THE RANDOM VIBRATIONS OF A STRING*

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1. Introduction. In the theory of linear electrical networks one encounters systems of equations having the form

$$\sum_{k=1}^{n} (L_{jk}q_k'' + R_{jk}q_k' + G_{jk}q_k) = \sum_{k=1}^{n} E_{jk}, \qquad j = 1, \cdots, n \tag{1}$$

where the q_k represent charges, L_{jk} inductances, R_{jk} resistances, G_{jk} reciprocals of capacitance, the E_{jk} are random e.m.f's**, and primes denote differentiations with respect to time.

The theory of such systems of equations, quite carefully examined during the war years in applications to noise in electrical networks, turns out also to be applicable to the various mechanical systems—in particular, it is applicable to the system of a vibrating string with fixed end points. A general theory for the system of equations (1) has been developed by Uhlenbeck and Wang [2]. Some of the results which we shall obtain in this article have been derived without making use of the general theory. We shall make comparison with these results and in addition we shall derive several more results. The results of Uhlenbeck and certain of his co-authors [1, 5], have been derived directly from the differential equation of motion for the string. In order to apply the general theory to the vibrating string, it is necessary to discretize the string. We therefore assume that the string† is made up of $n + 2$ particles, (2 fixed, n vibrating) of equal mass m harmonically bound together by means of massless springs. Furthermore let us assume that this system of particles has random forces acting on it and that as a result the system vibrates, the vibration taking place in a plane. As a last assumption we suppose that the vibration takes place in a viscous medium so that each of the particles undergoes a damped vibration.

When we have obtained our results for the discretized system, we can derive the results for the case of a continuous string by a limiting procedure, namely that of letting n the number of vibrating particles go to infinity while the total mass and length of the system remains constant.

In this article we shall derive the following:

1) the characteristic function of the sum of the squares of the deviations of the displacement of the particles from their given initial positions—also the corresponding characteristic function for the continuous string,

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**The word random used here in a heuristic sense obviously needs more careful definition and interpretation. A more precise investigation than can be given here is found in an article by Uhlenbeck and Wang (see [2]).

†We adhere to the assumptions made in the usual derivation of the differential equation of the string, i.e. small vibrations, constant tension throughout, etc.

2) the mean square deviation of the displacements of the n particles from some given initial distribution and the corresponding mean square deviation for the string,

3) the mean square deviation of a single particle of the system given that initially it had a certain displacement θ, and the corresponding limiting case.

2. Certain preliminary considerations. Let the displacement of the k-th particle of the system be denoted by y_k . Let $F_k(t)$ denote the random force acting on the k-th particle. By a simple application of Newton's law of mechanics we have the result

$$my_k'' + Ky_k' - \frac{\tau}{d}(y_{k+1} - 2y_k + y_{k-1}) = F_k(t), \tag{2}$$

where K is a quantity dependent on the damping and τ is the tension in the springs connecting the particles. The tension is assumed to be constant at all times. The quantity d is the distance between particles, also assumed constant.

In order to prepare ourselves for the limiting case of the string, it will be necessary to define a density μ in such a way that $\mu d = m$. Assuming the case of $n + 2$ particles (n vibrating, 2 fixed) it is easy to see that the following relations will hold

$$d = \frac{L}{n+1}, \qquad m = \frac{M}{n+1}, \qquad \mu = \frac{m}{d} = \frac{M}{L}, \tag{3}$$

where M is the total mass of the system and L is the total length for the limiting case of the string. Both M and L are given constants. Also, let $K = Bd$ where B is a constant dependent on the magnitude of the damping which is dependent on the viscosity of the medium in which vibration is taking place.

With the use of these relations (2) becomes

$$\mu d y_k'' + Bdy_k' - \frac{\tau}{d}(y_{k+1} - 2y_k + y_{k-1}) = F_k(t). \tag{4}$$

This equation is of the form (1) if the following relations hold:

$$L_{jk} = \delta_{jk}\mu d, \qquad R_{jk} = \delta_{jk}Bd, \qquad G_{jj} = 2r^2, \qquad G_{j(j+1)} = G_{J(J-1)} = -r^2, \tag{5}$$

where δ_{jk} is the Kronecker delta, equal to one if $j = k$ and zero otherwise, and $r^2 = \tau/d$.

From physical considerations it is evident that the quantity $B = \beta/\mu$ where β is the actual damping coefficient for the medium in which vibration is taking place.

As Uhlenbeck and Wang [1] have pointed out in their discussion of the system (1) the $2n$ variables $[q_1(t), q_2(t), \cdots, q_n(t); q_1'(t), q_2'(t), \cdots, q_n'(t)]$ or in our case the collection $[y_1(t), \cdots, y_n(t); y_1'(t), \cdots, y_n'(t)]$ will form a $2n$-dimensional Markoff process governed by the Fokker-Planck or generalized diffusion equation

$$\frac{\partial P}{\partial t} = -\sum_{j=1}^{2n} \frac{\partial}{\partial x_j}(A_j P) + \frac{1}{2}\sum_{j,k=1}^{2n} \frac{\partial^2}{\partial x_j\, \partial x_k}(D_{jk}P), \tag{6}$$

where P is the probability density associated with the distribution of the quantities

$[x_1, x_2, \cdots, x_{2n}]$ Here $x_1, x_2, \cdots, x_{2n}$ are the $2n$ variables describing the collection $[y_1(t), \cdots, y_n(t); y_1'(t), \cdots, y_n'(t)]$. As derived in the article

$$A_j = \sum_{k=1}^{2n} a_{jk} x_k , \tag{7}$$

where the a_{jk} are the elements obtained from the matrix

$$A = \begin{bmatrix} 0 & I \\ -L^{-1}G & -L^{-1}R \end{bmatrix}, \tag{8}$$

and the D_{jk} are obtained from the matrix*

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2\kappa T L^{-1} R L^{-1} \end{bmatrix} \tag{9}$$

In the matrices (8) and (9), L, R and G are themselves matrices consisting of the elements L_{jk} . R_{jk} and G_{jk} first encountered in the system of equations (1) and which for our purposes may be defined by (5).

In order to solve the Fokker-Planck equation it turns out to be convenient to make the linear transformation

$$z_j = \sum_{k=1}^{2n} c_{jk} x_k , \tag{10}$$

and it then follows that (6) becomes

$$\frac{\partial P}{\partial t} = -\sum_{j=1}^{2n} \lambda_j \frac{\partial}{\partial z_j}(z_j P) + \frac{1}{2} \sum_{j,k=1}^{2n} \sigma_{jk} \frac{\partial^2 P}{\partial z_j \, \partial z_k}, \tag{11}$$

where the σ_{jk} are obtained from the matrix $CDC^{\sim}$, C being the matrix which diagonalizes the matrix A and which thereby yields the eigenvalues λ_j $(j = 1, 2, \cdots, 2n)$.

It is easy to show that the eigenvalues satisfy the determinantal equation

$$\begin{vmatrix} Q & P & 0 & \cdots & 0 & 0 \\ P & Q & P & \cdots & 0 & 0 \\ 0 & P & Q & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P & Q \end{vmatrix} = 0, \tag{12}$$

$$Q \equiv \lambda^2 + \frac{B\lambda}{\mu} + \frac{2r^2}{\mu d}, \qquad P \equiv -\frac{r^2}{\mu d}$$

If we denote the determinant (12) by A_n we may show that the following difference equation is satisfied

*Here κ is Boltzmann's constant and T is the temperature of the medium in which the vibration is taking place.

$$A_n = QA_{n-1} - P^2A_{n-2} \tag{13}$$

subject to the boundary conditions $A_0 = 1$, $A_1 = Q$. Solving the difference equation by the usual methods we obtain

$$A_n = \frac{1}{(Q^2 - 4P^2)^{1/2}}\left[\left\{\frac{Q + (Q^2 - 4P^2)^{1/2}}{2}\right\}^{n+1} - \left\{\frac{Q - (Q^2 - 4P^2)^{1/2}}{2}\right\}^{n+1}\right] \tag{14}$$

yielding the evaluation of the determinant (12).

Upon finding the conditions under which $A_n = 0$, we obtain the equation

$$\lambda^2 + \frac{B\lambda}{\mu} + \frac{2r^2}{\mu d} = \frac{2r^2}{\mu d}\cos\frac{m\pi}{n+1}, \tag{15}$$

and so the $2n$ eigenvalues λ_j are given by

$$\lambda = -\frac{B}{2\mu} \pm \left[\frac{B^2}{4\mu^2} - \frac{4\tau}{\mu L^2}(n+1)^2\sin^2\frac{m\pi}{2n+2}\right]^{1/2} \qquad m = 1, \cdots, n \tag{16}$$

The evaluation of the matrix C which diagonalizes A is a little too long to present here but it is straight-forward. It turns out that the matrix C is given by*

$$\left.\begin{aligned} C_{p(m),\,k} &= (\lambda_{p(m)} + \beta)\sin\frac{km\pi}{n+1} \\ C_{p(m),\,n+k} &= \sin\frac{km\pi}{n+1} \end{aligned}\right\} \quad \begin{aligned} k &= 1, \cdots, n, \\ m &= 1, \cdots, n, \\ p_{(m)} &= 1, \cdots, 2n. \end{aligned} \tag{17}$$

After obtaining the matrix C, a simple evaluation of $CDC^{\sim}$, where D is given in (9), yields the σ_{jk} of the Fokker-Planck equation (11). This evaluation gives

$$\sigma_{jk} = \frac{2\kappa T\beta}{\mu d}\left(\sum_{m=1}^{n}\sin\frac{mj\pi}{n+1}\sin\frac{mk\pi}{n+1}\right), \tag{18}$$

where $\sigma_{jk} = \sigma_{j+1k}$ if j is odd and $\sigma_{jk} = \sigma_{jk+1}$ if k is odd. Evaluation of the sum in (18) shows that the σ_{jk} are given by the elements in

$$\left\|\begin{matrix} \alpha & \alpha & 0 & 0 & \cdots & 0 \\ \alpha & \alpha & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \alpha & \cdots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & \cdots & \alpha & \alpha \\ 0 & 0 & & \cdots & \alpha & \alpha \end{matrix}\right\|, \tag{19}$$

where $\alpha = \kappa T\beta(n+1)^2/\mu L$.

*The significance of the bracket in $p_{(m)}$ is that corresponding to the value 1 for m we have $p_{(m)}$ equal to 1 and also 2, corresponding to $m = 2$ we have $p_{(m)}$ equal to 3 and also 4 etc.

The transformed Fokker-Planck equation (11) has for its solution a $2n$-dimensional Gaussian distribution with the average values

$$\bar{z}_j = b_{j0} \exp (\lambda_j t) \tag{20}$$

and covariance

$$\langle (z_j - \bar{z}_j)(z_k - \bar{z}_k) \rangle_{\text{Average}} = \frac{-\sigma_{jk}}{\lambda_j + \lambda_k} [1 - \exp \{(\lambda_j + \lambda_k)t\}] \tag{21}$$

where b_{j0} are the initial values of the z_j .

3. The characteristic function. Using the results obtained thus far we may now write the probability density function for the system of the n vibrating particles using a well-known result from the theory of multivariate Gaussian distributions. The probability density has the form

$$P(z_1 , z_2 , \cdots , z_{2n}) = R_n \exp \left\{ -\frac{1}{2} \sum_{j,k} B_{jk}(z_j - \bar{z}_j)(z_k - \bar{z}_k) \right\}, \tag{22}$$

where the B_{jk} and R_n are constants determined from (20) and (21), and the values of λ are determined from (16).

We are now ready to obtain the characteristic function of the sum

$$\frac{1}{n+1} \sum_{j=1}^{n} (x_j - s_j)^2, \tag{23}$$

i.e. the characteristic function of the normalized* sum of the squares of the deviations of the displacements from some arbitrary initial values. This characteristic function, once we have obtained it, will at least in principle, enable us to obtain the characteristic function of

$$\frac{1}{L} \int_0^L [X(x, t) - S(x)]^2 \, dx \tag{24}$$

which is the corresponding limiting case of the finite sum (23). Once we have this we are theoretically able, by means of the Fourier Inversion Theorem of Probability, to obtain

$$Pr\left\{ \frac{1}{L} \int_0^L [X(x, t) - S(x)]^2 \, dx < \epsilon \right\} \tag{25}$$

that is the probability that the mean square deviation of the string from a given initial distribution be less than some given quantity ϵ.

The required characteristic function is defined by

$$F_n(\xi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R_n \exp \left[\frac{i\xi}{n+1} \sum_{j=1}^{n} (x_j - s_j)^2 - \frac{1}{2} \sum_{j,k} B_{jk}(z_j - \bar{z}_j)(z_k - \bar{z}_k) \right] dV \tag{26}$$

*This normalization is necessary if our results for the limiting case of the continuous string are to be meaningful.

where the integration element $dV = dz_1 \cdots dz_{2n}$. If the integration in (26) is performed, we obtain the result

$$\left\{\exp\left[2i\xi \sum_{k=1}^{n} U_k^2 V_k\right]\right\} \prod_{k=1}^{n} V_k , \tag{27}$$

where

$$U_k = \frac{1}{n+1} \sum_{m=1}^{n} (\bar{x}_m - s_m) \sin \frac{mk\pi}{n+1}, \tag{28}$$

and

$$V_k = [1 - P_{2k-1,2k}]^{-1} \tag{29}$$

Here, for example,

$$P_{12} = \frac{4i\xi\{B_{11} + 2B_{12} + B_{22}\}}{(\lambda_1 - \lambda_2)^2(n+1)^2(B_{11}B_{22} - B_{12}^2)} \tag{30}$$

with similar definitions for P_{34}, and in general for $P_{2k-1,2k}$. The B_{jk} are the coefficients in the probability density. If we make use of the values of B_{jk} in terms of the means and covariances (20) and (21) we may write the P's in convenient form as

$$P_{12} = \frac{4i\xi}{(\lambda_1 - \lambda_2)^2} \int_0^t (e^{\lambda_1 t} - e^{\lambda_2 t})^2 \, dt \tag{31}$$

with similar definitions for $P_{2k-1,2k}$.

It can be shown rigorously that the characteristic function (26) has a limit as $n \to \infty$ and that the value of this limit is

$$\left\{\exp\left[2i\xi \sum_{k=1}^{\infty} u_k^2 v_k\right]\right\} \prod_{k=1}^{\infty} v_k \tag{32}$$

where

$$u_k = \frac{1}{L} \int_0^L [A(x, t) - S(x)] \sin \frac{k\pi x}{L} \, dx \tag{33}$$

and

$$\lim_{n\to\infty} \prod_{k=1}^{n} V_k = \prod_{k=1}^{\infty} v_k \tag{34}$$

The details of the limiting procedure are not as obvious as they might appear on first sight since the quantity n occurs in each term of the series and products of (27).

4. Derivation of averages. In this section we shall be interested in obtaining certain averages indicating the behavior of our system of n harmonically bound particles. Once we have the averaged behavior of the finite number of particles, it is but a single step to the case of the average properties of the string.

We first derive the mean square deviation of the displacement of the k-th particle from some initial position, that is we wish

$$\langle x_k^2 \rangle_{\text{Average}} = \langle (x_k - \bar{x}_k)^2 \rangle_{\text{Average}} + \bar{x}_k^2 \tag{35}$$

From (10) and (17) we may obtain

$$\bar{x}_k = \frac{2}{n+1} \sum_{r=1}^{n} \left(\frac{\bar{z}_{2r-1} - \bar{z}_{2r}}{\lambda_{2r-1} - \lambda_{2r}}\right) \sin \frac{kr\pi}{n+1} \tag{36}$$

With the use of (36) we obtain

$$\left.\begin{aligned} \bar{x}_k^2 = {} & \sum_{r,s=1}^{n} (B_r(t)B_s(t)x_{r0}x_{s0} + e^{-\beta t/2}B_r(t)x_{r0}P_{s0} \sin \omega_s t) \\ & + \sum_{r,s=1}^{n} (\omega_r^{-1}e^{-\beta t/2}B_s(t)x_{s0}P_{r0} \sin \omega_r t + \omega_r^{-1}\omega_s^{-1}e^{-\beta t}P_{r0}P_{s0} \sin \omega_r t \sin \omega_s t) \end{aligned}\right\} \tag{37}$$

where

$$B_r(t) \equiv e^{-\beta t/2}\left(\cos \omega_r t + \frac{\beta}{2\omega_r} \sin \omega_r t\right) \tag{38}$$

and

$$\omega_r \equiv \left\{\frac{4\tau(n+1)^2}{\mu L^2} \sin^2 \frac{r\pi}{2n+2} - \frac{\beta^2}{4}\right\}^{1/2} \tag{39}$$

Similarly we may derive

$$\langle(x_k - \bar{x}_k)^2\rangle_{\text{Average}} = \frac{E\mu}{4La^2\beta} \sum_{r=1}^{n} \frac{\sin^2 kr\pi/(n+1)}{\sin^2 r\pi/(2n+2)} \{1 - U_r\} \tag{40}$$

where

$$U_r = \frac{\beta e^{-\beta t}}{4\omega_r^{\ 2}}\left(2\omega_r \sin 2\omega_r t - \beta \cos 2\omega_r t + 16a^2 \sin^2 \frac{r\pi}{2n+2}\right) \tag{41}$$

and

$$E = \frac{2\beta\kappa T}{\mu}, \qquad a^2 = \frac{\tau(n+1)^2}{\mu L^2}\,. \tag{42}$$

Putting (37) and (40) in (35) we obtain the result $\langle x_k^2\rangle_{\text{Average}}$ that is the mean square deviation for the system of n particles.

The quantities x_{r0}, P_{r0} represent initial values of the displacements and momenta of the n particles respectively. We now average over a canonical ensemble of systems having all possible initial displacements and momenta after a long time has elapsed. The probability density P, which is a function of time, as we have found it in (21) approaches a limiting value as $t \to \infty$ which we may call the stationary density. This probability density is given by

$$R \exp\left\{-\frac{\beta}{E}(P_{10}^2 + \lambda_1\lambda_2 x_{10}^2 + \cdots)\right\} \tag{43}$$

Using this stationary density or as it is usually called the density corresponding to the canonical distribution, we may examine the following question.

Given that a certain particle starts with the displacement θ at time zero, what is the mean square displacement of this particle after time t?

It is seen that what we want is the evaluation

$$\frac{\int \cdots \int \langle x^2 \rangle_{\text{Average}} \exp\left\{-\frac{\beta}{E}(P_{10}^2 + \lambda_1\lambda_2 x_{10}^2 + \cdots)\right\} dV}{\int \cdots \int \exp\left\{-\frac{\beta}{E}(P_{10}^2 + \lambda_1\lambda_2 x_{10}^2 + \cdots)\right\} dV} \tag{44}$$

The element dV is taken to be the element of volume in phase space $dP_{10} \cdots dP_{n0}$ $dx_{10} \cdots dx_{n0}$. Our condition that the initial displacement be of magnitude θ puts no restriction on the integration over the P but does put a restriction on the integration over the X. The restriction is that*

$$\theta \leq \sum_{r=1}^{n} x_{r0} \sin \frac{kr\pi}{n+1} \leq \theta + \epsilon \tag{45}$$

We must thus evaluate

$$\frac{\int \cdots \int x_{r0}x_{s0} \exp\left\{-\frac{\beta}{E}\sum_{r=1}^{n} \lambda_{2r-1}\lambda_{2r}x_{r0}^2\right\} dX}{\int \cdots \int \exp\left\{-\frac{\beta}{E}\sum_{r=1}^{n} \lambda_{2r-1}\lambda_{2r}x_{r0}^2\right\} dX} \tag{46}$$

where dX denotes the volume element $dx_{10} \cdots dx_{n0}$. To evaluate this we use a theorem of Hadamard [6]. We find the result

$$\frac{E}{2\beta} F_k + e^{-\beta t}\left\{\theta^2 - \frac{E}{2\beta} F_k\right\}\left\{\frac{\beta}{2}\frac{G_k(t)}{F_k} + \frac{1}{F_k}\frac{d}{dt} G_k(t)\right\}^2 \tag{47}$$

where

$$F_k \equiv \frac{2}{L} \sum_{r=1}^{n} \frac{\sin^2 kr\pi/(n+1)}{\lambda_{2r-1}\lambda_{2r}} \tag{48}$$

and

$$G_k(t) \equiv \frac{2}{L} \sum_{r=1}^{n} \frac{\sin \omega_r t}{\lambda_{2r-1}\lambda_{2r}\omega_r} \sin^2 \frac{kr\pi}{n+1} \cdot \tag{49}$$

We have found the results for the system of n particles but a rigorous mathematical discussion shows that the process of replacing each term by the limit as $n \to \infty$ and replacing the sum from 1 to n by the sum from 1 to ∞ is justifiable. The results thus obtained are the same as those obtained by Uhlenbeck and Van Lear [5] and Uhlenbeck and Ornstein [1] by entirely different methods. An advantage of the approach given here is that the results for the finite system as well as for the string are obtained.

One further simplification in the above results can be made by recognizing that F_k in (48) can be evaluated explicitly. We note that

$$\lambda_{2r-1}\lambda_{2r} = 4a^2 \sin^2 \frac{r\pi}{2n+2} \tag{50}$$

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*We use this restriction rather than $\theta = \Sigma_{r=1}^{n} x_{r0} \sin kr\pi/(n+1)$ because if we use the latter (46) becomes an indeterminate of the form 0/0. Instead we therefore use (45) and then let $\epsilon \to 0$.

so that

$$F_k \equiv \frac{1}{2La^2} \sum_{r=1}^{n} \frac{\sin^2 kr\pi/(n+1)}{\sin^2 r\pi/(2n+2)} \,. \tag{51}$$

An evaluation of this yields

$$F_k = \frac{k(n-k+1)}{La^2} \tag{52}$$

Thus the first term of (47) becomes

$$\frac{E}{2\beta} F_k = \frac{Ek(n-k+1)}{2La^2\beta} \tag{53}$$

and since

$$E = \frac{2\beta\kappa T}{\mu}, \tag{54}$$

we obtain in the limit as $n \to \infty$ since $k/(n+1) = x/L$ the term

$$\frac{\kappa LT}{\tau}\left(\frac{x}{L} - \frac{x^2}{L^2}\right) \tag{55}$$

This appears also in the results of Uhlenbeck and Van Lear for the limiting case of the string.

Bibliography

1. G. E. Uhlenbeck and L. S. Ornstein, *On the theory of the Brownian motion*, Phys. Rev. **36**, 823-841 (1930).
2. G. E. Uhlenbeck and M. C. Wang, *On the theory of the Brownian motion II*, Rev. Mod. Phys. **17**, 323-342 (1945).
3. S. Chandrasekhar, *Stochastic problems in physics and astronomy*, Rev. Mod. Phys. **15**, 1-81 (1943).
4. R. B. Barnes and S. Silverman, *Brownian motion*, Rev. Mod. Phys. **6**, 162-192 (1934).
5. G. E. Uhlenbeck and G. A. Van Lear, *The Brownian motion of strings and elastic rods*, Phys. Rev. **38**, 1583-1598 (1931).
6. J. Hadamard, *Cours d'analyse*, vol. 1, Libraire Scientifique J. Hermann, Paris, 1927, pp. 249-253.