}x)\to 0$. By assumption $\psi(n^{-1}x)\to 0$. Therefore $\psi(x)<\infty$. This shows that the four conditions are equivalent. Q.E.D.

THEOREM 6.6. *Let φ be a faithful normal semi-finite weight on $\mathcal{M}$ and let G be a group of automorphisms of $\mathcal{M}$ which leaves $\mathcal{Z}$ fixed and commutes with Σ. If there exists a G-invariant and Σ-invariant normal semi-finite weight ψ, with central support q, then $\varphi(q\cdot)$ is also G-invariant, provided that there is a set $\{p_i\}$ of central projections with $\Sigma p_i = 1$, such that $\psi(p_i\cdot)$ belongs to $O(\varphi(p_i\cdot))$ or $\varphi(p_i\cdot)$ belongs to $O(\psi(p_i\cdot))$ for each i.*

Proof. Restricting to $p_i\mathcal{M}p_i$ we may assume that $p_i = 1$. From Theorem 5.5 we have $\varphi(g(\cdot)) = \varphi(h_g\cdot)$ where h_g is affiliated with $\mathcal{Z}$ and from Theorem 5.12 $\psi = \varphi(h\cdot)$ where h is affiliated with $\mathcal{M}^\Sigma$. Since ψ is G-invariant we get

$$\begin{aligned}\varphi(h\cdot h_g x) = \psi(h_g x) = \psi(h_g g^{-1}(x)) &= \varphi(h\cdot h_g g^{-1}(x)) \\ &= \lim \varphi(h_\varepsilon h_g g^{-1}(x)) = \lim \varphi(g^{-1}(g(h_\varepsilon)\,h_g x)) \\ &= \lim \varphi(h_g^{-1} g(h_\varepsilon)\,h_g x) = \lim \varphi(g(h_\varepsilon)\,x),\end{aligned}$$

where $h_\varepsilon = h(1+\varepsilon h)^{-1}$. Defining $g(h) = \lim g(h_\varepsilon)$ and using the uniqueness of the Radon-Nikodym derivative we conclude that $g(h) = h_g\cdot h$.

Let e_ε be the spectral projection of h_g corresponding to $[1+\varepsilon, \infty[$. We want to show that $he_\varepsilon = 0$. To further this end, let f be a positive bounded monotone increasing function on $\mathbf{R}$. Then

$$g^n(f(he_\varepsilon)) = f(g^n(he_\varepsilon)) = f(h_g^n\cdot he_\varepsilon) \geqslant f((1+\varepsilon)^n he_\varepsilon)$$

since $h_g e_\varepsilon \geqslant (1+\varepsilon)e_\varepsilon$. Thus

$$\varphi(hf(h)) = \psi(f(h)) \geqslant \psi(f(he_\varepsilon)) = \psi(g^n(f(he_\varepsilon))) \geqslant \psi(f((1+\varepsilon)^n he_\varepsilon)) = \varphi(hf((1+\varepsilon)^n he_\varepsilon)). \qquad (*)$$

If instead we take f to be monotone decreasing then the analogous calculations yield

$$\varphi(hf(h)) \leqslant \varphi(hf((1+\varepsilon)^n he_\varepsilon)). \qquad (**)$$

Now let f_m be the characteristic function (increasing) for the set $[m, \infty[$. Then $\{f_m((1+\varepsilon)^n he_\varepsilon)\}$ increases to $[h]e_\varepsilon$ when $n\to\infty$. It follows from (*) that

$$\varphi(hf_m(h)) \geqslant \varphi(he_\varepsilon).$$

Assuming that ψ belongs to $O(\varphi)$ we have $\varphi(hf_m(h)) < \infty$ for large m, by Proposition 6.5. (ii), and $hf_m(h) \searrow 0$ so that $\varphi(hf_m(h)) \searrow 0$. Therefore $he_\varepsilon = 0$. Assuming instead that φ belongs to $O(\psi)$ we take f_m as the characteristic function (decreasing) of the set $]-\infty, m]$. Then

$$f_m((1+\varepsilon)^n he_\varepsilon) = (1-e_\varepsilon) + e_\varepsilon f_m((1+\varepsilon)^n h).$$

Since φ belongs to $O(\psi)$ and $\varphi = \psi(h^{-1}\cdot)$ (h must be non-singular) we see from Proposition

6.5. (ii) that if q_δ is the spectral projection of h corresponding to $[0, \delta]$ then $\varphi(q_\delta) < \infty$ for a small $\delta > 0$. For fixed m the sequence $\{f_m((1+\varepsilon)^n h)\}$ decreases to zero and $f_m((1+\varepsilon)^n h) \leqslant q_\delta$ for $m(1+\varepsilon)^{-n} \leqslant \delta$. Using (**) with $\varphi(e_\varepsilon \cdot)$ instead of φ we get

$$\varphi(he_\varepsilon f_m(h)) \leqslant \varphi(he_\varepsilon f_m((1+\varepsilon)^n h)) \searrow 0.$$

Thus $he_\varepsilon f_m(h) = 0$ for all m which implies that $he_\varepsilon = 0$.

Let q be the central support of ψ. Then q is the smallest central projection for which $(1-q)h = 0$. From the first part of the proof we see that $qe_\varepsilon = 0$ for all $\varepsilon > 0$. Therefore $h_g q \leqslant q$. Multiplying this inequality with h_g^{-1} we get $q \leqslant h_g^{-1} q$. Since these inequalities are valid for all g in G we conclude that $h_g q = q$, so that $\varphi(q\cdot)$ is G-invariant. Q.E.D.

COROLLARY 6.7. *If $\mathcal{M}$ is a semi-finite factor and G is a group of automorphisms of $\mathcal{M}$ which admits some G-invariant normal state then the trace on $\mathcal{M}$ is G-invariant.*

The next result generalizes a theorem due to N. Hugenholtz and E. Størmer (see [12] and [24]).

PROPOSITION 6.8. *Let G be a group of automorphisms of a factor $\mathcal{M}$ and suppose that G admits one and only one G-invariant normal state ω on $\mathcal{M}$. If Σ is a strongly continuous one-parameter group of automorphisms of $\mathcal{M}$ which commutes with G then either ω satisfies the* KMS *condition with respect to Σ or else no non-zero normal weight on $\mathcal{M}$ satisfies the* KMS *condition with respect to Σ.*

Proof. Since G and Σ commute each state of the form $\omega \circ \sigma_t$ is G-invariant. By the assumption on G this implies that $\omega \circ \sigma_t = \omega$ for all t, i.e. ω is Σ-invariant. If φ is a non-zero normal semi-finite weight on $\mathcal{M}$ which satisfies the KMS condition with respect to Σ then φ is faithful since its support belongs to $\mathcal{Z}$ by Lemma 5.1 and $\mathcal{M}$ is a factor. Therefore $\omega = \varphi(h\cdot)$ by Theorem 5.12. Since ω is finite it belongs to $O(\varphi)$ and therefore the G-invariance of ω implies the G-invariance of φ by Theorem 6.6. The uniqueness of the Radon-Nikodym derivative implies that $g(h) = h$ for each g in G. If h is not a scalar multiple of 1 then $h(1+h)^{-1}$ is not a scalar multiple of h. Put $k = \gamma h(1+h)^{-1}$ for a suitable $\gamma > 0$ such that $\varphi(k) = 1$. Then $\omega' = \varphi(k\cdot)$ is a normal state on $\mathcal{M}$ which is different from ω; but ω' is G-invariant since both φ and k are G-invariant, a contradiction. Thus h is a scalar multiple of 1 so that ω satisfies the KMS condition with respect to Σ. Q.E.D.

If one drops the assumption in Corollary 6.7 that the G-invariant functional is finite (or is O of the trace) then the trace need no longer be G-invariant, as the following example shows.

PROPOSITION 6.9. *There exists a factor $\mathcal{M}$ of type II_∞ and a group G of automorphisms of $\mathcal{M}$ such that the trace on $\mathcal{M}$ is not G-invariant, but there is a G-invariant faithful normal semi-finite weight on $\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be a factor of type II_∞ constructed in [4; pp. 130–136]. The Hilbert space $\mathfrak{H}$ on which $\mathcal{M}$ acts consists of the square integrable functions $\xi(\gamma, s)$ on $\mathbf{R} \times \mathbf{Q}$, with Lebesgue measure λ on $\mathbf{R}$ and the "counting" measure on the set $\mathbf{Q}$ of rational numbers in $\mathbf{R}$. For each f in $L^\infty_\lambda(\mathbf{R})$ define $\pi(f)$ on $\mathfrak{H}$ by $(\pi(f)\xi)(\gamma, s) = f(\gamma)\xi(\gamma, s)$. For each t in $\mathbf{Q}$ define $u(t)$ on $\mathfrak{H}$ by $(u(t)\xi)(\gamma, s) = \xi(\gamma + t, s + t)$. Then $\mathcal{M}$ is the von Neumann algebra generated by the operators $\pi(f)$ and $u(t)$; and each element x in $\mathcal{M}$ has a unique representation

$$x = \sum_t \pi(f_t^x)\, u(t).$$

The algebra $\mathcal{M}$ is a semi-finite factor and the trace τ on $\mathcal{M}_+$ is given by $\tau(x) = \int f_0^x(\gamma)\, d\gamma$.

Let $\mathbf{Q}^*$ denote the multiplicative group of non-zero rational numbers. For each r in $\mathbf{Q}^*$ define $v(r)$ on $\mathfrak{H}$ by $(v(r)\xi)(\gamma, s) = |r|^{\frac{1}{2}}\xi(\gamma r, sr)$. It is easily verified that the map $r \mapsto v(r)$ is a unitary representation of $\mathbf{Q}^*$ on $\mathfrak{H}$. Let G denote the corresponding transformation group on $\mathcal{B}(\mathfrak{H})$. Then for x in $\mathcal{M}$ we have

$$g_r(x) = v(r)\, x v(r)^* = \sum_t v(r)\, \pi(f_t^x)\, v(r)^*\, v(r)\, u(t)\, v(r)^*.$$

But for ξ in $\mathfrak{H}$ we have

$$\begin{aligned}
[v(r)\pi(f)v(r)^*\xi](\gamma, s) &= |r|^{\frac{1}{2}}[\pi(f)v(r)^*\xi](\gamma r, sr) \\
&= |r|^{\frac{1}{2}} f(\gamma r)[v(r)^*\xi](\gamma r, sr) = f(\gamma r)\xi(\gamma, s); \\
[v(r)u(t)v(r)^*\xi](\gamma, s) &= |r|^{\frac{1}{2}}[u(t)v(r)^*\xi](\gamma r, sr) \\
&= |r|^{\frac{1}{2}}[v(r)^*\xi](\gamma r + t, sr + t) = \xi(\gamma + r^{-1}t, s + r^{-1}t) \\
&= u(r^{-1}t)\xi(\gamma, s).
\end{aligned}$$

From these equations it follows that G is a group of automorphisms of $\mathcal{M}$. The trace τ is not G-invariant; for if $x \in \mathcal{M}_+$ then

$$\tau(g_r(x)) = \int f_0^x(\gamma r)\, d\gamma = r^{-1}\int f_0^x(\gamma)\, d\gamma = r^{-1}\tau(x).$$

Put $h = \pi(f^h)$, with $f^h(\gamma) = |\gamma|^{-1}$. Then h is a non-singular self-adjoint positive operator in $\mathfrak{H}$ affiliated with $\mathcal{M}$. By Proposition 4.2 the weight $\varphi = \tau(h\,\cdot)$ is faithful normal and semi-finite. For each x in $\mathcal{M}_+$ we have

$$\varphi(g_r(x)) = \tau(hg_r(x)) = \int f^h(\gamma)\, f_0^x(\gamma r)\, d\gamma$$
$$= \int f_0^x(\gamma r)\, |\gamma|^{-1} d\gamma = \int f_0^x(\gamma)\, |\gamma|^{-1} d\gamma = \varphi(x).$$

It follows that φ is G-invariant. Q.E.D.

If in the above example we let H be the group of inner automorphisms of $\mathcal{M}$ arising from unitaries $\pi(f)$ where $f \in L_\lambda^\infty(\mathbf{R})$ then scalar multiples of φ are the only normal semi-finite weights on $\mathcal{M}$ which are invariant under the automorphism group G' generated by G and H. For if φ' is a G'-invariant normal semi-finite weight on $\mathcal{M}$ and $\varphi' = \tau(h'\cdot)$ then since φ' is H-invariant, and since $\pi(L_\lambda^\infty(\mathbf{R}))$ is a maximal abelian subalgebra of $\mathcal{M}$ we must have $h' = \pi(f')$ for some f' in $L_\lambda^\infty(\mathbf{R})$. Since, furthermore, φ' is G-invariant we have

$$\int f'(\gamma)\, f_0^x(\gamma r)\, d\gamma = \varphi'(g_r(x)) = \varphi'(x) = \int f'(\gamma)\, f_0^x(\gamma)\, d\gamma$$

for each r in $\mathbf{Q}^*$. It follows that $|r|^{-1} f'(\gamma r^{-1}) = f'(\gamma)$ for each r in $\mathbf{Q}^*$ and almost all γ in $\mathbf{R}$. Define $f(\gamma) = |\gamma| f'(\gamma)$. Then $f(r\gamma) = f(\gamma)$ for almost all γ in $\mathbf{R}$ and all r in $\mathbf{Q}^*$. But the action of $\mathbf{Q}^*$ is ergodic on $\mathbf{R}$ with respect to Lebesgue measure; hence f is equal to a constant δ almost everywhere. It follows that $f'(\gamma) = \delta |\gamma|^{-1}$ so that $\varphi' = \delta\varphi$. This shows that Proposition 6.8 need not be true without the restriction that ω be finite. For with φ, G' and $\{\iota\}$ instead of ω, G and Σ in Proposition 6.9 we have an example where scalar multiples of φ are the only G'-invariant normal semi-finite weights on $\mathcal{M}$, yet there is a trace which satisfies the KMS condition with respect to the trivial group $\{\iota\}$.

We recall that a group G of automorphisms of a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathfrak{H}$ is said to be *unitarily implemented* on $\mathfrak{H}$ if there is a homomorphism $g \to u_g$ of G into the group of unitaries on $\mathfrak{H}$ such that $g(x) = u_g x u_g^*$ for each x in $\mathcal{M}$. (The representation of $\mathcal{M}$ on $\mathfrak{H}$ is *covariant*.) Our next theorems extend results of H. Halpern on the implementability of locally compact automorphism groups (see [9]).

THEOREM 6.10. *Let $\mathcal{M}$ be a von Neumann algebra and let φ be a faithful normal semi-finite weight on $\mathcal{M}$ with modular automorphism group Σ. Then each group G of automorphisms of $\mathcal{M}$ that commutes with Σ can be unitarily implemented on the Hilbert space $\mathfrak{H}$ of φ.*

Proof. As in the proof of Theorem 5.5 we have $\varphi \circ g = \varphi(h_g \cdot)$ where h_g is a non-singular positive operator affiliated with $\mathfrak{Z}$. Since we do not assume that G leaves $\mathfrak{Z}$ fixed we cannot conclude that the map $g \to h_g$ is a homomorphism. Let e_n be the spectral projection of h_g corresponding to $[n^{-1}, n]$. Then for each x in $\mathfrak{A}$ we have $g(h_g^{-\frac{1}{2}} e_n x)$ in $\mathfrak{A}$ and

$$\|\eta(g(h_g^{-\frac{1}{2}} e_n x))\|^2 = \varphi(g(h_g^{-1} e_n x^* x)) = \varphi(e_n x^* x) \nearrow \varphi(x^* x).$$

Moreover, for $m > n$

$$\|\eta(g(h_g^{-\frac{1}{2}} e_m x)) - \eta(g(h_g^{-\frac{1}{2}} e_n x))\|^2 = \varphi((e_m - e_n)\, x^* x).$$

It follows that the sequence $\{\eta(g(h_g^{-\frac{1}{2}} e_n x))\}$ converges in $\mathfrak{H}$ to an element which we shall denote by $\eta(g(h_g^{-\frac{1}{2}} x))$. The linear operator u_g' on $\eta(\mathfrak{A})$ defined by

$$u_g'(\eta(x)) = \eta(g(h_g^{-\frac{1}{2}} x))$$

extends to an isometry u_g of $\mathfrak{H}$. Since

$$\varphi(h_{g'g} x) = \varphi(g' g(x)) = \varphi(h_{g'} g(x)) = \varphi(g(g^{-1}(h_{g'}) x)) = \varphi(h_g \cdot g^{-1}(h_{g'}) x)$$

we have $h_{g'g} = h_g \cdot g^{-1}(h_{g'})$. Therefore

$$\begin{aligned} u_{g'g}\eta(x) &= \eta(g' g(h_{g'g}^{-\frac{1}{2}} x)) = \eta(g' g(h_g^{-\frac{1}{2}} \cdot g^{-1}(h_{g'}^{-\frac{1}{2}})\, x)) \\ &= \eta(g'(h_{g'}^{-\frac{1}{2}} \cdot g(h_g^{-\frac{1}{2}} x))) = u_{g'}\eta(g(h_g^{-\frac{1}{2}} x)) = u_{g'} u_g \eta(x), \end{aligned}$$

which shows that $g \mapsto u_g$ is a homomorphism of G into the group of unitaries on $\mathfrak{H}$. Since

$$u_g^* \eta(x) = \eta(g^{-1}(h_{g^{-1}}^{-\frac{1}{2}} x)) = \eta(h_g^{\frac{1}{2}} g^{-1}(x))$$

we finally have
$$u_g y u_g^* \eta(x) = u_g \eta(y h_g^{\frac{1}{2}} g^{-1}(x)) = g(y)\, \eta(x)$$

so that the representation $g \mapsto u_g$ is a unitary implementation of G on $\mathfrak{H}$. Q.E.D.

PROPOSITION 6.11. *Let G be a locally compact σ-weakly continuous group of automorphisms of $\mathfrak{M}$ which commutes with Σ and leaves $\mathfrak{Z}$ fixed. Then the unitary representation of G on $\mathfrak{H}$ constructed in Theorem 6.10 is strongly continuous.*

Proof. As in Theorem 6.2 let G_0 denote the normal subgroup of G under which φ is invariant. Then the set

$$F = \{g \in G \mid \varphi \circ g \leqslant \varphi\} = \bigcap_{x \geqslant 0} \{g \in G \mid \varphi(g(x)) \leqslant \varphi(x)\}$$

is closed in G (since φ is σ-weakly lower semi-continuous), and $G_0 = F \cap F^{-1}$. Hence G_0 is closed in G. It follows from Proposition 6.3 that G/G_0 is a locally compact abelian group with no non-zero compact subgroups. Therefore the connected component $\bar{H}$ of G/G_0 containing the identity is isomorphic to $\mathbf{R}^n$ for some $n \geqslant 0$ and $(G/G_0)/\bar{H}$ is discrete by the structure theorem for locally compact abelian groups (see [21; Theorem 2.4.1]). It suffices to prove continuity of the representation $g \mapsto u_g$ on the inverse image H of

$\bar{H}$ in G (since G/H is discrete). We may therefore assume that $G=H$, so that $G/G_0=\mathbf{R}^n$. Choose elements $g_1, \ldots, g_n$ in G such that the corresponding images $\bar{g}_1, \ldots, \bar{g}_n$ in G/G_0 form a basis for $\mathbf{R}^n$. Let $\varphi(g_k(\cdot))=\varphi(h_k\cdot)$, $1\leqslant k\leqslant n$, where h_k is affiliated with $\mathfrak{Z}$. If e_{km} denotes the spectral projection of h_k corresponding to $[m^{-1}, m]$, put $e_m=\wedge e_{km}$. Then $e_m\nearrow 1$. For each g in G with $\bar{g}=\Sigma\lambda_k\bar{g}_k$ in G/G_0, define $k_g=\prod h_k^{\lambda_k}$. Then $g\mapsto k_g e_m$ is a continuous homomorphism of G into the mutliplicative group of positive invertible elements of $\mathfrak{Z}e_m$. Since the homomorphism $g\mapsto h_g e_m$ is lower semi-continuous and since $h_g=k_g$ for all g in G with $\bar{g}=\Sigma\lambda_k\bar{g}_k$ such that all λ_k are dyadic rational numbers (i.e. on a dense subgroup of G) we conclude as in the proof of Proposition 5.7 that $h_g=k_g$ for all g in G.

Now take x and y in $\mathfrak{A}$ and z in $\mathfrak{A}_0$. Then

$$(u_g e_m\eta(x)\,|\,\eta((zy)^*))=\varphi(zyh_g^{-\frac{1}{2}}e_m g(x))=\varphi(yh_g^{-\frac{1}{2}}e_m g(x)\ \sigma_{-i}(z)),$$

using Lemma 3.5 with $\sigma_{-i}(z)$ and $yh_g^{-\frac{1}{2}}e_m g(x)$ instead of h and z. When $g\to\iota$ we have $h_g^{-\frac{1}{2}}e_m\to e_m$ uniformly, and since y and $\sigma_{-i}(z)$ belongs to $\mathfrak{A}$ we have $\varphi(yg(x)\sigma_{-i}(z))\to\varphi(yx\sigma_{-i}(z))$ as G is σ-weakly continuous on $\mathcal{M}$. Thus $u_g\to 1$ weakly on a dense set of vectors in $\mathfrak{H}$ which proves that $g\to u_g$ is weakly, hence strongly continuous. Q.E.D.

The main virtue of the preceding result is that it gives an explicit and canonical construction for the implementation of G. For whenever G is a separable locally compact group (and $\mathfrak{H}$ is infinite dimensional) we can represent $\mathcal{M}$ and G on $\mathfrak{H}\otimes L^2(G)$ as the induced covariant representation; and since this space has the same dimension as $\mathfrak{H}$ we can then pull back the covariant representation of $\mathcal{M}$ and G from $\mathfrak{H}\otimes L^2(G)$ to $\mathfrak{H}$ by a spatial isomorphism between $\mathcal{M}$ on $\mathfrak{H}\otimes L^2(G)$ and $\mathcal{M}$ on $\mathfrak{H}$. This isomorphism is, however, not unique.

7. Further applications of the Radon-Nikodym theorem

Our first application of the Radon-Nikodym theorem in this section provides a partial solution of the problem raised in [4; p. 52], whether each weight that respects monotone increasing limits is the sum of normal positive functionals. For the proof we shall need the following result which may have independent interest.

PROPOSITION 7.1. *For each faithful normal semi-finite weight φ on $\mathcal{M}$ there is a set $\{p_i\}$ of pairwise orthogonal projections from $\mathcal{M}^\Sigma$ with sum 1 such that p_i is the strong limit of an increasing sequence from $\mathfrak{m}_+$. In particular each $p_i\mathcal{M}p_i$ is σ-finite.*

Proof. Let $\{t_n\}$ be an enumeration of the rational numbers in $\mathbf{R}$ and for x in $\mathfrak{m}_+$ put

$$u_n=\left(n^{-1}+\sum_{k=1}^{n}\sigma_{t_k}(x)\right)^{-1}\sum_{k=1}^{n}\sigma_{t_k}(x).$$

Then $\{u_n\}$ is an increasing sequence in $\mathfrak{m}_+$ and converges strongly to a projection p (the union of the range projections of all the elements $\sigma_{t_k}(x)$). For each rational number t and each n there is an m such that

$$\{tt_k \mid k \leqslant n\} \subset \{t_k \mid k \leqslant m\}.$$

Therefore $\sigma_t(u_n) \leqslant u_m$. It follows that $\sigma_t(p) \leqslant p$ for each rational number; hence $p \in \mathcal{M}^\Sigma$. The algebra $p\mathcal{M}p$ is σ-finite. For if $\{q_j\}$ is a set of projections in $\mathcal{M}$ with $\Sigma q_j = p$ then for each n

$$\varphi(\sum u_n^{\frac{1}{2}} q_j u_n^{\frac{1}{2}}) = \varphi(u_n) < \infty.$$

Thus for all but countably many j's $u_n^{\frac{1}{2}} q_j u_n^{\frac{1}{2}} = 0$ for all n. But $u_n^{\frac{1}{2}} q_j u_n^{\frac{1}{2}} \to q_j$, and therefore the number of j's with $q_j \neq 0$ is countable.

Now let $\{p_i\}$ be a maximal family of pairwise orthogonal projections from $\mathcal{M}$ each of which is the strong limit of an increasing sequence from $\mathfrak{m}_+$. Suppose that $q = 1 - \Sigma p_i \neq 0$. Then $qxq \neq 0$ for some x in $\mathfrak{m}_+$ and $qxq \in \mathfrak{m}_+$ by Proposition 3.3. From the first part of the proof there is then a non-zero projection p in $\mathcal{M}^\Sigma$ which is the strong limit of an increasing sequence from $\mathfrak{m}_+$; and $p \leqslant q$. This contradicts the maximality of $\{p_i\}$. Hence $q = 0$. Q.E.D.

THEOREM 7.2. *Each normal weight on $\mathcal{M}$ is the sum of normal positive functionals.*

Proof. Let φ denote the weight and suppose that φ is faithful on $(1-p)\mathcal{M}(1-p)$ and semi-finite on $q\mathcal{M}q$ with $p \leqslant q$. If $\varphi = \Sigma \omega_i$ on $(q-p)\mathcal{M}_+(q-p)$ where each ω_i is a normal positive functional on $(q-p)\mathcal{M}(q-p)$ then put $\tilde{\omega}_i = \omega_i((q-p)\cdot(q-p))$ on $\mathcal{M}$ and choose a set $\{\omega_j\}$ of normal positive functionals on $(1-q)\mathcal{M}(1-q)$ such that $\Sigma\omega_j(x) = \infty$ for each x in $(1-q)\mathcal{M}_+(1-q)$ different from 0. Put $\tilde{\omega}_j = \omega_j((1-q)\cdot(1-q))$ on $\mathcal{M}$. Then $\varphi = \Sigma\tilde{\omega}_i + \Sigma\tilde{\omega}_j$ on $\mathcal{M}_+$. It follows that it is enough to prove the theorem assuming that φ is faithful and semi-finite.

Under this assumption there is by Proposition 7.1 a set $\{p_i\}$ of projections in $\mathcal{M}^\Sigma$ with $\Sigma p_i = 1$ such that each p_i is the strong limit of an increasing sequence from $\mathfrak{m}_+$. We have $\varphi = \Sigma\varphi(p_i\cdot)$ by Propositions 4.1 and 4.2. Hence without loss of generality we may assume that there is an increasing sequence $\{u_n\}$ in $\mathfrak{m}_+$ such that $u_n \nearrow 1$. Let $\{t_n\}$ be an enumeration of the rational numbers in $\mathbf{R}$. By [1; Lemmes 1.9 & 4.3] the set of normal positive functionals which are completely majorized by φ form an increasing net with limit φ. We can therefore by induction find a sequence $\{\omega_n\}$ of normal positive functionals such that (with $t_1 = 0$)

$$\omega_m(\sigma_{t_k}(\cdot)) \leqslant \omega_n < \varphi \quad \text{for } m < n \text{ and } k < n;$$

$$\omega_n(u_k) + n^{-1} \geqslant \varphi(u_k) \quad \text{for } k < n.$$

Put $\omega_n' = \omega_n - \omega_{n-1}$ (and $\omega_1' = \omega_1$). Then the weight $\psi = \Sigma\omega_n'$ is normal and $\psi \leqslant \varphi$. Moreover, $\psi(u_n) = \varphi(u_n)$ for all n. Since the set of elements x in $\mathfrak{M}_+$ such that $\psi(x) = \varphi(x) < \infty$ is a hereditary subcone of $\mathfrak{M}_+$, invariant under Σ, we have $\psi(x) = \varphi(x)$ for all x in $\cup\, u_n \mathfrak{M}_+ u_n$; so that ψ and φ are equal on a σ-weakly dense set in $\mathfrak{M}_+$. The construction of ψ implies that $\psi(\sigma_t(x)) \leqslant \psi(x)$ for each x in $\mathfrak{M}_+$ and all rational numbers t. Since ψ is σ-weakly lower semi-continuous on $\mathfrak{M}_+$ we obtain $\psi(\sigma_t(x)) \leqslant \psi(x)$ for all real t. This of course implies that ψ is Σ-invariant. Therefore $\psi = \varphi$ by Proposition 5.9. Q.E.D.

COROLLARY 7.3. (cf. [16; Theorem 3.1] and [1; Proposition 1.11]) *Each (norm) lower semi-continuous weight φ on a C^*-algebra A is the sum of positive functionals on A.*

Proof. Let B be the closure of the linear span of the set

$$\mathfrak{m} = \{x \in A_+ \,|\, \varphi(x) < \infty\}.$$

Then B is a hereditary C^*-subalgebra of A, and the restriction φ_B of φ to B is semi-finite. Let $\tilde{A}$ and $\tilde{B}$ denote the universal enveloping von Neumann algebras of A and B, respectively. Then $\tilde{B}$ is a σ-weakly closed hereditary subalgebra of $\tilde{A}$; hence $\tilde{B} = q\tilde{A}q$ for some projection q in $\tilde{A}$. By [1; Proposition 4.1] φ_B has an extension to a normal semi-finite weight $\tilde{\varphi}_B$ on $\tilde{B}$. We define a normal weight $\tilde{\varphi}$ on $\tilde{A}_+$ by

$$\tilde{\varphi}(x) = \begin{cases} \tilde{\varphi}_B(x) & \text{if } x \in \tilde{B}_+ \\ \infty & \text{if } x \notin \tilde{B}_+ \end{cases}.$$

If x is an element of A_+ then $x \in B_+$ if and only if $x = qxq$. Hence $\tilde{\varphi}$ is an extension of φ. From Theorem 7.2 $\tilde{\varphi}$ has a decomposition as a sum of positive functionals, and the restriction to A of this decomposition gives a decomposition of φ. Q.E.D.

The Radon-Nikodym theorem also gives an easy proof of the following result by the second author [25; Theorem 14.2].

THEOREM 7.4. *A von Neumann algebra $\mathfrak{M}$ is semi-finite if and only if there exists a faithful normal semi-finite weight on $\mathfrak{M}$ whose modular automorphism group is implemented by a strongly continuous one-parameter unitary group in $\mathfrak{M}$. In this case the modular group of any normal semi-finite weight on $\mathfrak{M}$ is implemented by a one-parameter unitary group in $\mathfrak{M}$.*

Proof. If $\mathfrak{M}$ is semi-finite then there is a faithful normal semi-finite trace τ on $\mathfrak{M}$. The modular group of τ is trivial. By Theorem 5.12 each normal semi-finite weight φ on $\mathfrak{M}$ is of the form $\tau(h\cdot)$ where h is affiliated with $\mathfrak{M}$ and by Theorem 4.6 the modular group of φ is given by $\sigma_t(x) = h^{it}xh^{-it}$ and consequently implemented by the one-parameter unitary group $\{h^{it}\}$ in $\mathfrak{M}$.

If, conversely, φ is a faithful normal semi-finite weight on $\mathfrak{M}$ such that the modular group is implemented by a one-parameter unitary group in $\mathfrak{M}$, then by Stone's theorem we have $\sigma_t(x) = h^{it}xh^{-it}$ where h is a non-singular self-adjoint positive operator affiliated with $\mathfrak{M}$. By Proposition 4.2 the weight $\tau = \varphi(h^{-1}\cdot)$ is faithful normal and semi-finite; and by Theorem 4.6 the modular automorphism group of τ is given by

$$\sigma_t^\tau(x) = h^{-it}\sigma_t(x)h^{it} = x.$$

It follows that τ is a trace; hence $\mathfrak{M}$ is semi-finite. Q.E.D.

COROLLARY 7.5. (cf. [26; Corollary 3]). *If φ is a faithful normal semi-finite weight on a von Neumann algebra $\mathfrak{M}$ of type* III *then there are no non-zero normal semi-finite weights on $\mathfrak{M}$ that satisfies the* KMS *condition with respect to the group $\{\sigma_{\beta t}\}$, $\beta \neq 1$.*

Proof. Suppose, to obtain a contradiction, that ψ was such a weight. Then $\beta \neq 0$ since $\mathfrak{M}$ is of type III. Therefore ψ is invariant under $\{\sigma_t\}$. Thus $\psi = \varphi(h\cdot)$ and

$$\sigma_{\beta t}(x) = h^{it}\sigma_t(x)h^{-it}$$

for all x in $\mathfrak{M}$. Put $\alpha = (\beta - 1)^{-1}$. Then

$$\sigma_t(x) = h^{i\alpha t}xh^{-i\alpha t}$$

so that the modular group of φ is implemented by a one-parameter unitary group in $\mathfrak{M}$. By Theorem 7.4 $\mathfrak{M}$ is semi-finite; a contradiction. Q.E.D.

We now assume that $\mathfrak{M}$ is semi-finite with a faithful normal semi-finite trace τ. Then by Theorem 5.12 each normal semi-finite weight φ on $\mathfrak{M}$ is of the form $\tau(h\cdot)$ where h is affiliated with $\mathfrak{M}$. Our next result provides a partial extension of S. Sakai's non-commutative Radon-Nikodym theorem [22].

PROPOSITION 7.6. *Suppose that φ and ψ are normal semi-finite weights on a semi-finite von Neumann algebra $\mathfrak{M}$ and $\psi \leqslant \varphi$. If there is a faithful normal semi-finite trace τ on $\mathfrak{M}$ such that φ (and ψ) belongs to $O(\tau)$ then there is a unique operator t in $\mathfrak{M}$ with $0 \leqslant t \leqslant 1$ such that $\psi = \varphi(t\cdot t)$.*

Proof. We may assume that φ is faithful. Put $\varphi = \tau(h\cdot)$ and $\psi = \tau(k\cdot)$. From Proposition 4.2 we have $k \leqslant h$. If φ belongs to $O(\tau)$ then $\tau(he_m) < \infty$ for large m, where e_m denotes the spectral projection of h corresponding to $[m, \infty]$. Therefore $\tau(e_m) \searrow 0$ as $e_m \searrow 0$ so that h and k are measurable operators in the sense of I. Segal. Since the measurable operators on $\mathfrak{H}$ form an algebra with involution under strong sum and strong product [23; Corollary 5.2] and since the correspondence between normal semi-finite weights and self-adjoint positive operators affiliated with $\mathcal{M}$ is a bijection, we see that the proposition is equivalent to solving the operator equation $k=t\cdot h\cdot t$ for a positive t in $\mathcal{M}$ under the assumption that $k\leq h$. For bounded h and k this problem was considered in [18], and the solution with unbounded measurable operators is obtained in the same way. Put $a=h^{\frac{1}{2}}\cdot k\cdot h^{\frac{1}{2}}$. Then $a\leq h^2$; hence $a^{\frac{1}{2}}\leq h$ (see for example [17]). There is then an operator x in $\mathcal{M}$ with $\|x\|\leq 1$ such that $a^{\frac{1}{2}}=x\cdot h^{\frac{1}{2}}$. Put $t=x^*x$. Then

$$h^{\frac{1}{2}}\cdot t\cdot h\cdot t\cdot h^{\frac{1}{2}}=(h^{\frac{1}{2}}\cdot t\cdot h^{\frac{1}{2}})^2=(a^{\frac{1}{2}})^2=h^{\frac{1}{2}}\cdot k\cdot h^{\frac{1}{2}}.$$

Since φ is faithful, h is non-singular; hence $t\cdot h\cdot t=k$. To show the uniqueness of the solution let s be a positive operator in $\mathcal{M}$ such that $s\cdot h\cdot s=t\cdot h\cdot t$. Then there is a partial isometry u in $\mathcal{M}$ such that $h^{\frac{1}{2}}\cdot s=u\cdot h^{\frac{1}{2}}\cdot t$, and u^*u is the range projection of $h^{\frac{1}{2}}\cdot t$. It follows that $h^{\frac{1}{2}}\cdot s\cdot h^{\frac{1}{2}}=u\cdot h^{\frac{1}{2}}\cdot t\cdot h^{\frac{1}{2}}$, and since the polar decomposition is unique this implies that u is the range projection of $h^{\frac{1}{2}}\cdot t$. Hence $h^{\frac{1}{2}}\cdot s=h^{\frac{1}{2}}\cdot t$, and since h is non-singular this implies that $s=t$. Q.E.D.

The above result need not be true if the functional φ does not belong to $\mathcal{O}(\tau)$ as shown by the following example.

PROPOSITION 7.7. *There exist two normal semi-finite weights φ and ψ on the von Neumann algebra $\mathcal{B}(\mathfrak{H})$ such that ψ is finite, φ is faithful and $\psi\leq\varphi$. However, there is no positive operator t in $\mathcal{B}(\mathfrak{H})$ such that $\psi=\varphi(t\cdot t)$.*

Proof. Let $\{\xi_n\}$ be an orthonormal basis for $\mathfrak{H}$. Denote by h the non-singular self-adjoint positive operator for which $h\xi_n=n\xi_n$, and let k denote the projection of $\mathfrak{H}$ on the subspace spanned by the vector $\eta=\Sigma n^{-1}\xi_n$. Put $\varphi=\mathrm{Tr}\ (h\cdot)$ and $\psi=\mathrm{Tr}\ (k\cdot)$. Since $k\leq 1\leq h$ we have $\psi\leq\varphi$. If t is a positive operator in $\mathcal{B}(\mathfrak{H})$ such that $\psi=\varphi(t\cdot t)$ then $k=\lim th_\varepsilon t$, where $h_\varepsilon=h(1+\varepsilon h)^{-1}$. Since $h_\varepsilon\geq(1+\varepsilon)^{-1}$ this gives $k\geq\lim\ (1+\varepsilon)^{-1}t^2$; hence $t^2\leq k$. Since k is a minimal projection, $t=\lambda k$ with $0\leq\lambda\leq 1$. But then

$$\begin{aligned}\|\eta\|^2=(k\eta\,|\,\eta)\geq\lambda^2(h_\varepsilon\eta\,|\,\eta)&=\lambda^2(\Sigma n^{-1}h_\varepsilon\xi_n\,|\,\Sigma m^{-1}\xi_m)\\&=\lambda^2(\Sigma n^{-1}n(1+\varepsilon n)^{-1}\xi_n\,|\,\Sigma m^{-1}\xi_m)\\&=\lambda^2\Sigma n^{-1}(1+\varepsilon n)^{-1}\geq\lambda^2\Sigma(\varepsilon n^2)^{-1}=\tfrac{1}{6}\pi^2\lambda^2\varepsilon^{-1},\end{aligned}$$

a contradiction. Q.E.D.

The following example shows that Proposition 5.9 need not be true for arbitrary normal semi-finite weights.

PROPOSITION 7.8. *There exist two normal semi-finite weights on the von Neumann algebra $\mathcal{B}(\mathfrak{H})$ such that $\varphi\leq\psi$ and $\varphi\neq\psi$; yet $\varphi=\psi$ on a σ-weakly dense $*$-subalgebra of $\mathcal{B}(\mathfrak{H})$.*

Proof. Let h and k be different self-adjoint extensions of the Laplacian $-d^2/dt^2$ on $\mathfrak{H}=L^2[0, 2\pi]$ such that $h \leq k$. For example h could be the extension corresponding to the boundary conditions $\xi(0)=\xi(2\pi)$ and $\xi'(0)=\xi'(2\pi)$ while k corresponds to the boundary conditions $\xi(0)=\xi(2\pi)=0$. Put $\varphi=\mathrm{Tr}\,(h\cdot)$ and $\psi=\mathrm{Tr}\,(k\cdot)$. Then $\varphi \leq \psi$ by Proposition 4.2, and $\varphi \neq \psi$. The set

$$\mathfrak{p} = \{x \in \mathcal{B}(\mathfrak{H})_+ \,|\, \varphi(x) = \psi(x) < \infty\}$$

is a hereditary cone in $\mathcal{B}(\mathfrak{H})_+$. If p is the projection on a finite dimensional subspace of $\mathfrak{H}$ with an orthonormal basis $\{\xi_k\}$ of C^∞-functions on $[0, 2\pi]$ vanishing at the points $\{0, 2\pi\}$ then

$$\varphi(p) = \mathrm{Tr}\,(hp) = \Sigma(h\xi_k\,|\,\xi_k) = \Sigma(k\xi_k\,|\,\xi_k) = \psi(p),$$

so that $p \in \mathfrak{p}$. Since 1 can be obtained as the strong limit of projections p it follows that $\mathfrak{p}$ is σ-weakly dense in $\mathcal{B}(\mathfrak{H})_+$. Q.E.D.

We shall finally use the Radon-Nikodym theorem to obtain information about unbounded operators on a Hilbert space $\mathfrak{H}$. With $\mathcal{M}=\mathcal{B}(\mathfrak{H})$ and Tr the usual trace on $\mathcal{M}$ we have a bijection between the set of self-adjoint positive operators in $\mathfrak{H}$ and normal semi-finite weights on $\mathcal{B}(\mathfrak{H})$. For semi-finite von Neumann algebras Murray–von Neumann and I. Segal have shown in [15] and [23] that the class of measurable operators affiliated with the algebra form a ring. If $\mathcal{M}=\mathcal{B}(\mathfrak{H})$, and $\mathfrak{H}$ is infinite dimensional then the only operators with essential dense domain are bounded, so that the Murray–von Neumann-Segal theory is not applicable (moreover, the densely defined closed operators on $\mathfrak{H}$ do not form a ring). However, for certain pairs of self-adjoint positive operators we can define a *strong sum*:

Let h and k be self-adjoint positive operators on $\mathfrak{H}$ and put $\varphi=\mathrm{Tr}\,(h\cdot)$ and $\psi=\mathrm{Tr}\,(k\cdot)$. If $\varphi+\psi$ is semi-finite then by Theorem 5.12 there is a unique self-adjoint positive operator, which we denote by $h\dot{+}k$, such that $\varphi+\psi=\mathrm{Tr}\,((h\dot{+}k)\cdot)$. It is clear from the definition that the strong sum is associative when it is defined, i.e. if $(h\dot{+}k)\dot{+}x$ is defined then $h\dot{+}(k\dot{+}x)$ is defined and $(h\dot{+}k)\dot{+}x=h\dot{+}(k\dot{+}x)$. Furthermore $h\dot{+}k=k\dot{+}h$ and $\alpha(h\dot{+}k)=\alpha h+\alpha k$ for $\alpha\geq 0$. If h and k are bounded then $h\dot{+}k=h+k$.

LEMMA 7.9. *Let $\{h_i\}$ be an increasing net of positive bounded operators on $\mathfrak{H}$. Then the following conditions are equivalent:*

(i) *There exists a self-adjoint positive operator h in $\mathfrak{H}$ such that $h_i \nearrow h$.*

(ii) *The set $\mathcal{D}=\{\xi\in\mathfrak{H}\,|\,\lim\,(h_i\xi\,|\,\xi)<\infty\}$ is a dense subspace of $\mathfrak{H}$.*

The operator h is unique and $\mathcal{D}=\mathcal{D}(h^{\frac{1}{2}})$.

Proof. (i)$\Rightarrow$(ii): As in § 4 define $h_\varepsilon=h(1+\varepsilon h)^{-1}$ for $\varepsilon>0$. By Lebesgue's monotone convergence theorem $\xi\in\mathcal{D}(h^{\frac{1}{2}})$ if and only if $\lim\,(h_\varepsilon\xi\,|\,\xi)<\infty$. Since $h_{i\varepsilon}\nearrow h_\varepsilon$ for each i it is immediate that $\mathcal{D}=\mathcal{D}(h^{\frac{1}{2}})$, and therefore $\mathcal{D}$ is dense in $\mathfrak{H}$.

(ii) $\Rightarrow$ (i): The inequality $\|h_i^{\frac{1}{2}}(\xi+\zeta)\|^2\leqslant 2(\|h_i^{\frac{1}{2}}\xi\|^2+\|h_i^{\frac{1}{2}}\zeta\|^2)$ shows that $\mathcal{D}$ is a vector space. Since $\{h_i\}$ is an increasing net there is a bounded operator k with $0\leqslant k\leqslant 1$ such that $(1+h_i)^{-1}\searrow k$. Suppose $k\zeta=0$ for some vector ζ in $\mathfrak{H}$. Then for each ξ in $\mathcal{D}$

$$\begin{aligned}|(\xi\,|\,\zeta)|^2&=|(\xi\,|\,(1+h_i)^{\frac{1}{2}}(1+h_i)^{-\frac{1}{2}}\zeta)|^2=|((1+h_i)^{\frac{1}{2}}\xi\,|\,(1+h_i)^{-\frac{1}{2}}\zeta)|^2\\&\leqslant((1+h_i)\xi\,|\,\xi)\,(1+h_i)^{-1}\zeta\,|\,\zeta)\to 0.\end{aligned}$$

Since $\mathcal{D}$ is dense, $\zeta=0$. Thus k is non-singular, and for each $\varepsilon>0$ we have

$$(1+h_i)_\varepsilon=(\varepsilon+(1+h_i)^{-1})^{-1}\nearrow(\varepsilon+k)^{-1}=(k^{-1})_\varepsilon,$$

so that $h_i\nearrow k^{-1}-1$. Q.E.D.

PROPOSITION 7.10. *Let h and k be self-adjoint positive operators in $\mathfrak{H}$.* Tr $(h\cdot)+$ Tr $(k\cdot)$ *is semi-finite if and only if the subspace $\mathcal{D}=\mathcal{D}(h^{\frac{1}{2}})\cap\mathcal{D}(k^{\frac{1}{2}})$ is dense in $\mathfrak{H}$; and in this case $\mathcal{D}=\mathcal{D}((h\dot{+}k)^{\frac{1}{2}})$ with*

$$\|h^{\frac{1}{2}}\xi\|^2+\|k^{\frac{1}{2}}\xi\|^2=\|(h\dot{+}k)^{\frac{1}{2}}\xi\|^2$$

for each ξ in $\mathcal{D}$.

Proof. If Tr $(h\cdot)+$ Tr $(k\cdot)$ is semi-finite then $h\leqslant h\dot{+}k$ and $k\leqslant h\dot{+}k$ so that $\mathcal{D}((h\dot{+}k)^{\frac{1}{2}})\subset\mathcal{D}(h^{\frac{1}{2}})\cap\mathcal{D}(k^{\frac{1}{2}})$. Thus $\mathcal{D}$ is dense in $\mathfrak{H}$.

Conversely, suppose that $\mathcal{D}$ is dense in $\mathfrak{H}$ and consider the increasing sequence $\{h_\varepsilon+k_\varepsilon\}$. We have $\lim\,((h_\varepsilon+k_\varepsilon)\xi\,|\,\xi)<\infty$ if and only if $\xi\in\mathcal{D}$. By Lemma 7.9 there is a self-adjoint positive operator x in $\mathfrak{H}$ such that $h_\varepsilon+k_\varepsilon\nearrow x$ and $\mathcal{D}(x^{\frac{1}{2}})=\mathcal{D}$. But then Tr $((h_\varepsilon+k_\varepsilon)\cdot)\nearrow$ Tr $(x\cdot)$ by Proposition 4.2; hence Tr $(x\cdot)=$ Tr $(h\cdot)+$ Tr $(k\cdot)$ and $x=h\dot{+}k$. For each ξ in $\mathcal{D}(=\mathcal{D}((h\dot{+}k)^{\frac{1}{2}}))$ we have

$$\|h^{\frac{1}{2}}\xi\|^2+\|k^{\frac{1}{2}}\xi\|=\lim\,((h_\varepsilon+k_\varepsilon)\xi\,|\,\xi)=\|(h\dot{+}k)^{\frac{1}{2}}\xi\|^2.\quad \text{Q.E.D.}$$

Instead of identifying a self-adjoint positive operator h on $\mathfrak{H}$ with its normal semi-finite weight Tr $(h\cdot)$ on $\mathcal{B}(\mathfrak{H})$ one often identifies the operator with its associated closed positive sesquilinear form $(h\cdot\,|\,\cdot)$ in $\mathfrak{H}$ (see [14; § 6.2]). The reader will have no difficulty in verifying that our definition of the strong sum $h\dot{+}k$ of two self-adjoint positive operators h and k gives the same result as the sum obtained by adding the two forms $(h\cdot\,|\,\cdot)$ and $(k\cdot\,|\,\cdot)$ as prescribed in [14; § 6.2.5].

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