# The rainbow connection of a graph is (at most) reciprocal to its minimum degree 

Michael Krivelevich * Raphael Yuster ${ }^{\dagger}$


#### Abstract

An edge-colored graph $G$ is rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph $G$, denoted by $r c(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow edgeconnected. We prove that if $G$ has $n$ vertices and minimum degree $\delta$ then $r c(G)<20 n / \delta$. This solves open problems from [5] and [3].

A vertex-colored graph $G$ is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph $G$, denoted by $\operatorname{rvc}(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. One cannot upper-bound one of these parameters in terms of the other. Nevertheless, we prove that if $G$ has $n$ vertices and minimum degree $\delta$ then $\operatorname{rvc}(G)<11 n / \delta$. We note that the proof in this case is different from the proof for the edgecolored case, and we cannot deduce one from the other.


## 1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the notation and terminology of [2]. The following interesting connectivity measure of a graph has recently attracted the attention of several researchers. An edge-colored graph $G$ is rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. Clearly, if a graph is rainbow edge-connected, then it is also connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow edge-connected; just color each edge with a distinct color. Thus, the following natural graph parameter was defined by Chartrand et al. in [4]. Let the rainbow connection of a connected graph $G$, denoted by $r c(G)$, be the smallest number of colors that are needed in order to make $G$ rainbow edge-connected.

An easy observation is that if $G$ has $n$ vertices then $r c(G) \leq n-1$, since one may color the edges of a given spanning tree with distinct colors (and leave the remaining edges uncolored). It

[^0]is easy to verify that $r c(G)=1$ if and only if $G$ is a clique, that $r c(G)=n-1$ if and only if $G$ is a tree, and that a cycle with $k>3$ vertices has rainbow connection $\lceil k / 2\rceil$. Also notice that, clearly, $\operatorname{rc}(G) \geq \operatorname{diam}(G)$ where $\operatorname{diam}(G)$ denotes the diameter of $G$. The parameter $r c(G)$ is monotone non-increasing in the sense that if we add an edge to a graph we cannot increase its rainbow connection.

Caro et al. [5] observed that $r c(G)$ can be bounded by a function of $\delta(G)$, the minimum degree of $G$. They have proved that if $\delta(G) \geq 3$ then $r c(G) \leq \alpha n$ where $\alpha<1$ is a constant and $n=|V(G)|$. They conjecture that $\alpha=3 / 4$ suffices and prove that $\alpha<5 / 6$ (a solution to this conjecture was recently announced by Zsolt Tuza). Clearly, we cannot obtain a similar result if we only assume that $\delta(G) \geq 2$. Just consider two vertex-disjoint triangles connected by a long path of length $n-5$. The diameter of this graph, as well as its rainbow connection, is $n-3$. More generally, it is proved in [5] that if $\delta=\delta(G)$ then $r c(G) \leq \frac{\ln \delta}{\delta} n\left(1+o_{\delta}(1)\right)$. An easier non-asymptotic bound $r c(G) \leq n \frac{4 \ln \delta+3}{\delta}$ is also proved there. They also construct an example of a graph $G$ with minimum degree $\delta$ for which $\operatorname{diam}(G)=\frac{3 n}{\delta+1}-\frac{\delta+7}{\delta+1}$. Naturally, they raise the open problem of determining the true behavior of $r c(G)$ as a function of $\delta(G)$. The lower bound construction suggests that the logarithmic factor in their upper bound may not be necessary and that, in fact $r c(G) \leq C n / \delta$ where $C$ is a universal constant.

If true, notice that for graphs with a linear minimum degree $\epsilon n$, this implies that $r c(G)$ is at most $C / \epsilon$. However, the result from [5] does not even guarantee the weaker claim that $r c(G)$ is a constant. This was proved recently by Chakraborty et al. in [3]. They prove that for every fixed $\epsilon>0$ there exists a constant $K=K(\epsilon)$ so that if $G$ is a connected graph with minimum degree at least $\epsilon n$ then $r c(G) \leq K$. We note that the constant $K=K(\epsilon)$ they obtain is a tower function in $1 / \epsilon$ and in particular extremely far from being reciprocal to $1 / \epsilon$.

Our main result in this paper determines the behavior of $r c(G)$ as a function of $\delta(G)$ and in particular resolves the above-mentioned open problem.

Theorem 1.1 $A$ connected graph $G$ with $n$ vertices has $r c(G)<20 n / \delta(G)$.
We note that the constant 20 obtained by our proof is not optimal and can be slightly improved with additional effort. However, by the construction from [5] one cannot expect to replace $C$ by a constant smaller than 3.

A vertex-colored graph $G$ is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph $G$, denoted by $\operatorname{rvc}(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. Obviously, we always have $\operatorname{rvc}(G) \leq n-2$ (except for the singleton graph), and $\operatorname{rvc}(G)=0$ if and only if $G$ is a clique. Also, clearly, $\operatorname{rvc}(G) \geq \operatorname{diam}(G)-1$ with equality if the diameter is 1 or 2 .

In some cases $\operatorname{rvc}(G)$ may be much smaller than $r c(G)$. For example, $r v c\left(K_{1, n-1}\right)=1$ while $r c\left(K_{1, n-1}\right)=n-1$. On the other hand, in some other cases, $r c(G)$ may be much smaller than
$\operatorname{rvc}(G)$. Take $n$ vertex-disjoint triangles and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph has $n$ cut-vertices and hence $\operatorname{rvc}(G) \geq n$. In fact, $\operatorname{rvc}(G)=n$ by coloring only the cut-vertices with distinct colors. On the other hand, it is not difficult to see that $r c(G) \leq 4$. Just color the edges of the $K_{n}$ with, say, color 1 , and color the edges of each triangle with the colors $2,3,4$. These examples show that there is no upper bound for one of the parameters in terms of the other. Nevertheless, we are able to prove a theorem analogous to Theorem 1.1 for the rainbow vertex-connected case.

Theorem 1.2 $A$ connected graph $G$ with $n$ vertices has $\operatorname{rvc}(G)<11 n / \delta(G)$.
In the next two sections prove Theorem 1.1 and Theorem 1.2, respectively.

## 2 Proof of Theorem 1.1

We start this section with several lemmas that are needed in order to establish Theorem 1.1. The first lemma is a simple consequence of Euler's Theorem.

Lemma 2.1 A graph with minimum degree $\delta$ has two edge-disjoint spanning subgraphs, each with minimum degree at least $\lfloor(\delta-1) / 2\rfloor$.

Proof: We can obviously assume that the graph is connected. As there are an even number of vertices with odd degree, we can add a matching to $G$ and obtain a (multi)graph $G^{\prime}$ which is Eulerian. By coloring the edges of an Eulerian cycle with alternating red and blue colors (starting, say, with a vertex $v$, with the color blue, and with a non-original edge incident with $v$ if there is such an edge) we obtain that for each vertex $u$ other than $v$, the number of red edges incident with $u$ is equal to the number of blue edges incident with $u$. At most one of these edges is not an original edge of $G$. For the vertex $v$, if the total number of edges of $G^{\prime}$ is odd we will have that the number of blue edges incident with $v$ is larger by two than the number of red edges incident with $v$. This difference of two is also at most the difference in $G$, since we started with a non-original edge incident with $v$ if there is such an edge.

A set of vertices $S$ of a graph $G$ is called a 2-step dominating set if every vertex of $V(G) \backslash S$ has either a neighbor in $S$ or a common neighbor with a vertex in $S$.

Lemma 2.2 If $H$ is a graph with $n$ vertices and minimum degree $k$, then $H$ has a 2 -step dominating set $S$ whose size is at most $n /(k+1)$.

Proof: Initialize $H_{0}=H, S=\emptyset$, and then for as long as $\Delta\left(H_{0}\right) \geq k$, take a vertex $v$ of degree at least $k$ in $H_{0}$, add it to $S$ and update $H_{0}$ by deleting $v$ and its neighbors from the vertex set of $H_{0}$. Observe that when the process has stopped each remaining vertex has lost in its degree and therefore has a neighbor in the set of deleted vertices. Since the latter is dominated by $S$, we have
that $S$ eventually dominates the whole of $V(G) \backslash S$ in two steps. Clearly the process lasted at most $n /(k+1)$ rounds.

Lemma 2.3 If $S$ is a 2-step dominating set of a connected graph $G$ then there is a set of vertices $S^{\prime} \supset S$ so that $G\left[S^{\prime}\right]$ is connected and $\left|S^{\prime}\right| \leq 5|S|-4$.

Proof: Let $c$ denote the number of connected components of $G[S]$. If $c=1$ we are done, as we may take $S^{\prime}=S$. Otherwise, consider a shortest path connecting two vertices in distinct components of $S$, say vertices $x$ and $y$. This path has at most four internal vertices, as if there were more then there would be a vertex on this path whose distance to any vertex of $S$ is at least 3, contradicting the fact that $S$ is a 2 -step dominating set. Thus, by adding four vertices to $S$ we can decrease the number of components. Henceforth, by adding at most $4(c-1)$ vertices to $S$ we obtain a set $S^{\prime} \supset S$ so that $G\left[S^{\prime}\right]$ is connected.
Proof of Theorem 1.1: Suppose that $G$ is a connected graph with $n$ vertices and minimum degree $\delta$. Set $k=\lfloor(\delta-1) / 2\rfloor$. We first apply Lemma 2.1 to obtain two edge-disjoint spanning subgraphs of $G$, denote $G_{1}$ and $G_{2}$, with $\delta\left(G_{i}\right) \geq k$. We next apply Lemma 2.2 to each of the $G_{i}$ to obtain a 2-step dominating set $S_{i}$ of $G_{i}$ with $\left|S_{i}\right| \leq n /(k+1)$ for $i=1,2$. Since $S_{i}$ is also a 2-step dominating set of $G$, and since $G$ is connected, we can apply Lemma 2.3 and obtain $S_{i}^{\prime} \supset S_{i}$ with $\left|S_{i}^{\prime}\right| \leq 5 n /(k+1)-4$ and so that $G\left[S_{i}^{\prime}\right]$ is connected. Now consider $S^{\prime}=S_{1}^{\prime} \cup S_{2}^{\prime}$. Notice that we have that either $G\left[S^{\prime}\right]$ is connected (this happens, for example, if $S_{1}^{\prime}$ and $S_{2}^{\prime}$ intersect) or else, using the fact that $S_{1}$ is a 2 -step dominating set, we can add at most one vertex to $S^{\prime}$ to obtain a set $S \supset S^{\prime}$ so that $G[S]$ is connected. In any case, we have constructed a set $S$ with $|S| \leq 10 n /(k+1)-7$ vertices so that $G[S]$ is connected and $S_{i} \subset S$ for $i=1,2$.

Let $T$ be a spanning tree of $S$. Let $W=V \backslash S$ and consider the following subsets of $W$. Let $D_{i} \subset W$ be the vertices of $W$ having a neighbor of $S_{i}$ for $i=1,2$ and notice that each vertex of $L_{i}=W \backslash D_{i}$ has a neighbor in $D_{i}$ for $i=1,2$. We color the edges of $G$ as follows. Each edge of $T$ receives a fresh distinct color. All edges between $S_{1}$ and $D_{1}$ belonging to $G_{1}$ receive the same fresh color. All edges between $D_{1}$ and $L_{1}$ belonging to $G_{1}$ receive the same fresh color. All edges between $S_{2}$ and $D_{2}$ belonging to $G_{2}$ receive the same fresh color. All edges between $D_{2}$ and $L_{2}$ belonging to $G_{2}$ receive the same fresh color. The remaining edges of $G$ may stay uncolored. The overall number of colors used is $|S|+3 \leq 10 n /(k+1)-4$.

It remains to show that the coloring makes $G$ rainbow edge-connected. Indeed, let $x, y$ be two vertices of $G$. If $x \in S$ and $y \in S$ then we can use the path in $T$ connecting them. Otherwise if $x \in S$ and $y \in D_{1} \cup D_{2}$ then we can use an edge of $G_{1}$ from $y$ to some vertex $z \in S$ and then the path in $T$ connecting $z$ and $x$. Otherwise if $x \in S$ then we must have $y \in L_{1} \cap L_{2}$ and hence we can use an edge of $G_{1}$ from $y$ to $z \in D_{1}$, an edge of $G_{1}$ from $z$ to $u \in S$ and the path from $u$ to $x$ in $T$. Otherwise, we may assume that both $x$ and $y$ are in $W$. If $x \in D_{1}$ and $y \in D_{2}$ then let $u \in S$ be a neighbor of $x$ so that $(x, u) \in E\left(G_{1}\right)$ and let $z \in S$ be a neighbor of $y$ so that $(y, z) \in E\left(G_{2}\right)$. These
two edges together with the path in $T$ connecting $u$ and $z$ form a rainbow path between $x$ and $y$. Otherwise if $x \in D_{1}$ and $y \in L_{2}$ then we can reduce to the last argument by adding another edge of $E\left(G_{2}\right)$ from $y$ to a vertex of $D_{2}$. Otherwise we can assume that both $x$ and $y$ are in $L_{1} \cap L_{2}$ and we can reduce to the last argument by adding another edge of $E\left(G_{1}\right)$ from $x$ to a vertex of $D_{1}$.

## 3 Proof of Theorem 1.2

The proof of Theorem 1.2 also requires us to find a relatively small 2-step dominating set. However, we need additional important requirement from it. We call a 2 -step dominating set $k$-strong if every vertex that is not dominated by it has at least $k$ neighbors that are dominated by it.

Lemma 3.1 If $H$ is a graph with $n$ vertices and minimum degree $\delta$, then $H$ has a $\delta / 2$-strong 2 -step dominating set $S$ whose size is at most $2 n /(\delta+2)$.

Proof: Initialize $H_{0}=H, S=\emptyset$, and then for as long as $\Delta\left(H_{0}\right) \geq \delta / 2$, take a vertex $v$ of degree at least $\delta / 2$ in $H_{0}$, add it to $S$ and update $H_{0}$ by deleting $v$ and its neighbors from the vertex set of $H_{0}$. Observe that when the process has stopped each remaining vertex has lost more than $\delta / 2$ in its degree and therefore has more than $\delta / 2$ neighbors in the set of deleted vertices. Clearly the process lasted at most $n /(\delta / 2+1)$ rounds.
Notice the obvious, but important fact: adding vertices to a 2 -step dominating set does not decrease its strength.

Lemma 3.2 If $G$ is a connected graph with minimum degree $\delta$ then it has a connected spanning subgraph with minimum degree $\delta$ and with less than $n(\delta+1 /(\delta+1))$ edges.

Proof: By deleting from $G$ edges that connect two vertices with degree greater than $\delta$ as long as there are any we obtain a spanning subgraph with minimum degree $\delta$ and less than $\delta n$ edges. Each connected component of this spanning subgraph has at least $\delta+1$ vertices. Thus, by adding back at most $n /(\delta+1)-1$ edges we can make it connected.

Proof of Theorem 1.2: The statement of the theorem is trivial for $\delta \leq 11$ so we assume that $\delta>11$. Suppose that $G$ is a connected graph with $n$ vertices and minimum degree $\delta$. By Lemma 3.2 we may assume that $G$ has less than $n(\delta+1 /(\delta+1))$ edges. We use Lemma 3.1 to construct a set $S$ which is a $\delta / 2$-strong 2 -step dominating set of size $|S| \leq 2 n /(\delta+2)$. From Lemma 2.3 we can add at most $4(|S|-1)$ vertices to $S$ and obtain $S^{\prime} \supset S$ so that $G\left[S^{\prime}\right]$ is connected and $S^{\prime}$ is also a $\delta / 2$-strong 2 -step dominating set. Observe that $\left|S^{\prime}\right| \leq 10 n / \delta-5$.

Let $W=V(G) \backslash S^{\prime}$ and consider the partition $W=D \cup L$ where $D$ is the set of vertices directly dominated by $S^{\prime}$ and $L$ is the set of vertices not dominated by $S^{\prime}$. Since $S^{\prime}$ is $\delta / 2$-strong, each $v \in L$ has at least $\delta / 2$ neighbors in $D$. We further partition $D$ into two parts $D_{1}$ and $D_{2}$ where $D_{1}$ are those vertices with at least $\delta(\delta+1)$ neighbors in $L$. Notice that $\left|D_{1}\right|<n / \delta$ since otherwise
$G$ would have had at least $n(\delta+1)$ edges, contradicting our assumption. We also partition $L$ into two parts $L_{1}$ and $L_{2}$ where $L_{1}$ are those vertices that have at least one neighbor in $D_{1}$.

We are now ready to describe our coloring. The vertices of $S \cup D_{1}$ are each colored with a distinct color. The vertices of $D_{2}$ are colored only with five fresh colors so that each vertex of $D_{2}$ chooses its color randomly and independently from all other vertices of $D_{2}$. The vertices of $L$ remain uncolored. The overall number of colors used is less than $11 n / \delta$.

It remains to show that, with positive probability, our coloring yields a rainbow vertex-connected graph. We first need to establish the following claim.

Claim 3.3 With positive probability, every vertex of $L_{2}$ has at least two neighbors in $D_{2}$ colored differently.

Consider a vertex $v \in L_{2}$. As it has no neighbor in $D_{1}$, it has at least $\delta / 2$ neighbors in $D_{2}$. Fix, therefore a set $X(v) \subset D_{2}$ of neighbors of $v$ with $|X(v)|=\lceil\delta / 2\rceil$. The probability of the bad event $B_{v}$ that all of the vertices of $X(v)$ receive the same color is $5^{-\lceil\delta / 2\rceil+1}$. As each vertex of $D_{2}$ has less than $\delta(\delta+1)$ neighbors in $L$ we have that the event $B_{v}$ is independent of all other events $B_{u}$ for $u \neq v$ but at most $(\delta(\delta+1)-1)\lceil\delta / 2\rceil$ of them. Since

$$
e \cdot 5^{-\lceil\delta / 2\rceil+1}((\delta(\delta+1)-1)\lceil\delta / 2\rceil+1)<1
$$

for all $\delta \geq 11$, we have by the Lovász Local Lemma (cf. [1]) that, with positive probability, none of the bad event $B_{u}$ hold.

Having proved the claim we can now fix a coloring of $D_{2}$ with five colors so that each vertex of $L_{2}$ has at least two neighbors in $D_{2}$ colored differently. We now show that this coloring, together with the coloring of $S^{\prime} \cup D_{1}$ with distinct colors, yields a rainbow vertex-connected graph. As $S^{\prime} \cup D_{1}$ is connected, and since each vertex of $D_{2}$ has a neighbor in $S^{\prime}$, we only need to show that pairs of vertices of $L$ have a rainbow path connecting them. Each $v \in L$ has (at least) two neighbors in $D$ colored differently. This is true for $v \in L_{2}$ as $v$ has two such neighbors already in $D_{2}$. This is also trivially true for $v \in L_{1}$ since the vertices of $D_{1}$ are colored distinctly, and with colors that are not one of the five colors used in $D_{2}$. Now let $u, v \in L$. Let $x \in D$ be a neighbor of $u$ and let $y \in D$ be a neighbor of $v$ whose color is different from the color of $x$. As there is a rainbow path from $x$ to $y$ whose internal vertices are only taken from $S^{\prime}$, the result follows.

## References

[1] N. Alon and J. H. Spencer, The Probabilistic Method, Second Edition, Wiley, New York, 2000.
[2] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics 184, Springer-Verlag 1998.
[3] S. Chakrborty, E. Fischer, A. Matsliah, and R. Yuster, Hardness and algorithms for rainbow connectivity, Proceedings of the $26^{\text {th }}$ International Symposium on Theoretical Aspects of Computer Science (STACS), Freiburg (2009), 243-254.
[4] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, Rainbow connection in graphs, Math. Bohem. 133 (2008) 85-98.
[5] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, On rainbow connection, Electronic Journal of Combinatorics 15 (2008), \#R57


[^0]:    *School of Mathematics, Tel Aviv University, Te; Aviv, Israel. Email: krivelev@post.tau.ac.il
    ${ }^{\dagger}$ Department of Mathematics, University of Haifa, Haifa 31905, Israel. Email: raphy@math.haifa.ac.il

