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THE RANK OF EXTREME POSITIVE OPERATORS  
ON POLYHEDRAL CONES

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In a recent paper on diagonals of convex sets [1] the authors have obtained a number of results describing the properties of this notion. It turned out that there is an intimate connection between diagonals of a polyhedral cone and linear dependence relations between the extreme rays of the cone.

One of the main results stated that an indecomposable polyhedral cone (of dimension  $n$  greater than one) has at least two diagonals; if it has exactly two diagonals then it is generated by  $n + 1$  extreme vectors which satisfy exactly one relation of linear dependence. Such cones are called minimal.

In the present paper we intend to describe completely the extreme rays of the cone  $P(C_1, C_2)$  of all operators  $A$  such that  $AC_1 \subset C_2$  in the case that both  $C_1$  and  $C_2$  are minimal cones.

The main result (Thm. (2,6)) says – roughly speaking – that the rank of such an extreme operator may assume any of the possible values within the natural boundaries with the exception of rank two.

**1. Preliminaries.** We shall use the notation and terminology of the paper [1]. In particular,  $\text{Hom}(E, F)$  will denote the set of all linear operators of the linear space  $E$  into the linear space  $F$ .

We shall need the following technical result.

**(1,1)** Suppose we are given two spaces  $E^{(1)}$  and  $E^{(2)}$  and two sets of vectors

$$p(1,1), \dots, p(1, v_1) \in E^{(1)}, \quad p(2,1), \dots, p(2, v_2) \in E^{(2)}.$$

Denote by  $R_1$  the linear space of all column vectors  $r$  of length  $v_1$  with coordinates  $r^T = (r(1,1), \dots, r(1, v_1))$  such that

$$\sum_{j=1}^{v_1} r(1, j) p(1, j) = 0.$$

Let  $R_2$  have a similar meaning with respect to the space  $E^{(2)}$  and the set of vectors  $p(2,1), \dots, p(2, v_2)$ .

Denote by  $D(\psi)$  the linear space of all  $v_2$  by  $v_1$  matrices  $B$  such that  $BR_1 \subset R_2$ . For each  $B \in D(\psi)$ , there exists an operator  $\psi(B) \in \text{Hom}(E^{(1)}, E^{(2)})$  such that

$$\psi(B) p(1, i) = \sum_{k=1}^{v_2} b_{ik} p(2, k).$$

The mapping  $\psi$  is a linear mapping of  $D(\psi)$  onto  $\text{Hom}(E^{(1)}, E^{(2)})$ . Its kernel consists of all  $v_2$  by  $v_1$  matrices all columns of which belong to  $R_2$ .

Suppose now we have chosen bases in  $E^{(1)}$  and  $E^{(2)}$  and that the vectors  $p(1, i)$  and  $p(2, j)$  are represented in these bases as column vectors respectively, of lengths  $n_1$  and  $n_2$ . Denote by  $P_1$  the  $n_1$  by  $v_1$  matrix

$$P_1 = (p(1,1), p(1,2), \dots, p(1, v_1)),$$

and similarly, by  $P_2$  the  $n_2$  by  $v_2$  matrix

$$P_2 = (p(2,1), p(2,2), \dots, p(2, v_2)).$$

If  $B \in D(\psi)$  is given and if  $\psi(B)$  is represented in these bases by the matrix  $M(\psi(B))$ , we have

$$(*) \quad MP_1 = P_2 B.$$

Conversely, if  $M$  is an  $n_2$  by  $n_1$  matrix for which there exists a  $v_2$  by  $v_1$  matrix  $B$  satisfying (\*), then  $B \in D(\psi)$  and  $\psi(B)$  is the operator whose matrix in the given bases is  $M$ .

Denote by  $C_1$  and  $C_2$  the convex cones generated respectively by the sets of vectors  $p(1, i)$  and  $p(2, j)$ . Then  $T \in P(C_1, C_2)$  if and only if  $T = \psi(B)$  for some nonnegative  $B \in D(\psi)$ .

Proof. May be left to the reader.

**2. Results.** We shall be dealing with minimal cones generated by relations of a particularly simple type. Set

$$N_1 = \{1, \dots, n_1 + 1\}, \quad N_2 = \{1, \dots, n_2 + 1\}$$

and consider two decompositions

$$N_1 = A_1 \cup B_1, \quad N_2 = A_2 \cup B_2$$

such that each of the four sets  $A_1, B_1, A_2, B_2$  contains at least two elements.

Assume that the vectors  $p(1, i)$ ,  $i \in N_1$ , satisfy exactly one (up to a factor) linear relation

$$\sum_{i \in A_1} p(1, i) - \sum_{i \in B_1} p(1, i) = 0.$$

Similarly, assume that the vectors  $p(2, i)$ ,  $i \in N_2$ , satisfy exactly one (up to a factor) linear relation

$$\sum_{i \in A_2} p(2, i) - \sum_{i \in B_2} p(2, i) = 0.$$

**(2,1) Proposition.** (i) Let  $A$  be a matrix of type  $n_2 + 1$  by  $n_1 + 1$ . Then  $A \in D(\psi)$  if and only if there exists a number  $\lambda(A)$  such that

$$\begin{aligned} \sum_{\sigma \in A_1} a_{i\sigma} - \sum_{\sigma \in B_1} a_{i\sigma} &= \lambda(A) \quad \text{for } i \in A_2, \\ &= -\lambda(A) \quad \text{for } i \in B_2. \end{aligned}$$

(ii) Given  $T \in P(C_1, C_2)$  there exists exactly one  $A \geq 0$  with  $\psi(A) = T$  which satisfies the following postulate: for each  $\sigma$  there exists a  $j \in A_1$  such that  $a_{j\sigma} = 0$ . The corresponding  $\lambda(A)$  will be denoted by  $\mu(T)$ .

(iii) According to (ii), there exists a one-to-one correspondence between operators in  $P(C_1, C_2)$  and  $n_1 + 1$  by  $n_2 + 1$  matrices which satisfy the conditions sub (i). The matrix corresponding in this way to the operator  $T$  will be called the canonical form of  $T$ .

*Proof.* According to (1,1) the inclusion  $A \in D(\psi)$  is equivalent to the inclusion  $BR_1 \subset R_2$ , in other words,

$$Br_1 = \lambda(B)r_2$$

where  $r_1$  and  $r_2$  are respectively the column vectors with coordinates  $r_{1i} = 1$  for  $i \in A_1$ ,  $r_{1i} = -1$  for  $i \in B_1$ ,  $r_{2j} = 1$  for  $j \in A_2$ ,  $r_{2j} = -1$  for  $j \in B_2$ . This proves (i).

Now consider an  $A \in P(C_1, C_2)$ ; there exist, for each  $\sigma \in N_1$ , nonnegative numbers  $\tilde{a}_{i\sigma}$  such that

$$T p(1, \sigma) = \sum_{j \in A_2} \tilde{a}_{j\sigma} p(2, j) + \sum_{j \in B_2} \tilde{a}_{j\sigma} p(2, j).$$

For each  $\sigma \in N_1$ , denote by  $\xi_\sigma$  the minimum of all  $\tilde{a}_{j\sigma}$ . For each  $\sigma \in N_1$  and each  $j \in A_2$ , set

$$a_{j\sigma} = \tilde{a}_{j\sigma} - \xi_\sigma;$$

for  $j \in B_2$ , set

$$a_{j\sigma} = \tilde{a}_{j\sigma} + \xi_\sigma.$$

Clearly, all  $a_{j\sigma}$  are nonnegative and, for each  $\sigma \in N_1$ , at least one  $a_{j\sigma}$ ,  $j \in A_2$ , is zero.

Since

$$\sum_{\sigma \in A_1} p(1, \sigma) - \sum_{\tau \in B_1} p(1, \tau) = 0,$$

it follows that

$$\sum_{\sigma \in A_1} \sum_{j \in A_2} a_{j\sigma} p(2, j) + \sum_{\sigma \in A_1} \sum_{j \in B_2} a_{j\sigma} p(2, j) - \sum_{\tau \in B_1} \sum_{j \in A_2} a_{j\tau} p(2, j) - \sum_{\tau \in B_1} \sum_{j \in B_2} a_{j\tau} p(2, j) = 0.$$

Rearranging this sum, we obtain

$$\sum_{j \in A_2} \left( \sum_{\sigma \in A_1} a_{j\sigma} - \sum_{\tau \in B_1} a_{j\tau} \right) p(2, j) + \sum_{j \in B_2} \left( \sum_{\sigma \in A_1} a_{j\sigma} - \sum_{\tau \in B_1} a_{j\tau} \right) p(2, j) = 0.$$

Since there exists only one relation for the  $p(2, j)$ , the relation just obtained must be a multiple of the relation

$$\sum_{j \in A_2} p(2, j) - \sum_{j \in B_2} p(2, j) = 0.$$

Hence there exists a number  $\lambda$  such that

$$\sum_{\sigma \in A_1} a_{j\sigma} - \sum_{\tau \in B_1} a_{j\tau} = \lambda \quad \text{if } j \in A_2, \quad \sum_{\sigma \in A_1} a_{j\sigma} - \sum_{\tau \in B_1} a_{j\tau} = -\lambda \quad \text{if } j \in B_2.$$

This proves the conditions sub (i).

Suppose now that  $a'_{j\sigma}$ , ( $j \in N_2$ ,  $\sigma \in N_1$ ) and  $\lambda'$  satisfy the conditions sub (i). Let  $\sigma$  be fixed. Then

$$\sum_{j \in A_2} (a_{j\sigma} - a'_{j\sigma}) p(2, j) + \sum_{j \in B_2} (a_{j\sigma} - a'_{j\sigma}) p(2, j) = 0.$$

Consequently, there exists an  $\eta_\sigma$  such that

$$a_{j\sigma} - a'_{j\sigma} = \eta_\sigma \quad \text{for each } j \in A_2, \quad a_{j\sigma} - a'_{j\sigma} = -\eta_\sigma \quad \text{for each } j \in B_2.$$

By (ii), there exist indices  $k(\sigma)$  and  $l(\sigma)$  such that

$$a_{k(\sigma), \sigma} = a_{l(\sigma), \sigma} = 0.$$

Since  $a_{k(\sigma), \sigma} \geq 0$ , it follows that  $\eta_\sigma \leq 0$ . Since  $a_{l(\sigma), \sigma} \geq 0$ , we have  $\eta_\sigma \geq 0$ . This proves the uniqueness of the numbers  $a_{i\sigma}$ . Uniqueness of  $\lambda$  follows from that of  $a_{i\sigma}$ .

To complete the proof, we shall apply Theorem (1,1) to the matrix  $B = (a_{\sigma j})$ . It is easy to verify that the relation  $BR_1 \subset R_2$  is fulfilled.

**(2,2)** Let  $A, B, C \in P(C_1, C_2)$  and suppose that  $A = B + C$ . Suppose that for some  $i \in N_1$

$$A p(1, i) = \sum_{j \in N_2} \alpha(j) p(2, j)$$

with nonnegative  $\alpha(j)$ ,

$$B p(1, i) = \sum_{j \in N_2} \beta(j) p(2, j), \quad C p(1, i) = \sum_{j \in N_2} \gamma(j) p(2, j).$$

Suppose further that there exist indices  $k \in A_2$  and  $j \in B_2$  such that  $\alpha(k) = \alpha(j) = 0$ . Then  $\alpha(s) = \beta(s) + \gamma(s)$  for all  $s \in N_2$ .

*Proof.* Denote by  $r$  the vector  $r = (r(1), \dots, r(n_1))$  with  $r(i) = -1$  for  $i \in A_2$  and  $r(i) = 1$  for  $i \in B_2$ . If we denote by  $\alpha, \beta, \gamma$  the vectors  $\alpha = (\alpha(1), \dots, \alpha(n_1))$ ,

$\beta = \dots, \gamma = \dots$  it follows from the relation  $A = B + C$  that

$$\beta + \gamma = \alpha + \xi r$$

for some  $\xi$ ; in particular, taking the  $j$ -th coordinate, we obtain

$$0 \leq \beta(j) + \gamma(j) = \xi.$$

For the  $k$ -th coordinate we obtain

$$0 \leq \beta(k) + \gamma(k) = -\xi = -\beta(j) + \gamma(j) \leq 0.$$

Thus  $\xi = 0$ . The rest is obvious.

Now we are able to formulate the main result of this section, a complete description of all extreme rays of  $P(C_1, C_2)$ .

**(2,3) Theorem.** *Let  $E^{(1)}$  and  $E^{(2)}$  be two linear spaces of dimensions  $n_1$  and  $n_2$  respectively. Let  $C_1$  and  $C_2$  be minimal cones in  $E^{(1)}$  and  $E^{(2)}$  respectively. Suppose that  $C_1$  is generated by the vectors*

$$p(1, 1), \dots, p(1, n_1 + 1)$$

and that  $C_2$  is generated by

$$p(2, 1), \dots, p(2, n_2 + 1).$$

Let us introduce the following abbreviations

$$N_i = \{1, 2, \dots, n_i + 1\} \quad \text{for } i = 1, 2.$$

Consider two decompositions

$$N_i = A_i \cup B_i, \quad i = 1, 2$$

such that each of the four sets  $A_1, A_2, B_1, B_2$  has at least two elements and the vectors  $p$  satisfy the following two relations

$$\sum_{j \in A_i} p(i, j) - \sum_{j \in B_i} p(i, j) = 0.$$

1° For each triple  $i \in A_1, j \in B_1, s \in N_2$  there exists an operator  $T$  such that

$$T p(1, i) = p(2, s), \quad T p(1, j) = p(2, s),$$

$$T p(1, k) = 0 \quad \text{for all } k \in N_1, \quad i \neq k \neq j.$$

Such operators will be called operators of type 1. The operator described above will be denoted by  $T_{sij}$ .

2° For each pair of decompositions

$$A_2 = \bigcup_{j \in A_1} A_{2j}, \quad B_2 = \bigcup_{j \in B_1} B_{2j}$$

such that at least two of the sets  $A_{2j}$  as well as at least two of the sets  $B_{2j}$  are non-void there exists an operator  $T$  satisfying the following postulates:

$$Tp(1, i) = \sum_{j \in A_{2i}} p(2, j) \quad \text{for } i \in A_1,$$

$$Tp(1, i) = \sum_{j \in B_{2i}} p(2, j) \quad \text{for } i \in B_1.$$

Such operators will be called operators of type 2.

3° for each pair of decompositions

$$A_2 = \bigcup_{j \in B_1} A_{2j}, \quad B_2 = \bigcup_{j \in A_1} B_{2j}$$

such that at least two of the sets  $A_{2j}$  as well as at least two of the sets  $B_{2j}$  are nonvoid there exists an operator  $T$  satisfying the following postulates:

$$Tp(1, i) = \sum_{j \in B_{2i}} p(2, j) \quad \text{for } i \in A_1,$$

$$Tp(1, i) = \sum_{j \in A_{2i}} p(2, j) \quad \text{for } i \in B_1.$$

Such operators will be called operators of type 3.

Then  $W \in \text{Hom}(E^{(1)}, E^{(2)})$  is an extreme element of  $P(C_1, C_2)$  if and only if it is a positive multiple of an operator of one of the three types described above.

*Proof.* The fact that the postulates above define operators and that these operators are distinct in an immediate consequence of the preceding theorem since the matrices defining the operators are given in the canonical form.

We prove first that operators of type 1, 2 and 3 are all extreme. To this purpose, we shall prove the following proposition:

Let  $B = (b_{ik})$  be an  $n_1 + 1$  by  $n_2 + 1$  matrix with nonnegative entries. Suppose that  $A$  is a matrix in canonical form such that  $\psi(A)$  is one of the operators of the three types described in the theorem. Suppose that  $B \leq A$ . Suppose that  $B \in D(\psi)$ ,  $Br_1 = \xi r_2$  where the column vectors  $r_1, r_2$  are defined by  $r_1 = (r_{1i})$ ,  $r_{1i} = 1$  if  $i \in A_1$ ,  $r_{1i} = -1$  if  $i \in B_1$ ,  $r_2 = (r_{2i})$ ,  $r_{2i} = 1$  if  $i \in A_2$ ,  $r_{2i} = -1$  if  $i \in B_2$ . Then  $B = \omega A$  for some  $\omega \geq 0$ .

*Proof.* Since  $0 \leq B \leq A$ ,  $b_{ik} = 0$  whenever  $a_{ik} = 0$ . Consider first operators of type 1. Thus there exist indices  $i \in A_1, j \in B_1$  and  $s \in N_2$  such that the only non-zero entries of the matrix  $A = (a_{pq})$  are  $a_{si} = a_{sj} = 1$ . Since  $Br_1 = \xi r_2$  and at most one row of  $B$  is non-zero, we have  $\xi = 0$ . Consequently,  $b_{si} - b_{sj} = \xi r_{2j} = 0$  which implies  $B = b_{si}A$ .

Now let  $A$  be a canonical matrix of an operator of type 2 corresponding to the decompositions

$$A_2 = \bigcup_{j \in B_1} A_{2j}, \quad B_2 = \bigcup_{j \in B_1} B_{2j}.$$

Let  $i \in N_2$  be a given index. It is easy to see that there exists exactly one index  $k(i)$  such that  $a_{i,k(i)} = 1$  and all remaining entries  $a_{i,l}$  are equal to zero. Indeed, if  $i \in A_2$  then  $k(i) = t$  if  $i \in A_{2t}$ ; if  $i \in B_2$  then  $k(i) = t$  if  $i \in B_{2t}$ . It follows from the relation  $B r_1 = \xi r_2$  that  $b_{i,k(i)} = \xi$  for all  $i \in N_2$ . Consequently,  $B = \xi A$ .

In the case of an operator of type 3 we obtain from  $B r_1 = \xi r_2$  similarly  $B = -\xi A$ . The proof of the proposition is complete.

Now let  $T$  be an operator of one of the three types described in the theorem and let  $A$  be the corresponding matrix in the canonical form. Suppose that  $T = T_1 + T_2$  where both  $T_1$  and  $T_2$  belong to  $P(C_1, C_2)$ . Let  $B_1$  and  $B_2$  be two nonnegative matrices defining  $T_1$  and  $T_2$  respectively. We observe first that the matrix  $A$ , being given in the canonical form, has at least two zero entries in each column, one with row index in  $A_2$  and one with row index in  $B_2$ . It follows from Lemma (2,2) that  $A = B_1 + B_2$ . Consequently,  $0 \leq B_1 \leq A$  so that  $B_1 = \omega A$  according to the auxiliary proposition just proved. Hence  $T_1 = \omega A$  which proves that  $A$  is extreme.

Denote by  $\mathcal{T}$  the set of all operators of type 1, 2 or 3. We have seen that every operator  $T$  of  $\mathcal{T}$  generates an extreme ray of the cone  $P(C_1, C_2)$ . To prove that every extreme ray is a multiple of some operator in  $\mathcal{T}$ , it suffices to prove that every element in  $P(C_1, C_2)$  may be written as a convex combination of elements of  $\mathcal{T}$ .

First of all, we shall dispose of the following case.

( $\alpha$ ) There exists a nonnegative matrix  $A$  such that  $T = \psi(A)$  and  $\lambda(A) = 0$ . Then it is easy to see that

$$T = \sum_{s \in N_2} \sum_{i \in A_1} \sum_{j \in B_1} \frac{a_{is} a_{js}}{\sum_{k \in A_1} a_{ks}} T_{sij}$$

where  $a_{pq}$  are the elements of the matrix  $A$  so that the assertion holds.

We shall use induction with respect to the number  $h(T)$  of positive entries in the canonical matrix  $A$  corresponding to  $T$ . If  $h(T) = 0$ , we have  $T = 0 \in \text{conv } \mathcal{T}$  trivially. Now let  $h(T) > 0$  and suppose the assertion proved for all operators with smaller  $h$ .

Now we shall distinguish three cases.

1°  $\mu(T) = 0$ ; in this case our assertion follows immediately from the preceding result ( $\alpha$ ).

2°  $\mu(T) > 0$ . Let  $A$  be the nonnegative matrix in canonical form such that  $\psi(A) = T$  whence  $\lambda(A) = \mu(T) > 0$ .

We shall distinguish two subcases:

21° there exists a non-zero entry  $a_{si}$  in  $A$  such that either  $(s, i) \in A_2 \times B_1$  or  $(s, i) \in B_2 \times A_1$ . Let first  $(s, i) \in A_2 \times B_1$ . It follows from the definition of  $\lambda(A)$  that

$$\sum_{r \in A_1} a_{sr} > \lambda(A) > 0$$



so that  $a_{sj} > 0$  for some  $j \in A_1$ . Since  $h(T - \min(a_{si}, a_{sj}) T_{sij}) < h(T)$ , we have, by the induction hypothesis,  $T - \min(a_{si}, a_{sj}) T_{sij} \in \text{conv } \mathcal{T}$ . Consequently,  $T \in \text{conv } \mathcal{T}$  as well. If  $(s, i) \in B_2 \times A_1$ , it follows from

$$-\sum_{r \in B_1} a_{sr} < -\lambda(A)$$

that  $a_{sj} > 0$  for some  $j \in B_1$ . Thus  $h(T - \min(a_{si}, a_{sj}) T_{sji}) < h(T)$  and the induction hypothesis applies as well.

22°  $\lambda(T) > 0$  and all non-zero entries  $a_{si}$  of  $A$  are in the union of the blocks  $A_2 \times A_1$  and  $B_2 \times B_1$ .

221° First we dispose of the following simpler case: All non-zero entries of  $B_2 \times B_1$  are concentrated in one column, with index  $k$ , say. According to 3° of Proposition (2,1) we have

$$a_{sk} = \lambda(A) \quad \text{for all } s \in B_2.$$

We shall denote by  $\tilde{A}$  the matrix obtained from  $A$  by changing the  $k$ -th column of  $A$  as follows

$$\tilde{a}_{sk} = \lambda(A) \quad \text{for } s \in A_2, \quad \tilde{a}_{sk} = 0 \quad \text{for } s \in B_2.$$

Since the difference  $\tilde{A} - A$  has non-zero entries in the  $k$ -th column only and this column is a multiple of the relation  $r_2$ , the matrix  $A$  generates the same operator  $T$ . At the same time,  $\tilde{A}$  is nonnegative and  $\lambda(\tilde{A}) = 0$ . It follows from ( $\alpha$ ) that  $T \in \text{conv } \mathcal{T}$ .

222° It remains to deal with the case where at least two columns with indices in  $B_1$  contain nonzero entries.

In order to use the induction hypothesis in this case it will be necessary to choose an operator of type 2.

Suppose we have found an operator  $T_0$  of type 2 such that  $\varepsilon A_0 \leq A$  for some positive  $\varepsilon$  where  $A_0$  is the canonical matrix of  $T_0$ . Then it is possible to find a positive  $\omega$  such that  $\omega A_0 \leq A$  and, for  $T_0 = \psi(A_0)$ ,

$$h(T - \omega T_0) < h(T).$$

By induction hypothesis, we have then  $T - \omega T_0 \in \text{conv } \mathcal{T}$  so that  $T \in \text{conv } \mathcal{T}$  as well. This shows that it will be sufficient to find an operator  $T_0$  with the properties mentioned above. In other words, we are to find two decompositions:

$$A_2 = \bigcup_{k \in A_1} A_{2k}$$

and

$$B_2 = \bigcup_{k \in B_1} B_{2k}$$

with the following properties:

1° each of these decompositions contains at least two nonvoid sets;

2° the following two implications hold:

whenever  $s \in A_{2k}$  then  $a_{sk} \neq 0$ ;

whenever  $s \in B_{2k}$  then  $a_{sk} = 0$ .

The rest of this proof is devoted to the construction of these decompositions.

We shall define the decomposition

$$A_2 = \bigcup_{k \in A_1} A_{2k}$$

as follows:

For each  $k \in A_1$  denote by  $V_k$  the set

$$V_k = \{s \in A_2; a_{sk} \neq 0\}$$

and set

$$A_{2k} = V_k \setminus \bigcup\{V_t; t \in A_1, t < k\}.$$

Clearly the sets  $A_{2k}$  are disjoint. Since  $\lambda(A) > 0$ , each row of  $A$  contains at least one non-zero entry. Consequently,  $\bigcup_{k \in A_1} A_{2k} = A_2$ . We shall show now that at least two sets of decomposition  $\{A_{2k}\}$  are nonvoid. To see that it suffices to observe that none of the sets  $V_k$  can fill the whole set  $A_2$ , the matrix  $A$  being in the canonical form. It is obvious from the construction that the decomposition possesses property 2°.

To define the decomposition

$$B_2 = \bigcup_{k \in B_1} B_{2k},$$

we shall distinguish two cases:

Consider first the case that each row of  $A$  with row index in  $B_2$  contains at most one non-zero entry.

We set then  $B_{2k} = \{s \in B_2; a_{sk} \neq 0\}$ . Since  $\lambda(A) > 0$ , each row contains at least, and thus exactly one non-zero entry. Consequently, the union of the  $B_{2k}$  is the whole of  $B_2$  and the  $B_{2k}$  are disjoint. At the same time, at least two of them are nonvoid because the non-zero entries of  $A$  in  $B_2 \times B_1$  are not all contained in one column.

It remains to deal with the case that some row, with row index  $t \in B_2$  say, contains at least two non-zero entries, say  $a_{ti} \neq 0$ ,  $a_{tj} \neq 0$ ,  $i, j \in B_1$ . We define then for  $k \in B_1$

$$\tilde{W}_k = \{s \in B_2; s \neq t, a_{sk} \neq 0\}$$

and set

$$\tilde{B}_{2k} = \tilde{W}_k \setminus \bigcup\{\tilde{W}_r; r \in B_1, r < k\}.$$

Now we set  $B_{2k} = \tilde{B}_{2k}$  for all  $k \in B_1$ , different from both  $i$  and  $j$ . To define  $B_{2i}$  and  $B_{2j}$ , we shall distinguish two cases:

If  $\tilde{B}_{2i} = B_2 \setminus \{t\}$ , we set

$$B_{2i} = \tilde{B}_{2i}, \quad B_{2j} = \{t\}.$$

If  $\tilde{B}_{2i} \neq B_2 - \{t\}$ , hence is properly contained in  $B_2 \setminus \{t\}$ , we set

$$B_{2i} = \tilde{B}_{2i} \cup \{t\}, \quad B_{2j} = \tilde{B}_{2j}.$$

It follows from this construction that the properties 1° and 2° are satisfied in this case as well. The proof is complete.

3°  $\mu(T) < 0$ . In this case, the proof may be effected analogously to that of case 2°; one has only to use operators of type 3 instead of operators of type 2.

The attentive reader may have observed that one can proceed also as follows: The number  $\mu(T)$  depends on the choice of one subset of the decomposition  $N_1 = A_1 \cup B_1$  as  $A_1$  and the other as  $B_1$ . If we interchange the notation, the number  $\mu(T)$  will be changed to  $-\mu(T)$  and the classes of operators of type 2 and 3 will be interchanged, too. Consequently, the validity of the assertion in the case  $\mu(T) < 0$  follows from the validity of case 2°.

**(2,4) Theorem.** Denote by  $r(T)$  the dimension of the range of the operator  $T$ . Then

1°  $r(T) = 1$  for each operator of type 1;

2° if  $T$  is an operator of type 2 corresponding to the decompositions  $\{A_{2j}\}$ ,  $j \in A_1$  and  $\{B_{2j}\}$ ,  $j \in B_1$  then  $r(T) = K_1 + K_2 - 1$  where  $K_1$  is the number of non-empty sets  $A_{2j}$  and  $K_2$  the number of non-empty sets  $B_{2j}$ .

3° if  $T$  is an operator of type 3 corresponding to the decompositions  $\{A_{2j}\}$ ,  $j \in B_1$  and  $\{B_{2j}\}$ ,  $j \in A_1$  then  $r(T) = K_1 + K_2 - 1$  where  $K_1$  is the number of non-empty sets  $A_{2j}$  and  $K_2$  the number of non-empty sets  $B_{2j}$ .

Proof. The assertion about the operator of type 1 is obvious. Let  $T$  be an operator of type 2 corresponding to the decompositions  $\{A_{2j}\}$ ,  $j \in A_1$  and  $\{B_{2j}\}$ ,  $j \in B_1$ . Suppose  $i \in A_1$  is such that  $A_{2i} \neq \emptyset$ . The vectors  $p_t$  with  $t \in N_1 \setminus \{i\}$  form a basis of the space  $E^{(1)}$ . The dimension of the range of  $T$  is equal to the number of linearly independent vectors among the vectors  $Tp_t$ ,  $t \in N_1 \setminus \{i\}$ . Denote by  $\tilde{A}_1$  the set of all  $j \in A_1$  such that  $A_{2j} \neq \emptyset$ ; hence  $i \in \tilde{A}_1$ . Similarly denote by  $\tilde{B}_1$  the set of all  $j \in B_1$  such that  $B_{2j} \neq \emptyset$ . It is easy to see that the vectors  $Tp_j$ ,  $j \in \tilde{A}_1 \cup \tilde{B}_1 \setminus \{i\}$  are linearly independent and form a basis of the range of  $T$ . Thus  $r(T) = K_1 + K_2 - 1$  as asserted.

The proof of 3° is analogous.

**(2,5) Lemma.** Let  $C_1$  and  $C_2$  be two non-empty sets, let  $1 \leq r \leq \min(|C_1|, |C_2|)$ . Then there exists a decomposition  $C_2 = \bigcup_{j \in C_1} C_{2j}$  with exactly  $r$  nonvoid components  $C_{2j}$ .

Proof. May be left to the reader.

**(2,6) Theorem.** Let  $C_1$  and  $C_2$  be two minimal cones in the spaces  $E^{(1)}$  and  $E^{(2)}$ , respectively. Denote by  $n_1$  and  $n_2$ , respectively the dimensions of  $C_1$  and  $C_2$ . Then  $P(C_1, C_2)$  is a polyhedral cone of dimension  $n_1 n_2$ . The set of all extreme operators of  $P(C_1, C_2)$  is described in Theorem (2,3). Extreme operators have rank either 1 or greater than 2. If  $T$  is an extreme operator then  $T$  has rank 1 if and only if  $T$  is a positive multiple of an operator of type 1. Operators of type 2 have rank greater than 2 and each number  $r$ ,  $3 \leq r \leq \min(k_1, k_2) + \min(l_1, l_2)$  may be obtained as the rank of some extreme operator of type 2. Operators of type 3 have also rank greater than 2 and each number  $r$ ,  $3 \leq r \leq \min(k_1, l_2) + \min(k_2, l_1)$  may be obtained as the rank of some extreme operator of type 3. Here,  $k_i = |A_i|$ ,  $l_i = |B_i|$ ,  $i = 1, 2$ .

*Proof.* We shall restrict ourselves to the proof that operators of type 2 realise all ranks between 3 and  $d = \min(k_1, k_2) + \min(l_1, l_2) - 1$ . Given  $r$  such that  $3 \leq r \leq d$  there exist numbers  $r_1, r_2$  such that  $2 \leq r_1 \leq \min(k_1, k_2)$ ,  $2 \leq r_2 \leq \min(l_1, l_2)$ ,  $r = r_1 + r_2 - 1$ . By Lemma (2,5) there exist decompositions  $\{A_{2j}\}$ ,  $j \in A_1$  of  $A_2$  and  $\{B_{2j}\}$ ,  $j \in B_1$ , of  $B_2$  with exactly  $r_1$  and  $r_2$  nonvoid components. It follows then from Theorem (2,4) that the corresponding extreme operator has rank  $r$ .

The case of operators of type 3 is analogous.

In the papers [2], [3] it was proved that if  $C$  is an indecomposable cone then the identity is an extreme operator of  $P(C, C)$ . This leads naturally to the conjecture: if both  $C_1$  and  $C_2$  are indecomposable then some of the extreme operators of  $P(C_1, C_2)$  has rank  $\min(n_1, n_2)$  where  $n_1, n_2$  are dimensions of  $C_1, C_2$  respectively. Theorem (2,6) may be used to disprove this conjecture. Indeed, if we take  $k_1 = 2$  and  $k_2 = \frac{1}{2}n_2$  (for  $n_2$  even), the maximal possible rank is  $2 + \frac{1}{2}n_2 - 1 = \frac{1}{2}n_2$  which will be less than  $\min(n_1, n_2)$  if  $n_1 > n_2 > 2$ .

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