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The ray method and the theory of edge waves

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Summary. A modification of the ray method including diffraction is outlined. The study is designed for the computation of the wavefields in 3-D inhomogeneous media containing such structural elements as pinch-outs, vertical and oblique contacts, faults and so on. The approach is based on the theory of edge waves. The total wavefield is considered as the superposition of two parts. The first part is described by the ray method. It has discontinuities because of its shadow boundaries. The second part is a superposition of two types of diffracted waves, caused by the edges and vertices of interfaces. This part smooths away the above-mentioned discontinuities, so that the total wavefield is a continuous one. The effects of multiple diffraction are considered. Of special importance is a mathematical form of amplitudes of diffracted waves, described with unified functions of eikonals. In fact, it allows all additional computations to be considered by finding the eikonals of diffracted waves.

1 Introduction

Of special importance for seismology is the ray method that allows wavefields to be computed efficiently in 3-D inhomogeneous media far from the source (Babich & Alekseyev 1958; Karal & Keller 1959; Červený, Molotkov & Pšenčík 1977, etc.). However, the method gives only the components of the wavefields connected with the energy flux along the ray tubes but not diffusion through their side walls. If the main part of the wavefield is formed by diffusion, it cannot be described by the ray method. The desire to adopt this method in similar situation has resulted in various modifications (Babich & Buldyrev 1972; Popov 1981; Červený 1983; Kennett 1984, etc.).

In the present paper a modification of the ray method for 3-D inhomogeneous block media is considered. The structural elements of interfaces in this type of media have sharp edges, the so-called diffracting edges (for example, the lines of pinch-outs vertical and oblique contacts of interfaces, faults and so on). The ray method does not give a continuous description of the wavefields in this type of media because of shadow boundaries. The main idea of the present modification is to smooth away the above discontinuities by the diffracted waves, scattering by the edges of interfaces, in such a way that the total wavefield is continuous. From a physical viewpoint, it is the same thing as to add the diffusion that is not considered by the ray method (Fock 1965). This principle is well-known in the classic theory of diffraction (Born & Wolf 1968) and in its modern modifications (Claerbout 1976; Trorey 1977; Hilterman 1982; Fertig & Müller 1979, etc.). However, there were no universal formulae to use the above-mentioned idea for improving the ray method.

The very core of the present approach is connected with the so-called boundary layer approximation. It allows us to put right the results of the ray method only within the neighbourhood of the shadow boundaries. It is just this kind of approximation that makes the final formulae universal and simple. The simplest way of getting these formulae is shown in this paper. It is based on assumptions concerning the analytic properties of the wavefields, but not the dynamic equations in any case. If a wave velocity is constant, the same formulae can be derived by the parabolic equation method or by an asymptotic analysis of the Kirchhoff integral. The formulae for the edge waves can be derived from a solution of the more general diffraction problem for the wedge-sharped structures as well. For details on this subject, see Klem-Musatov (1980, 1981a, b) and Aizenberg (1982).

Note that in the future monochromatic wavefields of an angular frequency ω will be considered. A time factor $\exp(-i\omega t)$ with $i^2 = -1$, where t is time, is omitted for convenience. Theoretical seismograms can be also computed in the time domain by the application of the Fourier transform.

2 Ray method

First of all, let us recollect the basic principles of the ray method. The model of the medium is considered as a combination of domains and interfaces. The functions, describing physical properties within the domains, are continuous and slowly changeable. A surface formed by points of discontinuity of any above functions is called an interface. The point of the interface is considered as a regular one if the above surface is continuous together with its first and second tangential derivatives. A part of the interface is considered as a regular one if any of its points is regular. The ray method allows us to describe only those components of the wavefield that are connected with reflections/transmissions at the regular parts of interfaces. The description has a form of superposition of the single waves

$$f = \sum_m f_m. \tag{1}$$

Let us give the main definitions related to the single wave f_m.

2.1 KINEMATICS

A ray is a space curve the tangential unit vector S_m of which complies with the differential equation:

$$\frac{d}{ds}\left(\frac{S_m}{c_m}\right) = \nabla \frac{1}{c_m} \tag{2}$$

where ds is a differential of the arc length, c_m is the wave velocity. This equation determines the ray single-valuedly, if its initial direction is given, and if a connection between the directions of incident and reflected/transmitted rays at the points of interface is given as well. The latter is expressed by Snell's law. In this law the geometry of the interface is usually characterized by the position of a normal to the interface. However, in this paper it is more convenient to achieve this by means of a tangential plane to the interface.

Let K_1 and K_2 be tangents to two arbitrary intersecting curves of the interface at the point of incidence. As a result we get the position of the tangential plane P at any regular point of interface. Let α_1 and α_2 be acute angles between the incident ray and the lines K_1 and K_2, respectively. Let β_1 and β_2 be acute angles between the reflected/transmitted ray and the same lines K_1 and K_2. Let Q be a plane that is normal to the line K_1 or K_2 at the point of incidence. The Snell's law can be expressed in the following way (Klem-Musatov 1980):

(1) the incident and the generated rays lie on different sides of the plane Q, (2) the directions of the above rays comply with the conditions:

$$\frac{\cos\beta_1}{c_m} = \frac{\cos\alpha_1}{c}, \quad \frac{\cos\beta_2}{c_m} = \frac{\cos\alpha_2}{c} \tag{3}$$

where c and c_m are the velocities of the incident and reflected/transmitted waves, respectively.

2.2 DYNAMICS

A single wave:

$$f_m = \Phi_m \exp(i\omega\tau_m) \tag{4}$$

is connected with a congruence of the rays S_m. Its eikonal τ_m complies with the differential equation:

$$\nabla\tau_m = S_m/c_m. \tag{5}$$

The formula (4) itself may represent a scalar wave (optics, acoustices) or a vector wave (elastodynamics, electrodynamics). In the first case, the ray amplitude Φ_m is a scalar one. In the second case,

$$\Phi_m = n_m \phi_m \tag{6}$$

where n_m is a unit vector of polarization, ϕ_m is the scalar. In an isotropic media the vector n_m coincides with the vector S_m (a longitudinal wave) or is perpendicular to S_m (a transverse wave).

The scalar amplitude Φ_m (or ϕ_m) complies with the so-called transport equation

$$2\nabla\tau_m \nabla\Phi_m + B_m \Phi_m = 0 \tag{7}$$

where the coefficient B_m depends on the kind of original accurate equations of optics, acoustics (or elastodynamics, electrodynamics). The solution of equation (7) is well known:

$$\Phi_m = \mathscr{H}_m L_m^{-1/2}, \quad L_m = \exp\left(\int_0^{\tau_m} c_m^2 B_m \, d\tau_m\right) \tag{8}$$

where integration must be performed along the ray. The choice of the constant $\mathscr{H}_m$ must comply with the boundary conditions. In fact, $\mathscr{H}_m$ is the product of reflection/transmission coefficients of the plane waves. Only the first term of the ray series is shown. As it will be seen later, the subsequent approach does not deal with the explicit formulae for the ray amplitude Φ_m.

3 Edge waves

We extend the theoretical basis using the ideas of the theory of diffraction. Let a certain line be formed by points of discontinuity of an interface or any of its first or second tangential derivatives. It is a common linear element of the regular parts of a single interface or several ones. This type of line is called an edge. A point of the edge is considered regular if the corresponding line is continuous together with its first tangential derivative. The edge is considered smooth if any of its points is regular.

Every single wavefield f_m exists within a connected domain of its continuity. This domain is called the primary illuminated zone. If the interfaces have edges, there may be a domain in which the wave f_m does not exist ($f_m \equiv 0$). This type of domain is called the primary shadow zone of the above mentioned wave. A singly-connected surface dividing these zones is called the primary shadow boundary. Let mn be the double number of each primary shadow boundary of the wave f_m. Let Ω_{mn}^{+} be a symbol of the primary shadow zone, formed by the mnth shadow boundary. Let Ω_{mn}^{-} be the symbol of the primary illuminated zone. The non-caustic shadow boundaries formed by the edges are considered.

We can see that shortcoming of equation (1) appears as discontinuities of the wavefields f_m at the primary shadow boundaries. Let us see how it can be put right.

3.1 KINEMATICS

We use a formal method to find the directions of the rays, generated at the points of an edge. Let the ray impinge on any regular point of the edge: the direction of a generated ray must comply with the Snell's law (3), it is necessary to fix the positions of a pair of the lines K_1 and K_2, i.e. to set the position of the plane P. One of two lines (for example, K_1) must be the tangent to the edge because it is a common linear element of the interfaces. However, there are no limitations in choosing the direction of the second line K_2. That is why, any plane, containing the tangent to the edge, may be considered as plane P. Let incident and generated rays make the acute angles α and β respectively with the tangent to the edge. Then Snell's law appears in the following form:

$$\frac{\cos\beta}{c_m} = \frac{\cos\alpha}{c}. \tag{9}$$

Note, the one-parameter set of generated rays complies with this condition.

The above-mentioned fact is known as the law of the edge diffraction (Keller 1962). It reads as follows. Let an incident ray make an acute angle α with a tangent to an edge. A set of generated rays forms a cone with its vortex at the point of incidence. Its apex angle is 2β, where β and α are connected under condition (9). The incident ray and the above-mentioned cone lie on opposite sides of the plane normal to the edge at the point of incidence. Obviously, this law holds true within a small neighbourhood of the point of incidence, in which it is possible to neglect the curvature of the rays.

Take S_{mn} to be a unit vector of the tangent to the ray. Let this ray comply with the condition (9) for that edge, which gives the mnth primary shadow boundary. Then the differential equation:

$$\frac{d}{ds}\left(\frac{S_{mn}}{c_m}\right) = \frac{1}{c_m} \tag{10}$$

determines the congruence of the edge diffracted rays.

3.2 AN INTEGRAL FORMULA

Let the wave

$$f_{mn} = \Phi_{mn} \exp(i\omega\tau_{mn}), \quad \nabla\tau_{mn} = S_{mn}/c_m \tag{11}$$

be connected with the mnth primary shadow boundary. The latter may be given implicitly by the equation $\tau_{mn} = \tau_m$. The wave (11) is called an edge diffracted wave. Now we have come to the description of diffracted waves.

Let τ_{mn}, η, ζ be the ray coordinates of the wave f_{mn}. Here η and ζ give a congruence of the diffracted rays, i.e. every pair of fixed values η = constant and ζ = constant gives a single ray. This pair of coordinates may be chosen in many different ways. Let the coordinate surface $\eta = 0$ coincide with the mnth primary shadow boundary $\tau_{mn} = \tau_m$, so that the primary shadow zone of the wave f_m coincides with the domain $\eta > 0$. The coordinate surfaces ζ = constant may be taken as arbitrary.

In the first place, let us take the case when the amplitude Φ_m of the wave (4) is a scalar one. Let the wave (4) be a function of the ray coordinates

$$f_m = \Phi_m(\tau_{mn}, \eta, \zeta) \exp[i\omega\tau_m(\tau_{mn}, \eta, \zeta)]. \tag{12}$$

Then this wavefield in the neighbourhood of its shadow boundary may be represented by the discontinuous function:

$$f_m = f_m(\tau_{mn}, \eta, \zeta) \quad \text{when} \quad \eta < 0, \quad f_m = 0 \quad \text{when} \quad \eta > 0 \tag{13}$$

which displays explicitly the shortcoming of the ray method.

Suppose, the expression (13) represents an analytic function of the variable η which allows us to make an analytic continuation into the complex plane of η for any permissible values τ_{mn} and ζ. In addition, $f_m \to 0$, when $|\eta| \to \infty$. Then we may construct the following integral

$$f_{mn} = \frac{1}{2\pi i} \int_L f_m(\tau_{mn}, \eta + \alpha, \zeta) \frac{d\alpha}{\alpha}. \tag{14}$$

Fig. 1 shows the contour L of integration.

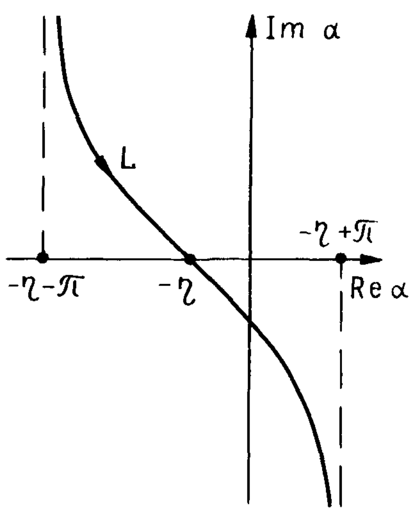


Figure 1. The contour of integration.

The integral (14) has the following properties. It is zero when $|\eta| \to \infty$. It has a discontinuity when $\eta = 0$. But the superposition of expressions (13) and (14) is a continuous and analytic function of η within the neighbourhood of the surface $\eta = 0$. If the function (13) is a solution of some linear differential equation (for example, the wave equation) within the domain $\eta < 0$, the superposition of (13) and (14) complies with the same equation for $\eta < 0$ as well as for $\eta > 0$.

3.3 AN EDGE DIFFRACTION COEFFICIENT

When $\omega \to \infty$, the asymptotic value of integral (14) is formed by contributions within a small neighbourhood of the saddle point $\alpha = -\eta$. Let us take the standard approximations at this point:

$$\Phi_m \approx \Phi_m(\tau_{mn}, 0, \zeta), \quad \tau_m \approx \tau_m(\tau_{mn}, 0, \zeta) + \frac{\eta^2}{2}\left(\frac{\partial^2 \tau_m}{\partial \eta^2}\right)_{\eta=0} \tag{15}$$

and use the following correlations:

$$\tau_m(\tau_{mn}, 0, \zeta) = \tau_{mn}, \quad \left(\frac{\partial^2 \tau_m}{\partial \eta^2}\right)_{\eta=0} = \frac{2}{\eta^2}(\tau_m - \tau_{mn}). \tag{16}$$

Then integral (14) may be written as

$$f_{mn} = q_{mn}\Phi_m W(w_{mn}) \exp(i\omega\tau_{mn}), \quad w_{mn} = \sqrt{2\omega(\tau_{mn} - \tau_m)/\pi}, \tag{17}$$

$q_{mn} = +1$ within Ω^+_{mn}, $q_{mn} = -1$ within Ω^-_{mn},

$$W(w) = \exp(-i\pi w^2/2) \cdot \int_{-i\pi w^2/2}^{\infty} t^{-1/2} \exp(-t)\, dt \tag{18}$$

where W may be regarded as an edge diffraction coefficient. If $\tau_{mn} < \tau_m$, we have $w_{mn} = ix$, $x = \sqrt{2\omega(\tau_m - \tau_{mn})/\pi}$, $W(ix) = \overline{W(x)}$, where $\bar{W}$ denotes a complex conjugate of W. In these formulae we may use the analytic continuation of the amplitude Φ_m and eikonal τ_m into the primary shadow zone by means of any type of parameterization of the space.

3.4 PROPERTIES OF THE FUNCTION $W(w)$

If $0 \leqslant w \leqslant \infty$, we have the following approximate formulae:

$$W(w) = W(0) + \frac{w}{\sqrt{2}} \exp(3\pi i/4), \quad W(0) = \frac{1}{2} \tag{19}$$

when $w \to 0$,

$$W(w) = \frac{\exp(i\pi/4)}{\pi w \sqrt{2}} + O(w^{-2}) \tag{20}$$

when $w \to \infty$

where O is the symbol of asymptotic estimation. Fig. 2 shows the graph of the function (18). Numbers above the curve give the values of w.

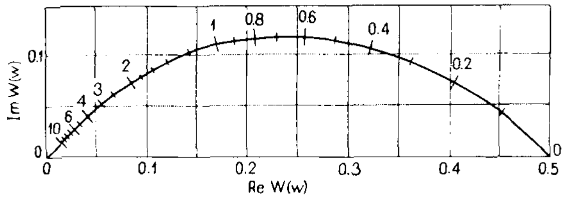


Figure 2. Function $W(w)$ of the real variable.

3.5 A BOUNDARY LAYER

Now we shall briefly discuss the type of approximation given by equation (17). Obviously, the above approach is approximate even in a high-frequency sense because of the disturbance of the boundary conditions at interfaces by the integral (14). To be more accurate, it would be necessary to add a certain term δf_{mn} to the integral (14) to satisfy the boundary conditions. In principle, this term can be found by using the geometrical theory of diffraction (for example, see Klem-Musatov 1980 for the case c_m = constant). It is essential that such an addition δf_{mn} would have no discontinuity at the primary shadow boundary $\eta = 0$. Using equation (20) allows us to estimate the value to the order of $\delta f_{mn} \sim O(\omega^{-1/2})$, except for grazing and critical regions.

Let us see how it would be connected with equation (17). According to (19)–(20), the amplitude of the edge wave changes its value rapidly from an asymptotic estimation O(1) at $w_{mn} = 0$ to $O(\omega^{-1/2})$ at $w_{mn} \gtrsim 2$. A domain of this rapid change forms a neighbourhood of the primary shadow boundary. It is called a boundary layer, with a position is determined by the inequality $w_{mn} \lesssim 2$. Within the boundary layer an inaccuracy $\delta f_{mn} \sim O(\omega^{-1/2})$ of equation (17) may be considered of no importance in comparison with O(1). Outside of the boundary layer the amplitude of the edge wave has the same asymptotic estimation $O(\omega^{-1/2})$ as δf_{mn}. It is clear that (17) fails here. More accurate analysis of the integral (14) would not improve the mentioned properties of this formulae. Thus, (17) gives a satisfying result only within the boundary layer. All this is quite enough for putting right the ray method.

Equation (17) has one other local property, which allows us to interpret the forthcoming results. According to (15), a value of Φ_m would be taken at the primary shadow boundary $\Phi_m = \Phi_m(\tau_{mn}, 0, \zeta)$. However, it is possible to consider Φ_m as a function of the free point $\Phi_m = \Phi_m(\tau_{mn}, \eta, \zeta)$ as well because a difference $\Phi_m(\tau_{mn}, \eta, \zeta) - \Phi_m(\tau_{mn}, 0, \zeta)$ is so small in comparison with Φ_m within the boundary layer. The real accuracy of the description of the edge wave is independent of the choice of the above versions. By the way, this is the reason why Φ_m may be continued analytically into the shadow zones.

3.6 COMPARISON WITH KNOWN RESULTS

We consider briefly, how the above matches the known ideas on the subject, if c_m = constant. In this case, (17) is an initial approximation of the more precise description of the edge wave by the successive approximation method in the form of an infinite series (Klem-Musatov 1980). Equation (17) complies with the so-called parabolic equation of transverse diffusion that describes diffusion of the wave energy out of the primary illuminated zone into the shadow zone. This process is only strong within a boundary layer. Outside it is damped by diffusion. Here it is replaced by a mechanism of transfer of wave energy along the rays. Thus, the above approach results in consideration of diffusion through the shadow boundaries.

On the other hand, (17) matches the ideas of the classic theory of diffraction. The superposition of (13) and (17) within the boundary layer can be written in the following form:

$$f_m + f_{mn} = f_m F(-q_{mn} w_{mn} \sqrt{\pi/2}),$$

$$F(z) = \pi^{-1/2} \exp(-i\pi/4) \int_{-\infty}^{z} \exp(ix^2)dx. \tag{21}$$

If c_m = constant, the same formula can be derived by asymptotic analysis of the Kirchhoff integral (Klem-Musatov 1980; Aizenberg 1982).

3.7 POLARIZATION

Now let us take the case when the amplitude of wave (4) is the vector (6). Let e_1, e_2, e_3 be the unit vectors of a certain immovable coordinate system (for example, the Cartesian one). Let us decompose the vector (6) over the above basis and represent the wave (4) in a form:

$$f_m = n_m \phi_m \exp(i\omega\tau_m) = \sum_{q=1}^{3} e_q f_m^{(q)}, \quad f_m^{(q)} = \phi_m^{(q)} \exp(i\omega\tau_m) \tag{22}$$

where $f_m^{(q)}$ are scalars. We represent the edge wave (11) over the same basis:

$$f_{mn} = \sum_{q=1}^{3} e_q f_{mn}^{(q)}, \quad f_{mn}^{(q)} = \phi_{mn}^{(q)} \exp(i\omega\tau_{mn}) \tag{23}$$

where $f_{mn}^{(q)}$ are the scalars. We use the same approach (12)–(17) for every scalar function $f_m^{(q)}(\tau_{mn}, \eta, \zeta)$ which was used for the function (12). It allows us to determine three scalar functions:

$$f_{mn}^{(q)} = q_{mn} \phi_m^{(q)} W(w_{mn}) \exp(i\omega\tau_{mn}) \qquad \text{with } q = 1, 2, 3. \tag{24}$$

Inserting (24) into (23) gives once again equation (17), where Φ_m is the vector (6).

This result has to be interpreted. Let n_{mn} be a unit vector of polarization of the edge wave f_{mn}. In accordance with the general theory this vector must coincide with S_{mn} (a longitudinal wave) or be perpendicular to S_{mn} (a transverse wave). But in (17) the vector n_{mn} coincides with $n_m(\tau_{mn}, 0, \zeta)$, which is out of the line of general theory. In other words, the above approach gives an inaccuracy $\delta n = n_{mn}(\tau_{mn}, \eta, \zeta) - n_m(\tau_{mn}, 0, \zeta)$. In fact, the real accuracy of the description of polarization is independent of the choice of any of the versions: $n_m(\tau_{mn}, 0, \zeta)$, $n_m(\tau_{mn}, \eta, \zeta)$ because the corresponding δn is of no importance in comparison with n_{mn} within the boundary layer. That is why, the vector Φ_m may be considered a function of the free point $\Phi_m(\tau_{mn}, \eta, \zeta)$ and then continued analytically into the shadow zones.

4 Tip waves

The point of break (or an end) of a smooth edge is called a tip. The common tip of several edges is a vertex. The sizes of edge wave domains are limited because of the tips. A single edge wave f_{mn} exists within the connected domain coinciding with the corresponding congruence of diffracted rays. This wavefield f_{mn} is continuous within its domain everywhere with the exception of the primary shadow boundary $\tau_{mn} = \tau_m$. This type of domain is called the secondary illuminated zone of the wave f_{mn}. A domain of absence of the wave ($f_{mn} \equiv 0$) is called the secondary shadow zone. One-connected surface dividing the above zones is called the secondary shadow boundary. It looks like the surface of a curvilinear cone which apex angle complies with the law of edge diffraction. Let mnp be the triple number of each secondary shadow boundary of the edge wave f_{mn}. The non-caustic shadow boundaries formed by tips are considered.

One can see a shortcoming of (17) appears as discontinuities of the wavefield f_{mn} at the secondary shadow boundaries. Let us see how it can be put right.

4.1 KINEMATICS

Let us use a formal way to find directions of rays arising from a tip. It has to do with Snell's law in the form (9). However, there are no limitations in choosing the directions of arising rays because the tip is not the linear element of interfaces. Any of the directions complies with the above law formally. This fact is formulated as the law of tip diffraction (Keller 1962). It reads as follows: the incident ray generates the rays leaving the tip in all directions.

Let S_{mnp} be a unit vector of the tangent to a ray. Let this ray comply with the law of tip diffraction at that tip, which gives the mnpth secondary shadow boundary. Then the differential equation

$$\frac{d}{ds}\left(\frac{S_{mnp}}{c_m}\right) = \nabla \frac{1}{c_m} \tag{25}$$

determines the congruence of tip diffracted rays.

4.2 AN INTEGRAL FORMULA

Let a wave

$$f_{mnp} = \Phi_{mnp} \exp(i\omega\tau_{mnp}), \qquad \nabla\tau_{mnp} = S_{mnp}/c_m \tag{26}$$

be connected with the mnpth secondary shadow boundary. The latter may be given implicitly by the equation $\tau_{mnp} = \tau_{mn}$. The wave (26) is called a tip diffracted wave. Let us divide the tip wave domain into separate parts. Suppose, the eikonal τ_{mn} may be continued analytically into the secondary shadow zone. Then the primary shadow boundary $\tau_{mn} = \tau_m$ and the secondary shadow boundary $\tau_{mnp} = \tau_{mn}$ divide the domain of the wave f_{mnp} into four parts. Let us give them the numbers: one, two, three and four, going round the line $\tau_{mnp} = \tau_{mn} = \tau_m$ clockwise or counter-clockwise, so that the shortest way from the fourth part to the first would coincide with the shortest way from the primary illuminated zone of the wave f_m to the primary shadow zone through the mnth primary shadow boundary. The first and third parts have common points only at the line $\tau_{mnp} = \tau_{mn} = \tau_m$. The second and fourth parts have the common points at the same line only. Let Ω^-_{mnp} be a symbol of the domain, formed by the first and third parts. Let Ω^+_{mnp} be the symbol of the domain, formed by the second and fourth parts. Let us denote the boundary between the first and second parts by $\mathscr{F}^+$, and between the third and fourth parts by $\mathscr{F}^-$. The union of $\mathscr{F}^+$ and $\mathscr{F}^-$ forms the secondary shadow boundary $\tau_{mnp} = \tau_{mn}$.

Let us represent the tip wave (26) in the form:

$$f_{mnp} = f^+ + f^-, \quad f^\pm = \Phi^\pm \exp(i\omega\tau_{mnp}). \tag{27}$$

Let the sum $f_{mn}+f^+$ be continuous at the boundary $\mathscr{F}^+$, and the sum $f_{mn}+f^-$ be continuous at the boundary $\mathscr{F}^-$. Under the above conditions we can find f^+ and f^- with the same way which is used for finding f_{mn}.

Let τ_{mnp}, $\psi^\pm$, σ be ray coordinates of the wave $f^\pm$. Here $\psi^\pm$ and σ give a congruence of the tip diffracted rays, i.e. every pair of fixed values $\psi^\pm$ = constant and σ = constant gives a single ray. Let $\psi^\pm$ vary into the number line interval $-\pi \leqslant \psi^\pm \leqslant \pi$. We choose $\psi^\pm$ in such a way that the surface $\psi^\pm=0$ would coincide with the surface $\mathscr{F}^\pm$, and the surface $\psi^\pm=\pi$ and $\psi^\pm=-\pi$ would coincide with $\mathscr{F}^\mp$. Then

$$\psi^\pm=\pm q_{mnp}\,|\psi^\pm| \quad \text{when} \quad |\psi^\pm|<\frac{\pi}{2}, \qquad \psi^\pm=q_{mnp}(|\psi^\pm|-\pi) \quad \text{when} \quad |\psi^\pm|>\frac{\pi}{2}, \tag{28}$$

$$q_{mnp}=+1 \quad \text{within} \quad \Omega^+_{mnp}, \qquad q_{mnp}=-1 \quad \text{within} \quad \Omega^-_{mnp}. \tag{29}$$

Let us consider $\sigma=0$ at the line $\tau_{mnp}=\tau_{mn}=\tau_m$.

In the first place, let us take the case when the amplitude Φ_{mn} of the wave (11) is a scalar one. Let the edge wavefield (11) be a function of the above ray coordinates

$$f_{mn}=\Phi_{mn}(\tau_{mnp},\psi^\pm,\sigma)\exp[i\omega\tau_{mn}(\tau_{mnp},\psi^\pm,\sigma)]. \tag{30}$$

Then in the neighbourhood of the secondary shadow boundary this wavefield may be represented by the discontinuous function

$$\tilde{f}_{mn}=f_{mn}(\tau_{mnp},\psi^\pm,\sigma) \quad \text{when} \quad -\pi<\psi^\pm<0, \quad \tilde{f}_{mn}=0 \quad \text{when} \quad 0<\psi^\pm<\pi \tag{31}$$

which displays explicitly a shortcoming of equation (17).

Suppose expression (31) represents an analytical function of the variable $\psi^\pm$ and allows us to make an analytical continuation into the complex plane of $\psi^\pm$ for any permissible values τ_{mnp} and σ. In addition, $\tilde{f}_{mn}\to 0$ when $|\psi^\pm|\xrightarrow{L}\infty$. Then we may construct the following integral:

$$\tilde{f}^\pm=\frac{1}{2\pi i}\int_L f_{mn}(\tau_{mnp},\psi^\pm+\alpha,\sigma)\frac{d\alpha}{\alpha}. \tag{32}$$

Fig. 1 shows the contour L of integration.

The properties of the same integral (14) have been discussed above. It has a discontinuity at $\psi^\pm=0$. However, the superposition of functions (31) and (32) is continuous at this point. The problem is that integral (32) has two extra discontinuities at $\psi^\pm=-\pi$ and $\psi^\pm=\pi$ because of the limited number line interval $-\pi\leqslant\psi^\pm\leqslant\pi$. To eliminate these discontinuities we take a periodic function of $\psi^\pm$, i.e.

$$f^\pm=\sum_{k=-\infty}^{\infty}\tilde{f}^\pm(\tau_{mnp},\psi^\pm+2\pi k,\sigma). \tag{33}$$

Inserting (32) into (33) and using the well-known formula

$$\sum_{k=-\infty}^{\infty}(z-2\pi k)^{-1}=\frac{1}{2}\cot\frac{z}{2} \tag{34}$$

we get:

$$f^\pm=\frac{1}{4\pi i}\int_L f_{mn}(\tau_{mnp},\psi^\pm+\alpha,\sigma)\cot\frac{\alpha}{2}\,d\alpha. \tag{35}$$

The integral (35) has the following properties. It is zero when $|\psi^\pm|\to\infty$. It has a discontinuity at $\psi^\pm=0$. But the superposition of (31) and (35) is a continuous and analytical function of $\psi^{\pm}$ in the neighbourhood of the surface $\psi^{\pm} = 0$. If the function (31) is a solution of some linear differential equation (for example, a wave equation) within the domain $-\pi < \psi^{\pm} < 0$, the superposition of (31) and (35) complies with the same equation within the whole domain $-\pi \leqslant \psi^{\pm} \leqslant \pi$.

4.3 A TIP DIFFRACTION COEFFICIENT

When $\omega \to \infty$, the asymptotic value of the integral (35) is formed by contributions within a small neighbourhood of the saddle point $\alpha = -\psi^{\pm}$. Let us take the standard approximation at this point

$$\Phi_{mn}(\tau_{mnp}, \psi^{\pm} + \alpha, \sigma) \approx \Phi_{mn}(\tau_{mnp}, 0, \sigma) \tag{36}$$

$$\tau_{mn}(\tau_{mnp}, \psi^{\pm}, \sigma) \approx \tau_{mnp} - A \sin^2 \psi^{\pm}; \quad A = \tau_{mnp} - \tau^* \approx \tau_{mnp} - \tau_m; \tag{37}$$

$$\tau^* = \tau_{mn}(\tau_{mnp}, \psi^{\pm}, \sigma) \quad \text{with} \quad |\psi^{\pm}| = \pi/2;$$

$$|\psi^{\pm}| = \arcsin \sqrt{\frac{\tau_{mnp} - \tau_{mn}}{\tau_{mnp} - \tau_m}}. \tag{38}$$

Let us note that expression (37) may be derived by using a similar method to that used for the case c_m = constant (Klem-Musatov 1981a). Then the integral (35) may be written as:

$$f^{\pm} = \Phi_{mn}(\tau_{mnp}, 0, \sigma)\, \tilde{\Psi}\, \exp(i\omega\tau_{mnp}),$$

$$\tilde{\Psi} = \frac{1}{4\pi i} \int_L \cot \frac{\alpha - \psi^{\pm}}{2} \cdot \exp[i\omega(\tau_m - \tau_{mnp}) \sin^2 \alpha]\, d\alpha. \tag{39}$$

To discuss the accuracy of this expression, we would repeat all that was said concerning equation (17). Equation (39) gives a satisfying description within the so-called boundary layer where the amplitude of a tip wave changes rapidly. Within this domain the amplitude Φ_m may be considered as a function of the free point $\Phi_m(\tau_{mnp}, \psi^{\pm}, \sigma)$. The real accuracy of description is independent of the choice of the versions:

$$\Phi_m(\tau_{mnp}, 0, \sigma) \quad \text{or} \quad \Phi_m(\tau_{mnp}, \psi^{\pm}, \sigma).$$

Consideration of expressions (17), (28), (38), (39) and using identical mathematical transformations (for details, see Klem-Musatov 1981a, b) allow us to write (27) in the form:

$$f_{mnp} = q_{mnp}\, \Phi_m H(\rho_{mnp}, \zeta_{mnp}) \exp(i\omega\tau_{mnp}), \tag{40}$$

$$H(\rho, \zeta) = W(\rho)\, \Psi(\rho, \zeta), \tag{41}$$

$$\Psi(\rho, \zeta) = \frac{\sin 2\zeta}{\pi} \int_0^1 \frac{\exp[i\pi\rho^2(x + x^{-1} - 2)/8]}{x^2 - 2x \cos 2\zeta + 1}\, dx, \tag{42}$$

$$\rho_{mnp} = \sqrt{2\omega(\tau_{mnp} - \tau_m)/\pi}, \quad \zeta_{mnp} = \arcsin \sqrt{\frac{\tau_{mnp} - \tau_{mn}}{\tau_{mnp} - \tau_m}}$$

where H may be regarded as a tip diffraction coefficient. In these formulae we may use the analytical continuation of the amplitude Φ_m and eikonals τ_m, τ_{mn} into the primary and secondary shadow zones by means of any type of parameterization of space.

4.4 PROPERTIES OF FUNCTION $\psi(\rho, \zeta)$

If $0 \leq \rho < \infty$, $0 \leq \zeta \leq \pi/2$, we have the following approximate formulae

$$\Psi(\rho,\zeta) = \Psi(0,\zeta) - \frac{i\rho^2 \sin\zeta}{8} \ln \frac{\pi\rho^2}{8} \quad \text{when} \quad \rho \to 0, \tag{43}$$

$$\Psi(\rho,\zeta) = \frac{1}{\pi\rho\sqrt{2}} \left(\frac{1}{\zeta} - \cot\zeta\right) \exp(5\pi i/4) + W(\rho\zeta) + O(\rho^{-2}) \quad \text{when} \quad \rho \to \infty \tag{44}$$

$$\Psi(0,\zeta) = \frac{1}{2} - \frac{\zeta}{\pi}, \quad \Psi(\rho,0) = \frac{1}{2}, \quad \Psi(\rho,\pi/2) = 0 \tag{45}$$

where O is a symbol of asymptotic estimation. The point $\rho = 0$ is the essential special point because the value of the function depends on the direction of the way down to this point. However, the total wavefield at this point is determined single-valuedly. Let us give the corresponding result.

Every single wavefield f_m has only two primary shadow boundaries within the small neighbourhood of the line $\tau_{mnp} = \tau_{mn} = \tau_m$, i.e. at $\rho = 0$. Let us mark their numbers as $n = A$ and $n = b$. Let γ_m be the dihedral angle between the tangent planes to the mA th and mb th primary shadow boundaries at a point of the line $\tau_{mnp} = \tau_{mn} = \tau_m$ with $n = A$ and $n = b$. This angle must be taken within but not outside the primary illuminated zone. Then at this point the following equality exists:

$$f_m + f_{mA} + f_{mb} + f_{mAp} + f_{mbp} = f_m \gamma_m / 2\pi. \tag{46}$$

Fig. 3 shows the graphs of modulus and argument of equation (42). Numbers at the curves give the values of ζ expressed in degrees.

4.5 COMPARISON WITH KIRCHOFF'S THEORY

It was shown (Aizenberg 1982) that (40) matches the classic theory of diffraction. If c_m = constant, the tip wave can be found out by asymptotic analysis of Kirchoff's integral in the form (40) where

$$H(\rho,\zeta) = \frac{\rho\cos\zeta}{2\pi} \int_{\rho\sin\zeta}^{\infty} \frac{\exp[i\pi(x^2 - \rho^2\sin^2\zeta)/2]}{x^2 + \rho^2\cos^2\zeta} \, dx. \tag{47}$$

It has been shown both numerically and analytically that this function is identical to the product (41). It allows us to represent the superposition of (17) and (40) in the form:

$$f_{mn} + f_{mnp} = f_m G(q_{mn}\sqrt{\pi/2}\,\rho\cos\zeta, q_{mnp}\sqrt{\pi/2}\,\rho\sin\zeta), \tag{48}$$

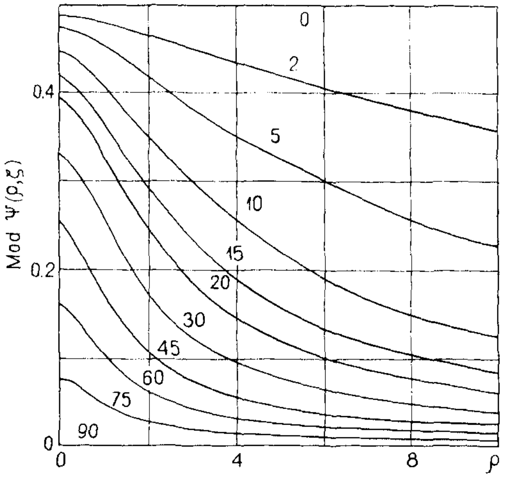
$$G(A,b) = \frac{A}{2\pi} \int_b^{\infty} \frac{\exp[i(x^2 + A^2)]}{x^2 + A^2} \, dx \tag{49}$$

where $G(A, b)$ is the so-called generalized Fresnel integral (Clemmow & Senior 1953).

4.6 POLARIZATION

Now let us take the case when the amplitude of the wave (11) is the vector (23). Let us represent the tip wave (26) in the form:

$$\mathbf{f}_{mnp} = \sum_{q=1}^{3} \mathbf{e}_q f_{mnp}^{(q)}, \quad f_{mnp}^{(q)} = \phi_{mnp}^{(q)} \exp(i\omega\tau_{mnp}) \tag{50}$$



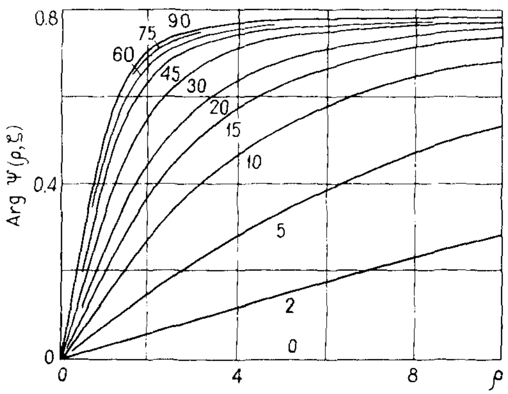


Figure 3. Modulus and argument of function $\Psi(\rho, \zeta)$.

where $f_{mnp}^{(q)}$ are scalars. Let us use the same approach (27)–(40) for every function $\phi_{mn}^{(q)}(\tau_{mnp}, \psi^{\pm}, \sigma)$ which has been used for the scalar case. It allows us to determine three scalar functions

$$f_{mnp}^{(q)} = q_{mnp}\, \phi_m^{(q)} H(\rho_{mnp}, \zeta_{mnp}) \exp(i\omega\tau_{mnp}) \quad \text{with} \quad q = 1, 2, 3. \tag{51}$$

Inserting (51) into (50) gives again equation (40) where Φ_m is the vector (6).

This result may be interpreted in the same way as for the edge wave above. Let n_{mnp} be a unit vector of polarization of the tip wave f_{mnp}. In accordance with the general theory, this vector must coincide with S_{mnp} (a longitudinal wave) or be perpendicular to S_{mnp} (a transverse wave). But in (40) the vector n_{mnp} coincides with n_m. In fact, the real accuracy of description is independent of this discrepancy because the latter is of no importance within the boundary layer. The vector Φ_m may be considered a function of a free point of space and continued analytically into the shadow zones.

5 Multiple diffraction

Here we shall outline in principle the generalization of the above approach to many interfaces. We take the wavefield (1) to be given by the formulae of the ray method. The problem is to include all the edge and tip waves which can be generated by propagation of the wavefield (1). As shown above, the diffracted waves are originated by the primary shadow boundaries of the field (1). However, when propagated, the diffracted waves may be reflected and transmitted at interfaces. If the interfaces have edges, new shadow boundaries in the reflected/transmitted diffracted waves must emerge. Hence, the new diffracted waves must arise. These phenomena are known as multiple diffraction (Felsen & Marcuvitz 1973). The formulae above allow us to consider reflection/transmission as well as multiple edge/tip diffraction, because their derivation is not connected with any assumption concerning special forms of the function (13).

5.1 MAIN PRINCIPLES

Let us outline the main points of description of the above mentioned phenomena. Obviously, the directions of reflected/transmitted diffracted rays at regular parts of interfaces must comply with Snell's law. The corresponding diffracted wavefields are determined again by (17) and (40). Using these formulae, reflection/transmission must be taken into account to compute the eikonals τ_{mn} and τ_{mnp}, note, there is no need to use reflection/transmission coefficients for amplitudes of diffracted waves because the former are considered in the term Φ_m. The multiple diffraction may be considered by the above approach if the term f_m is regarded as the incident diffracted wave. Obviously, directions of multiply edge/tip diffracted rays comply with the laws of edge/tip diffraction. The multiply diffracted wavefields are described again with formulae such as (17) and (40).

As we see, the problem results in a recursive use of the formulae above, which require the enumeration of branching sequences of multiply diffracted waves; there are not difficulties when computing, but it is inconvenient for an analytical description. That is why we use below a symbolic system of description, displaying the recurrent use of the above formulae, but giving no particular way of wave sequence enumeration.

5.2 A FORMAL DESCRIPTION

First, let us take the case of single diffraction described in the previous sections. In this case, the modified formula of the ray method, considering corresponding edge and tip waves, can be written symbolically in the following way:

$$f = \sum_m \left[f_m + \sum_{mn} \left(f_{mn} + \sum_{mnp} f_{mnp} \right) \right]. \tag{52}$$

To generalize this formula for multiple diffraction, let us rewrite it as follows:

$$f = \sum_m D_m f_m, \tag{53}$$

$$D_m = \delta_m + \sum_{mn} \left(W_{mn} + \sum_{mnp} H_{mnp} \right), \tag{54}$$

$$W_{mn} = \delta_{mn} q_{mn} W(w_{mn}) \exp[i\omega(\tau_{mnp} - \tau_m)],$$

$$H_{mnp} = \delta_{mnp} q_{mnp} H(\rho_{mnp}, \zeta_{mnp}) \exp[i\omega(\tau_{mnp} - \tau_m)],$$

$$\delta_m = 1 \quad \text{within} \quad \Omega_m, \quad \delta_m = 0 \quad \text{outside} \quad \Omega_m,$$

$$\delta_{mn} = 1 \quad \text{within} \quad \Omega_{mn}, \quad \delta_{mn} = 0 \quad \text{outside} \quad \Omega_{mn},$$

$$\delta_{mnp} = 1 \quad \text{within} \quad \Omega_{mnp}, \quad \delta_{mnp} = 0 \quad \text{outside} \quad \Omega_{mnp}$$

where Ω_m, Ω_{mn} and Ω_{mnp} are the corresponding domains of the existence of the waves f_m, f_{mn} and f_{mnp}.

The gist of this form of description is that we may regard the term D_m as an operator for the generation of diffracted waves. Obviously, it is easy to write a similar operator for the generation of multiply diffracted waves. Then multiple diffraction will result in the formation of a product of corresponding operators. For brevity, let us use the following system of designation: let ν be the number of acts of reflection/transmissions in the course of formation of a given wave, and let $f_{m(\nu)}$, $f_{m(\nu)n(\nu)}$ and $f_{m(\nu)n(\nu)p(\nu)}$ be correspondingly the incident, edge and tip waves, taking part in the νth act. Let us consider $f_{m(\nu)} = f_{m(\nu-1)n(\nu-1)}$ if the incident wave is the edge one, and $f_{m(\nu)} = f_{m(\nu-1)n(\nu-1)p(\nu-1)}$ if the incident wave is the tip one. Then we may use (17) and (40) or (54) for describing any generated diffracted waves with $\nu > 1$ if we consider

$$m = m(\nu), \quad mn = m(\nu)n(\nu), \quad mnp = m(\nu)n(\nu)p(\nu). \tag{55}$$

Using the above equations recursively for $\nu = 2, 3, \ldots N$ gives

$$f = \sum_{m(1)} \left(\prod_{\nu=1}^{N} D_{m(\nu)} \right) f_{m(1)} \tag{56}$$

where the case of the single diffraction $\nu = 1$ is described above by (53). Here Π is a product symbol. The term $D_{m(\nu)}$ is determined by (54), where (55) has to be taken into consideration.

6 Remarks

In conclusion, let us mention that there are many examples of mathematical modelling of wavefields in typical structures by the above method. There are theoretical seismograms for several types of pinch-out and low-amplitude faults (Klem-Musatov 1980), interfaces of complex forms (Aizenberg & Klem-Musatov 1980) and 3-D systems of intersecting faults (Klem-Musatov, Aizenberg & Klem-Musatova 1982).

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