

The random interchange process on the hypercube

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Abstract

We prove the occurrence of a phase transition accompanied by the emergence of cycles of diverging lengths in the random interchange process on the hypercube.

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1 Introduction

The interchange process is defined on a finite graph. With any edge is associated the transposition of its endvertices. The outcomes of the interchange process consist of sequences of random transpositions and the main questions of interest deal with the cycle structure of the random permutation that is obtained as the composition of these transpositions. As the number of random transpositions increases, a phase transition may occur that is indicated by the emergence of cycles of diverging lengths involving a positive density of vertices.

The most relevant graphs are regular graphs with an underlying “geometric structure” like a finite cubic box in \mathbb{Z}^d with edges between nearest neighbours. But the problem of proving the emergence of long cycles is out of reach for now and recent studies have been devoted to simpler graphs such as trees [4, 12] and complete graphs [13, 5, 6]. (Note also the intriguing identities of Alon and Kozma based on the group structure of permutations [3].) The motivation for the present article is to move away from the complete graph towards \mathbb{Z}^d . We consider the hypercube $\{0, 1\}^n$ in the large n limit and establish the occurrence of a phase transition demonstrated by the emergence of cycles larger than $2^{(\frac{1}{2}-\varepsilon)n}$. Our proof combines the recent method of Berestycki [5], which was used for the complete graph but is valid more generally, with an estimate of the rate of splittings that involves the isoperimetric inequality for hypercubes.

Besides its interest in probability theory, the random interchange process appears in studies of quantum spin systems [14], see also [11] for a review. Cycle lengths and cycle correlations give information on the magnetic properties of the spin systems. The setting is a bit different, though. First, the number of transpositions is not a fixed parameter, but a Poisson random variable. Second, there is an additional weight of the form $\theta^{\#\text{cycles}}$

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with $\theta = 2$ in the case of the spin $\frac{1}{2}$ quantum Heisenberg model. The first feature is not a serious obstacle, but the second feature turns out to be delicate. Notice that Björnberg recently obtained results about the occurrence of macroscopic cycles on the complete graph in the case $\theta > 1$ [7, 8]. Correspondence between random transpositions and quantum models when $\theta = 3, 4, \dots$, and with a more general class of random loop models, can be found in [1, 15]. We expect that our hypercube results can be extended to these situations as well, but it may turn out not to be entirely straightforward.

2 Setting and results

Let $G_n = (Q_n, E_n)$ be a graph whose $N = 2^n$ vertices form a *hypercube* $Q_n = \{0, 1\}^n$ with edges joining nearest-neighbours—pairs of vertices that differ in exactly one coordinate, $E_n = \{\{x, y\} : x, y \in Q_n, |x - y|_1 = 1\}$, $|E_n| = \frac{Nn}{2}$.

Let Ω_n be the set of infinite sequences of edges in E_n . For $t \in \mathbb{N}$ by $\mathcal{F}_{n,t}$ we denote the σ -algebra generated by the first t elements of the sequence. Further, for $t \in \mathbb{N}$ we use $\Omega_{n,t}$ to denote the set of sequences of t edges $e = (e_1, \dots, e_t)$, where $e_s \in E_n$ for all $s = 1, \dots, t$. The σ -algebra $\mathcal{F}_{n,t}$ will be identified with the total σ -algebra over $\Omega_{n,t}$. For an event $A \in \mathcal{F}_{n,t}$ we set

$$\mathbb{P}_n(A) = |A| \left(\frac{2}{Nn}\right)^t,$$

i.e. edges are chosen independently and uniformly from E_n .

Using τ_e to denote the transposition of the two endvertices of an edge $e \in E_n$, we can view the sequence $e \in \Omega_{n,t}$ as a series of *random interchanges* generating a *random permutation* $\sigma_t = \tau_{e_t} \circ \tau_{e_{t-1}} \circ \dots \circ \tau_{e_1}$ on Q_n . For any $\ell \in \mathbb{N}$, let $V_t(\ell)$ be the random set of vertices that belong to permutation cycles of lengths greater than ℓ in σ_t .

We start with the straightforward observation that only small cycles occur in σ_t when t is small. It is based on the fact that the random interchange model possesses a natural percolation structure when viewing any edge contained in e as opened. The probability that a particular edge remains closed by the time t is $(1 - \frac{1}{Nn/2})^t$. Since the set of vertices of any cycle must be contained in a single percolation cluster, only small cycles occur when percolation clusters are small.

Theorem 2.1. *Let $c < 1/2$ and $\epsilon > 0$. Then there exists n_0 such that*

$$\mathbb{P}_n(|V_t(\kappa n)| = 0) > 1 - \epsilon \kappa^{-3/2}$$

for all $t \leq cN$, all $\kappa \geq \frac{2 \ln 2}{(1-2c)^2}$, and all $n > n_0$.

Proof. In view of the above mentioned percolation interpretation of the random interchange model, the claim follows from the fact that the percolation model on the hypercube graph Q_n is subcritical for $p = 2c/n$ with $c < 1/2$ and the size of the largest cluster is of the order n (see [2]). The value $p = 2c/n$ corresponds to $t = cN$ implying that the probability of any particular edge to be open is $1 - (1 - \frac{2}{Nn})^{cN} \sim \frac{2c}{n}$. The claim of the theorem follows from [9, Theorem 9]. In particular, the last displayed inequality in its proof can be reinterpreted as a claim that

$$\mathbb{E}_n(|V_t(\kappa n)|) \leq \epsilon(n) \kappa^{-3/2} \tag{2.1}$$

with $\epsilon(n) \rightarrow 0$ as $n \rightarrow \infty$ whenever $\kappa > \frac{2 \ln 2}{(1-2c)^2}$. □

Our main result addresses the emergence of long cycles for large times, $t > N/2$. We expect that cycles of order N occur for all large times; here we prove a weaker claim: cycles larger than $N^{\frac{1}{2}-\epsilon}$ occur for a “majority of large times”.

Theorem 2.2. *Let $c > \frac{1}{2}$ and let (Δ_n) be a sequence of positive numbers such that $\Delta_n n / \log n \rightarrow \infty$ as $n \rightarrow \infty$. Then there exist $\eta(c) > 0$ and n_0 such that for all $n > n_0$, all $T > cN$, and all $a > 0$, we have*

$$\frac{1}{\Delta_n T} \sum_{t=T+1}^{\lfloor (1+\Delta_n)T \rfloor} \mathbb{E}_n \left(\frac{|V_t(N^a)|}{N} \right) \geq \eta(c) - a.$$

For $c > 1$, we can take $\eta(c) = \frac{1}{2}(1 - \frac{1}{c})$.

Let us observe that the highest achievable value of the exponent a is just below $1/2$; this can be accomplished only with c becoming large. But we expect that the size of the long cycles is of order N . In fact, one can formulate a precise conjecture, namely that the joint distribution of the lengths of long cycles is Poisson-Dirichlet. This was proved in the complete graph [13], and advocated in \mathbb{Z}^d with $d \geq 3$ [11].

The proof of Theorem 2.2 can be found in Section 3.4; it is based on a series of lemmas obtained in the next section.

We can choose $\Delta_n \equiv \Delta > 0$, rather than a sequence that tends to 0. In this case, Theorem 2.2 takes a simpler form, which perhaps expresses the statement ‘long cycles are likely’ more directly.

Corollary 2.3. *Let $a \in (0, 1/2)$, $\Delta > 0$, and $\epsilon_1 \in (0, \frac{1}{2} - a)$. Then there exists $c > 1$ and $\epsilon_2 > 0$ such that for n large enough we have*

$$\frac{1}{\Delta T} \sum_{t=T+1}^{\lfloor (1+\Delta)T \rfloor} \mathbb{P}_n \left(\frac{|V_t(N^a)|}{N} \geq \epsilon_1 \right) \geq \epsilon_2$$

for all $T > cN$.

Proof. This follows from Theorem 2.2 and Markov’s inequality. Namely,

$$\begin{aligned} \mathbb{P}_n \left(\frac{|V_t(N^a)|}{N} \geq \epsilon_1 \right) &= 1 - \mathbb{P}_n \left(1 - \frac{|V_t(N^a)|}{N} \geq 1 - \epsilon_1 \right) \\ &\geq 1 - \frac{1}{1 - \epsilon_1} \mathbb{E}_n \left(1 - \frac{|V_t(N^a)|}{N} \right) \\ &\geq 1 - \frac{1 - \eta(c) + a}{1 - \epsilon_1}. \end{aligned} \tag{2.2}$$

This is positive for $\epsilon_1 < \eta(c) - a$. □

3 Occurrence of long cycles

3.1 Number of cycles vs number of clusters

Cycle structure and percolation properties are intimately related, and we will rely on Berestycki’s key observation that the number of cycles remains close to the number of clusters [5]. Let N_t denote the random variable for the number of cycles of the random permutation σ_t at time t , and \tilde{N}_t the number of clusters of the underlying percolation model. Notice that $N_t \geq \tilde{N}_t$.

Let us consider the possible outcomes when a new random transposition arrives at time t . There are three possibilities; the endpoints of a new edge e_t are either both in the same cycle of σ_{t-1} (and thus also in the same cluster), or in the same cluster but in different cycles, or in different clusters. Correspondingly, we are distinguishing three events:

- S_t , a splitting of a cycle where $N_t = N_{t-1} + 1$ and $\tilde{N}_t = \tilde{N}_{t-1}$. Indeed, a splitting of any cycle necessarily occurs within the same percolation cluster.

- M_t , a merging of two cycles within the same cluster: $N_t = N_{t-1} - 1$ and $\tilde{N}_t = \tilde{N}_{t-1}$.
- \tilde{M}_t , a merging of two cycles in distinct clusters: $N_t = N_{t-1} - 1$ and $\tilde{N}_t = \tilde{N}_{t-1} - 1$.

Also, let $I_t = S_t \cup M_t$ be the event where the endpoints of the edge e_t belong to the same cluster. Notice that $\tilde{M}_t = I_t^c$. Obviously, the three events above are mutually disjoint and cover all outcomes,

$$\Omega_{n,t} = S_t \cup M_t \cup \tilde{M}_t. \tag{3.1}$$

Notice that

$$N_t - \tilde{N}_t = \sum_{i=1}^t (\mathbb{1}_{S_i} - \mathbb{1}_{M_i}). \tag{3.2}$$

A key in the proof of Theorem 2.2 is the isoperimetric inequality of the hypercube Q_n . Namely, for any set $A \subset Q_n$, the number $|E(A)|$ of edges of G_n whose both end-vertices are in A is

$$|E(A)| \leq \frac{1}{2}|A| \log |A|. \tag{3.3}$$

Here (and elsewhere in this paper) \log is always meant as the logarithm of base 2. See [10] for the proof of the bound in this form. It implies a lower bound on the number $|E(A|A^c)|$ of edges connecting A with its complement $A^c = Q_n \setminus A$, namely

$$|E(A|A^c)| \geq |A|(n - \log |A|). \tag{3.4}$$

We are not referring to this inequality in this article, but we found it useful in discussions.

Theorem 2.2 would follow from the following lemma once its assumption is proven.

Lemma 3.1. *Assume that $\mathbb{P}_n(S_t) \geq \lambda$ with $\lambda \in (0, 1)$. Then*

$$\mathbb{E}_n \left(\frac{|V_t(N^a)|}{N} \right) \geq \frac{\lambda - a}{1 - a}$$

for any $a \in (0, \lambda)$.

Proof. Let \mathcal{C}_{t-1} denote the set of cycles at time $t - 1$. Since the total number of edges is $Nn/2$, and at most $\frac{1}{2} \sum_{C \in \mathcal{C}_{t-1}} |C| \log |C|$ edges cause a splitting, we have

$$\mathbb{P}_n(S_t | \mathcal{C}_{t-1}) \leq \frac{1}{Nn} \sum_{C \in \mathcal{C}_{t-1}} |C| \log |C|. \tag{3.5}$$

It follows that

$$\begin{aligned} \lambda &\leq \mathbb{P}_n(S_t) = \mathbb{E}_n(\mathbb{P}_n(S_t | \mathcal{C}_{t-1})) \\ &\leq \frac{1}{Nn} \mathbb{E}_n \left(\sum_{C \in \mathcal{C}_{t-1}} |C| \log |C| \right) \\ &= \frac{1}{Nn} \mathbb{E}_n \left(\sum_{C \in \mathcal{C}_{t-1}: |C| \leq N^a} |C| \log |C| \right) + \frac{1}{Nn} \mathbb{E}_n \left(\sum_{C \in \mathcal{C}_{t-1}: |C| > N^a} |C| \log |C| \right) \\ &\leq \frac{a}{N} \mathbb{E}_n \left(\sum_{C \in \mathcal{C}_{t-1}: |C| \leq N^a} |C| \right) + \frac{1}{N} \mathbb{E}_n \left(\sum_{C \in \mathcal{C}_{t-1}: |C| > N^a} |C| \right). \end{aligned} \tag{3.6}$$

Using $\sum_{C \in \mathcal{C}_{t-1}} |C| = N$ and $\sum_{C \in \mathcal{C}_{t-1}: |C| > N^a} |C| = |V_t(N^a)|$, we get the lemma. \square

What remains to be done is to establish a lower bound on the probability for an edge to connect vertices within a cycle and thus splitting it. We will get it by combining lower bounds on the probability $\mathbb{P}_n(I_t)$ for an edge to connect vertices within one cluster and on the rate $\mathbb{P}_n(S_t)/\mathbb{P}_n(I_t)$ for those actually connecting vertices within a cycle.

As it turns out, we can verify the latter lower bound only in a mean sense, averaging over an interval $[T, T + L]$ where T is large. The ratio L/T can be chosen to vanish but not too fast. We will use the following corollary, whose proof is essentially a verbatim repetition of the proof above.

Corollary 3.2. *Assume that for some $T, L \in \mathbb{N}$, and $\lambda \in (0, 1)$, we have*

$$\frac{1}{L} \sum_{t=T+1}^{T+L} \mathbb{P}_n(S_t) \geq \lambda.$$

Then

$$\frac{1}{L} \sum_{t=T+1}^{T+L} \mathbb{E}_n \left(\frac{|V_t(N^a)|}{N} \right) \geq \frac{\lambda - a}{1 - a}$$

for any $a \in (0, \lambda)$.

3.2 Lower bound on the probability of I_t

Here we show that, if the time is large enough, there is a positive probability that the vertices of a random edge belong to the same cluster. Equivalently, we need an upper bound on the probability of the event $\widetilde{M}_t = I_t^c$ that two clusters are merging. The first lemma applies to $c > \frac{1}{2}$; the second lemma is restricted to $c > 1$ but it gives an explicit bound. Let \widetilde{V}_t denote the largest percolation cluster after t random transpositions.

Lemma 3.3. *Assume that $\mathbb{E}_n(|\widetilde{V}_t|) > c_0 N$ for some constant $c_0 > 0$. Then there exists $c' > 0$ such that*

$$\mathbb{P}_n(I_t) > c' c_0 (1 - o(1)).$$

Proof. It is based on [2, Remark 2], which states that there exist $\varepsilon > 0$ and $c' > 0$ such that

$$\mathbb{P}_n(|W_t| > N - N^{1-\varepsilon}) = 1 - o(1), \tag{3.7}$$

where W_t is the set of vertices which have at least $c'n$ neighbours in \widetilde{V}_t . By only considering edges within the largest cluster, we obtain

$$\mathbb{P}_n(I_t) \geq \frac{c'}{N} \mathbb{E}_n(|\widetilde{V}_t \cap W_t|). \tag{3.8}$$

Using $|\widetilde{V}_t \cap W_t| \geq |\widetilde{V}_t| - |W_t^c|$, the lemma follows. □

Remark: In [2] the authors use their Remark 2 as an indication that for $t > \frac{1}{2}N$ the second largest cluster is of size $o(N)$. Actually, this has been proven in [9, Theorem 31] where it was shown that the size of the second largest cluster is of the order at most $n/(2c - 1)^2$ (we adhere here to our notation with critical $c = 1/2$). The claim (3.7) thus follows immediately, combining [2, Lemma 3] — which states the same for the set of all vertices that have at least $c'n$ neighbours in clusters of size at least n^2 , with [9, Theorem 31] — which implies that this set actually coincides with W_t . (Notice that in both [2] and [9, Theorem 31], the results are actually formulated for percolation clusters on the hypercube with probability of an edge being occupied chosen as $p = 2c/n$.)

We state and prove the next lemma for the hypercube, but it actually holds for any finite graph.

Lemma 3.4. *Let $t \in \mathbb{N}$ and $\delta \in (0, 1)$. Then*

$$\mathbb{P}_n(\widetilde{M}_t) \leq N/t + t^{-(1-\delta)/2} + \exp(-t^\delta/2).$$

Proof. We recall that $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the filtration associated with the process of adding edges (i.e. \mathcal{F}_t contains information about the first t edges), and define the random variables

$$p_t = \mathbb{E}_n(\mathbb{1}_{\widetilde{M}_t} | \mathcal{F}_{t-1}). \tag{3.9}$$

Simple but crucial observations are that for any $t \in \mathbb{N}$, we have

$$1 \geq p_t \geq p_{t+1} \quad \text{and} \quad \sum_{i=1}^t \mathbb{1}_{\widetilde{M}_i} \leq N. \tag{3.10}$$

Indeed, adding an edge decreases the chance of next merging and the total number of mergings is smaller than the size of the graph. Let us define the process $\{X_t\}_{t \geq 0}$ by

$$X_t = \sum_{i=1}^t (\mathbb{1}_{\widetilde{M}_i} - p_i). \tag{3.11}$$

One verifies that it is a martingale and $|X_{t+1} - X_t| \leq 1$. By the Azuma inequality we have $\mathbb{P}_n(X_t \leq -t^{(1+\delta)/2}) \leq \exp(-t^{1+\delta}/(2t)) = \exp(-t^\delta/2)$. Hence,

$$\begin{aligned} 1 - \exp(-t^\delta/2) &\leq \mathbb{P}_n\left(\sum_{i=1}^t (\mathbb{1}_{\widetilde{M}_i} - p_i) \geq -t^{(1+\delta)/2}\right) \\ &= \mathbb{P}_n\left(\sum_{i=1}^t \mathbb{1}_{\widetilde{M}_i} \geq -t^{(1+\delta)/2} + \sum_{i=1}^t p_i\right). \end{aligned} \tag{3.12}$$

Using (3.10) we estimate further,

$$\begin{aligned} 1 - \exp(-t^\delta/2) &\leq \mathbb{P}_n\left(\sum_{i=1}^t \mathbb{1}_{\widetilde{M}_i} \geq -t^{(1+\delta)/2} + tp_t\right) \\ &\leq \mathbb{P}_n(N + t^{(1+\delta)/2} \geq tp_t). \end{aligned} \tag{3.13}$$

In other words $\mathbb{P}_n(p_t \geq N/t + t^{-(1-\delta)/2}) \leq \exp(-t^\delta/2)$. Finally,

$$\begin{aligned} \mathbb{P}_n(\widetilde{M}_t) &= \mathbb{E}_n(\mathbb{1}_{\widetilde{M}_t}) \\ &= \mathbb{E}_n(p_t) \\ &\leq \mathbb{P}_n(p_t \geq N/t + t^{-(1-\delta)/2}) + N/t + t^{-(1-\delta)/2} \\ &\leq \exp(-t^\delta/2) + N/t + t^{-(1-\delta)/2}. \end{aligned} \quad \square$$

3.3 Lower bound on the rate $\mathbb{P}_n(S_t)/\mathbb{P}_n(I_t)$

Let us begin with a bound on the probability of unfavourable splittings that result in short cycles. We define the event $S_t^{\leq k} \subset S_t$ as those splittings that result in producing a cycle of length less than or equal to k (or in two such cycles).

Lemma 3.5. *For any $n, t, k \in \mathbb{N}$ we have*

$$\mathbb{P}_n(S_t^{\leq k}) \leq \frac{2 \log(2k)}{n}.$$

Proof. Given an arbitrary configuration $e \in \Omega_{n,t}$ yielding a collection of cycles covering Q_n , we can find a family of sets $\{A_i\}, A_i \subset Q_n$ such that

- (a) $|A_i| \leq 2k$;
- (b) $\sum_i |A_i| \leq 2N$;

(c) $\mathbb{1}_{S_t^{\leq k}} \leq \sum_i \mathbb{1}_{e_t \in E(A_i)}$, where $e_t \in E_n$ is the random edge at time t .

Indeed, to each cycle of length less than or equal to $2k$, we define A_i to be its support. For a cycle of length $\ell > 2k$, we label its vertices consecutively by natural numbers (starting from an arbitrary one) identifying the labels $j, \ell + j, 2\ell + j, \dots, j = 1, \dots, \ell$. Denoting $m = \lfloor \ell/k \rfloor$, notice that $mk \leq \ell < (m + 1)k$ and $\ell + k < (m + 2)k < 2\ell$. We cover the cycle by the following collection of intervals

$$(1, \dots, 2k), (k + 1, \dots, 3k), \dots, ((m - 1)k + 1, \dots, (m + 1)k), (mk + 1, \dots, \ell + k)$$

if $mk < \ell$. In the case $mk = \ell$, the last interval is skipped and the collection ends with $((m - 1)k + 1, \dots, (m + 1)k)$. Clearly, the length of all intervals is either $2k$ or, for the last one, $\ell + k - mk < 2k$ thus (a) holds. Further, (b) is implied by the fact that any site of the cycle is covered exactly twice. Moreover, any pair $j_1 < j_2$ such that $j_1 \in (1, \dots, \ell)$ and $j_2 - j_1 < k$ is necessarily contained in at least one of above intervals. Namely, if $j_1 \in (rk + 1, \dots, (r + 1)k)$ (resp. $j_1 \in (mk + 1, \dots, \ell)$ for the last interval if $mk < \ell$), then $j_1, j_2 \in (rk + 1, \dots, (r + 2)k)$ (resp. $j_1, j_2 \in (mk + 1, \dots, \ell + k)$). As a result, (c) is verified and thus we get

$$\mathbb{P}_n(S_t^{\leq k}) \leq \sum_i \mathbb{P}_n(e_t \in E(A_i)). \tag{3.14}$$

The number of edges $E(A_i)$ induced by A_i is, according to the isoperimetric inequality (3.3), bounded by $\frac{1}{2}|A_i| \log |A_i|$. Given that the number of all edges in Q_n is $\frac{Nn}{2}$, we get $\mathbb{P}_n(e_t \in E(A_i)) \leq \frac{|A_i| \log |A_i|}{Nn} \leq |A_i| \frac{\log(2k)}{Nn}$. Using also that $\sum_i |A_i| \leq 2N$, we get the claimed bound

$$\mathbb{P}_n(S_t^{\leq k}) \leq \sum_i |A_i| \frac{\log(2k)}{Nn} \leq 2N \frac{\log(2k)}{Nn} = \frac{2 \log(2k)}{n}. \tag{3.15} \quad \square$$

The bound from Lemma 3.5 can be used to show that the number of cycles N_t does not depart too far from the number of clusters \tilde{N}_t .

Lemma 3.6. *There exists n_1 such that for $n \geq n_1$ and any $t \in \mathbb{N}$ we have*

$$\mathbb{E}_n(N_t - \tilde{N}_t) \leq t \frac{3 \log(4n)}{n}.$$

Proof. Let $N_t^{\leq 2n}$ (resp. $N_t^{>2n}$) denote the number of cycles shorter or equal to $2n$ (resp. longer than $2n$). Obviously $N_t = N_t^{\leq 2n} + N_t^{>2n}$ and thus

$$\begin{aligned} \mathbb{E}_n(N_t - \tilde{N}_t) &= \mathbb{E}_n(N_t^{\leq 2n} - \tilde{N}_t) + \mathbb{E}_n(N_t^{>2n}) \\ &\leq \mathbb{E}_n\left(\sum_{i=1}^t \mathbb{1}_{S_i^{\leq 2n}}\right) + \mathbb{E}_n(N_t^{>2n}). \end{aligned} \tag{3.16}$$

To bound $\mathbb{E}_n(N_t^{>2n})$, we simply use that $N_t^{>2n} \leq N/(2n) \leq t \frac{\log(2n)}{n}$ once $t > \frac{N}{2 \log(2n)}$. On the other hand, for $t \leq \frac{N}{2 \log(2n)}$ we get $\mathbb{E}_n(N_t^{>2n}) \leq 1/n$ once n is sufficiently large. Indeed, observe that $N_t^{>2n} \leq \frac{|V_t(2n)|}{2n}$. Hence, we can use (2.1) with $c = 1/\log(2n)$ allowing to choose $\kappa = 2 > \frac{2 \ln 2}{(1-2c)^2}$.

The result then follows from Lemma 3.5. □

Lemma 3.7. *For any $T, L \in \mathbb{N}$ and any $n \geq n_1$ (with n_1 the constant from Lemma 3.6), we have*

$$\sum_{t=T+1}^{T+L} \mathbb{P}_n(S_t) \geq \frac{1}{2} \sum_{t=T+1}^{T+L} \mathbb{P}_n(I_t) - \frac{3}{2} T \frac{\log(4n)}{n}.$$

Proof. We have

$$\begin{aligned} \sum_{t=T+1}^{T+L} \mathbb{1}_{S_t} &= \frac{1}{2} \sum_{t=T+1}^{T+L} (\mathbb{1}_{S_t} - \mathbb{1}_{M_t}) + \frac{1}{2} \sum_{t=T+1}^{T+L} (\mathbb{1}_{S_t} + \mathbb{1}_{M_t}) \\ &= \frac{1}{2} (N_{T+L} - \tilde{N}_{T+L} - N_T + \tilde{N}_T) + \frac{1}{2} \sum_{t=T+1}^{T+L} \mathbb{1}_{I_t}. \end{aligned} \tag{3.17}$$

The claim follows by taking expectations, using that $N_{T+L} - \tilde{N}_{T+L} \geq 0$, and applying Lemma 3.6 for the expectation of $N_T - \tilde{N}_T$. \square

3.4 Proof of Theorem 2.2

We check the condition of Corollary 3.2 with $\lambda = \eta(c)$, where

$$\eta(c) = \begin{cases} \frac{1}{2}c'c_0 & \text{if } c \in (\frac{1}{2}, 1], \\ \frac{1}{2}(1 - \frac{1}{c}) & \text{if } c > 1. \end{cases} \tag{3.18}$$

By Lemma 3.7 and Lemmas 3.3 with 3.4, we have

$$\frac{1}{L} \sum_{t=T+1}^{T+L} \mathbb{P}_n(S_t) \geq \frac{1}{2} \frac{1}{L} \sum_{t=T+1}^{T+L} \mathbb{P}_n(I_t) - \frac{3}{2} \frac{T \log(4n)}{L n} \geq \eta(c) - o(1) \tag{3.19}$$

once we choose $L = \Delta_n T$ with $\Delta_n n / \log n \rightarrow \infty$. Theorem 2.2 now follows from Corollary 3.2, since $\frac{\eta(c)-a}{1-a} > \eta(c) - a$; this actually allows to neglect the corrections $o(1)$. \square

References

- [1] M. Aizenman, B. Nachtergaele, *Geometric aspects of quantum spin states*, Comm. Math. Phys. 164, 17–63 (1994). MR-1288152
- [2] M. Ajtai, J. Komlos, E. Szemerédi, *Largest random component of a k-cube*, Combinatorica 2, 1–7 (1982). MR-0671140
- [3] G. Alon, G. Kozma, *The probability of long cycles in interchange processes*, Duke Math. J. 162, 1567–1585 (2013). MR-MR3079255
- [4] O. Angel, *Random infinite permutations and the cyclic time random walk*, Discrete Math. Theor. Comput. Sci. Proc., 9–16 (2003). MR-2042369
- [5] N. Berestycki, *Emergence of giant cycles and slowdown transition in random transpositions and k-cycles*, Electr. J. Probab. 16, 152–173 (2011). MR-2754801
- [6] N. Berestycki, G. Kozma, *Cycle structure of the interchange process and representation theory*, Bull. Soc. Math. France 143, 265–280 (2015). MR-3351179
- [7] J.E. Björnberg, *Large cycles in random permutations related to the Heisenberg model*, Electr. Commun. Probab. 20, no. 55, 1–11 (2015). MR-3384113
- [8] J.E. Björnberg, *The free energy in a class of quantum spin systems and interchange processes*, preprint, arXiv:1512.06986 (2015).
- [9] B. Bollobás, Y. Kohayakawa, T. Łuczak, *The evolution of random subgraphs of the cube*, Random Struct. Algor. 3, 55–90 (1992). MR-1139488
- [10] B. Bollobás, I. Leader, *Exact face-isoperimetric inequalities*, Europ. J. Combin. 11, 335–340 (1990). MR-1067204
- [11] C. Goldschmidt, D. Ueltschi, P. Windridge, *Quantum Heisenberg models and their probabilistic representations*, in Entropy and the Quantum II, Contemp. Math. 552, 177–224 (2011); arXiv:1104.0983. MR-2868048
- [12] A. Hammond, *Sharp phase transition in the random stirring model on trees*, Probab. Theory Rel. Fields 161, 429–448 (2015). MR-3334273

- [13] O. Schramm, *Compositions of random transpositions*, Isr. J. Math. 147, 221–243 (2005). MR-2166362
- [14] B. Tóth, *Improved lower bound on the thermodynamic pressure of the spin 1/2 Heisenberg ferromagnet*, Lett. Math. Phys. 28, 75–84 (1993). MR-1224836
- [15] D. Ueltschi, *Random loop representations for quantum spin systems*, J. Math. Phys. 54, 083301 (2013). MR-3135479

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