THE RAMSEY NUMBER FOR HYPERGRAPH CYCLES II.

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Abstract. Let $C_n^{(3)}$ denote the 3-uniform *tight cycle*, that is the hypergraph with vertices $v_1, \ldots, v_n$ and edges $v_1v_2v_3$, $v_2v_3v_4$, $\ldots$, $v_{n-1}v_nv_1$, $v_nv_1v_2$. We prove that the smallest integer $N = N(n)$ for which every red-blue coloring of the edges of the complete 3-uniform hypergraph with N vertices contains a monochromatic copy of $C_n^{(3)}$ is asymptotically equal to $4n/3$ if n is divisible by 3, and $2n$ otherwise. The proof uses the regularity lemma for hypergraphs of Frankl and Rödl.

1. Introduction

Given a k-uniform hypergraph H, $k \geq 2$, the *Ramsey number* $r(H)$ is the smallest integer N such that every red-blue coloring of the edges of the complete k-uniform hypergraph $K_N^{(k)}$ with N vertices yields a monochromatic copy of H. A classical result in graph Ramsey theory ([1, 2, 10]) states that for $k = 2$ and $n \geq 5$ the Ramsey number of the graph cycle C_n with n vertices is

$$r(C_n) = \begin{cases} \frac{3}{2}n - 1 & \text{if } n \text{ is even,} \\ 2n - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, the Ramsey numbers for graph cycles depend strongly on the parity of n.

In this paper we continue our study of Ramsey numbers for 3-uniform hypercycles, initiated in [5]. There are various definitions of a cycle in a 3-uniform hypergraph. Given a suitably labeled set of vertices $\{v_1, \ldots, v_n\}$, a *loose* cycle has the edge set $\{v_1v_2v_3, v_3v_4v_5, v_5v_6v_7, \ldots, v_{n-1}v_nv_1\}$, while the *tight* cycle, denoted henceforth by $C_n^{(3)}$, has the edge set

$$\{v_1v_2v_3, v_2v_3v_4, v_3v_4v_5, \ldots, v_{n-1}v_nv_1, v_nv_1v_2\}.$$

In [5] we proved that the Ramsey number for the n-vertex loose cycle, n even, is asymptotic to $5n/4$. (Note that loose cycles do not exist for n odd.)

Here an analogous problem is investigated for the tight cycles. So far, the only known value of the Ramsey number for a tight cycle is $r(C_4^{(3)}) = 13$ (see [8]). Asymptotically, it turns out that the Ramsey number for the tight cycle is larger than that for the loose cycle, and depends on whether n is divisible by 3. Thus in this respect, tight cycles behave more like graph cycles than loose cycles do. Our aim is to prove the following theorem.

Theorem 1.1.

(a) *For every integer $n \geq 1$ and $i = 0, 1, 2$,*

$$r(C_{3n+i}^{(3)}) \geq \begin{cases} 4n - 1 & \text{if } i = 0, \\ 6n + 2i - 1 & \text{if } i \neq 0. \end{cases}$$

(b) *Let $\eta > 0$ be given. Then for all sufficiently large n and $i = 0, 1, 2$,*

$$r(C^{(3)}_{3n+i}) \leq \begin{cases} (4+\eta)n & \text{if } i = 0, \\ (6+\eta)n & \text{if } i \neq 0. \end{cases}$$

In the next section we prove the lower bounds and outline the proofs of the upper bounds. Their complete proofs are deferred to Section 5.

2. Lower bounds and the outline of the main proof

Most of the work in proving Theorem 1.1 lies in the upper bounds. In this section, we begin by establishing the lower bounds (Theorem 1.1(a)), and then we sketch the main ideas needed for Theorem 1.1(b), which include a notion of connectedness for 3-uniform hypergraphs. Since all hypergraphs considered in this paper are 3-uniform, we will more concisely call them *hypergraphs.*

2.1. **Proof of lower bounds.** The first lower bound is based on relation between cycles and matchings. Let $M^{(3)}_n$ be a 3-uniform $3n$-vertex matching, that is, a hypergraph consisting of n disjoint edges. Observe that $C^{(3)}_{3n}$ contains $M^{(3)}_n$, and so $r(C^{(3)}_{3n}) \geq r(M^{(3)}_n)$.

Proof of Theorem 1.1*(a).* To prove that $r(C^{(3)}_{3n}) \geq 4n-1$, partition the vertex set of $K^{(3)}_{4n-2}$ into two parts, X and Y, where $|X| = 3n-1$, $|Y| = n-1$, and color all edges inside X red and all other edges blue. It is easily seen that this coloring contains no monochromatic $M^{(3)}_n$, and thus no monochromatic copy of $C^{(3)}_{3n}$. (Unlike in the case of graphs, the above extremal coloring is not unique. For another one, see Example 1 in Subsection 2.2.)

To prove that $r(C^{(3)}_{3n+i}) \geq 6n+2i-1$, $i = 1, 2$, partition the vertex set of $K^{(3)}_{6n+2i-2}$ into two parts, X and Y, where $|X| = |Y| = 3n+i-1$, and color red [blue] all edges with an odd [even] number of elements in X. An edge containing a vertex of X and a vertex of Y is called *crossing.*

Suppose that there is a red copy C of $C^{(3)}_{3n+i}$ in such a coloring. Since $|X| < 3n+i$, at least one edge of C is crossing. But then, by the definition of a tight cycle, every edge of C is crossing, that is, every edge of C contains one vertex of X and two of Y. This means that every third vertex of C belongs to X, which is impossible when $i \neq 0$. □

Note that the first construction in the above proof implies that $r(M^{(3)}_n) \geq 4n-1$, and so, in view of Theorem 1.1, $r(M^{(3)}_n)$ and $r(C^{(3)}_{3n})$ are asymptotically equal. In fact, it is easy to prove that $r(M^{(3)}_n) = 4n-1$.

2.2. **Paths, pseudo-paths and connectedness.** A *(tight) path* is a hypergraph with vertices $v_1, \ldots, v_{p+2}$ and edges $v_1v_2v_3$, $v_2v_3v_4$, …, $v_pv_{p+1}v_{p+2}$. The pairs (v_1, v_2) and (v_{p+2}, v_{p+1}) are called the endpairs of the path. (Note the reverse order of the latter pair which emphasizes the symmetry of the path.) The *length* of a path on $p+2$ vertices is equal to p, the number of edges.

A *pseudo-path* in a hypergraph H is a sequence $(e_1, \ldots, e_p)$ of not necessarily distinct edges of H such that $|e_i \cap e_{i+1}| = 2$ for each $i = 1, \ldots, p-1$. In particular, the edges of every path can be ordered (in two ways) to form a pseudo-path. If $(e_1, \ldots, e_p)$ is a pseudo-path in H then we say that e_1 and e_p are *connected in H by a pseudo-path.* Unlike for paths, this defines an equivalence relation and we call the equivalence classes the *components* of H.

A hypergraph H is *connected* if every two edges $e, f \in H$ are connected by a pseudo-path. Note that there are several ways to define connectedness in hypergraphs (cf. [5]), but in this paper we will always mean the one defined above. A sub-hypergraph H' of H is *externally connected* (in H) if every two edges $e, f \in H'$ are connected in H by a pseudo-path. In other words, there is a component C of H that contains H'.

Example 1. Consider a 3-uniform hypergraph with vertex set $V = X \cup Y$, $X, Y \neq \emptyset$, and a red-blue coloring where every edge with an odd intersection with X is colored red and all other edges are colored blue. Then, the red sub-hypergraph has two components, one consisting of all edges contained in X, the other formed by all edges with one vertex in X and two in Y.

Clearly, every red tight cycle must be entirely contained in one of these two components, a fact utilized already in the proof of Theorem 1.1(a), $i \neq 0$. Moreover, with $|X| = |Y| = 2n - 1$ this yields an alternative "extremal coloring" in the proof of Theorem 1.1(a), $i = 0$. Indeed, neither of the two red components contains a cycle of length $3n$. As a matter of fact, none of them contains an externally connected matching of size n.

2.3. **Monochromatic matchings in colorings of almost complete hypergraphs.** The basic idea of our proof, similar to that given by Łuczak [6] and Figaj and Łuczak [3] (see also [5]), is to apply to the colored complete (hyper)graph the regularity lemma, find in the cluster (hyper)graph a large structure of a certain type, and use this structure to obtain a long, monochromatic cycle.

Thus, a crucial role in the proof of Theorem 1.1(b) is played by the two following Ramsey-type results on externally connected matchings. We state them now, but their proofs are deferred to the end of the paper.

Lemma 2.1. *For every $\eta > 0$ there exist $\delta > 0$ and s_0 such that the following holds. Let K be a hypergraph with $t = (4+\eta)s$ vertices, $s \geq s_0$, and at least $(1-\delta)\binom{t}{3}$ edges. Then, for every red-blue coloring $K = K_{\mathrm{red}} \cup K_{\mathrm{blue}}$, either K_{red} or K_{blue} contains an externally connected matching $M_s^{(3)}$.*

The proof, given in Section 8, is so technically involved that, for the sake of the reader, it is preceded in Section 6 by its "idealized" version with $\eta = \delta = 0$. There we will prove that the Ramsey number $r(M_s^{(3)}) = 4s - 1$ does not increase when the matching is requested to be externally connected in one of the colors (cf. Theorem 6.1).

To deal with the case $i \neq 0$, we will need the following modification of Lemma 2.1.

Lemma 2.2. *For every $\eta > 0$ there exist $\delta > 0$ and s_0 such that the following holds. Let K be a hypergraph with $t = (6+\eta)s$ vertices, $s \geq s_0$, and at least $(1-\delta)\binom{t}{3}$ edges. Then, for every red-blue coloring $K = K_{\mathrm{red}} \cup K_{\mathrm{blue}}$, either K_{red} or K_{blue} contains an externally connected union of a matching $M_s^{(3)}$ and a cycle $C_4^{(3)}$ or $C_5^{(3)}$.*

Why does the size of the largest monochromatic, externally connected matching found in a red-blue colored K go down from $t/4$ to $t/6$, if it has to be accompanied by a copy of $C_4^{(3)}$ or $C_5^{(3)}$? The answer can be provided by the second construction in the proof of Theorem 1.1(a) (see Section 2.1). Indeed, that construction yields a coloring of $K_{6s+2i-2}$ without any externally connected, monochromatic copy of a vertex-disjoint union of $M_{s_1}^{(3)}$ and $C_{3s_2+i}^{(3)}$, $s = s_1 + s_2$, $i = 1, 2$. Although in Lemma 2.2 we do not assume that a copy of $C_4^{(3)}$ or $C_5^{(3)}$ has to be disjoint from the matching, it can be reduced to the disjoint case by disregarding at most five edges of the matching. This small loss does not affect the asymptotics of Lemma 2.2.

The proof of Lemma 2.2 is based on Lemma 2.1 and quite similar to its proof, but even more technical. The full version can be be found in the Appendix.

2.4. **Outline of the proof of upper bounds.** We first consider the case of $C_{3n}^{(3)}$. Let $K_N^{(3)} = H_{\mathrm{red}} \cup H_{\mathrm{blue}}$, where $N \sim 4n$, be a red-blue coloring of the edges of the complete 3-uniform hypergraph $K_N^{(3)}$.

We apply simultaneously, to both H_{red} and H_{blue}, the hypergraph regularity lemma (Theorem 3.2) with suitably chosen parameters, and obtain a vertex partition $V = V_1 \cup \ldots \cup V_t$, $|V_i| \sim N/t$, such that for almost all triples $\{i, j, k\}$ one of the induced sub-hypergraphs, $H_{\mathrm{red}}[V_i \cup V_j \cup V_k]$ or $H_{\mathrm{blue}}[V_i \cup V_j \cup V_k]$, is "well structured", that is, enjoys high regularity and large density (see Section 5 for details).

It will be proved in Section 4 that a "well structured" hypergraph contains a long path (Lemma 4.6), in our case of length almost $3N/t$. We will build a monochromatic copy of $C_{3n}^{(3)}$ mostly out of such paths, coming from about $t/4$ vertex disjoint "well-structured" hypergraphs. Thus, it is crucial to find about $t/4$ disjoint, but mutually connected, "well-structured" sub-hypergraphs in one color.

To this end, let $K_{\rm red}$ and $K_{\rm blue}$ be two auxiliary hypergraphs on the vertex set $\{1,2,\dots,t\}$, whose edges are those triples $\{i,j,k\}$ for which, respectively, $H_{\rm red}[V_i\cup V_j\cup V_k]$ or $H_{\rm blue}[V_i\cup V_j\cup V_k]$ contains a "well structured" sub-hypergraph. Set $K = K_{\rm red}\cup K_{\rm blue}$ and note that $|K|\sim\binom{t}{3}$. We call K *the cluster hypergraph* and the edges of K the cluster edges.

By Lemma 2.1 either $K_{\rm red}$ or $K_{\rm blue}$ (say, $K_{\rm red}$) contains an externally connected matching $M = M_s^{(3)}$ of size $s\sim t/4$. Next, using Lemma 4.6, we will find a long path in each sub-hypergraph $H_{\rm red}[V_i,V_j,V_k]$, where $\{i,j,k\}\in M$. These paths are disjoint and have total length of about $(t/4)\times(3N/t) = 3N/4\sim 3n$ (in fact, $3n-O(1)$).

To connect the long paths together into a red cycle of length $3n$, we will construct in $H_{\rm red}$ short paths (length $O(1)$) between the endpairs of long paths, being guided by the pseudo-paths linking in $K_{\rm red}$ the cluster edges of $M_s^{(3)}$ (in reality, we build the short paths first).

The case of $C_{3n+i}^{(3)}$, $i=1,2$, requires just one modification: in addition to an externally connected, monochromatic matching in K, we will need a copy of a cycle of length not divisible by three in the same color. This is provided by Lemma 2.2, which guarantees in either $K_{\rm red}$ or $K_{\rm blue}$ the existence of an externally connected sub-hypergraph which is a union of $M_s^{(3)}$, $s\sim t/6$, and a copy of either $C_4^{(3)}$ or $C_5^{(3)}$. Due to the presence of a cluster cycle of length not divisible by three we will be able to adjust the length of the final cycle to be equal one or two modulo three (by running once or twice around the cluster cycle – see Section 5 for more details).

In the next section we introduce the regularity of hypergraphs and present a corresponding regularity lemma. In Section 4 we prove the existence of paths of prescribed length in quasi-random hypergraphs (Lemma 4.6), one of the two main ingredients of the proof of Theorem 1.1(b). In Section 5 we put together the main proof, and, finally, in Sections 6-8 we provide the proofs of the second crucial ingredient, Lemmas 2.1 and 2.2.

3. Regularity of hypergraphs

In this section we describe the regularity lemma for hypergraphs established in [4], in a modified version presented in [9]. To do this we will need to refer to the notion of ϵ-regularity for graphs, the key idea in Szemerédi's Regularity Lemma [11].

3.1. **Graph regularity.** Let G be a bipartite graph with vertex classes X and Y and let $0\le d\le 1$. For $X'\subseteq X$ and $Y'\subseteq Y$, we write $E_G(X',Y')$ for the set of edges of G that have one end in X' and the other in Y'. The *density* $d_G(X',Y')$ of G over the pair (X',Y') is defined by

$$d_G(X',Y') = \frac{|E_G(X',Y')|}{|X'||Y'|}\,.$$

Let $\epsilon>0$. We say that G is (d,ϵ)-regular, if for all $X'\subseteq X$ and $Y'\subseteq Y$ with $|X'|\ge\epsilon|X|$ and $|Y'|\ge\epsilon|Y|$, we have

$$|d_G(X',Y')-d|<\epsilon\,.$$

We say that G is ϵ*-regular* if it is (d,ϵ)-regular with $d=d_G(X,Y)$.

3.2. **Hypergraph regularity.** We now turn to hypergraph regularity. A triple (P^{12},P^{13},P^{23}) of bipartite graphs with vertex sets $V_1\cup V_2$, $V_1\cup V_3$ and $V_2\cup V_3$, or equivalently, the 3-partite graph $P=P^{12}\cup P^{13}\cup P^{23}$ itself, will be referred to as a *triad*.

In what follows we often need to focus on the set of edges of a hypergraph H that are also vertex sets of triangles in a fixed triad P with $V(P)\subseteq V(H)$. We denote by $\mathrm{Tr}(P)$ the family of the vertex sets of the triangles in the graph P, and set $\mathrm{tr}(P)=|\,\mathrm{Tr}(P)|$. Thus for any P, $\mathrm{Tr}(P)$ is a 3-uniform hypergraph on the same vertex set as P. Moreover, $\mathrm{Tr}(P)$ is 3-partite in the sense that every edge intersects each set V_1, V_2 and V_3.

Further, we define the notion of the *density* of H with respect to P as

$$d_H(P) = \frac{|H \cap \mathrm{Tr}(P)|}{|\mathrm{Tr}(P)|}.$$

Similarly, for every r-tuple of triads $\vec{\mathcal{Q}} = (\mathcal{Q}(1), \mathcal{Q}(2), \dots, \mathcal{Q}(r))$, let

$$d_H(\vec{\mathcal{Q}}) = \frac{\left|H \cap \bigcup_{p=1}^r \mathrm{Tr}(\mathcal{Q}(p))\right|}{\left|\bigcup_{p=1}^r \mathrm{Tr}(\mathcal{Q}(p))\right|}.$$

Note that in the definition above, the sets of triangles $\mathrm{Tr}(\mathcal{Q}(p))$ need not be pairwise disjoint.

Next, we define the notion of regularity for 3-uniform hypergraphs. Given a triad $P = P^{12} \cup P^{13} \cup P^{23}$, by a *sub-triad* we mean a triad $Q = Q^{12} \cup Q^{13} \cup Q^{23}$ where

$$Q^{12} \subseteq P^{12}, \quad Q^{13} \subseteq P^{13}, \quad Q^{23} \subseteq P^{23}.$$

Definition 3.1. Let $\delta > 0$ and $\alpha > 0$, and let r be a positive integer. Further, let H be a 3-uniform hypergraph with $V(H) \supseteq V(P)$.

- We say that H is (α, δ, r)-*regular* with respect to a triad P if for every r-tuple of sub-triads $\vec{\mathcal{Q}} = (Q(1), Q(2), \dots, Q(r))$ satisfying $\left|\bigcup_{p=1}^{r} \mathrm{Tr}(\mathcal{Q}(p))\right| > \delta |\mathrm{Tr}(P)|$, we have $|d_H(\vec{\mathcal{Q}}) - \alpha| < \delta$.
- We say that H is (δ, r)-*regular* with respect to P if it is (α, δ, r)-regular with $\alpha = d_{\mathcal{H}}(P)$.
- A triad P with respect to which H is (δ, r)-regular will be called (δ, r)-*regular* Otherwise, it will be called (δ, r)-*irregular*.
- Moreover, if each graph P^{12}, P^{13}, P^{23} of an (α, δ, r)-regular triad $P = P^{12} \cup P^{13} \cup P^{23}$ is $(1/\ell, \epsilon)$-regular, then we call the pair (H, P) an $(\alpha, \delta, \ell, r, \epsilon)$-*regular complex*.

Observe that if H^c is the complement of H then $d_H(\vec{\mathcal{Q}}) = 1 - d_{H^c}(\vec{\mathcal{Q}})$. Consequently, if H is (α, δ, r)-regular, then H^c is $(1-\alpha, \delta, r)$-regular with respect to the same triad P.

3.3. **Regularity Lemma for Hypergraphs.** We now state the regularity lemma for 3-uniform hypergraphs from [4] in a simplified form presented in [9] (see Lemma 4.1 and Remark 4.1 there). We write $K(U, W)$ for the complete bipartite graph with vertex sets U and W.

Theorem 3.2 (Regularity Lemma for Hypergraphs)**.** *For every $\delta > 0$, every integer t_0, all integer-valued functions $r = r(t, \ell)$, and all decreasing sequences $\varepsilon(\ell) > 0$, there exist constants T_0, L_0 and N_0 such that every 3-uniform hypergraph H with at least N_0 vertices admits a partition Π consisting of an auxiliary vertex set partition $V(H) = V_0 \cup V_1 \cup \dots \cup V_t$, where $t_0 \le t < T_0$, $|V_0| < t$ and $|V_1| = |V_2| = \dots = |V_t|$, and, for each pair i, j, $1 \le i < j \le t$, a partition $K(V_i, V_j) = \bigcup_{a=1}^{\ell} P_a^{ij}$, where $1 \le \ell < L_0$, satisfying the following conditions:*

(i) *all graphs P_a^{ij} are $(1/\ell, \varepsilon(\ell))$-regular,*
(ii) *H is (δ, r)-regular with respect to all but at most $\delta \ell^3 t^3$ triads $(P_a^{hi}, P_b^{hj}, P_c^{ij})$.*

Note that the conclusions of Theorem 3.2 hold for the complement H^c of H as well.

Since the outcome of the regularity lemma may be overwhelming, we simplify the picture a little bit by selecting only one graph P_a^{ij} from each $K(V_i, V_j)$.

Claim 3.3. *Given the partition produced by Theorem 3.2, there exists a family $\mathcal{P}$ of bipartite graphs $P^{ij} = P_{a_{ij}}^{ij}$, one between each pair (V_i, V_j), where $1 \le i < j \le t$, such that H is (δ, r)-regular with respect to all but at most $2\delta t^3$ triads (P^{hi}, P^{hj}, P^{ij}).*

Proof. We apply the probabilistic method. For all $1 \le i < j \le t$, choose an index $a_{ij} \in \{1, 2, \dots, \ell\}$ independently and uniformly at random. The selected indices determine a (random) family $\mathcal{P}$ of $\binom{t}{2}$ bipartite graphs. By condition (ii) of Theorem 3.2, the expected number of (δ, r)-irregular triads of $\mathcal{P}$ is at most $\delta t^3 \ell^3 (1/\ell)^3 = \delta t^3$, and hence, by Markov's inequality, the probability that there are more than $2\delta t^3$ such triads is less than $1/2$. Thus, there exists a selection $\mathcal{P}$ with fewer than $2\delta t^3$ (δ, r)-irregular triads. $\square$

4. A long, long path

Our goal in this section is to find tight hyperpaths of given lengths connecting two designated edges of P in an $(\alpha, \delta, \ell, r, \varepsilon)$-complex (H, P), as defined in Definition 3.1. To distinguish the hypergraph edges from the graph edges, in this section the former will be called *hyperedges*. On the other hand, as in the whole paper, we will use the name "path" instead of "hyperpath".

4.1. Short paths.

Recall that a tight path of length m was defined as a hypergraph with vertices $\{v_1, \dots, v_{m+2}\}$ and the m hyperedges $v_1v_2v_3, \dots, v_mv_{m+1}v_{m+2}$. We call the (ordered) pairs (v_1, v_2) and (v_{m+2}, v_{m+1}) the *endpairs* of the path, while the vertices $v_3, \dots, v_m$ are called *internal vertices*. Two paths are said to be *internally disjoint* if they do not share any internal vertex.

Note that the endpairs of a 3-uniform path are ordered pairs of vertices. However, in a 3-partite 3-uniform hypergraph H on vertex set $V_1 \cup V_2 \cup V_3$, we may designate one cyclic orientation, say $V_1 \to V_2 \to V_3 \to V_1$, as canonical, and view the endpairs of paths as unordered pairs of vertices, or simply the edges of the underlying graph P. Then saying that *a path goes from e to f* is not ambiguous and means that the endpairs of the paths are the edges e and f directed by the canonical ordering. For example, let $e = ab$ and $f = cd$ be two edges, where $a, d \in V_1$, $b, c \in V_2$. Then, under the above canonical orientation, a path going from edge e to edge f is a path with the endpairs (a, b) and (c, d).

Definition 4.1. With the convention that ijk is the canonical cyclic orientation, we say that an ordered pair of edges (e, f), where $e \in P^{ij}$, is of type 1 if $f \in P^{jk}$, of type 2 if $f \in P^{ik}$, and of type 3 if $f \in P^{ij}$. We denote the type of (e, f) by type(e, f).

Thus, every path from e to f has some length m such that

$$m \equiv \text{type}(e, f) \pmod 3.$$

Definition 4.2. Let e_1, e_2 be two edges of P and x, y be two integers. We say that e_1 *reaches* e_2 *within H in x steps and in y ways* if there exist at least y internally disjoint paths in H of length x from e_1 to e_2.

Let

$$\gamma_0 = \frac{\alpha^4}{5000\ell^7}.$$

For an edge $e \in P$ we denote by $\text{Four}^+(e, H)$ the set of those edges of P, which *a*re reached from e within H in four steps and in $\gamma_0 n$ ways, and by $\text{Four}^-(e, H)$ the set of all edges of P which *r*each e within H in four steps and in $\gamma_0 n$ ways (see Figure 4.1). Owing to the canonical orientation in which all paths proceed, the sets $\text{Four}^+(e, H)$ and $\text{Four}^-(e, H)$ are contained in different subgraphs P^{ij}, and thus are disjoint.

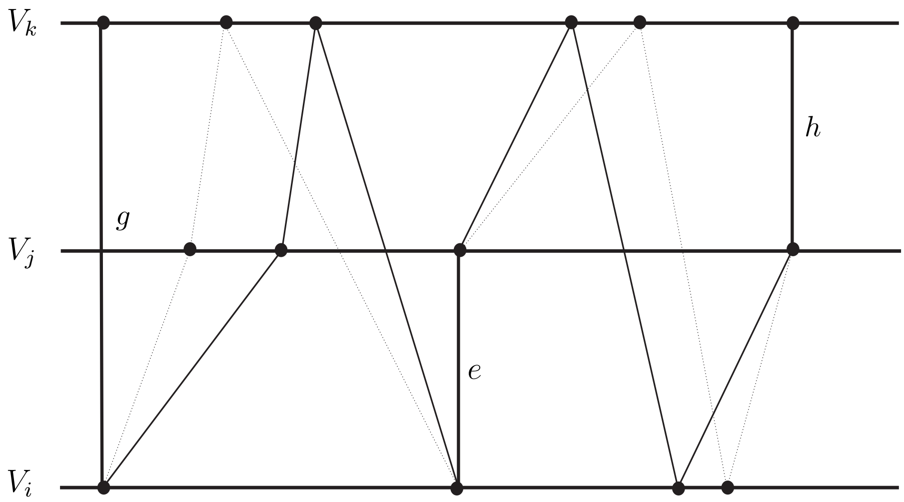

FIGURE 4.1. The fourth neighborhoods of e ($g \in \text{Four}^-(e,H)$, $h \in \text{Four}^+(e,H)$)

In [7] the following result is proved . For a subset $S \subset V(H)$ a path $Q \subset H$ is called *S-avoiding* if $V(Q) \cap S = \emptyset$. Given a graph G with $V(G) = V(H)$, we denote by $H - G$ the sub-hypergraph of H obtained by removing from H all hyperedges containing at least one edge of G. Finally, let

$$R_0 = \left\{ e \in P \ : \ \min\left\{|\text{Four}^+(e,H)|, |\text{Four}^-(e,H)|\right\} < \frac{\alpha^4}{2000} \times \frac{n^2}{\ell} \right\}.$$

Theorem 4.3 ([7]). *For each $\alpha \in (0,1)$ there exists $\delta > 0$ and sequences $r(\ell)$, $\varepsilon(\ell)$, and $n_0(\ell)$ such that for all integers $\ell \geq 1$ the following holds: if (H,P) is an $(\alpha,\delta,\ell,r(\ell),\varepsilon(\ell))$-complex with $|V_1| = |V_2| = |V_3| = n > n_0(\ell)$, then there is a subgraph P_0 of at most $27\sqrt{\delta}n^2/\ell$ edges of P such that*

(i) *for all $e \in P \setminus P_0$*

$$\min\left(|\text{Four}^+(e,H-P_0)|, |\text{Four}^-(e,H-P_0)|\right) \geq \left(\frac{\alpha^4}{2000}\right)\frac{n^2}{\ell},$$

and

(ii) *for every ordered pair of disjoint edges $(e,f) \in (P \setminus R_0) \times (P \setminus R_0)$, $e \cap f = \emptyset$, and for every set $S \subset V(H) \setminus (e \cup f)$ of size $|S| \leq n/\log n$, there is in H an S-avoiding path from e to f of length $9 + \text{type}(e,f)$.* □

Part (i) above is Lemma 4.2 in [7], while part (ii) is Theorem 3.4(ii) in [7] (see also Remark 4.3 there). Now we formulate a useful corollary of Theorem 4.3.

Corollary 4.4. *For each $\alpha \in (0,1)$ there exists $\delta > 0$ and sequences $r(\ell)$, $\varepsilon(\ell)$, and $n_0(\ell)$ such that for all integers $\ell \geq 1$ the following holds: if (H,P) is an $(\alpha,\delta,\ell,r(\ell),\varepsilon(\ell))$-complex with $|V_1| = |V_2| = |V_3| = n > n_0(\ell)$, then there is a subgraph P_0 of at most $27\sqrt{\delta}n^2/\ell$ edges of P such that*

(i) *for all $e \in P \setminus P_0$*

$$|\text{Four}^+(e,H)| \geq \left(\frac{\alpha^4}{2000}\right)\frac{n^2}{\ell},$$

and

(ii) *for every ordered pair of disjoint edges* $(e,f) \in (P\setminus P_0)\times(P\setminus P_0)$, $e\cap f=\emptyset$, *and for every set* $S\subset V(H)\setminus(e\cup f)$ *of size* $|S|\le n/\log n$, *there is in* H *an* S*-avoiding path from* e *to* f *of length* $9+\mathrm{type}(e,f)$. □

Proof. Part (i) follows trivially from Theorem 4.3(i), because $\mathrm{Four}^+(e,H)\supseteq \mathrm{Four}^+(e,H-P_0)$. To prove part (ii), observe that, by definition of R_0 and Theorem 4.3(i), we have $R_0\subseteq P_0$, and thus $(P\setminus R_0)\times(P\setminus R_0)\subseteq(P\setminus P_0)\times(P\setminus P_0)$. Hence, part (ii) follows from Theorem 4.3(ii). □

Let us conclude this subsection with an observation that, for a small decrease in the size of S, the path length in Corollary 4.4(ii) may be specified to be any integer from $\{10,\dots,17\}$.

Claim 4.5. *Under the assumptions of Corollary* 4.4, *for every ordered pair of disjoint edges* $(e,f)\in(P\setminus P_0)\times(P\setminus P_0)$, $e\cap f=\emptyset$, *for every set* $S\subset V(H)\setminus(e\cup f)$ *of size* $|S|\le n/\log n-12$, *and for each* $m\in\{10,\dots,17\}$, $m=\mathrm{type}(e,f)(\mathrm{mod}\ 3)$, *there is in* H *an* S*-avoiding path from* e *to* f *of length* m.

Proof. In view of Corollary 4.4(ii), we may assume that $m\ge 13$. In this case will apply Corollary 4.4(ii) twice. First we find in H an S-avoiding path Q_1 from e to f of length $m_0=10,11$, or 12, depending on the type of (e,f). Note that $m_0\equiv m(\mathrm{mod}\ 3)$, and thus $m-m_0$ is divisible by three.

Consider the initial segment Q_1' of Q_1 of length $m-m_0$, and call its other endpair e' (note that $\mathrm{type}(e',f)=\mathrm{type}(e,f)$). Now, find in H an $(S\cup V(Q_1')\setminus e')$-avoiding path Q_2 from e' to f of length m_0. Then, the concatenation $Q_1'+Q_2$ forms in H an S-avoiding path from e to f of length m. □

4.2. **Long paths.** It was shown in [7] that $(\alpha,\delta,\ell,r,\varepsilon)$-complexes contain long paths. Here we strengthen that result by showing that, in fact, most pairs of edges of the underlying graph P are connected in H by paths of any given, feasible length m, for a wide range of m.

Lemma 4.6. *For each* $\alpha\in(0,1)$ *there exists* $\delta>0$ *and sequences* $r(\ell)$, $\varepsilon(\ell)$, *and* $n_1(\ell)$ *with the following property: for all integers* $\ell\ge1$, *if* (H,P) *is a* $(d_H(P),\delta,\ell,r(\ell),\varepsilon(\ell))$*-complex with* $d_H(P)\ge\alpha$ *and* $|V_1|=|V_2|=|V_3|=n>n_1(\ell)$, *then there is a subgraph* P_0 *of at most* $27\sqrt{\delta}n^2/\ell$ *edges of* P *such that for all ordered pairs of disjoint edges* $(e,f)\in(P\setminus P_0)\times(P\setminus P_0)$, *for every set* $S\subset V(H)\setminus(e\cup f)$, $|S|<n/(\log n)^2$, *and for all integers* m *from the range*

$$10\le m\le(1-\delta^{1/4})(3n),$$

with $m=type(e,f)(\mathrm{mod}\ 3)$, *there is in* H *an* S*-avoiding path from* e *to* f *of length* m.

Proof. Note that unlike in Claim 4.5, here we need to construct a possibly very long path from e to f. This will be achieved by a repeated application of Corollary 4.4(i). There is a minor, but irritating difference, however, in the set-ups of Corollary 4.4 and Lemma 4.6: in the former, the hypergraph density was roughly equal to α, while now we have a hypergraph H satisfying $d_H(P)\ge\alpha$. To circumvent this technical obstacle, we consider a random sub-hypergraph $H_R\subset H$, where each hyperedge of H is present independently with probability $\alpha/d_H(P)$. By Chernoff's bound, the pair (H_R,P) is an $(\alpha,2\delta,\ell,r(\ell),\varepsilon(\ell))$-complex. Clearly, if H_R contains the desired path then so does H. By resetting $H:=H_R$ and $\delta:=\delta/2$, we thus reduce Lemma 4.6 to the instance when (H,P) is an $(\alpha,\delta,\ell,r(\ell),\varepsilon(\ell))$-complex.

Given α, let $\delta>0$ and the sequences $r(\ell)$, $\varepsilon_1(\ell)$, and $n_0(\ell)$ be such that Corollary 4.4 holds with $\delta'=4\delta^{\frac14}$ in place of δ, $r(\ell)$, $\varepsilon_1(\ell)$ in place of $\varepsilon(\ell)$, and $n_0(\ell)$. Set $\varepsilon(\ell)=\delta^{\frac14}\varepsilon_1(\ell)$. Assume also that

$$27\sqrt{4\delta^{\frac14}}<\frac{\alpha^4}{2000}\ . \tag{4.1}$$

We will prove Lemma 4.6 with the above choice of $\delta, r(\ell)$ and $\varepsilon(\ell)$, and with a choice of $n_1(\ell)\ge n_0(\ell)$ such that for all $\ell\ge1$ and $n\ge n_1(\ell)$ all inequalities encountered in the proof below hold true. Let (H,P) be an $(\alpha,\delta,\ell,r(\ell),\varepsilon(\ell))$-complex and $P_0=P_0(H)$ be given by Corollary 4.4, where $|V_1|=|V_2|=|V_3|=n>n_1=n_1(\ell)$. Let us fix an ordered pair of disjoint edges $(e,f)\in$

$(P\setminus P_0)\times(P\setminus P_0)$, and a set $S\subset V(H)\setminus(e\cup f)$, $|S|<n/(\log n)^2$. Finally, fix an integer m from the range $10\le m\le(1-\delta^{1/4})(3n)$, with $m\equiv \mathit{type}(e,f)(\mathrm{mod}\ 3)$.

Our goal is to show that there exists an S-avoiding path from e to f of length m. Without loss of generality, let us assume that type$(e,f)=3$, $e=ab\in P^{12}$ and $f=cd\in P^{12}$, where $a,d\in V_1$ and $b,c\in V_2$.

The plan is to first grow, by recursive application of Corollary 4.4(i), two disjoint S-avoiding paths Q_e and Q_f of equal length m', one from e, the other from f, until their total length $2m'$ reaches roughly m. Then, making sure that $10\le m-2m'\le 17$, we will use Claim 4.5 to connect the other endpairs of these two paths to form in H an S-avoiding path from e to f of length precisely m.

The two "parallel" paths will be grown recursively, in increments of four, using the property of the sets $\mathrm{Four}^+(e',H)$ and $\mathrm{Four}^+(f',H)$, where e' and f' will denote the current endpairs. Thus, we must take care to always choose the extending paths so that the new endpairs are outside the exceptional set P_0 of the current sub-hypergraph. To this end, at any given step of this procedure, we will have to consider two sub-hypergraphs defined as follows.

Given two disjoint paths, Q_e from e and Q_f from f, of equal length m', let $H'=H'(Q_e,Q_f)$ be the sub-hypergraph obtained from H by deleting all vertices of Q_e and Q_f, except for the last four from each path (if $m'\le 4$, we set $H'=H$). Further, let the sub-hypergraph $H''=H''(Q_e,Q_e)$ be obtained from H by deleting all vertices of Q_e and Q_f (no exceptions). Set also $P'=P[V(H')]$ and $P'=P[V(H'')]$. As long as

$$\big|V(Q_e)\cup V(Q_f)\big|=2m'<\big(1-\delta^{1/4}\big)(3n),$$

the hypergraphs H' and H'' have at least $\delta^{1/4}n$ vertices in each set V_i, $i=1,2,3$, and so, the pairs (H',P') and (H'',P'') are $(\alpha,4\delta^{\frac14},\ell,r,\varepsilon/\delta^{\frac14})$-complexes (see, e.g., [9], Fact 4.2). Let P_0' and P_0'' be the subgraphs of P' and P'', respectively, guaranteed by Corollary 4.4.

As a next step in the proof of Lemma 4.6, we show that two long paths can be grown from e and f. Their length m', due to the chosen method of construction, will be a multiple of four.

Fact 4.7. *For every $0\le m'<\frac12(1-\delta^{1/4})(3n)$, m' divisible by four, there exists in H a pair of disjoint S-avoiding paths Q_e and Q_f of length m', originating from e and f, respectively, and such that their other endpairs are not in P_0' .*

Proof. We proceed by induction on m'. There is nothing to prove for $m'=0$. Let Q_e and Q_f be a pair of disjoint S-avoiding paths, one from e and the other from f, of the same length $m'\ge 0$, m' divisible by four, and such that their other endpairs, e' and f', are not in P_0'. (If $m'=0$, we set $e'=e$ and $f'=f$.) We will now show how to extend Q_e and Q_f to a new pair of paths Q_e' and Q_f' of length $m'+4$, thus completing the inductive step. (The reader may be guided throughout by Figure 4.2.)

Noticing that $|V(H'')|<|V(H')|$ and $\varepsilon/\delta^{\frac14}=\varepsilon_1(\ell)$, by Corollary 4.4 applied to H'' we have

$$|P_0''|\le 27\sqrt{4\delta^{\frac14}}\,\frac{\lceil|V(H'')|/3\rceil^2}{\ell}<27\sqrt{4\delta^{\frac14}}\,\frac{\lfloor|V(H')|/3\rfloor^2}{\ell}\ . \tag{4.2}$$

On the other hand, by Corollary 4.4(i) applied to H' and by the fact that $e'\in P_0'$, we infer that the edge e' reaches in four steps at least

$$\frac{\alpha^4}{2000}\frac{\lfloor|V(H')|/3\rfloor^2}{\ell}$$

other edges of P'. Therefore, since $n>n_1$, by (4.2) and (4.1), e' reaches in four steps at least $|P_0''|+4n$ other edges of P', where the term $4n$ takes care of all edges adjacent to the two vertices of the set

$$T_e=V(H')\cap V(Q_e)\setminus e'.$$

Consequently, there is at least one edge $e'' \in P''\setminus P_0''$, reached from e' by at least $\gamma_0|V(H'')|$ internally disjoint paths in H' of length four. Thus, since $n > n_1$, at least one of them avoids $S \cup T_e$, and we may extend Q_e by four vertices, so that the new path Q_e' ends in $e'' \notin P_0''$.

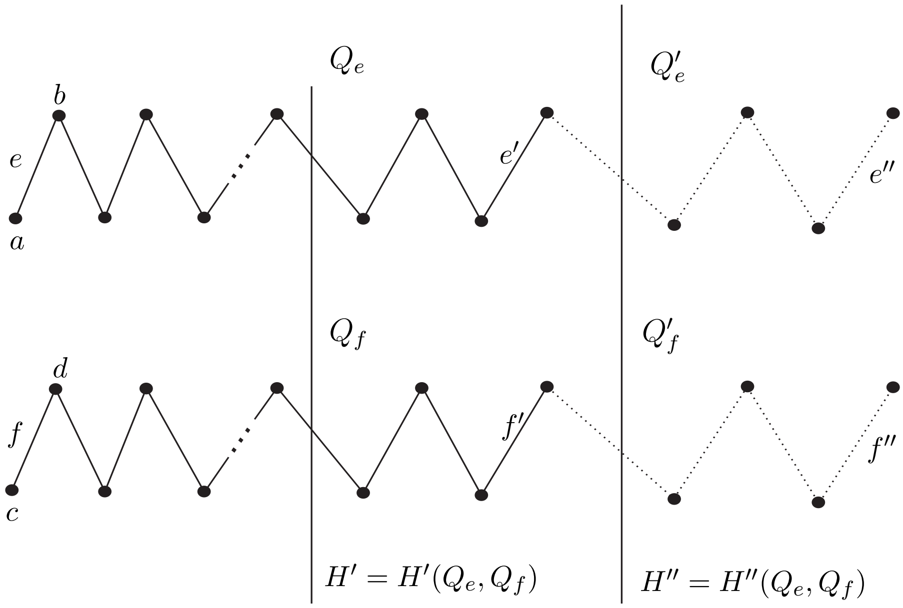

FIGURE 4.2. Growing hyperpaths from e and f (illustration to the proof of Lemma 4.6).

We now similarly extend Q_f by four vertices, so that the new path Q_f' is disjoint from Q_e', avoids S, and ends in $f'' \notin P_0''$. Since $H'' = H'(Q_e', Q_f')$, and so $P_0'' = P_0(H'(Q_e', Q_f'))$, the pair of paths (Q_e', Q_f') satisfies all conditions required in Fact 4.7. □.

Now comes the final, gluing part of the proof of Lemma 4.6. First, we have to choose the right length m' of the paths Q_e and Q_f guaranteed by Fact 4.7. Since their total length $2m'$ is divisible by eight, it is convenient to represent m in the form

$$m = 8k + h,$$

where $0 \le h \le 7$. Note that in view of Claim 4.5, there is nothing to prove when $k = 1$, or $k = 2$ and $h \le 1$. If $k \ge 2$ and $h \ge 2$, we need $m' = 4(k-1)$ because then $m - 2m' = 8 + h \in \{10, \ldots, 15\}$. Similarly, when $k \ge 3$ and $h \le 1$, we need $m' = 4(k-2)$ (this time $m - 2m' = 16$ or 17).

Let

$$T_f = V(H') \cap V(Q_f) \setminus f'.$$

We connect e' and f' by a path $Q_{e'f'}$ in H' of length precisely $m - 2m' \in \{10, \ldots, 17\}$, which avoids the set $S \cup T_e \cup T_f$. This follows from Claim 4.5 above. The concatenation $Q_e + Q_f + Q_{e'f'}$ forms in H an S-avoiding path from e to f of length m, as required. □

5. PROOF OF THEOREM 1.1(B)

In Sections 5.1-5.4 we prove Theorem 1.1(b) for $C_{3n}^{(3)}$ and then, in Section 5.5, we explain how to adjust the proof to obtain Theorem 1.1(b) in the remaining cases of $C_{3n+1}^{(3)}$ and $C_{3n+2}^{(3)}$.

5.1. **The choice of constants and the use of the regularity lemma.** Let $\eta > 0$ be given. Set $\alpha = 1/2$ and let δ', $r(\ell)$, $\epsilon(\ell)$, $n_1(\ell)$ be as guaranteed by Lemma 4.6. Let $\delta'' = \delta(\eta/2)$ and $s_0 = s_0(\eta/2)$ be given by Lemma 2.1. Envisioning an application of Theorem 3.2, we set

$$\delta = \min\left\{\frac{\delta'}{2}, \frac{\delta''}{40}\right\}, \tag{5.1a}$$

$$t_0 = \max\left\{\delta^{-100}, 5s_0\right\}, \tag{5.1b}$$

and

$$r(t,\ell) = r(\ell). \tag{5.1c}$$

Theorem 3.2 yields integers L_0, T_0, N_0 from which we derive

$$N_1 = \max\left\{2T_0 \max_{\ell \le L_0} n_1(\ell), N_0\right\}.$$

Now, for an arbitrary $n > \frac{1}{4}N_1$, consider a red-blue coloring $K_N^{(3)} = H_{\text{red}} \cup H_{\text{blue}}$, where $N = (4+\eta)n > N_1 \ge N_0$.

We apply the hypergraph regularity lemma (Theorem 3.2) with parameters given by (5.1a)-(5.1c) to H_{red} (and H_{blue}), yielding a partition Π satisfying conditions (i) and (ii) of Theorem 3.2. In particular, this determines the values of t and ℓ. Note that $|V_1| = |V_2| = \cdots = |V_t| > (N-T_0)/T_0 > n_1(\ell)$.

By Claim 3.3, setting $\varepsilon = \varepsilon(\ell)$, there exists a family $\mathcal{P}$ of $(1/\ell, \varepsilon)$-regular, bipartite graphs $P^{ij} = P^{ij}_{a_{ij}}$ between pairs (V_i, V_j), where $1 \le i < j \le t$, such that H_{red} (and, by complement, H_{blue}) is $(\delta, r(t,\ell))$-regular with respect to all but at most $2\delta t^3$ triads $P^{ijk} = P^{ij} \cup P^{jk} \cup P^{ik}$. Setting $r = r(t,\ell)$, we will more concisely call these triads (δ, r)-regular.

Note that if P^{ijk} is a (δ, r)-regular triad then

$$(H_{\text{red}}, P^{ijk}) \text{ is a } (d_{H_{\text{red}}}(P^{ijk}), \delta, \ell, r, \varepsilon)\text{-complex}$$

and

$$(H_{\text{blue}}, P^{ijk}) \text{ is a } (d_{H_{\text{blue}}}(P^{ijk}), \delta, \ell, r, \varepsilon)\text{-complex.}$$

Moreover, since

$$d_{H_{\text{red}}}(P^{ijk}) + d_{H_{\text{blue}}}(P^{ijk}) = 1, \tag{5.2}$$

either $d_{H_{\text{red}}}(P^{ijk}) \ge 1/2$ or $d_{H_{\text{blue}}}(P^{ijk}) \ge 1/2$. (This is what we meant in Section 2.4 by a "well structured" sub-hypergraph.)

5.2. **Finding a monochromatic pseudo-path in K.** We construct the cluster hypergraph K with the vertex set $\{1,\dots,t\}$, and the edge set consisting of all triples $\{i,j,k\}$ such that the triad P^{ijk} is (δ, r)-regular. Note that K contains at least

$$\binom{t}{3} - 2\delta t^3 > (1-\delta'')\binom{t}{3}$$

edges, where the inequality follows by (5.1a).

With the ultimate goal of finding a monochromatic cycle $C_n^{(3)}$, we first design a "big picture" route (as a pseudo-path in K) that the monochromatic cycle will eventually follow.

To this end, define a red-blue coloring $K = K_{\text{red}} \cup K_{\text{blue}}$ of the cluster hypergraph K, by including $\{i,j,k\} \in K_{\text{red}}$ if

$$d_{H_{\text{red}}}(P^{ijk}) \ge 1/2$$

and $\{i,j,k\} \in K_{\text{blue}}$ otherwise. By (5.2), this coloring is well defined.

By Lemma 2.1 with $\eta/2$ in place of η, there exists in K_{red}, say, a connected matching $M = \{h_1,\dots,h_s\}$ of size $s = t/(4+\eta/2)$. Let Q_i, $i = 1,\dots,s-1$, be a shortest pseudo-path in K_{red}

from h_i to h_{i+1}. Note that the edges of each Q_i are all distinct, and thus the length ℓ_i of Q_i satisfies the bound $\ell_i \le \binom{t}{3}$, which is independent of n.

Given two pseudo-paths P and Q, where the last edge of P coincides with the first edge of Q, $P+Q$ stands for the concatenation of P and Q. The pseudo-path

$$Q = Q_1 + \cdots + Q_{s-1} = (e_1, \ldots, e_p)$$

will serve as "a frame" for the long red cycle in H_{red}.

5.3. **Creating a short monochromatic cycle in H.** For every $i = 1, \ldots, p$, let $P^i = P^{e_i}$ be the triad corresponding to a cluster edge e_i. Recall that all these triads are (δ, r)-regular. Let $P_0^i \subset P^i$ be the subgraph of P^i (of prohibited edges) given by Lemma 4.6 applied to the complex (H_{red}, P^i), and, for $i = 1, \ldots, p-1$, set

$$B^i = (P^i \setminus P_0^i) \cap (P^{i+1} \setminus P_0^{i+1}).$$

Choose mutually distinct edges f_i, $g_i \in B^i$ for $1 \le i \le p-1$. The bound on $|P_0^i|$ from Lemma 4.6 ensures that for sufficiently large n this is possible.

In the next step of our construction, applying repeatedly Claim 4.5, we create a short cycle C in H_{red} of length divisible by 3. To this end, we connect by disjoint paths of length 10, 11, or 12, f_1 to f_2 to $f_3 \ldots$ to f_{p-1} to g_{p-1} and then, "backward", g_{p-1} to $g_{p-2} \ldots$ to g_1 to f_1.

For the passages from f_{p-1} to g_{p-1} and from g_1 to f_1, we choose the triads P^p and P^1, respectively, while for all $i = 1, \ldots, p-2$, the paths from f_i to f_{i+1} and from g_{i+1} to g_i use the triad P^{i+1}.

We have a choice of the direction around P^2 in which we connect f_1 to f_2, but then all other directions are determined. For the types to be well defined (cf. Definition 4.1), we need to designate one orientation around each triad as canonical. For convenience, we declare canonical the orientation consistent with the direction in which our paths proceed.

Note that for each $i = 1, \ldots, p-2$, the paths from f_i to f_{i+1} and from g_{i+1} to g_i go in the same, canonical by now, direction around P^{i+1}. Hence,

$$\text{type}(f_i, f_{i+1}) + \text{type}(g_{i+1}, g_i) = 1 + 2 = 0 (\text{mod } 3). \tag{5.3}$$

Since also

$$\text{type}(g_1, f_1) = \text{type}(f_{p-1}, g_{p-1}) = 0 (\text{mod } 3),$$

the obtained short cycle C has length divisible by 3.

To keep the paths disjoint, we apply Claim 4.5 with the set S collecting the vertices of the so far constructed paths. Since $|S| \le 12(2p) < n/\log n$, the assumptions on the size of S in Claim 4.5 are satisfied. For future reference, we denote by R_1 the just created short path from g_1 to f_1, by R_{i+1}, $i = 1, \ldots, p-2$, the paths from f_i to f_{i+1}, and by R_p the path from f_{p-1} to g_{p-1}.

5.4. **Creating a monochromatic cycle of length $3n$.** Preparing for the final step, let

$$I = \{1, \ell_1, \ell_1 + \ell_2 - 1, \ldots, p\}.$$

Observe that $|I| = s$ and that $M = \{h_1, \ldots, h_s\} = \{e_i : i \in I\}$.

To complete the proof, we replace the short paths R_i, $i \in I$, in C by disjoint, long paths with the same endpairs as the R_i's, which lie in the same triads (and thus, have the same length modulo 3 as the R_i's), in such a way that the total length of the obtained cycle is $3n$.

Specifically, let m' be the length of C, minus the sum of the lengths of all paths R_j with $j \in I$. Furthermore, for each $i \in I$, $i \neq p$, let

$$m_i = \left\lfloor \frac{3n - m'}{s} \right\rfloor + x_i,$$

where $x_i = 0, 1$, or 2, so that

$$m_i - \text{type}(f_i, f_{i+1}) = 0 (\text{mod } 3).$$

For each $i \in I$, $i \neq 1, p$, we apply Lemma 4.6 to the complex $(H_{\text{red}}, P^{i+1})$, with $e = f_i$, $f = f_{i+1}$, $S = V(C) \setminus (e \cup f)$ (note that $|S| = O(1)$), and with $m = m_i$. As a result, we obtain paths T_i

from f_i to f_{i+1} of length m_i, $i \in I$, $i \neq 1, p$, and, similarly, a path T_1 from f_1 to g_1 of length m_1. To achieve precisely the length $3n$ for the final cycle, we take a path T_p from f_{p-1} to g_{p-1} of length

$$m_p = 3n - \left(m' + \sum_{i \in I \setminus \{p\}} m_i \right).$$

This is possible, because for large n

$$10 \leq m_p \leq \frac{3n}{s} + O(1) \leq (1 - \delta^{1/4}) 3 \left\lfloor \frac{N}{t} \right\rfloor,$$

and Lemma 4.6 can again be applied. Since the edges of M are vertex-disjoint, the paths T_i do not interfere with each other.

5.5. **Adjustment to lengths $3n+1$ and $3n+2$.** In order to prove the second part of Theorem 1.1(b), we first choose the constants in the same way as in Section 5.1, then apply the hypergraph regularity lemma (Theorem 3.2) to the red-blue colored $K_{(6+\eta)n} = H_{\text{red}} \cup H_{\text{blue}}$, from which we obtain the cluster hypergraph K.

Next, we color the edges of K with red and blue as in Section 5.2 and then use Lemma 2.2 to find, say, in K_{red} a connected union of a matching $M = \{h_1, \ldots, h_s\}$ of size $s = t/(6 + \eta/2)$ and a copy D of $C_4^{(3)}$ or $C_5^{(3)}$. Below we consider only the case when $D = C_4^{(3)}$, leaving the other case to the reader.

We use the approach from Section 5.3 to obtain a red copy of $C_{3n+1}^{(3)}$ [or $C_{3n+2}^{(3)}$]. Let, as before, Q_i, $i = 1, \ldots, s-1$, be a shortest red pseudo-path from h_i to h_{i+1}, and, in addition, let Q_s be the shortest red pseudo-path from h_s to an edge of D. The pseudo-path

$$Q = Q_1 + \cdots + Q_s = (e_1, \ldots, e_p)$$

will now serve as a frame for the desired red cycle in H_{red}.

We define P^i, P_0^i, B^i and mutually distinct edges $f_i, g_i \in B^i$ for $1 \leq i \leq p-1$ as before. Relying on Claim 4.5, we construct first the short paths as before, except that now the path R_p from f_{p-1} to g_{p-1} has to be of length equal to 1 [or 2] modulo 3. To ensure this, we build R_p out of 4 pieces, one in each triad constituting D, each piece connecting a pair of edges of type 1 [or 2].

More specifically, let $V(D) = \{a, b, c, d\}$, where $e_p = \{a, b, c\}$ and $\{a, b\} \subset e_{p-1}$. Let us choose disjoint, typical (that is, not belonging to respective prohibited subgraphs P_0^{xyz}) edges from the intersections of consecutive triads: $f_{bc} \in P^{abc} \cap P^{bcd}$, $f_{cd} \in P^{bcd} \cap P^{cda}$, and $f_{da} \in P^{cda} \cap P^{dab}$.

By Claim 4.5, going around each triad alphabetically, there are internally disjoint paths of length 10, connecting f_{p-1} to f_{bc} to f_{cd} to f_{da} to g_{p-1}. This settles the case $i = 1$. For $i = 2$, we build paths of length 11, connecting f_{p-1} to f_{da} to f_{cd} to f_{bc} to g_{p-1}.

Finally, using Lemma 4.6, some s paths R_i, corresponding to the edges of M, are replaced by long paths T_i, in exactly the same way as in Section 5.4. Of course, we now adjust the length of the last path, so that the length of the resulting cycle is exactly $3n+1$ [or $3n+2$].

6. Matchings in components (idealized)

In this section we prove a version of Lemma 2.1 with $\eta = \delta = 0$. There are two reasons for doing this. Firstly, we exhibit here all essential ingredients of the real proof given in Section 8, not hidden under the burden of tedious estimations. Secondly, the result we present here is interesting in its own right, as dealing with a "connected" version of the classical Ramsey number $r(M_s^{(3)}) = 4s - 1$. It turns out that this Ramsey number is not affected by the additional restriction that the matching must be contained in a monochromatic component. Interestingly, besides the extremal coloring of $K_{4s-2}^{(3)}$ described in the proof of Theorem 1.1(a), which prevents any monochromatic matching of size s, there is another one which contains monochromatic matchings of size s, but not externally connected (see Example 1 in Section 2.2).

Theorem 6.1. *In every red-blue coloring of the complete* 3*-uniform hypergraph* $K^{(3)}_{4s-1} = K_{\rm red} \cup K_{\rm blue}$*, either* $K_{\rm red}$ *or* $K_{\rm blue}$ *contains an externally connected matching* $M^{(3)}_s$*.*

The connectedness and components of a hypergraph H were defined in Section 2.2. Denote by ∂H the set of all pairs xy for which there exists z such that $xyz \in H$ (∂H is usually referred to as the *shadow* of H). We find it convenient to view ∂H as both a graph and a set of pairs of the vertices of H. Observe that

$$\partial H' \cap \partial H'' = \emptyset \text{ for any two distinct components } H', H'' \text{ of } H. \tag{6.1}$$

In particular, any two edges of the same color (say red), sharing two vertices must be in the same red component.

Set $t = 4s-1$, $K = K^{(3)}_t$, $V = V(K)$, and consider an arbitrary red-blue coloring $K = K_{\rm red} \cup K_{\rm blue}$. Our goal is to find $M^{(3)}_s$ in some component of $K_{\rm red}$ or $K_{\rm blue}$. We start our proof with two observations.

Observation 6.2. For every $x \in V$ there exists a monochromatic component C such that $\{xy\colon y \in V \setminus \{x\}\} \subseteq \partial C$.

Proof. Let $K_{\rm red}(x) := \{yz\colon xyz \in K_{\rm red}\}$ and $K_{\rm blue}(x) := \{yz\colon xyz \in K_{\rm blue}\}$. Since every edge of K is colored by only one color, $K_{\rm blue}(x)$ is the complement of $K_{\rm red}(x)$, and consequently, one of these two graphs must be connected. Suppose that $K_{\rm red}(x)$ is connected. Then, for every two vertices $y, z \in V \setminus \{x\}$ there is a path $y = x_1, x_2, \dots, x_k = z$ in $K_{\rm red}(x)$ which corresponds to a red pseudo-path $e_1, e_2, \dots, e_{k-1}$, where $e_i = xx_ix_{i+1}$, $i = 1, \dots, k-1$. This pseudo-path connects xy with xz in $K_{\rm red}$, and hence, there is a red component C such that $xy, xz \in \partial C$. □

For each $x \in V$ let us choose arbitrarily one component satisfying the condition in Observation 6.2 and denote it by C_x. Let $V_{\rm red} = \{x \in V\colon C_x \text{ is red}\}$ and $V_{\rm blue} = \{x \in V\colon C_x \text{ is blue}\}$. Note that $V = V_{\rm red} \cup V_{\rm blue}$ and these two sets are disjoint.

Observation 6.3. If $V_{\rm red} \neq \emptyset$ ($V_{\rm blue} \neq \emptyset$, respectively), then there is a red component S (a blue component A) such that $C_x = S$ for every $x \in V_{\rm red}$ ($C_x = A$ for every $x \in V_{\rm blue}$).

Proof. This observation is trivial if $|V_{\rm red}| = 1$. Suppose $|V_{\rm red}| \geq 2$ and let $x, x' \in V_{\rm red}$. Then $xx' \in \partial C_x \cap \partial C_{x'}$, and, by (6.1), we have $C_x = C_{x'}$. □

Components A and S will play a special role, and we will refer to them as *azure* (A) and *scarlet* (S).

The next two claims form a mechanism to build an externally connected matching in one color given an externally connected matching of the same size in the other color (see Lemma 6.7). Clearly, the colors in their statements can be interchanged.

Claim 6.4. *Let* $X = \{x, y, z, a, b, c, d\} \subset V$ *be a set of seven vertices. Suppose that* xyz *is an edge of some red component* $C_{\rm red}$ *and* $ya, zb \in \partial C_{\rm blue}$ *for some blue component* $C_{\rm blue}$*. Then at least one of the following holds.*

1. *X contains two disjoint edges of $C_{\rm red}$,*
2. *there is an edge $e \subset X$ in $C_{\rm blue}$ such that $|e \cap \{a, b, c\}| = 1$ and $|e \cap \{x, y, z\}| = 2$,*
3. *X contains two disjoint edges of $C_{\rm blue}$.*

Proof. Suppose that neither (1) nor (2) holds. Then both $xya \in C_{\rm red}$ and $xzb \in C_{\rm red}$, and, consequently, $ya, zb \in \partial C_{\rm red}$. Thus, if zbc or yad were red, they would belong to $C_{\rm red}$. Since $xya \in C_{\rm red}$, this implies that the edge zbc has to be blue, and thus $zbc \in C_{\rm blue}$ (because $zb \in \partial C_{\rm blue}$). Similarly, since $xzb \in C_{\rm red}$, the edge yad has to be blue, and thus $yad \in C_{\rm blue}$ (because $ya \in \partial C_{\rm blue}$), yielding (3). □

Claim 6.5. *Let $X = \{u, v, w, x, y, z, a, b, c\} \subset V$ be a set of nine vertices. Suppose that uvw and xyz are edges of some red component C_{red} and $ya, zb, vb, wc \in \partial C_{\text{blue}}$ for some blue component C_{blue}. Then at least one of the following holds.*

1. *X contains three disjoint edges of C_{red},*
2. *there is an edge e in C_{blue} such that $|e \cap \{a, b, c\}| = 1$ and either $|e \cap \{x, y, z\}| = 2$, or $|e \cap \{u, v, w\}| = 2$,*
3. *there are two disjoint edges e_1, $e_2 \in C_{\text{blue}}$ such that for $i = 1, 2$, the edge e_i intersects each of $\{x, y, z\}$, $\{u, v, w\}$, and $\{a, b, c\}$ in one vertex.*

Proof. If (2) does not hold, then the edges xzb, vwc, uwc and yza are all red (because $ya, zb, wc \in \partial C_{\text{blue}}$), and thus in C_{red} (because $xyz, uvw \in C_{\text{red}}$). Consider the edges yua and xvb. If either of them is red, then it has to be in C_{red} (because $ya, xb \in \partial C_{\text{red}}$), yielding (1), as xzb, vwc, and yua are disjoint and in C_{red}, and so are uwc, yza and xvb. If both yua and xvb are blue, then they belong to C_{blue} (because $ya, vb \in \partial C_{\text{blue}}$). Hence (3) holds. □

Remark 6.6. Note that for the proofs of Claims 6.4 and 6.5 it is not essential that K is a complete hypergraph. In the case of Claim 6.4, we just need to assume that all triples of vertices within X, intersecting simultaneously $\{x, y, z\}$ and $\{a, b, c, d\}$, are edges of K. In the case of Claim 6.5, all triples of vertices within X, having two vertices in $\{u, v, w, x, y, z\}$ and one in $\{a, b, c\}$, must be edges of K. This observation will be used in the full proof of Lemma 2.1 in Section 8.

Our last preliminary result relies heavily on the two previous claims. Essentially, it says that given a maximal matching in a red component, one can construct a matching in a blue component of roughly the same size.

Lemma 6.7 (The Mirror Lemma)**.** *Let M be a largest matching in a red component C_{red} and let P be a set of at least $|M| + 3$ vertices outside M. Assume further that for some blue component C_{blue} and for every $e \in M$, the bipartite induced subgraph $\partial C_{\text{blue}}[e, P]$ of ∂C_{blue} contains $K_{2,|P|-1}$. Moreover, setting $G = \partial C_{\text{blue}}[V(M), P]$, let J be an arbitrary, non-empty subset of P such that*

$$J \supseteq \big\{v \in P : \deg_G(v) < |V(M)|\big\}.$$

Then there exists a matching $M' \subset C_{\text{blue}}$ such that either

(i) $|M'| = |M|$,
(ii) $|V(M') \cap P| \le |M|$, *and*
(iii) $(P \setminus V(M')) \cap J \neq \emptyset$,

or

(iv) $|M'| = |M| + 1$, *and*
(v) $|V(M') \cap P| \le |M| + 3$.

Proof. Let $M'' \subset C_{\text{blue}}$ be a largest matching such that

(6.2)
- $|V(M'') \cap P| \le |M''|$,
- $V(M'')$ intersects at most $|M''|$ edges of M,
- $(P \setminus V(M'')) \cap J \neq \emptyset$.

We claim that $|M''| \ge |M| - 1$. Indeed, suppose $|M''| \le |M| - 2$. It follows that there exist $e_1, e_2 \in M$ so that $(e_1 \cup e_2) \cap V(M'') = \emptyset$. Set $P'' = P \setminus V(M'')$. Since

$$|P''| = |P| - |P \cap V(M'')| \ge |M| + 3 - (|M| - 2) = 5,$$

one can choose $a, b, c \in P''$ so that $\partial C_{\text{blue}}[e_i, \{a, b, c\}] \supset K_{2,3}$ for $i = 1, 2$, and $(P'' \setminus \{a, b, c\}) \cap J \neq \emptyset$. This is always possible, because at most one vertex of P can be excluded for each e_1 and e_2, and these excluded vertices have to belong to J. (If no vertex is excluded then we can simply choose a, b, and c so that a vertex of J remains in $P'' \setminus \{a, b, c\}$.)

Claim 6.5, applied to $X = e_1 \cup e_2 \cup \{a,b,c\}$, implies that we can either enlarge M in $C_{\rm red}$ (if (1) of Claim 6.5 occurs) or M'' in $C_{\rm blue}$ with conditions (6.2) preserved (if (2) or (3) of Claim 6.5 occurs), yielding a contradiction with the choice of M or M'', respectively.

Hence $|M''| \geq |M| - 1$. If $|M''| \geq |M|$, we are done. Otherwise, let $xyz \in M$ be such that $\{x,y,z\} \cap V(M'') = \emptyset$. Since

$$|P''| \geq |M| + 3 - (|M| - 1) = 4,$$

one can choose $a,b,c \in P''$ so that $\partial C_{\rm blue}[e,\{a,b,c\}] \supset K_{2,3}$ and $(P'' \setminus \{a,b,c\}) \cap J \neq \emptyset$. We apply Claim 6.4 to the set $X = \{x,y,z,\ a,b,c,d\}$, where $d \in P'' \setminus \{a,b,c\}$ is arbitrary. By the maximality of M in $C_{\rm red}$, (1) cannot hold. If (2) holds, we enlarge M'' by adding the edge e, obtaining a matching M' satisfying conditions (i), (ii), and (iii). If conclusion (3) holds, we enlarge M'' by adding two disjoint edges, obtaining a matching M' satisfying conditions (iv) and (v). □

Proof of Theorem 6.1. Let M be a largest matching among all matchings contained in S or A. Without loss of generality we assume that $\emptyset \neq M \subset S$. This implies that $V_{\rm red} \neq \emptyset$, but $V_{\rm blue}$ might be empty. Suppose that

$$1 \leq m = |M| \leq s - 1 \tag{6.3}$$

and set

$$R = V_{\rm red} \setminus V(M) \text{ and } B = V_{\rm blue} \setminus V(M). \tag{6.4}$$

Note that $R \cap B = \emptyset$,

$$t = 4s - 1 = 3m + |R \cup B|, \tag{6.5}$$

and consequently, using also (6.3),

$$|R \cup B| = 4s - 1 - 3m \geq s + 2 \geq m + 3 \geq 4. \tag{6.6}$$

Observation 6.8. All edges in $R \cup B$ with at least one vertex in R are blue, and therefore in the same blue component $C_{\rm blue}$. Furthermore, if $B \neq \emptyset$, then $C_{\rm blue} = A$.

Proof. Note that any red edge with at least one vertex in R is in the scarlet component S and, if disjoint from $V(M)$, could be used to enlarge M. Hence, all edges from the set $T = \{e \subset R \cup B : e \cap R \neq \emptyset\}$ must be blue. Moreover, every pair of edges from T is connected by a pseudo-path in T, and thus, they all belong to the same blue component. The second part follows because any blue edge containing a vertex from $V_{\rm blue}$ also contains a pair from A (see Observation 6.3). □

For the rest of the proof we distinguish three cases. In each of them, the Mirror Lemma plays a central role. However, we need its technical conclusion (iii) only in the third case.

Case 1: $B = \emptyset$

In this case, $R \cup V(M) = V$ and thus $|R| = t - 3m \geq m + 3 \geq 4$. Denote by $C_{\rm blue}$ the blue component guaranteed by Observation 6.8.

Observation 6.9. For every edge $e \in M$, the bipartite induced subgraph $\partial C_{\rm blue}[e,R]$ of $\partial C_{\rm blue}$ contains $K_{2,|R|-1}$ as a subgraph.

Proof. Suppose there is an edge $xyz \in M$ and two vertices $a,b \in R$ such that xa and $yb \notin \partial C_{\rm blue}$. Let $c,d \in R \setminus \{a,b\}$ (recall that $|R| \geq 4$). Note that, by Observation 6.8, $ac, bd \in \partial C_{\rm blue}$, and thus edges xac and ybd must be red.

Since $ax, by \in \partial S$, we have that $xac, ybd \in S$. Consequently, $(M \setminus \{xyz\}) \cup \{xac, ybd\}$ is a red matching in S larger than M – a contradiction. □

Now we apply Lemma 6.7 with $P = R$ (recall that $|R| \geq m+3$), obtaining a matching $M' \subset C_{\text{blue}}$ either of size m and with $|V(M') \cap R| \leq m$, or of size $m+1$ and with $|V(M') \cap R| \leq m+3$. Note that by (6.3)

$$|R \setminus V(M')| \geq \begin{cases} 4s - 1 - 3m - m \geq 3(s-m) \text{ in the former case,} \\ 4s - 1 - 3m - (m+3) \geq 3(s-m-1) \text{ in the latter case.} \end{cases}$$

This allows us in either case to enlarge M' to size s. Indeed, since all edges contained in R are in C_{blue} (cf. Observation 6.8), we can greedily find $s-m$ or $s-m-1$, respectively, disjoint edges from C_{blue} and add them to M'.

Case 2: $R = \emptyset$

In this case, $B \cup V(M) = V$ and thus $|B| = t - 3m \geq m + 3 \geq 4$. Since $B \neq \emptyset$, the azure component component A exists. Furthermore, by the definition of V_{blue} and (6.4), we know that for every $e \in M$ the graph $\partial A[e, B]$ is the complete bipartite graph. Thus, by the Mirror Lemma applied with $P = B$, we obtain a matching $M' \subset A$ of size $|M'| = m$ and such that $|V(M') \cap B| \leq m$. (A matching of size $m+1$ in the azure component A is impossible by our choice of M.)

Note that $|B \setminus V(M')| \geq 4s - 1 - 3m - m \geq 3$. We claim that $R' := V_{\text{red}} \setminus V(M') = \emptyset$. Indeed, suppose that $R' \neq \emptyset$. Take any three vertices $a, b, c \in B \setminus V(M')$ and $d \in R'$ (observe that $d \notin B \setminus V(M')$ because $R' \subset V_{\text{red}}$ in this case). Since $ab \in \partial A$ (because $a \in V_{\text{blue}}$), both abc and abd are red (otherwise we could enlarge M' to size $m+1$). But $ad \in \partial S$ (because $d \in V_{\text{red}}$), therefore $abd \in S$ and, consequently, $abc \in S$. Since $\{a, b, c\} \cap V(M) = \emptyset$, we can enlarge M, which is a contradiction.

Thus $R' = \emptyset$ and we are back in Case 1 with the colors red and blue interchanged and M replaced by M'.

Case 3: $|B|, |R| \geq 1$

Set $P = R \cup B$ and note that, by (6.6), we have $|P| \geq m + 3$. Since $B \neq \emptyset$, the blue component guaranteed by Observation 6.8 is $C_{\text{blue}} = A$. In particular, for all pairs of vertices $a, b \in P$ we have $ab \in \partial A$.

Observation 6.10. For every $e \in M$, the bipartite induced subgraph $\partial A[e, P]$ of ∂A contains $K_{2,|P|-1}$ as a subgraph.

Proof. The proof follows the lines of the proof of Observation 6.9. Suppose there is an edge $xyz \in M$ and two vertices $a, b \in P$ such that $xa, yb \notin \partial A$. Note that, in fact, $a, b \in R$ because $\partial A[e, B]$ is the complete bipartite graph. Recall that $|P| \geq 4$ by (6.6), and choose arbitrarily $c, d \in P \setminus \{a, b\}$. Since $ac, bd \in \partial A$, edges xac and ybd must be red.

On the other hand, by the definition of V_{red}, we also have $ax, by \in \partial S$, so $xac, ybd \in S$. Hence, $(M \setminus \{xyz\}) \cup \{xac, ybd\}$ is a red matching in S larger than M – a contradiction. □

We apply the Mirror Lemma with $C_{\text{red}} = S$, $C_{\text{blue}} = A$, $P = R \cup B$, and $J = R$. Let M' be a matching in A satisfying conclusions (i)-(iii) (again, option (iv)-(v) is excluded by the choice of M). We have

$$|P \setminus V(M')| \geq 4s - 1 - 3m - m \geq 3.$$

By conclusion (iii), we can choose $a, b, c \in P \setminus V(M')$ so that $c \in R$. Hence, the pair $ac \in \partial A \cap \partial S$, and consequently, $abc \in S$ if it is red and $abc \in A$ if it is blue. Also $\{a, b, c\}$ is disjoint from both $V(M)$ and $V(M')$. Thus, either we obtain a matching of size $|M| + 1$ in S, or a matching of size $|M'| + 1 = |M| + 1$ in A, contradicting the maximality of M among all matchings contained in S or A. □

7. Matchings and short cycles in components (idealized)

In this section we prove a version of Lemma 2.2 with $\delta = 0$, and with the term ηs replaced by $\Omega(\sqrt{s})$. The main reason for doing this is, similarly to the previous section, to show the ideas of the proof clearly and without tiring calculations. A complete proof of Lemma 2.2 is not included in this paper, but can be found in the Appendix.

Theorem 7.1. *There exists* c_0 *such that the following holds. Let* $s \geq c_0^2$ *and let* K *be the complete 3-uniform hypergraph with* $t \geq 6s + c_0\sqrt{s}$ *vertices. Then, for every red-blue coloring* $K = K_{\text{red}} \cup K_{\text{blue}}$*, either* K_{red} *or* K_{blue} *contains an externally connected union of a matching* $M_s^{(3)}$ *and a cycle* $C_4^{(3)}$ *or* $C_5^{(3)}$.

Please note that the above theorem determines only the asymptotic value of the Ramsey number for a connected union of a matching $M_s^{(3)}$ of size s and a copy of $C_4^{(3)}$ or $C_5^{(3)}$ (we do not require them to be disjoint). At this point we do not know whether the lower bound of $6s + 2i - 1$ given in Sections 2.1 and 2.3 is optimal.

Proof. Let $c_0 = 25\sqrt{7}$, $s \geq c_0^2$, and let K be the complete 3-uniform hypergraph with $t = 6s + c_0\sqrt{s}$ vertices. For simplicity, we assume that $6s + c_0\sqrt{s}$ is an integer and note that $t \leq 7s$. Suppose that for an arbitrary red-blue coloring $K = K_{\text{red}} \cup K_{\text{blue}}$

$$\text{no monochromatic component contains } M_s^{(3)} \text{ and } C_4^{(3)} \text{ or } C_5^{(3)}. \tag{7.1}$$

Recall that the sets V_{red} and V_{blue}, and the scarlet component S and the azure component A were defined in Section 6. We distinguish two complementary cases, and in each of them we obtain a contradiction to (7.1) or its consequence, (7.2) below. In each case we use the fact that $r(C_4^{(3)}) = 13$ (see [8]).

Case 1: $|V_{\text{red}}|, |V_{\text{blue}}| \geq s$.

In this case we are able to prove Theorem 7.1 even with $t = 6s - 1$ and $s \geq 37$. We first prove that each of S and A contains a matching $M_s^{(3)}$.

Observation 7.2. $M_s^{(3)} \subset A$ and $M_s^{(3)} \subset S$.

Proof. Partition the set of vertices $V(K) := V$ into sets V', V'_{red}, V'_{blue} such that $V'_{\text{red}} \subset V_{\text{red}}$, $|V'_{\text{red}}| = s$, $V'_{\text{blue}} \subset V_{\text{blue}}$, $|V'_{\text{blue}}| = s$, and $V' = V \setminus (V'_{\text{red}} \cup V'_{\text{blue}})$.

Since $|V'| \geq 6s - 1 - 2s \geq 4s - 1$, Theorem 6.1 applied to the induced red-blue coloring $K_{\text{red}}[V'] \cup K_{\text{blue}}[V']$ of $K[V']$ implies that there exists a matching $M = M_s^{(3)}$ in a component (say red) C_{red} of K_{red}. (This is true because each component of any sub-hypergraph of K_{red} is contained in some component of K_{red}.)

By (7.1) we know that $C_4^{(3)} \not\subset C_{\text{red}}$. Consequently, for each edge $xyz \in M$ and any vertex $a \in V'_{\text{blue}}$, at least one of the edges xya, xza, yza must be blue and also in A, since $a \in V'_{\text{blue}}$. Thus, using all s vertices of V'_{blue} and s edges of M, we greedily find a matching of size s in A. Using (7.1) again, we have $C_4^{(3)} \not\subset A$. Replacing C_{red} with A, V_{blue} with V'_{red}, A with S, and interchanging colors red and blue in the argument above, we obtain a matching of size s in S. □

In view of Observation 7.2, it follows from (7.1) that

$$C_4^{(3)} \not\subset A \text{ and } C_4^{(3)} \not\subset S. \tag{7.2}$$

Observation 7.3. For every pair of vertices $xy \in \binom{V_{\text{red}}}{2}$ there exist at most twelve vertices $z \in V_{\text{blue}}$ such that xyz is blue (and therefore in A).

Proof. Suppose there is a pair $xy \in \binom{V_{\text{red}}}{2}$ and 13 vertices $z_1, \ldots, z_{13} \in V_{\text{blue}}$ so that $xyz_i \in A$ for $i = 1, 2, \ldots, 13$. Since $r(C_4^{(3)}) = 13$, the sub-hypergraph induced in K by $z_1, \ldots, z_{13}$ contains a monochromatic copy $\mathcal{C}$ of $C_4^{(3)}$.

On the one hand, all pairs z_iz_j are in ∂A, because $z_i, z_j \in V_{\text{blue}}$. Therefore, if $\mathcal{C}$ was blue then $\mathcal{C} \subset A$ – a contradiction to (7.2). On the other hand, all edges xyz, where $z \in V(\mathcal{C})$, are in A by our assumption. In order to avoid a copy of $C_4^{(3)}$ in A, one of the edges xzz', yzz', where $z, z' \in V(\mathcal{C})$, must be red. Since $x, y \in V_{\text{red}}$, such an edge is in S, and we have $zz' \in \partial S$. Hence, if $\mathcal{C}$ was red, then $\mathcal{C} \subset S$ – again a contradiction to (7.2). □

Observation 7.4. Every triple of vertices in V_{red} is blue and, consequently, $\binom{V_{\text{red}}}{3} \subset C'_{\text{blue}}$ for some blue component C'_{blue}.

Proof. By Observation 7.3, for all $x, y, z \in V_{\text{red}}$, there are at most 3×12 vertices $a \in V_{\text{blue}}$ so that one of the edges xya, xza, yza is blue. Since $|V_{\text{blue}}| \geq s \geq 37$, we can select a vertex $a \in V_{\text{blue}}$ so that $xya, xza, yza \in S$. We must have xyz blue to avoid $C_4^{(3)}$ in S. □

We can clearly interchange colors red and blue in Observations 7.3 and 7.4 and obtain that $\binom{V_{\text{blue}}}{3} \subset C'_{\text{red}}$ for some red component C'_{red}. Since one of V_{red}, V_{blue} must contain at least $\lceil t/2 \rceil \geq 3s$ vertices, we find greedily both a copy of $M_s^{(3)}$ and a copy of $C_4^{(3)}$, in either C'_{red} or C'_{blue}, contradicting (7.1). □

Case 2: $|V_{\text{red}}| < s$ **or** $|V_{\text{blue}}| < s$**.**

By symmetry, we may assume that $|V_{\text{red}}| < s$ and $|V_{\text{blue}}| > 5s + c_0\sqrt{s}$. We first prove that the azure component A contains a matching $M_s^{(3)}$ whose vertex set is in V_{blue}. Again, this is true even for $t = 6s - 2$.

Observation 7.5. There exists a matching $M_A = M_s^{(3)} \subset A$ with $V(M_A) \subset V_{\text{blue}}$.

Proof. Let $V_{\text{blue}} = V' \cup V''$ be a partition of V_{blue} such that $|V'| = s$. Since $|V''| \geq 6s-2-(s-1)-s \geq 4s - 1$, Theorem 6.1 applied to the induced 2-coloring $K_{\text{red}}[V''] \cup K_{\text{blue}}[V'']$ of $K[V'']$ implies that there exists a matching $M = M_s^{(3)}$ in a monochromatic component of $K[V'']$ (which is contained in some monochromatic component C in K).

If C is blue, then it must be A, because V'' is a subset of V_{blue}, and we are done. Hence assume $C = C_{\text{red}}$ is red. By (7.1), we have $C_4^{(3)} \not\subset C_{\text{red}}$. To avoid $C_4^{(3)}$ in C_{red}, for each edge $xyz \in M$ and any vertex $a \in V'$, at least one of the edges xya, xza, yza must be a blue edge, and, consequently, also in A, because $a \in V' \subset V_{\text{blue}}$. Thus, using all s vertices $a \in V'$ and s edges of M, we greedily find a matching M_A of size s in A. Clearly, $V(M_A) \subset V' \cup V'' = V_{\text{blue}}$. □

In view of Observation 7.5 and the assumption (7.1), we know that $C_4^{(3)} \not\subset A$. We distinguish two subcases. In the first one we assume that almost all pairs of vertices from V_{blue} are contained in the shadows of at most two red components.

Subcase 2a. There exist two red components C^1_{red} and C^2_{red} such that

$$\left| \binom{V_{\text{blue}}}{2} \setminus (\partial C^1_{\text{red}} \cup \partial C^2_{\text{red}}) \right| < 6t. \tag{7.3}$$

We now prove a series of observations. Recall that by Observation 6.3 the scarlet component S exists whenever $V_{\text{red}} \neq \emptyset$. We now show that in that case one of C^1_{red} and C^2_{red} equals S or can be replaced by S.

Observation 7.6. If $V_{\text{red}} \neq \emptyset$, then there exists a red component C_{red} such that

$$\left| \binom{V_{\text{blue}}}{2} \setminus (\partial C_{\text{red}} \cup \partial S) \right| = \left| \binom{V}{2} \setminus (\partial C_{\text{red}} \cup \partial S) \right| < 24t. \tag{7.4}$$

Proof. Note that $\binom{V}{2} \setminus \binom{V_{\rm blue}}{2} \subset \partial S$. If $|\partial C^1_{\rm red}| \le 18t$ holds, then with $C_{\rm red} = C^2_{\rm red}$ we have

$$\left|\binom{V}{2} \setminus (\partial C_{\rm red} \cup \partial S)\right| \overset{(7.3)}{<} 6t + |\partial C^1_{\rm red}| \le 24t.$$

Hence, suppose that $|\partial C^1_{\rm red}| > 18t$ and $|\partial C^2_{\rm red}| > 18t$. We claim that there exist vertices $u, v, w \in V_{\rm blue}$ such that $uv \in \partial C^1_{\rm red}$, $uw \in \partial C^2_{\rm red}$, and $vw \in \partial C^1_{\rm red} \cup \partial C^2_{\rm red}$.

This follows from a simple graph-theoretic fact.

Fact 7.7. *Let the edges of the complete graph K_n be partitioned into three sets E_1, E_2, E_3 so that, with $e_i = |E_i|$, $i = 1, 2, 3$, we have $\min\{e_1, e_2\} > 3e_3$. Then there exists a triangle with at least one edge in E_1, at least one edge in E_2 and no edge in E_3.*

Proof. Since the average degree in E_3 is $2e_3/n$, there is a vertex u such that $\deg_{E_1}(u) + \deg_{E_2}(u) \ge n - 1 - 2e_3/n$. If $\deg_{E_1}(u), \deg_{E_2}(u) > \sqrt{e_3}$, then there is a non-$E_3$ edge between the neighborhoods $N_{E_1}(u)$ and $N_{E_2}(u)$, completing a desired triangle.

Suppose now that, say, $\deg_{E_1}(u) \le \sqrt{e_3}$. If there is an edge $xy \in E_1$ with $x \in N_{E_1}(u)$ and $y \in N_{E_2}(u)$, then u, x, y is a triangle with the desired triangle. Otherwise, the number of edges of E_1 not contained in $N_{E_2}(u)$ is at most

$$\deg_{E_1}(u) + \binom{\deg_{E_1}(u)}{2} + \deg_{E_3}(u) \times n \le \frac{1}{2}(\sqrt{e_3} + e_3) + 2e_3 < 3e_3 < e_1.$$

Hence, there is an edge of E_1 with both endpoints in $N_{E_2}(u)$, yielding again a desired triangle. $\square$

We apply Fact 7.7 to $E_1 := \partial C^1_{\rm red}$, $E_2 := \partial C^2_{\rm red}$ and $E_3 := \binom{V_{\rm blue}}{2} \setminus (\partial C^1_{\rm red} \cup \partial C^2_{\rm red})$ (note that the assumptions hold).

Take any $x \in V_{\rm red}$ and vertices $u, v, w \in V_{\rm blue}$ such that $uv \in \partial C^1_{\rm red}$, $uw \in \partial C^2_{\rm red}$, and $vw \in \partial C^1_{\rm red} \cup \partial C^2_{\rm red}$. Since all three pairs of vertices contained in any red edge are in the shadow of the same red component, uvw must be a blue edge and hence in A. To avoid a copy of $C_4^{(3)}$ in A, at least one of the edges uvx, uwx, vwx must be a red edge, say uvx. Since $uv \in \partial C^1_{\rm red}$ and $xu \in \partial S$, we have $C^1_{\rm red} = S$ and the proof is completed by setting $C_{\rm red} = C^2_{\rm red}$ and recalling (7.3). $\square$

From now on we assume that

$$\left|\binom{V_{\rm blue}}{2} \setminus (\partial C^1_{\rm red} \cup \partial C^2_{\rm red})\right| < 24t, \tag{7.5}$$

and that $C^1_{\rm red} = S$, if S exists.

Observation 7.8. Every set $X \subset V_{\rm blue}$ with $|X| \ge 25\sqrt{t}$ contains a copy of $C_4^{(3)}$ in $C^1_{\rm red}$ or $C^2_{\rm red}$.

Proof. Let $X \subset V_{\rm blue}$ with $|X| \ge 25\sqrt{t}$ be given. Note that by (7.5)

$$\left|(\partial C^1_{\rm red} \cup \partial C^2_{\rm red}) \cap \binom{X}{2}\right| \ge \binom{|X|}{2} - 24t > \frac{11}{24}|X|^2.$$

Thus, by the Turán Theorem, there is a complete graph K_{13} in $\partial C^1_{\rm red} \cup \partial C^2_{\rm red}$. Let X_0 be the vertex set of one such K_{13}. Since $r(C_4^{(3)}) = 13$, the set $\binom{X_0}{3}$ contains a monochromatic copy $\mathcal{C}$ of $C_4^{(3)}$. It cannot be blue because all pairs of vertices of X are in ∂A and so $\mathcal{C}$ would be in A. Hence, $\mathcal{C}$ must be red and, thus, in $C^1_{\rm red}$ or $C^2_{\rm red}$ because $\binom{X_0}{2} \subset \partial C^1_{\rm red} \cup \partial C^2_{\rm red}$. $\square$

Now, we are ready to finish the proof of Theorem 7.1 in Subcase 2a. Recall that $c_0 = 25\sqrt{7}$ and $t \le 7s$. Suppose first that $V_{\rm red} = \emptyset$. By Observation 7.8, every set of $25\sqrt{t}$ vertices in $V_{\rm blue} = V$ contains a copy of $C_4^{(3)}$ in $C^1_{\rm red}$ or $C^2_{\rm red}$. Hence, we can find greedily, by taking one edge from a copy of $C_4^{(3)}$ and reusing the remaining vertex, a matching of size

$$\big(t - 25\sqrt{t}\big)/3 \ge \big(6s + c_0\sqrt{s} - 25\sqrt{7s}\big)/3 \ge 2s$$

in $C^1_{\rm red}\cup C^2_{\rm red}$. Thus, there is an index $i\in\{1,2\}$ such that $C^i_{\rm red}$ contains $M_s^{(3)}$ as well as a copy of $C_4^{(3)}$.

Assume now that $V_{\rm red}\neq\emptyset$ and, thus, S exists and $C^1_{\rm red}=S$. We know (see Observation 7.5) that A contains a matching M_A, $V(M_A)\subset V_{\rm blue}$, of size s but no $C_4^{(3)}$. As in the proof of Observation 7.2, for every vertex $x\in V_{\rm red}$ and each edge $e\in M_A$, there exists a edge $f\in S$ so that $x\in f$ and $|e\cap f|=2$. Hence, we can find a matching of size $|V_{\rm red}|<s$ in S that uses exactly $2|V_{\rm red}|$ vertices of $V_{\rm blue}$. After this, we use the greedy procedure from the previous paragraph and find a matching in $S\cup C^2_{\rm red}$ of size $(|V_{\rm blue}|-2|V_{\rm red}|-25\sqrt{t})/3$. Combining these two matchings and the fact that $|V_{\rm blue}|+|V_{\rm red}|=|V|=t$ yields a matching in $S\cup C^2_{\rm red}$ of size

$$|V_{\rm red}|+\big(|V_{\rm blue}|-2|V_{\rm red}|-25\sqrt{t}\big)/3=\big(t-25\sqrt{7s}\big)/3\geq 2s,$$

as before. Consequently, either S or $C^2_{\rm red}$ contains $M_s^{(3)}$. Note that at least one edge of this matching comes from a copy of $C_4^{(3)}$ in S or $C^2_{\rm red}$. Thus, in either case, we have $M_s^{(3)}$ and $C_4^{(3)}$ in the same red component.

Subcase 2b. Inequality (7.3) does not hold for any two red components $C^1_{\rm red}$ and $C^2_{\rm red}$.

We will first show that in this case the red components can be grouped into three large sets. To this end, we need the following simple fact. (We will only need part (b) now; part (a) will be used twice in Section 8.)

Fact 7.9. *For given numbers $a_1\geq a_2\geq\cdots\geq a_k\geq 0$, let $N=a_1+\cdots+a_k$.*

(a) *Let $d\geq 2N/3$ and $k\geq 2$. If $a_1\leq d$, then there exists $1\leq\ell_0\leq k-1$ such that $N-d\leq\sum_{i=1}^{\ell_0}a_i\leq d$.*

(b) *Let $N\geq 5r$ and $k\geq 3$. If $a_1+a_2\leq N-2r$, then there exist $1\leq\ell_1<\ell_2\leq k-1$ such that $\sum_{i=1}^{\ell_1}a_i\geq r$, $\sum_{i=\ell_1+1}^{\ell_2}a_i\geq r$, and $\sum_{i=\ell_2+1}^{k}a_i\geq r$.*

Proof. (a) Define $\ell_0=\min\big\{\ell:\sum_{i=1}^{\ell}a_i\geq N-d\big\}$. If $\ell_0=1$ then we are done. Otherwise, $a_{\ell_0}\leq a_1<N-d$, and so

$$N-d\leq\sum_{i=1}^{\ell_0}a_i\leq(N-d)+a_{\ell_0}<2(N-d)\leq d.$$

(b) If $a_1\geq a_2\geq r$, take $\ell_1=1$ and $\ell_2=2$. If $a_1\geq r$ but $a_2<r$, take $\ell_1=1$ and define $\ell_2=\min\{\ell:\sum_{i=2}^{\ell}a_i\geq r\}$. Then,

$$\sum_{i=1}^{\ell_2}a_i=a_1+\sum_{i=2}^{\ell_2-1}a_i+a_{\ell_2}<a_1+r+a_2\leq N-r,$$

and so, $\sum_{i=\ell_2+1}^{k}a_i\geq r$ as well. Finally, if $a_2\leq a_1<r$, define $\ell_1=\min\big\{\ell:\sum_{i=1}^{\ell}a_i\geq r\big\}$ and $\ell_2=\min\big\{\ell:\sum_{i=\ell_1+1}^{\ell}a_i\geq r\big\}$. Then

$$\sum_{i=1}^{\ell_1}a_i\leq\sum_{1}^{\ell_1-1}a_i+a_1\leq 2r$$

and, similarly, $\sum_{i=\ell_1+1}^{\ell_2} a_i \le 2r$. Hence,

$$\sum_{i=\ell_2+1}^{k} a_i \ge N - 4r \ge r.$$

□

Now we can prove the following consequence of negating (7.3).

Observation 7.10. There exists a partition $\binom{V}{2} = F^1 \cup F^2 \cup F^3$ such that

(i) F^1, F^2, F^3 are pairwise disjoint,
(ii) $|F^i[V_{\rm blue}]| \ge 3t$ for $i = 1, 2, 3$,
(iii) for every red component $C_{\rm red}$ there exists $i \in \{1, 2, 3\}$ such that $\partial C_{\rm red} \subset F^i$.

Proof. The shadows of all red components, intersected by $\binom{V_{\rm blue}}{2}$, form a partition of $\binom{V_{\rm blue}}{2}$ into disjoint sets of pairs. (Each pair that is not in any red edge is in a partition class by itself.) Let $a_1 \ge a_2 \ge \dots$ be the sizes of these partition classes. If (7.3) does not hold for any two red components then $a_1 + a_2 \le \binom{|V_{\rm blue}|}{2} - 6t$ and, by Fact 7.9(b) with $N = \binom{|V_{\rm blue}|}{2}$ and $r = 3t$, the a_i's can be grouped into three sums, each at least $3t$. Let the corresponding three sets of pairs, forming a partition of $\binom{V_{\rm blue}}{2}$, be denoted by $\tilde{F}^i$, $i = 1, 2, 3$. Then the conclusion follows with F^i's being arbitrary extensions of $\tilde{F}^i$'s such that for each red component $C_{\rm red}$ if $\partial C_{\rm red} \cap \binom{V_{\rm blue}}{2} \subset \tilde{F}_i$ then $\partial C_{\rm red} \subset F_i$. □

For convenience, set $\tilde{F}^i = F^i[V_{\rm blue}]$ and $\deg_i(v) = \deg_{\tilde{F}^i}(v)$, $i = 1, 2, 3$, $v \in V_{\rm blue}$.

Observation 7.11. For every vertex $v \in V_{\rm blue}$ there is an index $i \in \{1, 2, 3\}$ so that $\deg_i(v) = 0$.

Proof. Suppose that there is a vertex $v \in V_{\rm blue}$ such that $\deg_i(v) > 0$ for all $i = 1, 2, 3$. Denote by U_i the neighborhood of v in $\tilde{F}^i$ and take any three vertices $u_i \in U_i$, $i = 1, 2, 3$.

Since the pairs vu_1, vu_2, vu_3 belong to the shadows of distinct red components, all edges vu_iu_j, $1 \le i < j \le 3$, are blue and thus in the azure component A (because $v \in V_{\rm blue}$).

Consequently, since there is no $C_4^{(3)}$ in A, the edge $u_1u_2u_3$ must be red. Thus, all pairs of vertices $u_i \in U_i$ and $u_j \in U_j$, $i \ne j$, are in the shadow of the same red component. Without loss of generality we may assume that $u_iu_j \in \tilde{F}^1$.

Take any three vertices u_i, u_i', u_j, such that $u_i, u_i' \in U_i$ and $u_j \in U_j$. Since the edges vu_iu_j and $vu_i'u_j$ are both in A and $C_4^{(3)} \not\subset A$, either vu_iu_i' is red or $u_iu_i'u_j$ is red. In the first case, $u_iu_i' \in \tilde{F}^i$, while in the second case $u_iu_i' \in \tilde{F}^1$.

From this it follows that all pairs of $\tilde{F}^i$ are contained in $\{v\} \cup U_i$, $i = 2, 3$. Since $|\tilde{F}^i| \ge 3t > \deg_i(v)$, there exist vertices $u_i, u_i' \in U_i$ so that $u_iu_i' \in \tilde{F}^i$, $i = 2, 3$.

If all four edges induced by $\{u_2, u_2', u_3, u_3'\} \subset V_{\rm blue}$ were blue, we would have $C_4^{(3)}$ in A – a contradiction. Hence, at least one of them is red, say $u_2u_2'u_3$. Since $u_2u_3 \in \tilde{F}^1$, we have $u_2u_2' \in \tilde{F}^1$. But then $u_2u_2' \in \tilde{F}^1 \cap \tilde{F}^2$ – a contradiction with Observation 7.10(i). □

For $1 \le i < j \le 3$, let $W_{ij} = \{v \in V_{\rm blue} \colon \deg_i(v) > 0, \deg_j(v) > 0\}$. Next, we prove that W_{12}, W_{13}, and W_{23} have each at least two vertices.

Observation 7.12. $|W_{ij}| \ge 2$ for $1 \le i < j \le 3$.

Proof. By symmetry, we can restrict ourselves to the case $i = 1$ and $j = 2$. Since $|\tilde{F}^i| > 3t$, $i = 1, 2$, there is a matching M_i of size four in $|\tilde{F}^i|$. Let $u_1u_1' \in M_1$ and $u_2u_2' \in M_2$ be vertex disjoint. Since the copy of $C_4^{(3)}$ induced by $\{u_1, u_1', u_2, u_2'\}$ cannot be blue, at least one of its edges must be red. However then at least one pair from u_1u_2, u_1u_2', $u_1'u_2$, $u_1'u_2'$ is in $\tilde{F}^1$ or $\tilde{F}^2$. This implies that at least one of these vertices is adjacent to an edge of $\tilde{F}^1$ and an edge of $\tilde{F}^2$ and, thus, belongs to W_{12}. Now we remove that vertex and find another pair of disjoint edges, one from M_1, the other from M_2. Repeating the above reasoning, we obtain another vertex in W_{12}, completing the proof. $\square$

Let $w_{12}, w'_{12} \in W_{12}$, $w_{13}, w'_{13} \in W_{13}$, $w_{23}, w'_{23} \in W_{23}$. Clearly, by Observation 7.11 for all $1 \le i < j \le 3$ and $1 \le i' < j' \le 3$, $\{i, j\} \ne \{i', j'\}$, the pairs $w_{ij}w_{i'j'}$, $w'_{ij}w_{i'j'}$, $w_{ij}w'_{i'j'}$, $w'_{ij}w'_{i'j'}$ are from F^ℓ, where $\ell = \{i, j\} \cap \{i', j'\}$.

We show now that the sub-hypergraph H induced in K by vertices w_{12}, w'_{12}, w_{13}, w'_{13}, w_{23}, $w'_{23} \in V_{\text{blue}}$ contains a copy of $C_5^{(3)}$ in the azure component A.

Since $F^1 \cap F^2 = \emptyset$, we may assume that the pair $w_{12}w'_{12}$ is *not* contained in F^1. Also, at least one edge of the sub-hypergraph of K induced by vertices $w_{12}, w'_{12}, w_{13}, w'_{13}$ must be red (or we have $C_4^{(3)}$ in the azure component).

Edges $w_{12}w'_{12}w_{13}$ and $w_{12}w'_{12}w'_{13}$ must be blue because $w_{12}w_{13}, w_{12}w'_{13} \in F^1$ and $w_{12}w'_{12}$ does not belong to F^1. Hence, either $w_{12}w_{13}w'_{13}$ or $w'_{12}w_{13}w'_{13}$ is red, and the pair $w_{13}w'_{13}$ must lie in F^1 (and, consequently, not in F^3). Using the same argument we infer that $w_{23}w'_{23}$ belongs to F^3 and $w_{12}w'_{12}$ to F^2.

Observe now that all edges of the form $v_{12}v_{13}v_{23}$, where $v_{ij} \in \{w_{ij}, w'_{ij}\}$, are blue because the pairs contained in them belong to different F^i's and the shadow of every component is contained in a unique F^i. Moreover, the edges $w_{12}w'_{12}w_{13}$, $w_{12}w'_{12}w'_{13}$, $w_{13}w'_{13}w_{23}$, $w_{13}w'_{13}w'_{23}$, $w_{23}w'_{23}w_{12}$, and $w_{23}w'_{23}w'_{12}$ must be blue as well because, again, all the pairs contained in any red edge belong to the shadow of the same red component (and to a unique F^i), which is not the case here.

Therefore, all edges $w_{12}w_{23}w_{13}$, $w_{23}w_{13}w'_{13}$, $w_{13}w'_{13}w'_{23}$, $w'_{13}w'_{23}w_{12}$, and $w'_{23}w_{12}w_{23}$ of the cycle $C_5^{(3)}$ on vertices $w_{12}, w_{23}, w_{13}, w'_{13}, w'_{23}$ are colored blue and belong to the azure component. $\square$

8. Matchings in components (the real thing)

In this section we prove Lemma 2.1. Since the hypergraph K appearing in Lemma 2.1 is almost complete, we will be guided by the proof of Theorem 6.1 presented in Section 6. However, it will be convenient to replace K with a large sub-hypergraph K_1 with a more regular structure. Its existence is guaranteed by the following simple lemma.

For a vertex x in a hypergraph H, let $N_H(x) = \{y : xy \in \partial H\}$. For two vertices x, y, let $N_H(x, y) = \{z : xyz \in H\}$. Note that if $y \in N_H(x)$ (equivalently, $x \in N_H(y)$), then $N_H(x, y) \ne \emptyset$. We call all such pairs xy of vertices *active*. Thus, the active pairs in H are exactly those pairs of vertices which belong to the shadow ∂H of H.

Lemma 8.1. *Fix $\delta > 0$ and set $\delta_1 = 10\delta^{1/6}$. Let K be a 3-uniform hypergraph with t vertices and at least $(1-\delta)\binom{t}{3}$ edges. Then K contains a sub-hypergraph K_1 with $t_1 \ge (1-\delta_1)t$ vertices such that every vertex x of K_1 is in an active pair and for all active pairs xy we have $|N_{K_1}(x, y)| \ge (1-\delta_1)t_1$.*

A (fairly standard) proof of Lemma 8.1 can be found in [5] (see Lemma 4.1 therein).

Proof of Lemma 2.1. We may assume that $\eta < 1$. Given $0 < \eta < 1$, define $\delta = \eta^6 10^{-24}$. For any hypergraph K on $t = (4+\eta)s$ vertices and with at least $(1-\delta)\binom{t}{3}$ edges, let K_1 be the sub-hypergraph of K satisfying the conclusions of Lemma 8.1 with $\delta_1 = 10\delta^{1/6} = \eta/1000$. In particular, using the bound $t < 5s$, we get

$$t_1 = |V(K_1)| \ge (1-\delta_1)t \ge t - 5\delta_1 s = t - (\eta/200)s > (4 + \eta/2)s.$$

Since every monochromatic component of K_1 is contained in a monochromatic component of K, it is enough to show the conclusion of Lemma 2.1 for K_1. For the clarity of our presentation we will reset $K := K_1$, $\delta := \delta_1$ and $\eta := 2\eta$. Equivalently, we will assume that K has $t = (4+\eta)s$ vertices, $0 < \eta < 1/2$, every vertex x of K is in an active pair, and for all active pairs xy

$$|N_K(x, y)| \ge (1-\delta)t, \tag{8.1}$$

where $\delta = \eta/500$.

Since every x is in an active pair, it follows from (8.1) that for all $x \in V(K)$,

$$|N_K(x)| = |\{y : xy \in \partial K\}| \geq (1-\delta)t + 1. \tag{8.2}$$

Let $V = V(K)$ and fix a coloring $K = K_{\text{red}} \cup K_{\text{blue}}$. Our ultimate goal is to show that either in K_{red} or in K_{blue} there is an externally connected matching $M_s^{(3)}$. We begin with some preliminary results. Our first observation establishes for every $x \in V$ the existence of a dominant monochromatic component C_x the shadow of which contains most pairs of vertices xy. (For the complete hypergraph K this was done in Observation 6.2.)

Observation 8.2. For every vertex $x \in V$ there exists a monochromatic component C_x such that

$$\big|\{y \in V : xy \in \partial C_x\}\big| \geq (1-\delta)t. \tag{8.3}$$

The observation will follow from a simple graph theoretic result.

Fact 8.3. *Let G be a graph with n vertices and minimum degree d. If $n > d \geq 3n/4$, then for every red-blue coloring of the edges of G there is a monochromatic component with at least $d+1$ vertices.*

Proof. Let $G = G_{\text{red}} \cup G_{\text{blue}}$ be a red-blue coloring of the edges of G. Suppose that no component of G_{red} has more than d vertices. Then, by Fact 7.9(a) in Section 7 there is a partition $V(G) = V_1 \cup V_2$, where

$$n - d \leq |V_1| \leq \frac{n}{2} \leq |V_2| \leq d \text{ and } E_{G_{\text{red}}}(V_1, V_2) = \emptyset.$$

Observe that in G_{blue} every vertex of V_2 has a neighbor in V_1 and every vertex of V_1 has more than $|V_2|/2$ neighbors in V_2. Thus, the graph G_{blue} is connected, and so there is a blue component on all $n \geq d+1$ vertices. □

Proof of Observation 8.2. Note that $\delta < 1/4$ and that, by (8.1), for every vertex $x \in V$ the graph $K(x) = \{yz : xyz \in K\}$ has minimum degree at least $(1-\delta)t \geq 3t/4$ (and at most t vertices). The coloring $K = K_{\text{red}} \cup K_{\text{blue}}$ induces a coloring $K(x) = K_{\text{red}}(x) \cup K_{\text{blue}}(x)$ which, by Fact 8.3, contains a monochromatic component with at least $(1-\delta)t$ vertices. Consequently, there is a monochromatic component C in K such that ∂C contains at least $(1-\delta)t$ pairs xy. □

For each $x \in V$ let us choose arbitrarily one component satisfying the condition in Observation 8.2 and denote it by C_x. Let

$$V_{\text{red}} = \{x \in V : C_x \text{ is red}\} \text{ and } V_{\text{blue}} = \{x \in V : C_x \text{ is blue}\}.$$

Observation 8.2 tells us that $V = V_{\text{red}} \cup V_{\text{blue}}$ and this union is disjoint by the definition of V_{red} and V_{blue}.

Our next result says that for most $x \in V_{\text{red}}$, as well as for most $x \in V_{\text{blue}}$, the components C_x are the same. (For the complete hypergraph K this is Observation 6.3.)

Observation 8.4. If $|V_{\text{red}}| \geq 6\delta t$ ($|V_{\text{blue}}| \geq 6\delta t$, respectively) then there is a red component C_{red} (a blue component C_{blue}) so that $C_x = C_{\text{red}}$ ($C_x = C_{\text{blue}}$) for all but at most $2\delta t$ vertices $x \in V_{\text{red}}$ ($x \in V_{\text{blue}}$).

Proof. Consider a graph G defined on V_{red} by putting an edge between x and y whenever $xy \in \partial C_x \cap \partial C_y$ (note that by (6.1) this means that $C_x = C_y$). By Observation 8.2 every vertex "spoils" at most δt edges, and thus $|E(G)| \geq \binom{v}{2} - v\delta t$, where $v = |V_{\text{red}}|$. Our goal is to show that G has a component of order at least $v - 2\delta t$. Suppose this is not true. Then, by Fact 7.9(a) in Section 7 with $d = v - 2\delta t$, there is a partition $V_{\text{red}} = V_1 \cup V_2$ with

$$2\delta t \leq |V_1|, |V_2| \leq v - 2\delta t \text{ and } E_G(V_1, V_2) = \emptyset,$$

which yields at least $2\delta t(v - 2\delta t) > v\delta t$ edges in the complement of G – a contradiction. □

If $|V_{\text{red}}| \geq 6\delta t$, we define the *scarlet* component S as the (unique) red component C_{red} guaranteed by Observation 8.4 and set

$$V'_{\text{red}} = \{x \in V_{\text{red}} : C_x = S\}.$$

Then

$$|V'_{\text{red}}| \geq |V_{\text{red}}| - 2\delta t \geq 4\delta t.$$

If $|V_{\text{red}}| < 6\delta t$ then e say that the scarlet component does not exist and $V'_{\text{red}} = \emptyset$. Similarly, when $|V_{\text{blue}}| \geq 6\delta t$, we define the *azure* component A and the set

$$V'_{\text{blue}} = \{x \in V_{\text{blue}} : C_x = A\}.$$

Then

$$|V'_{\text{blue}}| \geq |V_{\text{blue}}| - 2\delta t \geq 4\delta t,$$

and $V'_{\text{blue}} = \emptyset$ if $|V_{\text{blue}}| < 6\delta t$. We also set

$$V' = V'_{\text{red}} \cup V'_{\text{blue}}.$$

Since $\delta < 1/12$,

$$|V'| = |V'_{\text{red}}| + |V'_{\text{blue}}| \geq t - 8\delta t. \tag{8.4}$$

For each $x \in V'_{\text{red}}$, set

$$\partial S(x) = \big|\{y \in V : xy \in \partial S\}\big|,$$

and for each $x \in V'_{\text{blue}}$, set

$$\partial A(x) = \big|\{y \in V : xy \in \partial A\}\big|.$$

By Observation 8.2 and the definitions of S and A we have

$$|\partial S(x)|, |\partial A(x)| \geq (1-\delta)t. \tag{8.5}$$

Our last preliminary result is the Mirror Lemma (cf. Lemma 6.7) adjusted to non-complete hypergraphs.

Lemma 8.5 (The Blurred Mirror Lemma)**.** *Let M be a largest matching in a red component C_{red} and let $P \subset V$, where $P \cap V(M) = \emptyset$ and $|P| \geq |M| + 30\delta t$. Assume further that for some blue component C_{blue} and for every $e \in M$, the bipartite induced subgraph $\partial C_{\text{blue}}[e, P]$ of ∂C_{blue} contains $K_{2,|P|-9\delta t-1}$. Then there exists a matching $M' \subset C_{\text{blue}}$ such that*

(i) *$|M'| \geq |M|$ and*
(ii) *$|V(M') \cap P| \leq |M| + 4\delta t$.*

Proof. Let $M'' \subset C_{\text{blue}}$ be a largest matching such that

(8.6)
- $|V(M'') \cap P| \leq |M''|$ and
- $V(M'')$ intersects at most $|M''|$ edges of M.

We first claim that $|M''| \geq |M| - 4\delta t$. Indeed, suppose $|M''| \leq |M| - 4\delta t$. We will show that there exist $e_1, e_2 \in M$ and $a, b, c \in P'' := P \setminus V(M'')$ such that $(e_1 \cup e_2) \cap V(M'') = \emptyset$ and the set $X = e_1 \cup e_2 \cup \{a, b, c\}$ satisfies the assumptions of Claim 6.5 (see Remark 6.6).

From the second part of (8.6) and our supposed bound on $|M''|$, it follows that there exist at least $4\delta t$ edges of M disjoint from $V(M'')$. Let $e_1 = uvw \in M$ be any such edge. Below we suppress the dependence on K and write $N(x)$ for the neighborhood of x in the shadow of K, and $N(x,y)$ for the neighborhood of x, y in K. By (8.2),

$$\big|V \setminus (N(u) \cap N(v) \cap N(w))\big| \leq 3t - |N(u)| - |N(v)| - |N(w)| \leq 3\delta t$$

and so, there exists $e_2 = xyz \in M$ such that $e_2 \cap V(M'') = \emptyset$ and every pair of vertices $p, q \in e_1 \cup e_2 = \{u, v, w, x, y, z\}$ is active.

By the first part of (8.6) and our bounds on $|P|$ and $|M''|$, we have

$$|P''| = |P| - |P \cap V(M'')| \geq |M| + 30\delta t - (|M| - 4\delta t) = 34\delta t.$$

Among the vertices of P'' at most $18\delta t+2$ do not belong to the bipartite cliques $K_{2,|P|-9\delta t-1}$ between e_i, $i=1,2$, and P, guaranteed by the assumptions. Also, by (8.1),

$$\left|P''\setminus\bigcap_{p,q}N(p,q)\right|\le\binom{6}{2}\delta t=15\delta t,$$

where the intersection is taken over all pairs of vertices $p,q\in e_1\cup e_2$. Since $(34-18-15)\delta t-2\ge 3$, one can choose $a,b,c\in P''$ so that

(a) $\partial C_{\rm blue}[e_i,\{a,b,c\}]\supset K_{2,3}$ for $i=1,2$, and
(b) all triples of vertices having two vertices in $\{u,v,w,x,y,z\}$ and one in $\{a,b,c\}$ are edges of K.

Thus, we can apply Claim 6.5 (see Remark 6.6) to the set $X=e_1\cup e_2\cup\{a,b,c\}$. But then we can either enlarge M in $C_{\rm red}$ (if (1) of Claim 6.5 occurs) or M'' in $C_{\rm blue}$ with conditions (8.6) preserved (if (2) or (3) of Claim 6.5 occurs), yielding a contradiction with the choice of M or M'', respectively.

Hence $|M''|\ge|M|-4\delta t$. If $|M''|\ge|M|$, we are done. Otherwise, we repeat the following procedure which keeps enlarging M'' by increments of two until its size reaches $|M|$ (for convenience, we assume that $|M|-|M''|$ is even). Let the current matching be denoted by M', $|M'|<|M|$. It is important that in each step we will

- not delete any edge of M', that is, $M''\subseteq M'$,
- add to $V(M')$ at most four vertices of P, and
- maintain the second part of (8.6).

Since there are $(|M|-|M''|)/2$ steps, for the final M' we have

$$|V(M')\cap P|\le|M''|+2(|M|-|M''|)=|M|+(|M|-|M''|)\le|M|+4\delta t,$$

so (ii) holds. Now we describe a single step of the procedure. Let $e=xyz\in M$ be such that $e\cap V(M')=\emptyset$. Denote by P_0 the set of at most $9\delta t+1$ vertices of P which do not belong to the bipartite clique $K_{2,|P|-9\delta t-1}$ between e and P, guaranteed by the assumptions.

Set $P'=P\setminus(V(M')\cup P_0)$. Similarly to the above,

$$|P'|\ge|M|+30\delta t-(|M|+4\delta t)-|P_0|\ge 16\delta t.$$

Set $N_1=P'\cap N(x,y)\cap N(x,z)\cap N(y,z)$. By (8.1), $|N_1|>(16-3)\delta t=13\delta t$. Let $a\in N_1$ and set $N_2=N_1\cap N(a,x)\cap N(a,y)\cap N(a,z)$. We have, again by (8.1), $|N_2|>10\delta t$. Similarly, for every $b\in N_2$ and every $c\in N_3=N_2\cap N(b,x)\cap N(b,y)\cap N(b,z)$, we have

$$|N_3\cap N(c,x)\cap N(c,y)\cap N(c,z)|>4\delta t\ge 1.$$

Thus, one can choose $a,b,c,d\in P'$ so that

(a) $\partial C_{\rm blue}[e,\{a,b,c\}]\supset K_{2,3}$
(b) all triples of vertices within $\{x,y,z,a,b,c,d\}$ intersecting simultaneously $\{x,y,z\}$ and $\{a,b,c,d\}$ are edges of K.

We apply Claim 6.4 (see Remark 6.6) to the set $X=e\cup\{a,b,c,d\}$. By the maximality of M in $C_{\rm red}$ and the maximality of M'' with respect to (8.6) in $C_{\rm blue}$ (note that $V(M'')\cap X=\emptyset$), conclusions (1) and (2) of Claim 6.4 cannot hold. Thus, (3) holds, which allows us to enlarge M' by adding the edges e_1 and e_2 guaranteed by Claim 6.4(3). Note that, indeed, in a single step we have used four vertices of P and one edge of M. □

We are now ready to complete the proof of Lemma 2.1. Since $\delta<1/12$, in view of Observation 8.4, either the scarlet component S or the azure component A (or both) does exist.

Let M be a matching of maximum size in K among all matchings that lie in S or A. Without loss of generality we assume that $\emptyset\ne M\subset S$. This implies that $|V'_{\rm red}|\ge 4\delta t$, but $V'_{\rm blue}$ might be empty, that is, the azure component A might not exist. Suppose that

$$1\le m=|M|\le s-1$$

and set

$$R = V'_{\text{red}} \setminus V(M) \text{ and } B = V'_{\text{blue}} \setminus V(M). \tag{8.7}$$

According to this definition, if $B \neq \emptyset$, then $V'_{\text{blue}} \neq \emptyset$, and consequently, the azure component A does exist. Note that $R \cap B = \emptyset$ and

$$t = (4+\eta)s = 3m + |R \cup B| + |V \setminus V'|, \tag{8.8}$$

and, using $m < s$ and (8.4),

$$|R \cup B| \geq (4+\eta)s - 3m - |V \setminus V'| \geq (1+\eta)s - 8\delta t. \tag{8.9}$$

Observe that by (8.9) and our choice of δ, whenever one of the sets R or B has size at most $5\delta t$ then the other one has size at least

$$(1+\eta)s - 13\delta t \geq m + 30\delta t.$$

We first show the following variant of Observation 6.8.

Observation 8.6. If $|R| \geq 2\delta t$, then all edges $xyz \in K[R \cup B]$ with $x \in R$ and $xy \in \partial S$ belong to the same blue component C_{blue}. Furthermore, if also $|B| \geq 2\delta t$, then $C_{\text{blue}} = A$.

Proof. First note that any red edge $xyz \in K[R \cup B]$ with $xy \in \partial S$ would be in S and disjoint from $V(M)$, and thus it could be added to M, contradicting the maximality of M. Hence, every such edge is blue. Let $x, y, z \in R \cup B$ and xy and xz be two pairs in ∂S. Since $|B \cup R| \geq 2\delta t$ and the pairs xy and xz are active, by (8.1) there is $w \in R \cup B$ such that $xyw \in K$ and $xzw \in K$. Hence, both edges are blue and in the same blue component. Now, by (8.5), the subgraph $\partial S[R]$ has minimum degree at least $|R| - \delta t \geq |R|/2$ and, thus, it is connected. This implies that all pairs $xy \in \partial S$ such that $x \in R$ and $y \in R \cup B$ are in the shadow of the same blue component C_{blue}.

To prove the second part, notice that if both $|R|, |B| \geq 2\delta t$ then, again by (8.5), the number of edges of ∂S with one endpoint in R and the other in B is more than $|R||B|/2$, and the same is true for the edges of ∂A. Hence, there is a pair $x \in R$ and $y \in B$ such that $xy \in \partial S \cap \partial A$. It follows that $C_{\text{blue}} = A$. □

For the rest of the proof of Lemma 2.1 we distinguish three cases analogous to the three cases considered in the proof of Theorem 6.1.

Case 1: $|B| \leq 5\delta t$

Denote by C_{blue} the blue component guaranteed by Observation 8.6.

Observation 8.7. For every edge $e \in M$, the bipartite induced subgraph $\partial C_{\text{blue}}[e, R]$ of ∂C_{blue} contains $K_{2,|R|-3\delta t-1}$ as a subgraph.

Proof. Let $e = xyz \in M$. By (8.2), at least $|R| - 3\delta t$ vertices $a \in R$ are such that all three pairs xa, ya and za are active. Let the set of such vertices be denoted by R_e.

Suppose that $\partial C_{\text{blue}}[e, R_e]$ contains no copy of $K_{2,|R_e|-1}$. Then there exist two vertices $a, b \in R_e$ such that, say, $ya, zb \notin \partial C_{\text{blue}}$. Since $|R| \geq 2\delta t + 5$, by (8.5) and (8.1) there are $c, d, u \in R \setminus \{a, b\}$ such that $ac, bd \in \partial S$ and $yac, zbd, uac, ubd \in K$. By Observation 8.6, $uac, ubd \in C_{\text{blue}}$ and thus $ac, bd \in \partial C_{\text{blue}}$. Hence, the edges yac and zbd must be red. Consequently, $yac, zbd \in S$ and $(M \setminus \{xyz\}) \cup \{yac, zbd\}$ is a matching in S larger than M – a contradiction. □

Now we apply Lemma 8.5 with $C_{\text{red}} = S$, C_{blue} and $P = R$ (recall that $|R| \geq m + 30\delta t$), obtaining a matching $M' \subset C_{\text{blue}}$ of size $|M'| := m' \geq m$ and with $|V(M') \cap R| \leq m + 4\delta t$.

If $m' \geq s$, we are done. Otherwise, by (8.9) and (8.4), we have

$$|R \setminus V(M')| \geq (4+\eta)s - 3m - 13\delta t - (m + 4\delta t) \geq 3(s - m') + 3\delta t.$$

This allows us to enlarge M' to size s by adding blue edges contained in $R \setminus V(M')$. Indeed, by (8.1) and (8.5), we can greedily find $s - m'$ disjoint edges $xyz \in K[R]$ with $xy \in \partial S$. Since all such edges belong to C_{blue} (cf. Observation 8.6), we can add them to M' obtaining a matching of size s in a blue component.

Case 2: $|R| \le 5\delta t$

By our assumptions and (8.9), $B \neq \emptyset$ and thus the azure component A exists.

Observation 8.8. For every edge $e \in M$, the bipartite induced subgraph $\partial A[e,B]$ of ∂A contains $K_{2,|B|-9\delta t-1}$ as a subgraph.

Proof. Fix an edge $xyz \in M$. By (8.1), at least $|B| - 3\delta t$ vertices $a \in B$ are such that all three pairs xa, ya and za are active. Let the set of such vertices be denoted by B_e. Call a vertex $a \in B_e$ *friendly* to x if $xa \in \partial S \cup \partial A$ and let B_x be the subset of B_e containing all unfriendly vertices to x.

Claim 8.9. $|B_x| < 2\delta t$

Proof. Suppose that $|B_x| \ge 2\delta t$, recall that $|V'_{\text{red}}| \ge 4\delta t$ (since S exists), and consider the bipartite induced subgraphs G_S and G_A of ∂S and ∂A, respectively, with vertex set $B_x \cup V'_{\text{red}}$. Assume for simplicity that $|B_x| = 2\delta t$ and $|V'_{\text{red}}| = 4\delta t$, taking subsets if necessary. Recalling that $B_x \subset V'_{\text{blue}}$, by (8.5), $|G_S| \ge 4(\delta t)^2$ and $|G_A| \ge 6(\delta t)^2$, and consequently, $|G_S \cap G_A| \ge 2(\delta t)^2$. Let $a \in B_x$ have degree at least δt in $G_S \cap G_A$. Then, by (8.1) and the definition of B_e, one can find a vertex $u \in V'_{\text{red}}$ such that $xau \in K$ and $au \in G_S \cap G_A \subset \partial S \cap \partial A$, which contradicts the assumption that a is unfriendly to x, no matter how xau is colored. □

Set $B'_e = B_e \setminus (B_x \cup B_y \cup B_z)$. It is sufficient to show that $\partial A[e, B'_e]$ contains a copy of $K_{2,|B'_e|-1}$. Suppose it does not. Then there exist two vertices $a, b \in B'_e$ such that, say, $ya, zb \notin \partial A$ (and thus, they must be in ∂S). Since $|B| \ge 2\delta t + 4$, by (8.5) and (8.1), there are $c, d, \in B \setminus \{a, b\}$ such that $ac, bd \in \partial A$ and $yac, zbd \in K$. Hence, the edges yac and zbd must be red. Consequently, $yac, zbd \in S$ and $(M \setminus \{xyz\}) \cup \{yac, zbd\}$ is a matching in S larger than M – a contradiction. □

We apply Lemma 8.5 with $C_{\text{red}} = S$, $C_{\text{blue}} = A$ and $P = B$ (recall that $|B| \ge m + 30\delta t$) and obtain a matching $M' \subset A$ of size $|M'| = m$ and $|V(M') \cap B| \le m + 4\delta t$. (A matching larger than m in the azure component A is impossible by our choice of M.)

We claim that $R' := V'_{\text{red}} \setminus V(M') = \emptyset$. Indeed, suppose that $d \in R'$. Since

$$|B \setminus V(M')| \ge s + 18\delta t - (m + 4\delta t) \ge 14\delta t \ge 2\delta t + 3,$$

by (8.5) and (8.1) we can find vertices $a, b, c \in B \setminus V(M')$ such that $ad \in \partial S$, $ab \in \partial A$, and $abd, abc \in K$. Then both abc and abd are red (or we can enlarge M'). But $ad \in \partial S$, therefore $abd \in S$ and, consequently, $abc \in S$. Since $\{a, b, c\} \cap V(M) = \emptyset$, we can enlarge M in S, which is a contradiction.

Thus $|R'| = 0$ and we are back in Case 1 with the colors red and blue interchanged and M replaced by M'.

Case 3: $|B|, |R| \ge 5\delta t$

Set $P = R \cup B$. In this case not only the azure component A exists, but also the blue component C_{blue} guaranteed by Lemma 8.6 is A.

Observation 8.10. For every edge $e \in M$, the bipartite induced subgraph $\partial A[e,B]$ of ∂A contains $K_{2,|P|-9\delta t-1}$ as a subgraph.

Proof. The proof follows the lines of the proof of Observation 8.8. Fix an edge $xyz \in M$. By (8.2), at least $|P| - 3\delta t$ vertices $a \in P$ are such that all three pairs xa, ya and za are active. Let the set of such vertices be denoted by P_e.

Recall that a vertex $a \in P_e \cap B$ *friendly* to x if $xa \in \partial S \cup \partial A$ and let B_x be the subset of unfriendly vertices of $P_e \cap B$. We have shown in Claim 8.9 that $|B_x| \le 2\delta t$. Set $P'_e = P_e \setminus (B_x \cup B_y \cup B_z)$ and suppose that $\partial A[e, P'_e]$ contains no copy of $K_{2,|P'_e|-1}$. Thus, there exist two vertices $a, b \in P'_e$ such that, say, $ya, zb \notin \partial A$. But then, combining arguments from the proofs of Observations 8.7 and 8.8 (each of a and b can be in R or B), one can show that there exist vertices $c, d \in P$ such that $yac, zbd \in S$. Consequently, $(M \setminus \{xyz\}) \cup \{yac, zbd\}$ is a red matching in S larger than M – a contradiction. □

We apply Lemma 8.5 with $C_{\text{red}} = S$, $C_{\text{blue}} = A$ and $P = (R \cup B) \setminus \{a, b\}$ where $a \in R, b \in B$ and ab is an active pair. Let M' be a matching in A satisfying conclusions (i) and (ii) of Lemma 8.5. By the maximality of M, we have $|M'| = m$ and, by (ii) and (8.9),

$$|P \setminus V(M')| \geq s + 30\delta t - 8\delta t - (m + 4\delta t) \geq 18\delta t.$$

By (8.1) and (8.5), we can choose $c \in P \setminus V(M')$ so that $ac \in \partial S$, $bc \in \partial A$ and $abc \in K$. Consequently, $abc \in S$ if it is red and $abc \in A$ if it is blue. Also abc is disjoint from both $V(M)$ and $V(M')$. Thus, either we obtain a matching of size $m+1$ in S, or a matching of size $|M'|+1 = m+1$ in A, contradicting the maximality of M among all matchings contained in S or A. □

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Appendix A. Matchings and short cycles in components (the real thing)

We may assume that $\eta < 1/14$. Given $0 < \eta < 1/14$, similarly to the proof of Lemma 2.1, by Lemma 8.1, we will assume that K has $t = (6 + 14\eta)s$ vertices, every vertex x of K is in an active pair, and for all active pairs xy

$$|N_K(x, y)| \geq (1 - \delta)t, \tag{A.1}$$

where $\delta = (\eta^6 10^{-24})^2$.

Since every x is in an active pair, it follows from (A.1) that for all $x \in V(K)$,

$$|N_K(x)| = \big|\{y : xy \in \partial K\}\big| \geq (1 - \delta)t + 1. \tag{A.2}$$

Let $V := V(K)$ and suppose there exists a coloring $K = K_{\text{red}} \cup K_{\text{blue}}$ such that

$$\text{no monochromatic component contains } M_s^{(3)} \text{ and either } C_4^{(3)} \text{ or } C_5^{(3)}. \tag{A.3}$$

For each $x \in V$ let us choose arbitrarily one component satisfying the condition in Observation 8.2 and denote it by C_x. Let

$$V_{\text{red}} = \{x \in V : C_x \text{ is red}\} \text{ and } V_{\text{blue}} = \{x \in V : C_x \text{ is blue}\}.$$

Observation 8.2 tells us that $V = V_{\text{red}} \cup V_{\text{blue}}$ and this union is disjoint by the definition of V_{red} and V_{blue}.

If $|V_{\text{red}}| \geq 6\delta t$, we define the scarlet component S as the (unique) red component C_{red} guaranteed by Observation 8.4 and set $V'_{\text{red}} = \{x \in V_{\text{red}} : C_x = S\}$. Then

$$|V'_{\text{red}}| \geq |V_{\text{red}}| - 2\delta t \geq 4\delta t.$$

If $|V_{\text{red}}| < 6\delta t$ then we say that the scarlet component does not exist and $V'_{\text{red}} = \emptyset$. Similarly, when $|V_{\text{blue}}| \geq 6\delta t$, we define the azure component A and the set $V'_{\text{blue}} = \{x \in V_{\text{blue}} : C_x = A\}$. Then

$$|V'_{\text{blue}}| \geq |V_{\text{blue}}| - 2\delta t \geq 4\delta t,$$

and $V'_{\text{blue}} = \emptyset$ if $|V_{\text{blue}}| < 6\delta t$. We also set

$$V' = V'_{\text{red}} \cup V'_{\text{blue}}.$$

Since $\delta < 1/12$, we have

$$|V'| = |V'_{\text{red}}| + |V'_{\text{blue}}| \geq t - 8\delta t. \tag{A.4}$$

For each $x \in V'_{\text{red}}$, set $\partial S(x) = \big|\{y \in V : xy \in \partial S\}\big|$, and for each $x \in V'_{\text{blue}}$, set $\partial A(x) = \big|\{y \in V : xy \in \partial A\}\big|$. By Observation 8.2 and the definitions of S and A we have

$$|\partial S(x)|, |\partial A(x)| \geq (1 - \delta)t. \tag{A.5}$$

We distinguish two complementary cases, and in each of them we obtain a contradiction to (A.3) or its consequence, (A.6) below. In both cases we again use the fact that $r(C_4^{(3)}) = 13$ (see [8]).

Case 1: $|V'_{\text{red}}|, |V'_{\text{blue}}| \geq (1 + 2\eta)s$.

We first prove that each of S and A contains a matching $M^{(3)}_{(1+\eta)s}$.

Observation A.1. $M^{(3)}_{(1+\eta)s} \subset A$ and $M^{(3)}_{(1+\eta)s} \subset S$.

Proof. Partition the set of vertices V into sets U, U_{red}, U_{blue} such that $U_{\text{red}} \subset V'_{\text{red}}$, $|U_{\text{red}}| = (1+2\eta)s$, $U_{\text{blue}} \subset V'_{\text{blue}}$, $|U_{\text{blue}}| = (1 + 2\eta)s$, and $U = V \setminus (U_{\text{red}} \cup U_{\text{blue}})$.

Since $|U| \geq t - 2(1 + 2\eta)s \geq (4 + \eta)(1 + 2\eta)s$, Lemma 2.1 applied to the induced coloring $K_{\text{red}}[U] \cup K_{\text{blue}}[U]$ of $K[U]$ implies that there exists a matching $M = M^{(3)}_{(1+2\eta)s}$ in some component (say red) C_{red} of K_{red}.

Let $M' \subset M$, $U' \subset U_{\text{blue}}$ be arbitrary, $|M'| \geq 5\delta t$ and $|U'| \geq 4\delta t$. Firstly, we claim that there is an edge $xyz \in M'$ and a vertex $a \in U'$ such that xya, xza, $yza \in K$ and xa, ya, $za \in \partial A$. This is a consequence of the following fact applied with $X = M'$, $Y = U'$, $t = 3\delta t$, $s = \delta t$.

Fact A.2. *Let B_1 and B_2 be two bipartite graphs with the same bipartition $X \cup Y$. If $\deg_{B_1}(x) \geq |Y| - t$ for all $x \in X$, $\deg_{B_2}(y) \geq |X| - s$ for all $y \in Y$, and $(|X| - s)(|Y| - t) > st$, then there exist $x \in X$, $y \in Y$ such that $xy \in B_1 \cap B_2$.*

In particular, this is true for $|X| > 2s$ and $|Y| > 2t$.

Proof. Clearly, $(|X|-s)(|Y|-t) > st$ implies

$$|B_1| + |B_2| \geq |X|(|Y|-t) + |Y|(|X|-s) > |X||Y|,$$

from which the fact follows. □

Secondly, by (A.3), we know that $C_4^{(3)} \not\subset C_{\text{red}}$, hence at least one of the edges xya, xza, yza must be blue and also in A.

Thus, using all but at most δs vertices of U_{blue} and all but at most $5\delta s$ edges of M, we greedily find a matching of size $(1+2\eta)s - 5\delta s > (1+\eta)s$ in A. Using (A.3) again, we have $C_4^{(3)} \not\subset A$. Replacing C_{red} with A, U_{blue} with U_{red}, A with S, and interchanging colors red and blue in the argument above, we obtain a matching of size $(1+\eta)s$ in S. □

In view of Observation A.1, it follows from (A.3) that

$$C_4^{(3)} \not\subset A \text{ and } C_4^{(3)} \not\subset S. \tag{A.6}$$

Observation A.3. For every active pair of vertices $xy \in \binom{V'_{\text{red}}}{2}$ there exist at most $80\delta t$ vertices $z \in V'_{\text{blue}}$ such that xyz is blue.

Proof. Suppose there is an active pair $xy \in \binom{V'_{\text{red}}}{2}$ and a set Z of $1 + 80\delta t$ vertices $z \in V'_{\text{blue}}$ such that xyz is blue. We show that this implies the existence of vertices $z_i \in V'_{\text{blue}}$, $i = 1, 2, \ldots, 13$, such that

(i) $z_i z_j z_k \in K$ for $1 \leq i < j < k \leq 13$,
(ii) $z_i z_j \in \partial S \cap \partial A$ for $1 \leq i < j \leq 13$.

Since $r(C_4^{(3)}) = 13$, the sub-hypergraph induced in K by $z_1, \ldots, z_{13}$ contains a monochromatic copy $\mathcal{C}$ of $C_4^{(3)}$. By (ii), if $\mathcal{C}$ was blue then $\mathcal{C} \subset A$ – a contradiction to (A.6), and if $\mathcal{C}$ was red then $\mathcal{C} \subset S$ – a contradiction to (A.6) again.

We choose $z_1 \in Z$ so that $xz_1, yz_1 \in \partial S$. By (A.5), at most $2\delta t$ vertices of Z cannot be selected. For $i = 2, \ldots 13$, we choose $z_i \in Z$ so that $xz_i, yz_i \in \partial S$, $z_i z_j \in \partial A$ for $j = 1, \ldots, i-1$, and $z_i z_j z_k \in K$ for $1 \leq j < k \leq i-1$.

By (A.1) and (A.5), at most $2\delta t + (i-1)\delta t + \binom{i-1}{2}\delta t$ vertices of Z are ineligible for selection. Since $2\delta t + (i-1)\delta t + \binom{i-1}{2}\delta t \leq (2 + 12 + 66)\delta t = 80\delta t < |Z|$, we can always pick z_i.

Vertices $z_1, \ldots, z_{13}$ clearly satisfy (i) above. All edges xyz_i, where $i = 1, \ldots, 13$, are blue, and all pairs $z_i z_j \in \partial A$, where $1 \leq i < j \leq 13$, by our assumption. In order to avoid a copy of $C_4^{(3)}$ in A, one of the edges $xz_i z_j$, $yz_i z_j$ must be red for every $1 \leq i < j \leq 13$. Since $xz_i, xz_j, yz_i, yz_j \in \partial S$, such an edge is in S, and we have $z_i z_j \in \partial S$. Thus, (ii) holds as well. □

Observation A.4. Every edge contained in V'_{red} is blue and, consequently, $K \cap \binom{V'_{\text{red}}}{3} \subset C'_{\text{blue}}$ for some blue component C'_{blue}.

Proof. By Observation A.3, for all $xyz \in K \cap \binom{V'_{\text{red}}}{3}$, there are at most $3 \times 80\delta t$ vertices $a \in V'_{\text{blue}}$ so that one of the edges xya, xza, yza is blue. Moreover, by (A.1) and (A.5), there are at most $3\delta t + 3\delta t$ vertices $a \in V'_{\text{blue}}$ so that either one of the triples xya, xza, yza is not in K or one of the pairs xa, ya, za is not in ∂S.

Since $|V'_{\text{blue}}| \geq (1+2\eta)s > 86\delta t$, we can select a vertex $a \in V'_{\text{blue}}$ so that $xya, xza, yza \in S$ (all triples are red and, thus, in S because $xa, ya, za \in \partial S$). We must have xyz blue to avoid $C_4^{(3)}$ in S.

Clearly, any two blue edges with two common vertices are in the same blue component. If $z_1 z_2 z_3$ and $z_3 z_4 z_5$ are two blue edges in V'_{red}, then by (A.1) and since $|V'_{\text{red}}| \geq (1+2\eta)s$, there is a vertex $a \in V'_{\text{red}}$ so that $z_2 z_3 a, a z_3 z_4 \in K$. Hence, $z_2 z_3 a$ and $a z_3 z_4$ are blue and $z_1 z_2 z_3$ and $z_3 z_4 z_5$ are in the same blue component.

Finally, if $z_1z_2z_3$ and $z_4z_5z_6$ are two disjoint blue edges in V'_{red}, we can again find a vertex $a \in V'_{\mathrm{red}}$ so that $z_2z_3a, az_4z_5 \in K$. These two edges are blue and in the same blue component as observed above, therefore, $z_1z_2z_3$ and $z_4z_5z_6$ are in the same blue component as well. $\square$

We can clearly interchange colors red and blue in Observations A.3 and A.4 and obtain that $K \cap \binom{V'_{\mathrm{blue}}}{3} \subset C'_{\mathrm{red}}$ for some red component C'_{red}. Since one of V'_{red}, V'_{blue} must contain at least $\lceil t/2 \rceil \geq (3+6\eta)s$ vertices, we find greedily both a copy of $M_s^{(3)}$ and a copy of $C_4^{(3)}$, in either C'_{red} or C'_{blue}, contradicting (A.3). $\square$

Case 2: $|V'_{\mathrm{red}}| < (1+2\eta)s$ or $|V'_{\mathrm{blue}}| < (1+2\eta)s$.

By symmetry, we may assume that $|V'_{\mathrm{red}}| < (1+2\eta)s$ and $|V'_{\mathrm{blue}}| > (5+11\eta)s$. We first prove that the azure component A contains a matching $M^{(3)}_{(1+\eta)s}$ whose vertex set is in V'_{blue}.

Observation A.5. There exists a matching $M_A = M^{(3)}_{(1+\eta)s} \subset A$ with $V(M_A) \subset V'_{\mathrm{blue}}$.

Proof. Let $V'_{\mathrm{blue}} = U \cup U'$ be a partition of V'_{blue} such that $|U| = (1+2\eta)s$. Since $|U'| \geq (5+11\eta)s - (1+2\eta)s \geq (4+\eta)(1+2\eta)s$, Theorem 6.1 applied to the induced 2-coloring $K_{\mathrm{red}}[U'] \cup K_{\mathrm{blue}}[U']$ of $K[U']$ implies that there exists a matching $M = M^{(3)}_{(1+2\eta)s}$ in a monochromatic component of $K[U']$ (which is contained in some monochromatic component C in K).

Suppose that C is blue, but $C \neq A$ (otherwise this proof is finished). By (A.1) and (A.5), for a given $x_1x_2x_3 \in M$, all but at most $3\delta t + 3\delta t$ triples $y_1y_2y_3 \in M$ are such that $x_iy_j \in \partial A$ for every $1 \leq i,j \leq 3$, and all triples of the form $x_ix_jy_k$ are the edges of K. From this we conclude that all but at most $3\delta t|M|$ pairs $x_1x_2x_3, y_1y_2y_3 \in M$ are such that $x_iy_j \in \partial A$ for every $1 \leq i,j \leq 3$, and the subgraph of K induced on $\{x_1,x_2,x_3,y_1,y_2,y_3\}$ is $K_6^{(3)}$.

For each such a pair $x_1x_2x_3, y_1y_2y_3 \in M$, all the edges of the form $x_ix_jy_k$ and $x_iy_jy_k$ must be red and in the same red component because $x_ix_j, y_jy_k \in \partial C \neq \partial A$ and $x_iy_k \in \partial A$.

A moment's thought yields that there exist $s+2$ edges $x_{i1}x_{i2}x_{i3} \in M$ such that $x_{i1}x_{i2}x_{i-1j}, x_{i1}x_{i2}x_{(i+1)j}, x_{i2}x_{i3}$ $x_{i1}x_{i3}x_{(i-1)j}$, and $x_{i1}x_{i3}x_{(i+1)j}$ are red edges of K for all $i = 1,\dots,s+2$. It follows that these edges are in the same red component. Furthermore, $x_{i1}x_{i2}x_{(i+1)3}$, $x_{i3}x_{(i+1)1}x_{(i+1)2}$, where $i = 1,3,5,\dots$, form a matching of size s with a copy of $C_4^{(3)}$ on vertices $x_{12}, x_{13}, x_{22}, x_{23}$. This is, however, a contradiction to (A.3).

Hence assume $C = C_{\mathrm{red}}$ is red. By (A.3), we have $C_4^{(3)} \not\subset C_{\mathrm{red}}$. By Fact A.2 (see also the proof of Observation A.1), for arbitrary subsets $M' \subset M$, $U'' \subset U_{\mathrm{blue}}$ be, each of size $5\delta t$, there is an edge $xyz \in M'$ and a vertex $a \in U''$ such that xya, xza, $yza \in K$ and xa, ya, $za \in \partial A$. To avoid $C_4^{(3)}$ in C_{red}, at least one of the edges xya, xza, yza must be a blue edge, and, consequently, also in A.

Thus, using $(1+2\eta)s$ vertices in U and $(1+2\eta)s$ edges of M, we greedily find a matching M_A of size $(1+2\eta)s - 5\delta s > (1+\eta)s$ in A. Clearly, $V(M_A) \subset U \cup U' = V'_{\mathrm{blue}}$. $\square$

In view of Observation A.5 and the assumption (A.3), we know that $C_4^{(3)} \not\subset A$. We distinguish two subcases. In the first one we assume that almost all pairs of vertices from V'_{blue} are contained in the shadows of at most two red components.

Subcase 2a. There exist two red components C^1_{red} and C^2_{red} such that

$$\left|\binom{V'_{\mathrm{blue}}}{2} \setminus (\partial C^1_{\mathrm{red}} \cup \partial C^2_{\mathrm{red}})\right| < 2\sqrt[3]{\delta}t^2. \tag{A.7}$$

We now prove a series of observations. Recall that the scarlet component S exists whenever $V'_{\mathrm{red}} \neq \emptyset$. We now show that in that case either one of C^1_{red} and C^2_{red} equals S or can be replaced by S.

Observation A.6. If $V'_{\text{red}} \neq \emptyset$, then there exists a red component C_{red} such that

$$\left|\binom{V'_{\text{blue}}}{2} \setminus (\partial C_{\text{red}} \cup \partial S)\right| < 4\sqrt[6]{\delta}t^2. \tag{A.8}$$

Proof. If $|\partial C^1_{\text{red}} \cap \binom{V'_{\text{blue}}}{2}| \leq 3\sqrt[6]{\delta}t^2$ holds, then with $C_{\text{red}} = C^2_{\text{red}}$ we have

$$\left|\binom{V'_{\text{blue}}}{2} \setminus (\partial C_{\text{red}} \cup \partial S)\right| \overset{(A.7)}{<} 2\sqrt[3]{\delta}t^2 + \left|\partial C^1_{\text{red}} \cap \binom{V'_{\text{blue}}}{2}\right| \leq 4\sqrt[6]{\delta}t^2.$$

Hence, suppose that $\left|\partial C^1_{\text{red}} \cap \binom{V'_{\text{blue}}}{2}\right| > 3\sqrt[6]{\delta}t^2$ and $\left|\partial C^2_{\text{red}} \cap \binom{V'_{\text{blue}}}{2}\right| > 3\sqrt[6]{\delta}t^2$. We claim that there exist vertices $u, v, w \in V'_{\text{blue}}$ and $a \in V'_{\text{red}}$ such that $uv \in \partial C^1_{\text{red}} \cap \partial A$, $uw \in \partial C^2_{\text{red}} \cap \partial A$, $vw \in \partial C^1_{\text{red}} \cup \partial C^2_{\text{red}}$, $ua, va, wa \in \partial S$, and $uvw, uva, uwa, vwa \in K$.

Indeed, since every graph G contains a vertex of degree at least $|G|/|V(G)|$ and since (A.7) holds, there are $2\delta t$ vertices $u \in V'_{\text{blue}}$ such that

$$|\partial C_1(u) \cap V'_{\text{blue}}| + |\partial C_2(u) \cap V'_{\text{blue}}| > |V'_{\text{blue}}| - 4\delta t. \tag{A.9}$$

Fix an arbitrary vertex $a \in V'_{\text{red}}$. By (A.5), $|\partial S(a)| \geq (1-\delta)t$, therefore, there exists a vertex u satisfying (A.9) such that $au \in \partial S$.

Since au is an active pair, all but at most $2\delta t^2$ pairs vw are such that $auv, auw, avw, uvw \in K$. Since $a \in V'_{\text{red}}$, all but at most δt^2 pairs vw satisfy $va, wa \in \partial S$. Similarly, since $u \in V'_{\text{blue}}$, all but at most δt^2 pairs vw satisfy $uv, uw \in \partial A$. Finally, by (A.7), there are at most $2\sqrt[3]{\delta}t^2$ pairs $vw \in \binom{V'_{\text{blue}}}{2}$ such that $vw \notin \partial C^1_{\text{red}} \cup \partial C^2_{\text{red}}$. Let $Z \subset \binom{V'_{\text{blue}}}{2}$ be the set of all such exceptional pairs. Then $|Z| \leq 3\sqrt[3]{\delta}t^2$.

If $|\partial C^1_{\text{red}}(u) \cap V'_{\text{blue}}| \cdot |\partial C^2_{\text{red}}(u) \cap V'_{\text{blue}}| > 3\sqrt[3]{\delta}t^2 \geq |Z|$, then there are $v \in \partial C^1_{\text{red}}(u) \cap V'_{\text{blue}}$ and $w \in \partial C^2_{\text{red}}(u) \cap V'_{\text{blue}}$ such that $vw \notin Z$.

Otherwise, we have $|\partial C^2_{\text{red}}(u) \cap V'_{\text{blue}}| \leq \sqrt{3}\sqrt[6]{\delta}t$ and (A.9) implies that $|\partial C^1_{\text{red}}(u) \cap V'_{\text{blue}}| > |V'_{\text{blue}}| - 2\sqrt[6]{\delta}t$. It follows that at most $|V'_{\text{blue}}| \cdot 2\sqrt[6]{\delta}t \leq 2\sqrt[6]{\delta}t^2$ pairs of $\partial C^2_{\text{red}}$ are not contained in $\binom{\partial C^1_{\text{red}}(u) \cap V_{\text{blue}}}{2}$. Since

$$\left|\partial C^2_{\text{red}} \cap \binom{V'_{\text{blue}}}{2}\right| > 3\sqrt[6]{\delta}t^2 > 2\sqrt[6]{\delta}t^2 + 13\sqrt[3]{\delta}t^2 \geq 2\sqrt[6]{\delta}t^2 + |Z|,$$

there are $v, w \in \partial C^1_{\text{red}}(u) \cap V'_{\text{blue}}$ such that $vw \in \partial C^2_{\text{red}} \setminus Z$. In both case, the vertices a, u, v, w are the ones we are looking for.

To avoid a copy of $C_4^{(3)}$ in A, one of $uvw, uva, uwa, vwa \in K$ must be red (because $uv, uw \in \partial A$). It cannot be uvw because $uv \in \partial C^1_{\text{red}}$ and $uw \in \partial C^2_{\text{red}}$. Thus at least one of the edges uva, uwa, vwa must be a red edge, say uwa. Since $uw \in \partial C^2_{\text{red}}$ and $au \in \partial S$, we have $C^2_{\text{red}} = S$ and the proof is completed by setting $C_{\text{red}} = C^2_{\text{red}}$ and recalling (A.7).

From now on we assume that

$$\left|\binom{V'_{\text{blue}}}{2} \setminus (\partial C^1_{\text{red}} \cup \partial C^2_{\text{red}})\right| < 4\sqrt[6]{\delta}t^2, \tag{A.10}$$

and that $C^1_{\text{red}} = S$, if S exists.

Observation A.7. Every set $X \subset V'_{\text{blue}}$ with $|X| \geq 25\sqrt[12]{\delta}t$ contains a copy of $C_4^{(3)}$ in C^1_{red} or C^2_{red}.

Proof. Let $X \subset V'_{\text{blue}}$ with $|X| \geq 25\sqrt[12]{\delta}t$ be given. Note that by (A.10), there is a subset $X' \subset X$, where $|X \setminus X'| \leq 8\sqrt[12]{\delta}t$ such that for each $u \in X'$ there are at most $\sqrt[12]{\delta}t$ vertices $v \in X'$ such that $uv \notin C^1_{\text{red}} \cup C^2_{\text{red}}$. Moreover, by (A.5), for each $u \in X'$, we have $|\partial A(u) \cap X'| \geq |X'| - \delta t$ and, by (A.1), for each active pair $uv \in \binom{X'}{2}$, we have $|N_K(u,v) \cap X'| \geq |X'| - \delta t$.

Since $|X \setminus X'| + 12\sqrt[12]{\delta}t + 12\delta t + \binom{12}{2}\delta t + 13 < 25\sqrt[12]{\delta}t \le |X|$, there exists a subset $X_0 \subset X'$ of 13 vertices such that $\binom{X_0}{3} \subset K$ and $\binom{X_0}{2} \subset A \cap (\partial C^1_{\rm red} \cup \partial C^2_{\rm red})$.

Since $r(C_4^{(3)}) = 13$, the set $\binom{X_0}{3}$ contains a monochromatic copy $\mathcal{C}$ of $C_4^{(3)}$. It cannot be blue because all pairs of vertices of X are in ∂A and so $\mathcal{C}$ would be in A. Hence, $\mathcal{C}$ must be red and, thus, in $C^1_{\rm red}$ or $C^2_{\rm red}$ because $\binom{X_0}{2} \subset \partial C^1_{\rm red} \cup \partial C^2_{\rm red}$. □

Now, we are ready to finish the proof of Lemma 2.2 in Subcase 2a. Suppose first that $V'_{\rm red} = \emptyset$. By (A.4), we have $t - 8\delta t < |V'_{\rm blue}| \le t \le 7s$. By Observation A.7, every set of $25\sqrt[12]{\delta}t$ vertices in $V'_{\rm blue}$ contains a copy of $C_4^{(3)}$ in $C^1_{\rm red}$ or $C^2_{\rm red}$. Hence, we can find greedily, by taking one edge from a copy of $C_4^{(3)}$ and reusing the remaining vertex, a matching of size

$$\frac{|V'_{\rm blue}| - 25\sqrt[12]{\delta}t}{3} \ge \frac{(6+14\eta)s - 8\delta t - 25\sqrt[12]{\delta}\cdot 7s}{3} \ge 2s$$

in $C^1_{\rm red} \cup C^2_{\rm red}$. Thus, there is an index $i \in \{1,2\}$ such that $C^i_{\rm red}$ contains $M_s^{(3)}$ as well as a copy of $C_4^{(3)}$.

Assume now that $V'_{\rm red} \neq \emptyset$ and, thus, S exists and $C^1_{\rm red} = S$. We know (see Observation A.5) that A contains a matching M_A, $V(M_A) \subset V_{\rm blue}$, of size $(1+\eta)s$ but no $C_4^{(3)}$. As in the proof of Observation A.1, for arbitrary subsets $M' \subset M_A$, $U \subset V'_{\rm red}$, where $|M'| \ge 5\delta t$ and $|U| \ge 4\delta t$, there exist an edge $e \in M_A$, vertex $x \in U$, and an edge $f \in S$ so that $x \in f$ and $|e \cap f| = 2$. Hence, we can find a matching of size $|V'_{\rm red}| - 4\delta t < s$ in S that uses exactly $2|V'_{\rm red}| - 8\delta t$ vertices of $V'_{\rm blue}$. After this, we use the greedy procedure from the previous paragraph and find a matching in $S \cup C^2_{\rm red}$ of size $(|V'_{\rm blue}| - 2|V_{\rm red}| + 8\delta t - 25\sqrt[4]{\delta}t)/3$. Combining these two matchings and the fact that $|V_{\rm blue}| + |V_{\rm red}| \ge t - 8\delta t$ yields a matching in $S \cup C^2_{\rm red}$ of size

$$\frac{|V'_{\rm red}| + \left(|V'_{\rm blue}| - 2|V'_{\rm red}| + 8\delta t - 25\sqrt[12]{\delta}t\right)}{3} \ge \frac{t - 25\sqrt[12]{\delta}\cdot 7s}{3} \ge 2s,$$

as before. Consequently, either S or $C^2_{\rm red}$ contains $M_s^{(3)}$. Note that at least one edge of this matching comes from a copy of $C_4^{(3)}$ in S or $C^2_{\rm red}$. Thus, in either case, we have $M_s^{(3)}$ and $C_4^{(3)}$ in the same red component.

Subcase 2b. Inequality (A.7) does not hold for any two red components $C^1_{\rm red}$ and $C^2_{\rm red}$.

Similarly to Section 7, we will first show that in this case the red components can be grouped into three large sets.

Observation A.8. There exists a partition $\binom{V}{2} = F^1 \cup F^2 \cup F^3$ such that

(i) F^1, F^2, F^3 are pairwise disjoint,
(ii) $|F^i[V'_{\rm blue}]| \ge \sqrt[3]{\delta}$ for $i = 1,2,3$,
(iii) for every red component $C_{\rm red}$ there exists $i \in \{1,2,3\}$ such that $\partial C_{\rm red} \subset F^i$.

The proof of this observation is the same as the proof of Observation 7.10 and we omit it here. For convenience, set $\tilde{F}^i = F^i[V'_{\rm blue}]$ and $\deg_i(v) = \deg_{\tilde{F}^i}(v)$, $i = 1,2,3$, $v \in V'_{\rm blue}$.

Observation A.9. For every vertex $v \in V_{\rm blue}$ there is an index $i \in \{1,2,3\}$ so that $\deg_i(v) < 7\delta t$.

Proof. Suppose that there is a vertex $v \in V_{\rm blue}$ such that $\deg_i(v) \ge 7\delta t$ for all $i = 1, 2, 3$. Denote by U_i the neighborhood of v in $\tilde{F}^i$. We will assume that $vu \in \partial S$ for all $u \in U_1 \cup U_2 \cup U_3$ since, by (A.5), only at most δt vertices do not satisfy this.

We call an edge $u_1u_2u_3 \in K$ v-good if $vu_1u_2, vu_1u_3, vu_2u_3 \in K$. Notice that by (A.1), for every $vu_1u_2 \in K$, all but at most $3\delta t$ vertices $u_3 \in U_3$ are such that $u_1u_2u_3 \in K$ is v-good. Hence, since

$|U_3| > 3\delta t + 1$, for every pair $u_1 \in U_1$, $u_2 \in U_2$, where $vu_1u_2 \in K$, there exists $u_3 \in U_3$ such that $u_1u_2u_3 \in K$ is v-good.

Suppose that a v-good edge $u_1u_2u_3 \in K$ also satisfies $u_i \in U_i$, $i = 1,2,3$. Since the pairs vu_1, vu_2, vu_3 belong to the shadows of distinct red components, all edges vu_iu_j, $1 \le i < j \le 3$, are blue and thus in the azure component A (because $vu_1, vu_2, vu_3 \in \partial A$). Consequently, since there is no $C_4^{(3)}$ in A, edge $u_1u_2u_3$ must be red.

Let $u_1u_2u_3, v_1v_2v_3 \in K$, where $u_i, v_i \in U_i$, $i = 1,2,3$, be two v-good edges. Since $|U_3| > 6\delta t$, there is a vertex $z_3 \in U_3$ so that $u_1u_2z_3$, $v_1v_2z_3$ are v-good edges. Similarly, since $|U_1| > 6\delta t$, there is a vertex $z_1 \in U_1$ so that $z_1u_2z_3$ and z_1v_2, z_3 are v-good. Hence, all v-good edges $u_1u_2u_3 \in K$, where $u_i \in U_i$, $i = 1,2,3$, are in the same red component.

Consequently, all pairs of vertices $u_i \in U_i$ and $u_j \in U_j$, $i \neq j$, $vu_iu_j \in K$, are in the shadow of the same red component. Without loss of generality we may assume that $u_iu_j \in \tilde{F}^1$.

Take any three vertices u_i, u_i' such that $u_i, u_i' \in U_i$ and $vu_iu_i' \in K$. Since $|U_j| > 3\delta t + 1$, there exists $u_j \in U_j$ such that $u_iu_i'u_j$ is a v-good edge. The edges vu_iu_j and $vu_i'u_j$ are both in A and $C_4^{(3)} \not\subset A$, hence either vu_iu_i' is red or $u_iu_i'u_j$ is red. In the first case, $u_iu_i' \in \tilde{F}^i$, while in the second case $u_iu_i' \in \tilde{F}^1$.

From (A.1) we have that at most $2\delta t^2$ pairs u_iu_i' of $\tilde{F}^i$ are such that $u_iu_i'v \notin K$. Since $|\tilde{F}^i| \ge \sqrt[3]{\delta}$, it follows that at least $\sqrt[3]{\delta}t^2/2$ pairs of $\tilde{F}^i$ are contained in $\{v\} \cup U_i$, $i = 2,3$. Hence, there exists a matching M_i in $\tilde{F}^i$ of size $5\delta t$. For each $u_iu_i' \in M_i$ there are at most $2\delta t$ pairs vv' such that either $u_iu_i'v$ or $u_iu_i'v' \notin K$. From Fact A.2 we obtain that there exist vertices $u_i, u_i' \in U_i$ so that $vu_iu_i' \in K$, $u_iu_i' \in \tilde{F}^i$, $i = 2,3$.

If all four edges induced by $\{u_2, u_2', u_3, u_3'\} \subset V_{\text{blue}}$ were blue, we would have $C_4^{(3)}$ in A – a contradiction. Hence, at least one of them is red, say $u_2u_2'u_3$. Since $u_2u_3 \in \tilde{F}^1$, we have $u_2u_2' \in \tilde{F}^1$. But then $u_2u_2' \in \tilde{F}^1 \cap \tilde{F}^2$ – a contradiction with Observation A.8(i). □

For $1 \le i < j \le 3$, let $W_{ij} = \{v \in V'_{\text{blue}} \colon \deg_i(v) \ge 7\sqrt{\delta}t, \deg_j(v) \ge 7\sqrt{\delta}t\}$. Next, we prove that W_{12}, W_{13}, and W_{23} have each at least $24\sqrt{\delta}t$ vertices.

Observation A.10. $|W_{ij}| \ge 24\sqrt{\delta}t$ for $1 \le i < j \le 3$.

Proof. By symmetry, we can restrict ourselves to the case $i = 1$ and $j = 2$. We shall show that $|W_{12}| \ge 24\sqrt{\delta}t$. Let $G_i := \tilde{F}^i \cap \partial A$, $i = 1,2$. Since $|\tilde{F}^i| > \sqrt[3]{\delta}t^2$, $i = 1,2$, and by (A.1), we have $|G_i| > \sqrt[3]{\delta}t^2 - 2\sqrt{\delta}t^2$, $i = 1,2$.

Hence, G_i, $i = 1,2$, contains a subgraph G_i' with minimum degree at least $(\sqrt[3]{\delta}t^2 - 2\sqrt{\delta}t^2)/8 > 1152\sqrt{\delta}t$. In G_i', $i = 1,2$, we find greedily matching M_i' of size at least $576\sqrt{\delta}t$. Finally, from M_1' and M_2' we select a submatchings $M_1 \subseteq M_1'$ and $M_2 \subseteq M_2'$ of size $t_0 = 144\sqrt{\delta}t$ each so that $M_1 \cup M_2$ is a matching as well. (Take such M_1, M_2 with $|M_1| = |M_2| = t$ maximum possible. If $t < t_0$, then $M_1 \cup M_2$ intersects at most $4t < 576\sqrt{\delta}t$ edges from each M_1' and M_2' and we can enlarge both M_1 and M_2)

For any pair $v_1v_1' \in M_1$, it follows from (A.1) that all but at most δt pairs $v_2v_2' \in M_2$ are such that the triples $v_1v_1'v_2, v_1v_1'v_2'$ are in K. Similarly, for any pair $v_2v_2' \in M_2$, all but at most δt pairs $v_1v_1' \in M_1$ are such that the triples $v_2v_2'v_1, v_2v_2'v_1'$ are in K.

Hence, at least $|M_1|(|M_2| - 2\delta t)/2$ pairs $v_1v_1' \in M_1$ and $v_2v_2' \in M_2$ are such that all triples in $\{v_1, v_1', v_2, v_2'\}$ are in K. Since the azure component contains no copy of $C_4^{(3)}$, one of these triples must be red, and, consequently, at least two of the pairs v_1v_2, $v_1'v_2$, v_1v_2', $v_1'v_2'$, must belong to a red component from $\mathcal{F}_1 \cup \mathcal{F}_2$.

Thus, we may assume that at least $|M_1|(|M_2| - 2\delta t)/2 \ge |M_1||M_2|/3$ edges joining vertices saturated by pairs from M_1 with the vertices saturated by M_2 are from $\tilde{F}^1$. Then at least $|M_2|/6 \ge 24\sqrt{\delta}t$ vertices saturated by the pairs from $M_2 \subseteq G_2$ are incident to at least $|M_1|/6 \ge 7\delta t$ pairs which are from $\tilde{F}^1$. Since each vertex saturated by M_2 has degree at least $1152\sqrt{\delta}t$ in $G_2 \subset \tilde{F}^2$, we have $|W_{12}| \geq 24\sqrt{\delta}t$. □

Now we claim that all except at most $14 \cdot \delta t^2$ pairs between W_{ij} and $W_{i'j'}$ must be from $\tilde{F}^\ell$, where $\ell = \{i,j\} \cap \{i',j'\}$. Indeed, suppose $\ell = 1$ and there are more than $14 \cdot \delta t^2$ pairs between W_{12} and W_{13} that are not from $\tilde{F}^1$. Then at least $7\delta t^2$ pairs must be from, say, $\tilde{F}^3$. Consequently, there is a vertex $v \in W_{12}$ incident with at least $7\delta t$ edges from $\tilde{F}^3$, yielding $\deg_3(v) \geq 7\delta t$. But $v \in W_{12}$ means $\deg_1(v), \deg_2(v) \geq 7\delta t$ - a contradiction with Claim A.9.

It follows that one can find vertices $w_{12}, w'_{12} \in W_{12}$, $w_{13}, w'_{13} \in W_{13}$, $w_{23}, w'_{23} \in W_{23}$, such that:

(i) the subgraph of K induced by $\{w_{12}, w'_{12}, w_{13}, w'_{13}, w_{23}, w'_{23}\}$ is $K_6^{(3)}$,
(ii) all pairs of vertices from $\{w_{12}, w'_{12}, w_{13}, w'_{13}, w_{23}, w'_{23}\}$ are in ∂A,
(iii) for $1 \leq i < j \leq 3$, $1 \leq i' < j' \leq 3$, $\{i,j\} \neq \{i',j'\}$, the pairs $w_{ij}w_{i'j'}$, $w'_{ij}w_{i'j'}$, $w_{ij}w'_{i'j'}$, $w'_{ij}w'_{i'j'}$ are from $\tilde{F}^\ell$, where $\ell = \{i,j\} \cap \{i',j'\}$.

Indeed, by Claim A.10, each of the sets W_{12}, W_{13}, W_{23} has size at least $24\sqrt{\delta}t$. By the previous paragraph, (A.1), and (A.5), there are 2 vertices $w_{12}, w'_{12} \in W_{12}$ and subsets $W'_{13} \subset W_{13}$, $W'_{23} \subset W_{23}$ such that $|W'_{13}|, |W'_{23}| \geq 19\sqrt{\delta}t$, all pairs $w_{12}x$, $w'_{12}x \in \partial A \cap \tilde{F}^1$ for every $x \in W'_{13}$, and all pairs $w_{12}x$, $w'_{12}x \in \partial A \cap \tilde{F}^2$ for every $x \in W'_{23}$.

Since all but at most $14 \cdot \delta t^2$ pairs between W'_{13} and W'_{23} must be from $\tilde{F}^3$, $|W'_{13}|, |W'_{23}| \geq 19\sqrt{\delta}t$, and by (A.1) and (A.5), there exists $w_{13}, w'_{13} \in W'_{13}$, $w_{23}, w'_{23} \in W'_{23}$ such that (i), (ii), and (iii) hold.

In the same one as in Section 7 (see page 30), we conclude that the sub-hypergraph H induced in K by vertices $w_{12}, w'_{12}, w_{13}, w'_{13}, w_{23}, w'_{23} \in V_{\text{blue}}$ contains a copy of $C_5^{(3)}$ in the azure component A. □

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