# The Rank of the Second Gaussian Map for General Curves 

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## Introduction

Let $X$ be a smooth, projective curve of genus $g$ and let $\mathcal{L}$ be a line bundle on $X$. Consider the product $X \times X$ with the projections $p_{1}, p_{2}$ to the factors and the natural morphism $p$ to the symmetric product $X(2)$. One has $p_{*}\left(p_{1}^{*} \mathcal{L} \otimes p_{2}^{*} \mathcal{L}\right)=$ $\mathcal{L}^{+} \oplus \mathcal{L}^{-}$, where $\mathcal{L}^{ \pm}$denotes the invariant and anti-invariant line bundles with respect to the involution $(x, y) \mapsto(y, x)$. One has $H^{0}\left(\mathcal{L}^{+}\right) \cong \operatorname{Sym}^{2} H^{0}(\mathcal{L})$ and $H^{0}\left(\mathcal{L}^{-}\right) \cong \wedge^{2} H^{0}(\mathcal{L})$. Restriction to the diagonal of $X(2)$ gives rise to the maps

$$
\mu_{\mathcal{L}, 1}: \operatorname{Sym}^{2} H^{0}(\mathcal{L}) \rightarrow H^{0}\left(\mathcal{L}^{\otimes 2}\right) \quad \text { and } \quad w_{\mathcal{L}, 1}: \wedge^{2} H^{0}(\mathcal{L}) \rightarrow H^{0}\left(\mathcal{L}^{\otimes 2} \otimes K_{X}\right)
$$

where $K_{X}$ is the canonical bundle of $X$. Both maps have a well-known geometric meaning. The former is given by considering the $\operatorname{map} \phi_{\mathcal{L}}: X \rightarrow \mathbb{P}^{r}:=\mathbb{P}\left(H^{0}(\mathcal{L})\right)^{*}$ defined by the complete linear series determined by $\mathcal{L}$ and by pulling forms of degree 2 in $\mathbb{P}^{r}$ back to $X$. The latter is given by considering the composition $\gamma$ of $\phi_{\mathcal{L}}$ with the Gauss map of $X$ to the Grassmannian of lines $\mathbb{G}(1, r)$ and by pulling forms of degree 1 in $\mathbb{P}^{\binom{(+1}{2}-1}$ back to $X$ via $\gamma$.

The maps $\mu_{\mathcal{L}, 1}$ and $w_{\mathcal{L}, 1}$ are the first instances of two hierarchies of maps $\mu_{\mathcal{L}, k}$ and $w_{\mathcal{L}, k}$, which are defined for all positive integers $k$ and are called by some authors higher Gaussian maps of $X$. They are inductively defined by iterated restrictions to the diagonal of $X(2)$. Precisely, for all $k \geq 2$ one has

$$
\begin{aligned}
& \mu_{\mathcal{L}, k}: \operatorname{ker}\left(\mu_{\mathcal{L}, k-1}\right) \rightarrow H^{0}\left(\mathcal{L}^{\otimes 2} \otimes K_{X}^{\otimes 2(k-1)}\right) \\
& w_{\mathcal{L}, k}: \operatorname{ker}\left(w_{\mathcal{L}, k-1}\right) \rightarrow H^{0}\left(\mathcal{L}^{\otimes 2} \otimes K_{X}^{\otimes(2 k-1)}\right)
\end{aligned}
$$

These maps are particularly interesting when $\mathcal{L} \cong K_{X}$, in which case we will simply denote them as $\mu_{k}$ and $w_{k}$. They are both defined at a general point of the moduli space of curves $\mathcal{M}_{g}$, and it is natural to suppose that they have some modular meaning. Indeed, $\mu_{1}$ is the codifferential, at the point corresponding to $X$, of the Torelli map $\tau: \mathcal{M}_{g} \rightarrow \mathcal{A}_{g}$, and Noether's theorem says that $\mu_{1}$ is surjective if and only if $X$ is nonhyperelliptic.

The map $w_{1}$ is called the Wahl map, and it is related to important deformation and extendability properties of the canonical image of the curve (cf. [BMé; W]). Because of this, it has been studied by various authors-too many to be quoted
here. One the most interesting results concerning the Wahl map is perhaps a theorem first proved by Ciliberto, Harris, and Miranda $[\mathrm{CiHM}]$ to the effect that $w_{1}$ is surjective, as expected, for a general curve of genus $g=10$ and $g \geq 12$. Moreover, this map is injective, as expected, for a general curve of genus $g \leq 8$ (cf. [CiM1]). Unexpectedly, however, the Wahl map is not of maximal rank for a general curve of genus $g=9,11$.

In general, all maps $\mu_{k}$ and $w_{k}$ are supposed to be meaningful in the geometry of curves, especially of curves with general moduli. Here we will look in particular at the map $\mu_{2}: \mathcal{I}_{2}\left(K_{X}\right) \rightarrow H^{0}\left(X, K_{X}^{\otimes 4}\right)$, where $\mathcal{I}_{2}\left(K_{X}\right)$ is the vector space of forms of degree 2 vanishing on the canonical model of $X$. From now on we will simply denote this map by $\mu$, and we will call it the second Gaussian map of $X$. This map was first considered by Green and Griffiths (see [Gr]), and its importance stems from its relation to the second fundamental form of the moduli space of curves $\mathcal{M}_{g}$ embedded in $\mathcal{A}_{g}$ via the Torelli map (cf. [CF1; CF2; CPT]).

Despite the unexpected behavior of the Wahl map for genus $g=9,11$, a reasonable working hypothesis is that the second Gaussian map $\mu$ should be of maximal rank for a general curve of any genus $g$. A dimension count shows that this is equivalent to saying that $\mu$ should be injective for a general curve of genus $g \leq 17$ and surjective if $g \geq 18$. So far, the best result in this direction has been proved by Colombo and Frediani in [CF3], where-by studying hyperplane sections of high genera of K3 surfaces-they show that $\mu$ is surjective for a general curve of genus $g>152$. For other interesting results concerning $\mu$, see also [CF2; CFPa].

In this paper, we prove the maximal rank property for every genus.
Theorem 1. The second Gaussian map $\mu: \mathcal{I}_{2}\left(K_{X}\right) \rightarrow H^{0}\left(X, K_{X}^{\otimes 4}\right)$ for $X$ a general curve of any genus $g$ has maximal rank; namely, it is injective for $g \leq 17$ and surjective for $g \geq 18$.

As shown in [CPT], the map $\mu$ has a lifting $\rho: \mathcal{I}_{2}\left(K_{X}\right) \rightarrow \operatorname{Sym}^{2}\left(H^{0}\left(K_{X}^{\otimes 2}\right)\right)$, which is the datum of the second fundamental form of the Torelli embedding at the point corresponding to $X$ in the nonhyperelliptic case. As proved in [CF2, Cor. 3.4], $\rho$ is injective for all nonhyperelliptic curves $X$. Our result shows that if $X$ is general then the image of $\rho$ is transversal to the kernel of the multiplication map $\operatorname{Sym}^{2}\left(H^{0}\left(K_{X}^{\otimes 2}\right)\right) \rightarrow H^{0}\left(K_{X}^{\otimes 4}\right)$.

The proof of Theorem 1 is by degeneration to a reducible nodal curve for which the limit of $\mu$, described in Section 1, has maximal rank. The theorem then follows by upper semicontinuity. We do not use graph curves here (i.e., the curves exploited in [CiHM]) because for them the limit of $\mu$ is more difficult to understand. We used instead a general binary curve-in other words, a stable curve of genus $g$ consisting of two rational components meeting at $g+1$ points that are general on both components. For such a curve $C$ we explicitly write down the ideal $\mathcal{I}_{2}\left(K_{C}\right)$ in Section 2. In Section 3 we describe the second Gaussian map for $C$ modulo torsion, and in Section 4 we deal with the torsion part. By direct computations performed with Maple (the script is presented and commented in the Appendix), we verify the injectivity for a general binary curve of genus $g \leq 17$
and the surjectivity for $g=18$. Finally, in Section 5, we proceed by induction on $g$ to complete the argument for $g \geq 19$.

The behavior of $\mu$, and its connection with the curvature of $\mathcal{M}_{g}$ in $\mathcal{A}_{g}$, indicates possible relations of the surjectivity of $\mu$ with the Kodaira dimension of $\mathcal{M}_{g}$ being nonnegative. This, we think, would be a great subject for future research. Also interesting is the study of the Gaussian maps $\mu_{k}, w_{k}$ for higher values of $k$. The maps $\mu_{k}$ are related to higher fundamental forms of the Torelli immersion of $\mathcal{M}_{g}$ in $\mathcal{A}_{g}$ at a nonhyperelliptic point. Are these maps also of maximal rank for a general curve?

In this paper we work over the complex field and use standard notation in algebraic geometry. In particular, if $X$ is a Gorenstein curve, then $\Omega_{X}^{1}$ will denote its sheaf of Kähler differentials and $K_{X}$ will denote its dualizing sheaf or canonical bundle, or a canonical divisor. In general, we will indifferently use sheaf, bundle, or divisor notation. We will often write $H^{i}(\mathcal{L})$ instead of $H^{i}(X, \mathcal{L})$ for cohomology spaces.

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## 1. The Second Gaussian Map for a Stable Curve

Let $X$ be a stable curve of genus $g$. We will denote by $\mathcal{I}_{2}\left(K_{X}\right)$ the vector space of forms of degree 2 vanishing on the canonical model of $X$. If $X$ is smooth, then the second Gaussian map $\mu: \mathcal{I}_{2}\left(K_{X}\right) \rightarrow H^{0}\left(X, K_{X}^{\otimes 4}\right)$ is locally defined as follows.

Fix a basis $\left\{\omega_{i}\right\}$ of $H^{0}\left(K_{X}\right)$, and write it in a local coordinate $z$ as $\omega_{i}=f_{i}(z) d z$. Let $Q \in \mathcal{I}_{2}\left(K_{X}\right)$ with $Q=\sum_{i, j} s_{i j} \omega_{i} \otimes \omega_{j}$, where the matrix $\left(s_{i j}\right)$ is symmetric. Since $\sum_{i, j} s_{i j} f_{i} f_{j} \equiv 0$, one has $\sum_{i, j} s_{i j} f_{i}^{\prime} f_{j} \equiv 0$. The local expression of $\mu(Q)$ is then (cf., e.g., [CF2])

$$
\begin{equation*}
\mu(Q)=\sum_{i, j} s_{i j} f_{i}^{\prime \prime} f_{j}(d z)^{4}=-\sum_{i, j} s_{i j} f_{i}^{\prime} f_{j}^{\prime}(d z)^{4} \tag{1}
\end{equation*}
$$

If $X$ is nodal, one can similarly define the second Gaussian map

$$
\mu: \mathcal{I}_{2}\left(K_{X}\right) \rightarrow H^{0}\left(X, \operatorname{Sym}^{2}\left(\Omega_{X}^{1}\right) \otimes K_{X}^{\otimes 2}\right)
$$

which is locally defined in a similar way as in (1). Precisely, let $\left\{\omega_{i}\right\}$ be a basis of $H^{0}\left(K_{X}\right)$. In local coordinates we can write $\omega_{i}=f_{i} \xi$, where $f_{i}$ is a regular function and $\xi$ is a local generator of the canonical bundle $K_{X}$. Then $\mu$ is locally defined by

$$
\begin{equation*}
\mu(Q)=-\sum_{i, j} s_{i j} d f_{i} d f_{j} \xi^{\otimes 2} \tag{2}
\end{equation*}
$$

Given a flat degeneration over a disc of a general curve to a stable curve $X$, the second Gaussian map for $X$ is the flat limit of the second Gaussian map for the general curve.

It is useful to describe in some detail the space $H^{0}\left(X, \operatorname{Sym}^{2}\left(\Omega_{X}^{1}\right) \otimes K_{X}^{\otimes 2}\right)$. We first remark that $\operatorname{Sym}^{2}\left(\Omega_{X}^{1}\right)$ has torsion $T$ supported at the nodes of $X$. Hence there is a short exact sequence

$$
0 \rightarrow T \rightarrow \operatorname{Sym}^{2}\left(\Omega_{X}^{1}\right) \rightarrow \mathcal{F}_{X} \rightarrow 0
$$

where $\mathcal{F}_{X}$ is a nonlocally free, rank-1, torsion-free sheaf on $X$.
Lemma 2. (a) For every node $p$ of $X, T_{p}$ is a 3-dimensional vector space; if the local equation of $X$ around $p$ is $x y=0$, then $T_{p}$ is spanned by $d x d y, x d x d y$, and $y d x d y$.
(b) If $X_{i}$ are the irreducible components of the normalization $\pi: \tilde{X} \rightarrow X$ of $X$, then

$$
\mathcal{F}_{X} \cong \bigoplus_{i} \pi_{*} K_{X_{i}}^{\otimes 2}
$$

Proof. Since $y d x=-x d y$, a local section of $\operatorname{Sym}^{2}\left(\Omega_{X}^{1}\right)$ around a node $x y=0$ can be uniquely written as $f(x)(d x)^{2}+g(x, y) d x d y+h(y)(d y)^{2}$, where $g(x, y)$ is linear. Then (a) is a local computation and (b) follows from (a).

As a consequence, since $K_{X \mid X_{i}}=K_{X_{i}}\left(D_{i}\right)$ for $D_{i}$ the divisor of nodes on $X_{i}$, it follows that

$$
\begin{equation*}
H^{0}\left(X, \operatorname{Sym}^{2}\left(\Omega_{X}^{1}\right) \otimes K_{X}^{\otimes 2}\right) \cong T \oplus \bigoplus_{i} H^{0}\left(X_{i}, K_{X_{i}}^{\otimes 4}\left(2 D_{i}\right)\right) \tag{3}
\end{equation*}
$$

here $T \cong \mathbb{C}^{3 \delta}$, with $\delta$ the number of nodes of $X$.

## 2. Canonical Binary Curves

Let $\left[x_{1}, \ldots, x_{g}\right]$ be homogenous coordinates in $\mathbb{P}^{g-1}, g \geq 3$. Let $p_{h}=[0, \ldots, 0,1$, $0, \ldots, 0$ ], with 1 at the $h$ th place, $1 \leq h \leq g$, be the coordinate points and let $u=$ $[1,1, \ldots, 1]$ be the unit point. Take two distinct rational normal curves $C_{1}, C_{2}$ in $\mathbb{P}^{g-1}$ passing through $p_{h}, 1 \leq h \leq g$, and $u$. Then $C_{1}$ and $C_{2}$ intersect transversally at these $g+1$ points and have no further intersection.

We may and will assume that $C_{k}, k=1,2$, is the closure of the image of the map $f_{k}$ given by

$$
\begin{equation*}
t \mapsto f_{k}(t)=\left[\frac{1}{t-\alpha_{k, 1}}, \frac{1}{t-\alpha_{k, 2}}, \ldots, \frac{1}{t-\alpha_{k, g}}\right] \tag{4}
\end{equation*}
$$

where $\alpha_{k, i} \in \mathbb{C}$ for $k=1,2$ and $i=1, \ldots, g$. In particular, $f_{k}\left(\alpha_{k, h}\right)=p_{h}, h=$ $1, \ldots, g$, and $f_{k}(\infty)=u$. For our purposes, the $\alpha_{k, i}$ will be general in $\mathbb{C}$. Actually, we will often consider them as indeterminates on $\mathbb{C}$.

The curve $C=C_{1} \cup C_{2}$ is the limit of a general canonical curve $X \subset \mathbb{P}^{g-1}$ of genus $g$, and $C$ is canonical, too; that is, $\mathcal{O}_{C}(1) \cong K_{C}$. The curve $C$ is usually called a canonical binary curve.

We sketch the proof of the following proposition, which is more than we need. Indeed, we will need only the quadratic normality of a general canonical binary curve $C$, which can be directly proved (see Remarks 4 and 8 ).

Proposition 3. A canonical binary curve $C=C_{1} \cup C_{2}$ is projectively normal.
Proof. The assertion is trivial for $g=3$, which is the minimum allowed value of $g$. So we may assume $g \geq 4$. By Theorem 1.2 in [S], it suffices to show that there are $g-2$ smooth points of $C$ spanning a $\mathbb{P}^{g-3}$ that meets $C$ scheme-theoretically at these $g-2$ points only. Choose $g-2$ general points on $C_{1}$ and let $\Lambda \cong \mathbb{P}^{g-3}$ be their span, which meets $C_{1}$ transversally at these points. We claim that $\Lambda$ does not meet $C_{2}$. Otherwise, choose $g-4$ general points on $C_{1}$ and project $C$ down to $\mathbb{P}^{3}$ from their span. The image of $C_{1}$ is a rational normal cubic $\Gamma_{1}$, whereas $C_{2}$ projects birationally (cf. [CaCi]) to a nondegenerate rational curve $\Gamma_{2}$ of degree $>3$; hence $\Gamma_{1}$ and $\Gamma_{2}$ are distinct. Moreover, the general secant line to $\Gamma_{1}$ would meet $\Gamma_{2}$, which is impossible by the trisecant lemma (see the focal proof in [ ChCi$]$ ).

Remark 4. The only information that we will need from Proposition 3 is that $C$ is quadratically normal, which is equivalent to

$$
\operatorname{dim}\left(\mathcal{I}_{2}\left(K_{C}\right)\right)=\binom{g-2}{2}
$$

The simple argument in the proof of Proposition 3 relies on Schreyer's result, which requires a careful analysis following the classical approach of Petri. The same result would follow by proving that the general hyperplane section of $C$ verifies the general position theorem (see [ACGH, p. 109]). This may be proved with the same argument as before, but we do not dwell on that here.

In case $C$ is a general binary curve, it is quite simple to prove that $C$ is quadratically normal. One way is to remark that the general trigonal binary curve is quadratically normal. For example, if $g=2 h$, embed $\mathbb{F}_{0}$ in $\mathbb{P}^{g-1}$ via the linear system of curves of type $(1, h-1)$. The general trigonal binary curve is the union of the images of a general curve of type $(1, h)$ and of a general curve of type $(2,1)$. The case $g$ odd is similar and is left to the reader.

We are now interested in explicitly describing the vector space $\mathcal{I}_{2}\left(K_{C}\right)$ of degree2 forms vanishing on $C$ (i.e., the domain of the map $\mu$ for $C$ ). The analysis we shall make provides another proof that the general binary curve $C$ is quadratically normal.

For $k=1,2$, set

$$
\begin{equation*}
A_{k}(t)=\prod_{i=1}^{g}\left(t-\alpha_{k, i}\right) \tag{5}
\end{equation*}
$$

For each $h=0, \ldots, g$, the coefficients $c_{k, h}$ of $t^{g-h}$ in $A_{k}(t)$ are, up to sign, the elementary symmetric functions

$$
c_{k, 0}=1, \quad c_{k, h}=(-1)^{h} \sum_{1 \leq i_{1}<i_{2}<\cdots<i_{h} \leq g} \alpha_{k, i_{1}} \alpha_{k, i_{2}} \cdots \alpha_{k, i_{h}} .
$$

Note that the index $h$ is the degree of $c_{k, h}$ as a polynomial in the $\alpha_{k, i}$.
Fix $k \in\{1,2\}$. Since $C_{k}$ passes through the coordinate points, the equation of a quadric $Q \subset \mathbb{P}^{g-1}$ containing $C_{k}$ has the form

$$
\begin{equation*}
\sum_{1 \leq i<j \leq g} s_{i j} x_{i} x_{j}=0 \tag{6}
\end{equation*}
$$

with the conditions

$$
P_{k}(t):=\sum_{1 \leq i<j \leq g} \frac{A_{k}(t)}{\left(t-\alpha_{k, i}\right)\left(t-\alpha_{k, j}\right)} s_{i j}=\sum_{n=0}^{g-2} P_{k, n} t^{n} \equiv 0
$$

where $P_{k}(t)$ is a polynomial in $t$ of degree $g-2$ whose coefficients are linear polynomials $P_{k, n}\left(s_{i j}\right)$ in the $s_{i j}, n=0, \ldots, g-2$. By expanding the product $A_{k}(t)$, one sees that the coefficients $p_{k, h ; i, j}$ of $s_{i j}$ in $P_{k, g-2-h}, h=0, \ldots, g-2$, are

$$
\begin{gather*}
p_{k, 0 ; i, j}=1, \quad p_{k, 1 ; i, j}=-\sum_{i_{1} \neq i, j} \alpha_{k, i_{1}}, \\
p_{k, h ; i, j}=(-1)^{h} \sum_{\substack{i_{1}<i_{2}<\ldots<i_{h} \\
\text { all } \neq i, j}} \alpha_{k, i_{1}} \alpha_{k, i_{2}} \cdots \alpha_{k, i_{h}}, \quad 2 \leq h \leq g-2, \tag{7}
\end{gather*}
$$

namely, the elementary symmetric functions (removing the $i$ and $j$ terms) up to sign. Again the index $h$ coincides with the degree of $p_{k, h ; i, j}$ as a homogeneous polynomial in the $\alpha_{k, i}$.

Consider also the polynomials

$$
Q_{k, n}\left(s_{i j}\right):=\sum_{1 \leq i<j \leq g}\left(\sum_{m=0}^{g-2-n} \alpha_{k, i}^{m} \alpha_{k, j}^{g-2-n-m}\right) s_{i j}, \quad n=0, \ldots, g-2,
$$

and let $q_{k, h ; i, j}=\sum_{m=0}^{h} \alpha_{k, i}^{m} \alpha_{k, j}^{h-m}$ be the coefficient of $s_{i j}$ in $Q_{k, g-2-h}, h=$ $0, \ldots, g-2$. In this case, too, the index $h$ coincides with the degree of $q_{k, h ; i, j}$ as a homogeneous polynomial in the $\alpha_{k, i}$.

Remark 5. The coefficient $q_{k, h ; i, j}$ of $s_{i j}$ in $Q_{k, g-2-h}$ can be recursively computed by

$$
\begin{gathered}
q_{k, 0 ; i, j}=1, \quad q_{k, 1 ; i, j}=\alpha_{k, i}+\alpha_{k, j} \\
q_{k, h ; i, j}=q_{k, 1 ; i, j} q_{k, h-1 ; i, j}-\alpha_{k, i} \alpha_{k, j} q_{k, h-2 ; i, j}, \quad 2 \leq h \leq g-2
\end{gathered}
$$

Note that all the monomials $\alpha_{k, j}^{m} \alpha_{k, i}^{h-m}, m=0, \ldots, h$-in particular, $\alpha_{k, i}^{h}$ and $\alpha_{k, j} \alpha_{k, i}^{h-1}$ —appear in $q_{k, h ; i, j}$ with coefficient 1 . Note also the recursive formula

$$
\begin{equation*}
q_{k, h ; i, j}=\alpha_{i} q_{k, h-1 ; i, j}+\alpha_{j}^{h}, \quad 1 \leq h \leq g-2 . \tag{8}
\end{equation*}
$$

We will need the following lemma.

Lemma 6. Fix $k \in\{1,2\}$. For each $n=0, \ldots, g-2$, one has

$$
\begin{equation*}
P_{k, n}=\sum_{m=0}^{g-2-n} c_{k, m} Q_{k, n+m} \tag{9}
\end{equation*}
$$

In particular, the linear system

$$
\begin{equation*}
P_{k, n}\left(s_{i j}\right)=0, \quad n=0, \ldots, g-2, \tag{10}
\end{equation*}
$$

in the $s_{i j}$ is equivalent to the linear system

$$
\begin{equation*}
Q_{k, n}\left(s_{i j}\right)=0, \quad n=0, \ldots, g-2 . \tag{11}
\end{equation*}
$$

Proof. One has $P_{k, g-2}=Q_{k, g-2}$ and $P_{k, g-3}=Q_{k, g-3}+c_{k, 1} Q_{k, g-2}$. Now we proceed by induction. Equation (9) is equivalent to

$$
\begin{equation*}
p_{k, h ; i, j}=\sum_{l=0}^{h} c_{k, l} q_{k, h-l ; i, j} \quad \text { for } h=0, \ldots, g-2 \tag{12}
\end{equation*}
$$

For $h=0,1$, (12) clearly holds. Since the index $k$ is fixed, we omit it. For $2 \leq$ $h \leq g-2$, one has

$$
\begin{aligned}
p_{h ; i, j}-c_{h} q_{0 ; i, j} & =\left(\alpha_{i}+\alpha_{j}\right) p_{h-1 ; i, j}-\alpha_{i} \alpha_{j} p_{h-2 ; i, j} \\
& =c_{h-1} q_{1 ; i, j}+\sum_{l=0}^{h-2} c_{l}\left(q_{h-l-1 ; i, j} q_{1 ; i, j}-\alpha_{i} \alpha_{j} q_{h-l-2 ; i, j}\right) \\
& =\sum_{l=0}^{h-1} c_{l} q_{h-l ; i, j},
\end{aligned}
$$

where the second equality follows by induction. This expression proves (12) and therefore (9). Since $c_{k, 0}=1$, the base change matrix between the $Q_{k, n}$ and the $P_{k, n}$ is unipotent triangular; hence it is invertible. The equivalence between (10) and (11) follows.

Next we can give the announced description of $\mathcal{I}_{2}\left(K_{C}\right)$.
Proposition 7. Let $g \geq 3$. For a general choice of $\alpha_{k, i}, 1 \leq k \leq 2,1 \leq i \leq g$, one has that:
(a) the linear system (11) has maximal rank $g-1$; and
(b) the linear system

$$
\begin{equation*}
Q_{1,0}\left(s_{i j}\right)=\cdots=Q_{1, g-2}\left(s_{i j}\right)=Q_{2,0}\left(s_{i j}\right)=\cdots=Q_{2, g-3}\left(s_{i j}\right)=0 \tag{13}
\end{equation*}
$$

has maximal rank $2 g-3$.
Proof. (a) Since the index $k$ is fixed, we drop it here. Let us consider the matrix

$$
U:=U\left(\alpha_{1}, \ldots, \alpha_{g}\right)=\left(q_{h ; i, j}\right)_{0 \leq h \leq g-2,1 \leq i<j \leq g}
$$

of size $(g-1) \times\binom{ g}{2}$, where the pairs $(i, j)$ are lexicographically ordered. We have to prove that there is a minor of $U$ of order $g-1$ that is not identically zero. We show this for the minor $D:=D\left(\alpha_{1}, \ldots, \alpha_{g}\right)$ determined by the first $g-1$ columns, indexed by $(1, i)$ with $2 \leq i \leq g$. This is true if $g=3$, so we proceed by induction on $g$. Look at $D$ as a polynomial in $\alpha_{g}$ : it has degree $g-2$ and the coefficient of $\alpha_{g}^{g-2}$ is $D\left(\alpha_{1}, \ldots, \alpha_{g-1}\right)$ (cf. Remark 5), which is nonzero by induction. This proves the assertion.

Equivalently, by subtracting from each row the previous one multiplied by $\alpha_{1}$ and using (8) (cf. Remark 5), one sees that $D$ is the Vandermonde determinant $V\left(\alpha_{2}, \ldots, \alpha_{g}\right)=\prod_{2 \leq i<j \leq g}\left(\alpha_{j}-\alpha_{i}\right)$ of $\alpha_{2}, \ldots, \alpha_{g}$.
(b) We use the same idea as in the proof of (a). Form a matrix

$$
Z:=Z\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g}
$$

of size $(2 g-3) \times\binom{ g}{2}$ by concatenating vertically $U$ (for $\left.k=1\right)$ and the matrix

$$
W:=W\left(\alpha_{2,1}, \ldots, \alpha_{2, g}\right)=\left(q_{2, h ; i, j}\right)_{1 \leq h \leq g-2,1 \leq i<j \leq g} .
$$

It suffices to prove that the minor $M:=M\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g}$ of $Z$ determined by the first $2 g-3$ columns, indexed by $(1, i),(2, j)$ with $2 \leq i \leq g$ and $3 \leq$ $j \leq g$, is not identically zero as a polynomial in the $\alpha_{k, i}$. This is clearly true for $g=3$, so we proceed by induction on $g$. Look at $M$ as a polynomial in $\alpha_{1, g}$ and $\alpha_{2, g}$ : one sees that the monomial $\alpha_{1, g}^{g-2} \alpha_{2, g}^{g-3}$ appears in $M$ with the coefficient $\left(\alpha_{2,2}-\alpha_{2,1}\right) M\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g-1}$, which is nonzero by induction; this proves the assertion.

Equivalently, looking at $M$ as a polynomial in $\alpha_{1,1}$, one sees that the coefficient of the monomial $\alpha_{1,1}^{g-2}$ is the product of two Vandermonde determinants: $V\left(\alpha_{2,2}, \ldots, \alpha_{2, g}\right) V\left(\alpha_{1,3}, \ldots, \alpha_{1, g}\right)$.

Remark 8. The solutions of the linear system (11), as well as those of (10), give us the quadrics containing the rational normal curve $C_{k}$, whereas the solutions of (13) give us the quadrics in $\mathcal{I}_{2}\left(K_{C}\right)$ for the binary curve $C=C_{1} \cup C_{2}$.

## 3. Binary Curves: The Second Gaussian Map Modulo Torsion

Let $C=C_{1} \cup C_{2}$ be a general binary curve. In this section we will consider the composition $v$ of the second Gaussian map for $C$ with the projection to the nontorsion part of $H^{0}\left(C, \operatorname{Sym}^{2}\left(\Omega_{C}^{1}\right) \otimes K_{C}^{\otimes 2}\right)($ cf. (3) in Section 1). Specifically, for $k=1,2$, we will look at the map

$$
v_{k}: \mathcal{I}_{2}\left(K_{C}\right) \rightarrow H^{0}\left(C_{k}, K_{C_{k}}^{\otimes 4}\left(2 D_{k}\right)\right),
$$

where $D_{k}$ is a divisor of degree $g+1$ on $C_{k}$; therefore, $v=\nu_{1} \oplus \nu_{2}$ and

$$
H^{0}\left(C_{k}, K_{C_{k}}^{\otimes 4}\left(2 D_{k}\right)\right) \cong H^{0}\left(\mathbb{P}^{1}, \mathcal{O}_{\mathbb{P}^{1}}(2 g-6)\right)
$$

The map $v_{k}$ can be explicitly written down by taking into account (2) and the description of the ideal $\mathcal{I}_{2}\left(K_{C}\right)$ (see Section 2). Precisely, let $Q \in \mathcal{I}_{2}\left(K_{C}\right)$ be of the form (6), where the $s_{i j}$ are solutions of (13). Then

$$
v_{k}(Q)=\sum_{1 \leq i<j \leq g} \frac{1}{\left(t-\alpha_{k, i}\right)^{2}\left(t-\alpha_{k, j}\right)^{2}} s_{i j}(d t)^{4} \in H^{0}\left(C_{k}, K_{C_{k}}^{\otimes 4}\left(2 D_{k}\right)\right)
$$

To look at this as a section of $H^{0}\left(\mathbb{P}^{1}, \mathcal{O}_{\mathbb{P}^{1}}(2 g-6)\right)$, we multiply by $A_{k}^{2}(t)$. Then

$$
\begin{equation*}
v_{k}(Q)=\sum_{1 \leq i<j \leq g} \frac{A_{k}^{2}(t)}{\left(t-\alpha_{k, i}\right)^{2}\left(t-\alpha_{k, j}\right)^{2}} s_{i j}=: R_{k}(t) \tag{14}
\end{equation*}
$$

is a polynomial in $t$ whose apparent degree is $2 g-4$. However, its coefficient of degree $2 g-4$ is $P_{k, g-2}$ and the one of degree $2 g-5$ is proportional to $P_{k, g-3}$, so they vanish and $R_{k}(t)$ has actual degree $2 g-6$.

Using this explicit description (14) of $\nu$, we asked Maple to compute its rank for low values of $g$ (see the Appendix for the Maple script). The result is as follows.

Proposition 9. The map $v$ has maximal rank for $g \leq 18$; in other words, $v$ is injective for $g \leq 10$ and is surjective for $11 \leq g \leq 18$.

Corollary 10. The second Gaussian map $\mu$ is injective for the general curve of genus $g \leq 10$.

## 4. Binary Curves: The Torsion

Let $C=C_{1} \cup C_{2}$ be a general binary curve as in Section 2. In (4) we may replace $f_{k}, 1 \leq k \leq 2$, with

$$
\begin{equation*}
A_{k}(t) f_{k}(t)=\left[\phi_{k, 1}(t), \ldots, \phi_{k, g}(t)\right], \quad \phi_{k, i}(t)=\frac{A_{k}(t)}{\left(t-\alpha_{k, i}\right)} \tag{15}
\end{equation*}
$$

Now we consider the restriction $\tau$ of the second Gaussian map for $C$ to $\operatorname{ker}(\nu)$, which lands in the torsion part $T$ of $H^{0}\left(C, \operatorname{Sym}^{2}\left(\Omega_{C}^{1}\right) \otimes K_{C}^{\otimes 2}\right)$ (cf. (3)). Once we take Lemma 2(a) into account, a direct computation shows that the composition of $\tau$ with the projection on the torsion part $T_{p_{h}}$ at the coordinate point $p_{h}$ is as follows: if $Q \in \operatorname{ker}(v)$ is of the form (6), then $Q$ is mapped to

$$
\begin{align*}
d x d y \sum_{i \neq j} s_{i j} \phi_{1, i}^{\prime}\left(\alpha_{1, h}\right) \phi_{2, j}^{\prime}\left(\alpha_{2, h}\right)+ & 2 x d x d y \sum_{i \neq j} s_{i j} \phi_{1, i}^{\prime \prime}\left(\alpha_{1, h}\right) \phi_{2, j}^{\prime}\left(\alpha_{2, h}\right) \\
& +2 y d x d y \sum_{i \neq j} s_{i j} \phi_{1, i}^{\prime}\left(\alpha_{1, h}\right) \phi_{2, j}^{\prime \prime}\left(\alpha_{2, h}\right), \tag{16}
\end{align*}
$$

where $s_{j i}=s_{i j}$ and where $x$ and $y$ are local coordinates around $p_{h}$ such that $C_{1}: y=0$ and $C_{2}: x=0$. The description of the torsion at the unitary point $u$ is similar. Replace $f_{k}$ by the parameterization $\frac{1}{t} f_{k}\left(\frac{1}{t}\right)$. Again a direct computation shows that the composition of $\tau$ with the projection on $T_{u}$ is

$$
\begin{align*}
& Q \mapsto d x d y \sum_{i \neq j} s_{i j} \alpha_{1, i} \alpha_{2, j}+2 x d x d y \sum_{i \neq j} s_{i j} \alpha_{1, i}^{2} \alpha_{2, j} \\
&+2 y d x d y \sum_{i \neq j} s_{i j} \alpha_{1, i} \alpha_{2, j}^{2} \tag{17}
\end{align*}
$$

where $s_{j i}=s_{i j}$ and where $x$ and $y$ are local coordinates around $u$ such that $C_{1}: y=0$ and $C_{2}: x=0$.

Consider the following commutative diagram with exact rows.


We asked Maple to compute the rank of the map $\tau$ for $11 \leq g \leq 18$ (see the script in the Appendix). Taking into account diagram (18), we obtain the following results.

Proposition 11. Let $C$ be a general binary curve of genus $g$. Then the maps $\tau$ and $\mu$ have maximal rank for $g \leq 18$ : they are injective for $g \leq 17$ and surjective for $g=18$.

Corollary 12. The map $\mu$ is injective for the general curve of genus $g \leq 17$ and is surjective for $g=18$.

## 5. The Induction Step

In this section we prove our main result-namely, the surjectivity of the second Gaussian map $\mu$ for the general curve of genus $\geq 18$.

Let $C \subset \mathbb{P}^{g-1}$ be a nodal canonical curve and let $p \in C$ be a node. Let $\tilde{C} \rightarrow C$ be the partial normalization of $C$ at $p$, and let $p_{1}, p_{2} \in \tilde{C}$ be the points over $p$. Note that the projection from $p$ maps $C$ to the canonical model of $\tilde{C}$ in $\mathbb{P}^{g-2}$; we will assume that this induces an isomorphism of $\tilde{C}$ to its canonical model. Consider the following diagrams.


Here $\tilde{T}$ is the torsion subsheaf of $\operatorname{Sym}^{2}\left(\Omega_{\tilde{C}}^{1}\right), \nu, \tau$ are the maps of diagram (18) for the curve $C$, and $\tilde{v}, \tilde{\tau}$ are the corresponding ones for $\tilde{C}$. The diagrams (19) are commutative and the horizontal sequences are exact, so the next lemma is clear.

Lemma 13. If $\tilde{v}$ and $\chi$ (resp., $\tilde{\tau}$ and $\tau_{p}$ ) are surjective, then $\nu$ (resp., $\tau$ ) is also surjective.

We apply this lemma to prove our next theorem.

Theorem 14. If $C=C_{1} \cup C_{2}$ is a general binary curve of genus $g \geq 18$, then $\mu$ is surjective for $C$.

Proof. The case $g=18$ has already been addressed by direct computation (cf. Proposition 11). We therefore proceed by induction on $g$ : the commutativity of diagram (18) and Lemma 13 show that it is enough to prove the surjectivity of $\chi$ and $\tau_{p}$, where $p$ is a node of $C$. We will do this for $p=u$ the unitary point.

In this situation, the map $v$ is the one $\nu_{1} \oplus \nu_{2}$ considered in Section 3. Hence $\chi=\chi_{1} \oplus \chi_{2}$, where $\chi_{k}$ is the composition of $v_{k}$ with the restriction to $\mathcal{O}_{2 p_{k}}, k=$ 1,2 . In local coordinates, $\chi_{k}(Q)$ is the pair formed by the constant term and the coefficient of the degree- 1 term of the Taylor expansion around $p$ of the polynomial $v_{k}(Q)$. In Section 3 we computed $v_{k}$ using a local coordinate $t$ on $C_{k}$. In this coordinate, the point $p=[1, \ldots, 1]$ corresponds to $t=\infty$. So if $Q \in \mathcal{I}_{2}\left(K_{C}\right)$ is of the form (6), with the $s_{i j}$ satisfying (13), then $\chi_{k}(Q)$ is the pair of coefficients of the highest degrees $2 g-6$ and $2 g-7$ of the polynomial $\nu_{k}(Q)$-that is, of the polynomial $R_{k}(t)$ given in (14). We denote these coefficients by $R_{k, 2 g-6}$ and $R_{k, 2 g-7}$, which are linear polynomials in the $s_{i j}$. We will now compute them.

We fix the index $k$ and then omit it. By expanding $A^{2}$ in (14), one sees that the coefficient of $s_{i j}$ in $R_{2 g-6}$ is

$$
4 p_{2 ; i, j}+\sum_{i_{1} \neq i, j} \alpha_{i_{1}}^{2}=4 p_{2 ; i, j}+n_{2}-\left(\alpha_{i}^{2}+\alpha_{j}^{2}\right)
$$

where $n_{2}=\sum_{m=1}^{g} \alpha_{m}^{2}$ is independent of $i, j$ and $p_{2 ; i, j}$ is the coefficient of $s_{i j}$ in $P_{k, g-4}$ (cf. (7)). By (10), this means that

$$
R_{2 g-6}=4 P_{g-4}+n_{2} P_{g-2}-\sum_{i<j}\left(\alpha_{i}^{2}+\alpha_{j}^{2}\right) s_{i j}=-\sum_{i<j}\left(\alpha_{i}^{2}+\alpha_{j}^{2}\right) s_{i j}
$$

Similarly, one sees that the coefficient of $s_{i, j}$ in $R_{2 g-7}$ is twice

$$
4 p_{3 ; i, j}-\sum_{\substack{i_{1} \neq i_{2} \\ \text { both } \neq i, j}} \alpha_{i_{1}}^{2} \alpha_{i_{2}}=4 p_{3 ; i, j}-n_{3}+n_{2} p_{1 ; i, j}-c_{1}\left(\alpha_{i}^{2}+\alpha_{j}^{2}\right)-\left(\alpha_{i}^{3}+\alpha_{j}^{3}\right)-q_{3 ; i, j},
$$

where $n_{3}=-\sum_{m=1}^{g} \alpha_{m}^{3}$ is independent of $i, j$. Therefore, taking into account (10) and (11), one has

$$
R_{2 g-7}=-2 c_{1} R_{2 g-6}-2 \sum_{i<j}\left(\alpha_{i}^{3}+\alpha_{j}^{3}\right) s_{i j}
$$

Form the matrix $Y:=Y\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g}$ of size $(2 g+1) \times\binom{ g}{2}$ obtained by concatenating vertically the matrix $Z$ in the proof of Proposition 7(b) and the matrix of size $4 \times\binom{ g}{2}$ whose rows are $\left(\alpha_{k, i}^{h}+\alpha_{k, j}^{h}\right)_{1 \leq i<j \leq g}$ with $1 \leq k \leq 2$ and $2 \leq h \leq 3$. In order to prove that $\chi$ is surjective, we must first prove that there is a minor of order $2 g+1$ of $Y$ that is not identically zero. We will do this for the minor $N:=N\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g}$ determined by the first $2 g+1$ columns indexed by $(1, i),(2, j)$, and $(3, \ell)$, where $2 \leq i \leq g, 3 \leq j \leq g$, and $4 \leq \ell \leq 7$.

This minor is nonzero for $g=7$, which we verified using Maple (see the script in the Appendix). Then we proceed by induction on $g$ and assume $g \geq 8$. The argument here is the same as the one in the proof of Proposition 7(b). Look at $N$ as a polynomial in $\alpha_{1, g}$ and $\alpha_{2, g}$ : the monomial $\alpha_{1, g}^{g-2} \alpha_{2, g}^{g-3}$ appears in $N$ with coefficient $\left(\alpha_{2,2}-\alpha_{2,1}\right) N\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g-1}$; this coefficient is nonzero by induction, proving that $\chi$ is surjective.

It remains to show that $\tau_{p}$ is surjective. This could be seen with a quick monodromy argument, but we prefer to present an argument in the same style as the ones made so far.

Recall that $\operatorname{ker}(v)$ is defined in $\mathcal{I}_{2}\left(K_{C}\right)$ by the vanishing of the polynomials $R_{k}(t), k=1,2$, whose coefficients of degree $\leq 2 g-8$ are polynomials in the $\alpha_{k, i}$ of degree $\geq 4$. By the description of the torsion at the unitary point given in (17), we need to show the rank maximality of the matrix $Y^{\prime}=Y^{\prime}\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g}$ of size $(2 g+4) \times\binom{ g}{2}$ obtained by concatenating vertically the above matrix $Y$ and the matrix of size $3 \times\binom{ g}{2}$ whose rows are $\left(\alpha_{1, i} \alpha_{2, j}+\alpha_{1, j} \alpha_{2, i}\right)_{1 \leq i<j \leq g}$, $\left(\alpha_{1, i}^{2} \alpha_{2, j}+\alpha_{1, j}^{2} \alpha_{2, i}\right)_{1 \leq i<j \leq g}$, and $\left(\alpha_{1, i} \alpha_{2, j}^{2}+\alpha_{1, j} \alpha_{2, i}^{2}\right)_{1 \leq i<j \leq g}$. We claim that the minor $N^{\prime}=N^{\prime}\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g}$ of $Y^{\prime}$ determined by the first $2 g+4$ columns-indexed by $(1, i),(2, j)$, and $(3, \ell)$, where $2 \leq i \leq g, 3 \leq j \leq$ $g$, and $4 \leq \ell \leq 10$-is nonzero for $g \geq 10$. We verify the case $g=10$ with Maple (see the script in the Appendix), and the induction is the same as before because the monomial $\alpha_{1, g}^{g-2} \alpha_{2, g}^{g-3}$ appears in $N^{\prime}$ again with coefficient $\left(\alpha_{2,2}-\alpha_{2,1}\right) N^{\prime}\left(\alpha_{k, i}\right)_{1 \leq k \leq 2,1 \leq i<j \leq g-1}$. This concludes the proof that $\tau_{p}$ is surjective and hence the proof of the theorem.

Corollary 15. The second Gaussian map $\mu$ is surjective for the general curve of genus $g \geq 18$.

## Appendix: Maple Script for Computations

Listed here is the Maple script we run. We explain it afterwards, for which purpose we have added line numbers at each five lines.

```
alpha[1]:=[3,12,21,29,37,41,43,46,54,62,65,72,81,85,89,94,97,105]:
alpha[2]:=[6,18,24,36,39,42,45,52,60,63,71,80,84,86,91,96,104,108]:
for g from 4 to 18 do
listsij:=[seq(seq(s[i,j],j=i+1..g),i=1..g)]:
for k from 1 to 2 do
    A[k]:=mul(t-alpha[k][i],i=1..g):
    R[k]:=add(add(s[i,j]*(A[k] 2)/((t-alpha[k][i])^2*(t-alpha[k][j])^2),
                j=i+1..g),i=1..g):
    end do:
Z:=linalg[matrix]([seq([seq(seq(add(alpha[1][i]^m*alpha[1][j]^(h-m),m=0..h),
                            j=i+1..g),i=1..g)],h=0..g-2),
                            seq([seq(seq(add(alpha[2][i] ^m*alpha[2][j]^(h-m),m=0..h),
                        j=i+1..g),i=1..g)],h=1..g-2)]):
Zref:=Gausselim(Z,'rO') mod 109:
15 printf("For g=%2d, one has dim I2(K)=%3d, ",g,nops(listsij)-r0):
EqsKerNu:=[seq(seq(primpart(coeff(R[k],t,n)),n=0.. 2*g-6),k=1..2)]:
```

```
    K:=Gausselim(linalg[stackmatrix] (Zref,
            linalg[genmatrix] (EqsKerNu,listsij)),'r1') mod 109:
    printf("dim Ker(nu)=%2d, corank(nu)=%d, ",nops(listsij)-r1,4*g-10-r1+r0):
20
25
    for h from 1 to g do
    tors[h,1]:=add(add(s[i,j]*(phi1e[1,i,h]*phi1e[2,j,h]
                            +phi1e[1,j,h]*phi1e[2,i,h]),j=i+1..g),i=1..g):
    tors[h,2]:=add(add(s[i,j]*(phi2e[1,i,h]*phi1e[2,j,h]
                                    +phi2e[1,j,h]*phi1e[2,i,h]),j=i+1..g),i=1..g):
    tors[h,3]:=add(add(s[i,j]*(phi1e[1,i,h]*phi2e[2,j,h]
                                    +phi1e[1,j,h]*phi2e[2,i,h]),j=i+1..g),i=1..g):
    end do:
    tors[0,1]:=add(add(s[i,j]*(alpha[1][i]*alpha[2][j]
                            +alpha[1][j]*alpha[2][i]),j=i+1..g),i=1..g):
tors[0,2]:=add(add(s[i,j]*(alpha[1][i] 2**alpha[2][j]
                            +alpha[1][j] 2*alpha[2][i]),j=i+1..g),i=1..g):
    tors[0,3]:=add(add(s[i,j]*(alpha[1][i]*alpha[2][j]^2
                            +alpha[1][j]*alpha[2][i]^2),j=i+1..g),i=1..g):
    EqsKerTau:=[seq(seq(primpart(tors[h,l]),l=1..3),h=0..g)]:
    Gausselim(linalg[stackmatrix](K,linalg[genmatrix](EqsKerTau,listsij)),'r2') mod 109:
    printf("dim ker(tau)=%d, corank(tau)=%2d\n",nops(listsij)-r2,3*g+3-r2+r1):
    if g=7 then
        N:=linalg[det](linalg[stackmatrix](linalg[delcols] (Z,16 . 21),
                        linalg[matrix] ([seq(seq([seq(seq(alpha[k][i]^h+alpha[k][j]^h,
                                    j=i+1..7),i=1..3)],h=2..3),k=1..2)]))):
    printf("For g= 7, the minor N is congruent to %d (mod 5)\n",N mod 5):
    elif g=10 then
    N2:=linalg[det](linalg[stackmatrix](linalg[delcols](Z,25..45),
            linalg[matrix]([seq(seq([seq(seq(alpha[k][i]^h+alpha[k][j]^h,
                                    j=i+1..10),i=1..3)],h=2..3),k=1..2)]),
            linalg[matrix]([[seq(seq(alpha[1][i]*alpha[2][j]
                        +alpha[1][j]*alpha[2][i],j=i+1..10),i=1..3)],
                        [seq(seq(alpha[1][i] ~2*alpha[2][j]
                        +alpha[1][j]^2*alpha[2][i],j=i+1..10),i=1..3)],
                [seq(seq(alpha[1][i]*alpha[2][j]^2
                        +alpha[1][j]*alpha[2][i]^2,j=i+1..10),i=1..3)]]))):
    printf("For g=10, the minor N' is congruent to %d (mod 23)\n",N2 mod 23):
end if:
end do:
```

In lines $1-2$, we define the $\alpha_{k, i}$ that will be used. We chose them randomly. In line 3 we start the main loop, which runs for $4 \leq g \leq 18$. In line 4 , we collect the unknowns $\left\{s_{i, j}\right\}_{1 \leq i<j \leq g}$ in the list listsij: there are $\binom{g}{2}$ of them. In lines 6-8 we define the polynomials $A_{k}(t)$ and $R_{k}(t)$ (cf. (5) and (14)).

In lines 10-13 we define the matrix Z associated to the linear system (13), whose solutions give us the quadrics in $\mathcal{I}_{2}\left(K_{C}\right)$ (cf. the proof of Proposition 7). In line 14, Maple computes the rank r0 of Z via Gaussian elimination, calculating modulo 109 to speed up computations. The resulting matrix in row echelon form is called Zref. In line with Proposition 7(b), Maple finds $r 0=2 g-3$ for each $g=4, \ldots, 18$. In line 15 , Maple prints out the genus $g$ and $\operatorname{dim}\left(\mathcal{I}_{2}\left(K_{C}\right)\right)=\binom{g}{2}-\mathrm{r} 0=\binom{g-2}{2}$.

In line 16, we collect in EqsKerNu the list of equations that determine $\operatorname{ker}(\nu)$ (cf. the definition (14) of $v$ in Section 3). In lines 17-18, Maple computes the rank $r 1$ of the linear system EqsKerNu $\cap \operatorname{ker}(\mathrm{Zref})$, again via Gaussian elimination modulo 109, and the resulting row echelon matrix is called K. Maple finds that r1 $=$ $\binom{g}{2}$ for $4 \leq g \leq 10$ and that $r 1=6 g-13$ for $11 \leq g \leq 18$. Therefore, the rank of $v$ is $r 1-r 0=\binom{g-2}{2}$ for $4 \leq g \leq 10$ and is $4 g-10$ for $11 \leq g \leq 18$. This proves Proposition 9.

In line 19 , Maple prints out the dimension of $\operatorname{ker}(v)$ and the corank of $v$; that is, $4 g-10-\mathrm{r} 1+\mathrm{r} 0$.

In lines 20-25, we define the first and second derivatives phi1 and phi2 of the $\phi_{k, i}$ (cf. (15)). We then define their evaluations phi1e and phi2e at the coordinate point $p_{h}$. In lines 26-33, we use these evaluations to compute the torsion at $p_{h}$, $h=1, \ldots, g$ (cf. (16)). In lines 34-39 we compute the torsion at the unit point $u$ (cf. (17)).

In lines 40 and 41, we collect in EqsKerTau the equations that determine $\operatorname{ker}(\tau)$ and Maple computes the rank $r 2$ of EqsKerTau $\cap \operatorname{ker}(\mathrm{K})$ via Gaussian elimination modulo 109 as before. Maple finds that $\mathrm{r} 2=\binom{g}{2}$ for $4 \leq g \leq 17$ and that $\mathrm{r} 2=$ 152 for $g=18$. Hence the rank of $\tau$ is $\mathrm{r} 2-\mathrm{r} 1=\left(g^{2}-13 g+26\right) / 2$ for $11 \leq$ $g \leq 17$ and is 57 for $g=18$. This proves Proposition 11 .

In line 42 , Maple prints out the the dimension of $\operatorname{ker}(\tau)$ and the corank of $\tau$; that is, $3 g+3-\mathrm{r} 2+\mathrm{r} 1$.

Finally, in lines 43-59, Maple computes the minors $N$ (when $g=7$ ) and $N^{\prime}$ (when $g=10$ ), which are needed in the proof of Theorem 14, and prints out that $N \bmod 5=4$ and $N^{\prime} \bmod 23=16$.

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