

The Rayleigh Hypothesis in the Theory of Diffraction by a Cylindrical Obstacle

PETER M. VAN DEN BERG AND JACOB T. FOKKEMA

Abstract—The Rayleigh hypothesis in the theory of scattering by a cylindrical obstacle of arbitrary cross section is investigated analytically. The hypothesis asserts that outside and on the obstacle the scattered field may be expanded in terms of outward-going wave functions of the circular cylinder. As such, it is analogous to the assumption made by Lord Rayleigh in his treatment of diffraction by a reflection grating. We show that the validity of the Rayleigh hypothesis is governed by the distribution of singularities in the analytic continuation of the exterior scattered field. Conditions are derived under which the Rayleigh hypothesis is rigorously valid. As examples, the elliptic cylinder and the perturbed circular cylinder are considered in detail.

I. INTRODUCTION

IN HIS WORK on sound [1] and in his dynamical theory of gratings [2] Lord Rayleigh put forward a theory of diffraction by a reflection grating, where he assumed the discrete set of upward-reflected, propagating, and evanescent spectral waves (together with the incident field) to be a complete description of the total field up to the boundary surface of the grating. Doubt as to the validity of the Rayleigh theory of gratings was first expressed by Deriugin [3] and Lippmann [4], [5] (see also [6]). Following Petit and Cadilhac [7]–[10], Millar [11], [12] investigated the conditions under which the Rayleigh hypothesis can be used for a sinusoidal grating. His basic problem involves the location of the singularities in the representation of the solution of the wave problem. Further progress as regards the location of these singularities has been made by Neviere and Cadilhac [13] and Millar [14]. A review of the relevant problem can be found in [15]. Recently the authors [16] have shown in a fairly simple way under which conditions the Rayleigh hypothesis holds rigorously for gratings of arbitrary profile. Further, Kalhor [17] has compared the numerical results of the diffracted field from a rectangular grating, obtained by an exact method and by Rayleigh's method.

In the present paper the Rayleigh hypothesis in the theory of scattering by a cylindrical body of arbitrary bounded cross section is investigated analytically. Under the hypothesis the scattered field outside and on the obstacle is expanded in terms of outward-going wave functions of the circular cylinder. A perturbation theory based upon this assumption was presented by Clemmow and Weston [18] and by Yeh [19]. Doubt as to this form of the Rayleigh theory was raised by Mullin, Sandburg, and Velline [20], Harrington [21], Bates [22], and by Wirgin [23]. The discussions about the Rayleigh hypothesis (this term was first coined by Burrows [30]) was continued by Bates, Burrows, and Millar [24]–[30]. Millar claimed that the Rayleigh hypothesis might be valid under

Manuscript received August 28, 1978; revised March 25, 1979. This work was supported in part by the Netherlands Organization for the Advancement of Pure Research.

The authors are with the Laboratory of Electromagnetic Research, Department of Electrical Engineering, Delft University of Technology, Delft, The Netherlands.

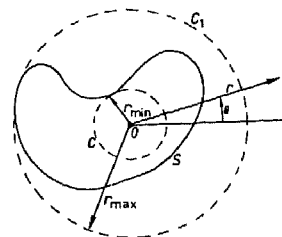


Fig. 1. Geometrical configuration of two-dimensional scatterer.

certain conditions, which he was, however, not able to fully discover [15], [31]. Further, Bolomey and Wirgin [32] compared numerically the Green's function and the Rayleigh theory of scattering from a cylinder with arbitrary cross section. In the present paper we show under which conditions the Rayleigh theory holds rigorously, when the obstacle is a cylindrical one and the problem is two-dimensional. As examples, the elliptic cylinder and the perturbed circular cylinder are considered in detail.

II. FORMULATION OF THE PROBLEM

We take a polar coordinate system (r, θ) with origin O inside the scatterer's cross section S . Let C be the circle with center O and radius r_{\min} inscribed in S and let C_1 be the corresponding circumscribed circle with radius r_{\max} (see Fig. 1). A field $u^i(r, \theta)$ is incident on S . The time-dependence factor $\exp(-i\omega t)$ is suppressed. At all points outside the circle C_1 the scattered field may be expressed in terms of cylindrical wave functions as [27]

$$u^s(r, \theta) = \sum_{n=-\infty}^{\infty} \tau_n u_n^s(r, \theta)$$

when $r > r_{\max}$, with

$$u_n^s(r, \theta) = H_n^{(1)}(k_0 r) \exp(in\theta), \quad (1)$$

in which $H_n^{(1)}$ is the Hankel function of first kind and order n , and k_0 is the wavenumber. We suppose that the total field $u = u^i + u^s$ vanishes on S (the Dirichlet boundary condition) though this is inessential for our procedure, and other boundary conditions could be imposed as well.

In our case the Rayleigh hypothesis amounts to the assumption that (1) represents the scattered field on and outside S and in particular also in the domain $r_{\min} < r < r_{\max}$. Let us write the Rayleigh solution of the boundary value problem ($u^s = -u^i$ on S) as

$$u_R^s(r, \theta) = \sum_{n=-\infty}^{\infty} T_n u_n^s(r, \theta) \quad (2)$$

when $r \geq r_{\min}$. Because of the uniqueness theorem (cf. Appendix), we claim that a necessary and sufficient condition for the Rayleigh assumption to hold is that the series (2) is an analytic function outside and up to S (see also Millar [11, p. 778]). Because of this uniqueness the factors T_n in (2) are identical to the factors τ_n in (1) for all n . Hence, under the assumption of analyticity, (2) is the exact solution. We remark that only uniqueness is investigated; no attempt is made to prove the existence of the solution. Even the proof of uniqueness in the Appendix is based on the existence of certain solutions.

The series (2) is an analytic function on and outside S , provided that the series can be expanded in a convergent power series in θ , $0 \leq \theta < 2\pi$, for $r = r_{\min}$. Since $w(\theta) = \exp(i\theta)$ can be expanded in a convergent power series in θ , the series (2) can be expanded in a convergent power series in θ for $r = r_{\min}$, if for $|w| = 1$ the Laurent series

$$u(w) = \sum_{n=-\infty}^{\infty} T_n H_n^{(1)}(k_0 r_{\min}) w^n \quad (3)$$

is a convergent one. This is the case provided that [33]

$$\limsup_{n \rightarrow \infty} |T_n H_n^{(1)}(k_0 r_{\min})|^{1/n} < 1,$$

and

$$\liminf_{n \rightarrow -\infty} |T_n H_n^{(1)}(k_0 r_{\min})|^{1/n} > 1. \quad (4)$$

Using the relation $H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z)$ and the asymptotic expression of the Hankel function for large order and finite argument (this latter expression is the dominant term in the ascending series of the Hankel function [34])

$$H_n^{(1)}(z) = -(i/\pi)(n-1)! (\frac{1}{2}z)^{-n}, \quad n \rightarrow \infty,$$

condition (4) can be rewritten as

$$\limsup_{n \rightarrow \infty} |T_n(n-1)!/\pi|^{1/n} (\frac{1}{2}k_0 r_{\min})^{-1} < 1,$$

and

$$\liminf_{n \rightarrow -\infty} |T_n(-n-1)!/\pi|^{1/n} (\frac{1}{2}k_0 r_{\min}) > 1. \quad (5)$$

As long as conditions (5) have been satisfied, the Rayleigh hypothesis holds. To verify (5) we need the behavior of $|T_n|$ as $|n| \rightarrow \infty$. This behavior of the scattering factors will be discussed in the next section.

III. BEHAVIOR OF THE SCATTERING FACTORS

If, now, the Rayleigh hypothesis were to hold exactly for some type of obstacle, at least the Rayleigh method ($u_R^s = -u^i$ on S) should lead to an analytic single-valued scattered-wave function upon approaching the boundary surface of the obstacle. We represent the surface of the obstacle as $r = \rho(\theta)$, where $\rho(\theta)$ is assumed to be an analytic function of the (real-valued) angular variable θ ($0 \leq \theta < 2\pi$). Let us define

$u^i(r, \theta)$ on S as $u^i(\theta)$, then the Rayleigh hypothesis yields

$$-u^i(\theta) = \sum_{n=-\infty}^{\infty} T_n u_n(\theta), \quad \text{Im}(\theta) = 0, \quad (6)$$

in which $u_n(\theta) = u_n^s(\rho(\theta), \theta) = H_n^{(1)}\{k_0 \rho(\theta)\} \exp(in\theta)$. Using again the asymptotic expressions of the Hankel functions (see the text below (4)), the dominant behavior of $|u_n(\theta)|$ as $|n| \rightarrow \infty$ is given by

$$\begin{aligned} \lim_{n \rightarrow \infty} |u_n| &= \pi^{-1}(n-1)! (\frac{1}{2}k_0)^{-n} |w_1|^n, \\ \lim_{n \rightarrow -\infty} |u_n| &= \pi^{-1}(-n-1)! (\frac{1}{2}k_0)^n |w_2|^n, \end{aligned} \quad (7)$$

where $w_1(\theta) = \{\rho(\theta)\}^{-1} \exp(i\theta)$ and $w_2(\theta) = \rho(\theta) \exp(i\theta)$. Let the maximum value of $|w_1(\theta)|$ along $\text{Im}(\theta) = 0$ be denoted by $|w_{1,\max}|$ and let the minimum value of $|w_2(\theta)|$ along $\text{Im}(\theta) = 0$ be denoted by $|w_{2,\min}|$. The series in (6) has to be a uniformly convergent series for real θ . This property restricts the behavior of $|T_n|$ as $|n| \rightarrow \infty$ to

$$\begin{aligned} \limsup_{n \rightarrow \infty} |T_n(n-1)!/\pi|^{1/n} (\frac{1}{2}k_0)^{-1} |w_{1,\max}| &< 1, \\ \liminf_{n \rightarrow -\infty} |T_n(-n-1)!/\pi|^{1/n} (\frac{1}{2}k_0) |w_{2,\min}| &> 1, \end{aligned} \quad (8)$$

in which $|w_{1,\max}| = |w_{2,\min}|^{-1}$.

More insight into the behavior of $|T_n|$ as $|n| \rightarrow \infty$ is obtained as follows. Let $\rho(\theta)$ be an analytic function of the complex variable θ in a domain A of the complex θ plane, containing the real axis. Let the incident field $u^i(\theta)$ be an analytic function of θ in A ; for plane-wave incidence, e.g., and analytic $\rho(\theta)$, this is true. Then (6) can be written as

$$-u^i(\theta) = \sum_{n=-\infty}^{\infty} T_n u_n(\theta), \quad \theta \in \mathcal{B}, \quad (9)$$

in which \mathcal{B} denotes the subdomain of A , where the series on the right side of (9) converges uniformly. The subdomain \mathcal{B} may not exist. In that case the Rayleigh hypothesis never holds. Because of (7), \mathcal{B} is also the domain where both series

$$u_1(\theta) = \sum_{n=1}^{\infty} T_n(n-1)! \pi^{-1} (\frac{1}{2}k_0)^{-n} w_1^n$$

and

$$u_2(\theta) = \sum_{n=-\infty}^{-1} T_n(-n-1)! \pi^{-1} (\frac{1}{2}k_0)^n w_2^n \quad (10)$$

converge uniformly. In order to investigate the behavior of $|T_n|$ as $|n| \rightarrow \infty$, we need the location of the boundary of \mathcal{B} in the complex θ plane. Let the real θ axis divide the domain \mathcal{B} into \mathcal{B}^+ and \mathcal{B}^- (see Fig. 2). Since $\rho(\theta)$ is real on the real θ axis, $\rho(\theta)$ takes on conjugate values at conjugate points (principle of reflection [35, p. 155], $\rho^*(\theta) = \rho(\theta^*)$). It is easily verified that

$$\begin{aligned} w_2^*(\theta) &= \rho^*(\theta) \exp(-i\theta^*) = \rho(\theta^*) \exp(-i\theta^*) \\ &= [\{\rho(\theta^*)\}^{-1} \exp(i\theta^*)]^{-1} = [w_1(\theta^*)]^{-1}, \end{aligned} \quad \theta \in A, \quad (11)$$

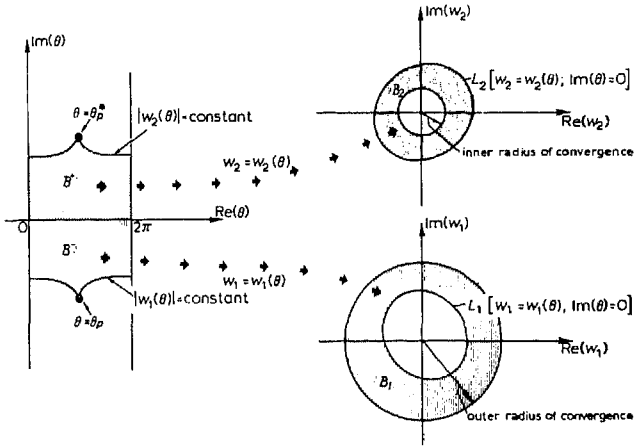


Fig. 2. Conformal transformations. θ_p is either a zero of $dw_1/d\theta$ or a singularity of $\rho(\theta)$.

in which the asterisk denotes the complex conjugate value. From (11) it follows that if the convergence properties of the first series of (10) determine the lower boundary of B^- , the convergence properties of the second series determine the upper boundary of B^+ . It is easy to verify that, on the real θ axis,

$$\begin{aligned} \frac{d|w_1|^2}{d\theta''} &= 2 \operatorname{Re} \{w_1^* dw_1/d\theta''\} = -2 \operatorname{Im} \{w_1^* dw_1/d\theta\} \\ &= -2 \operatorname{Im} \{w_1^* w_1 (i - \rho^{-1} d\rho/d\theta)\} \\ &= -2 |w_1|^2 \leq 0, \quad \text{for } \theta'' = \operatorname{Im}(\theta) = 0, \end{aligned} \quad (12)$$

in which θ'' denotes the imaginary part of θ . Hence, $|w_1|$ increases with decreasing $\operatorname{Im}\{\theta\}$ starting from the real θ axis. As a consequence, B^- determines the boundary of convergence of the first series of (10) and B^+ determines the boundary of convergence of the second series. An easy procedure to investigate the domain of convergence of the two series of (10) is to carry out the analysis in the complex w_1 plane and the complex w_2 plane. To this end we employ a conformal mapping of the domain A in the θ plane into the complex w_1 plane. Let the image of the arc $\{0 \leq \operatorname{Re}(\theta) < 2\pi, \operatorname{Im}(\theta) = 0\}$ be denoted by L_1 (see Fig. 2). Further, let B^- in the θ plane be mapped into the domain B_1 in the complex w_1 plane. Then, B_1 is the domain in the w_1 plane bounded by L_1 and the circle of convergence of the power series

$$v_1(w_1) = \sum_{n=1}^{\infty} T_n (n-1)! \pi^{-1} (\frac{1}{2} k_0)^{-n} w_1^n. \quad (13)$$

Note that L_1 is always located completely inside this circle of convergence, because of (8). Now, this circle of convergence passes through the singularity of $v_1(w_1)$ nearest to the origin. Let $\theta = \theta_p$ ($\operatorname{Im}(\theta) < 0$) correspond to this singularity, then $\theta = \theta_p$ is either a zero of $dw_1/d\theta$ (at this point the function $v_1(w_1)$ has a nonexistent derivative, since $dv_1/dw_1 = (dv_1/d\theta)/(dw_1/d\theta)$), or a nonanalytic point of $\rho(\theta)$ or a zero of $\rho(\theta)$. The zeros of $dw_1/d\theta$ follow from

$$\frac{dw_1}{d\theta} = w_1 \left(i - \frac{d\rho(\theta)/d\theta}{\rho(\theta)} \right) = 0,$$

OR

$$i - \frac{d\rho/d\theta}{\rho(\theta)} = 0, \quad \operatorname{Im}(\theta) < 0. \quad (14)$$

The radius of convergence of the power series in the right side of (13) is then given by [33]

$$\begin{aligned} \limsup_{n \rightarrow \infty} |T_n (n-1)! / \pi|^{1/n} (\frac{1}{2} k_0)^{-1} \\ = |w_1(\theta_p)|^{-1} = \left| \frac{\exp(i\theta_p)}{\rho(\theta_p)} \right|^{-1}. \end{aligned} \quad (15)$$

Using a conformal mapping of the domain A in the θ plane into the complex w_2 plane, a procedure similar to the one already used, but now with respect to the second series of (10), yields

$$\begin{aligned} \liminf_{n \rightarrow -\infty} |T_n (-n-1)!|^{1/n} (\frac{1}{2} k_0) \\ = |w_2(\theta_p^*)|^{-1} = \left| \frac{\exp(i\theta_p)}{\rho(\theta_p)} \right|. \end{aligned} \quad (16)$$

Formulas (15) and (16) specify the behavior of $|T_n|$ as $|n| \rightarrow \infty$, still provided that the Rayleigh hypothesis is used.

IV. VALIDITY OF THE RAYLEIGH HYPOTHESIS

From the two conditions in (5) and the behavior of $|T_n|$ for $n \rightarrow \infty$ and $n \rightarrow -\infty$ given in (15) and (16), respectively, we observe that the series (2) is uniformly convergent up to and including r_{\min} , if and only if

$$|r_{\min} w_1(\theta_p)| = |\{r_{\min}/\rho(\theta_p)\} \exp(i\theta_p)| > 1. \quad (17)$$

Equation (17), now, is the condition under which the Rayleigh hypothesis holds, if it ever does (see the text in Section II).

As an obvious example, we consider the scattering by an elliptic cylinder with semiaxes a, b ($a > b$). The origin of the coordinate system is taken at the center of the ellipse. Millar [28] has remarked that the expansion of the scattered field in terms of Mathieu functions converges on and within the cylinder up to the interfocal segment, and for any eccentricity, but when the series (2) is employed, the Rayleigh hypothesis holds only if the foci (at which the representation of the exterior scattered field is singular) lie inside the inscribed circle with $r_{\min} = b$. We now shall show that our condition (17) for the validity of the Rayleigh hypothesis indeed yields the same restriction on a/b . The representation of the ellipse in polar coordinates is

$$\begin{aligned} \rho(\theta) &= ab(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{-1/2}, \\ 0 &\leq \theta < 2\pi \quad (a > b). \end{aligned} \quad (18)$$

It can be shown that (14) has roots $\theta = \theta_p$ ($\operatorname{Im}(\theta_p) < 0$) that follow from

$$\tan(\theta_p) = -i \frac{b^2}{a^2} \quad \text{or} \quad \exp(2i\theta_p) = \frac{a^2 + b^2}{a^2 - b^2}. \quad (19)$$

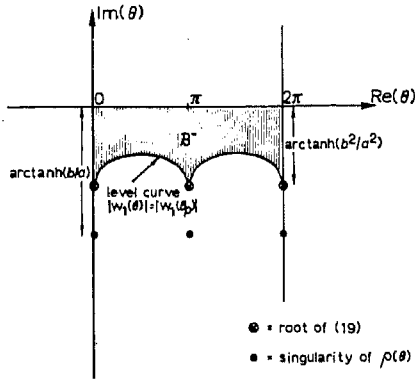


Fig. 3. Points of interest in complex θ plane concerning representation of ellipse ($a > b$).

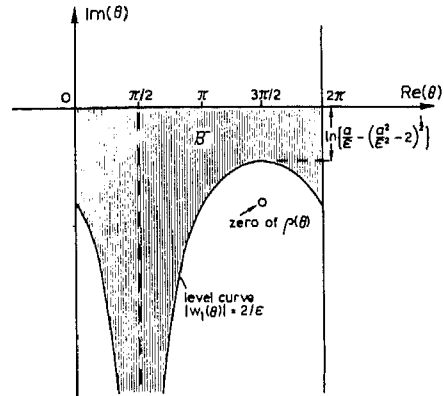


Fig. 4. Points of interest in complex θ plane concerning representation of perturbed circular cylinder $\rho(\theta) = a + \epsilon \sin \theta$.

The level curve $|w_1(\theta)| = |w_1(\theta_p)|$ in the complex θ plane¹ (see Fig. 3) corresponds to the circle of convergence of series (13) in the complex w_1 plane. We observe that the representation of the ellipse $\rho = \rho(\theta)$ in the complex θ plane has a singularity for $\text{Im}(\theta) < 0$ following from the equation $1/\rho(\theta) = 0$, or

$$\tan(\theta) = -ib/a.$$

But since $a > b$ and because of (19), it follows that this singularity is immaterial, since it lies beneath $|w_1(\theta)| = |w_1(\theta_p)|$ in the complex θ plane (see Fig. 3) and hence outside the domain of convergence of the expansion of the exterior scattered field. Squaring (17), using (18), expressing sine and cosine in terms of the tangent, and using (19) and some elementary calculations lead to the result that the Rayleigh hypothesis is valid if

$$b^2/(a^2 - b^2) > 1, \tag{20}$$

in which case $b > c = (a^2 - b^2)^{1/2}$, where c is half the inter-focal distance, hence if the foci lie inside the inscribed circle $r_{\text{min}} = b$.

As a second example we consider the scattering by a perturbed circular cylinder with representation

$$\rho(\theta) = a + \epsilon \sin(j\theta), \quad 0 \leq \theta < 2\pi, \tag{21}$$

in which $j = 1, 2, 3, \dots$. When $\epsilon = a$, $\rho(\theta)$ is not an analytic function of θ along $\text{Im}(\theta) = 0$. Then, the Rayleigh hypothesis never holds. Therefore, we restrict our considerations to smaller values of ϵ . Equation (14) then has roots for $\text{Im}(\theta) < 0$ that follow from

$$\exp(ij\theta_p) = i \frac{1 + \{1 + (j^2 - 1)(\epsilon/a)^2\}^{1/2}}{(j - 1)(\epsilon/a)}. \tag{22}$$

When $j = 1$, this root lies at infinity ($\theta_p = \frac{1}{2}\pi - i\infty$) in the complex θ plane. The value of w_1 at this point equals $2i/\epsilon$. The level curve $|w_1(\theta)| = 2/\epsilon$ is drawn in Fig. 4. This line corresponds to the circle of convergence in the complex w_1 plane. Hence, substituting in (17) the values $r_{\text{min}} = a - \epsilon$ and

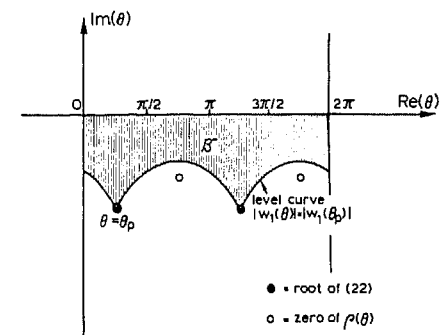


Fig. 5. Points of interest in complex θ plane concerning representation of perturbed circular cylinder $\rho(\theta) = a + \epsilon \sin(\theta)$.

$|w_1(\theta_p)| = 2/\epsilon$, it then follows that the Rayleigh hypothesis is valid when $0 \leq \epsilon/a < 2/3$.

When $j = 2$, the relevant level curve $|w_1(\theta)| = |w_1(\theta_p)|$ in the complex θ plane¹ ($\text{Im}(\theta) < 0$) is illustrated in Fig. 5. After a numerical calculation of $|w_1(\theta_p)|$, condition (17) leads to $0 \leq \epsilon/a < 0.265343$ when $j = 2$. For $j = 3$, we have three roots θ_p of (22), with $0 \leq \text{Re}\{\theta_p\} < 2\pi$. The level curve $|w_1(\theta)| = |w_1(\theta_p)|$ in the complex θ plane ($\text{Im}(\theta) < 0$) will then consist of three arcs (instead of two arcs in Fig. 5). A numerical evaluation of condition (17) leads to $0 \leq \epsilon/a < 0.166838$. In the same way we can for other values of j determine numerically the values of ϵ/a for which the Rayleigh hypothesis holds. The results are presented in Fig. 6. Typical obstacles for which the Rayleigh hypothesis is just not valid are illustrated in Fig. 7.

We note that it is possible for the Rayleigh hypothesis (with respect to the series in (2)) to be valid for one choice of origin but not for another, since $\rho(\theta)$ depends for a certain obstacle on the choice of the origin. In the next section we investigate this dependence on the choice of origin.

V. CHOICE OF ORIGIN

Let the obstacle in a polar coordinate system (r', θ') with origin O' be represented by $\rho' = \rho'(\theta')$ (see Fig. 8). In this primed coordinate system the Rayleigh hypothesis is the assumption that the scattered field can be represented as

$$u^s(r', \theta') = \sum_{n=-\infty}^{\infty} T_n' H_n^{(1)}(k_0 r') \exp(in\theta') \tag{23}$$

¹ This level curve follows from the solution of the transcendental equation $|w_1(\theta)| = |w_1(\theta_p)|$. In Figs. 3-5 we have roughly drawn this curve after numerical evaluation of this equation for a few points.

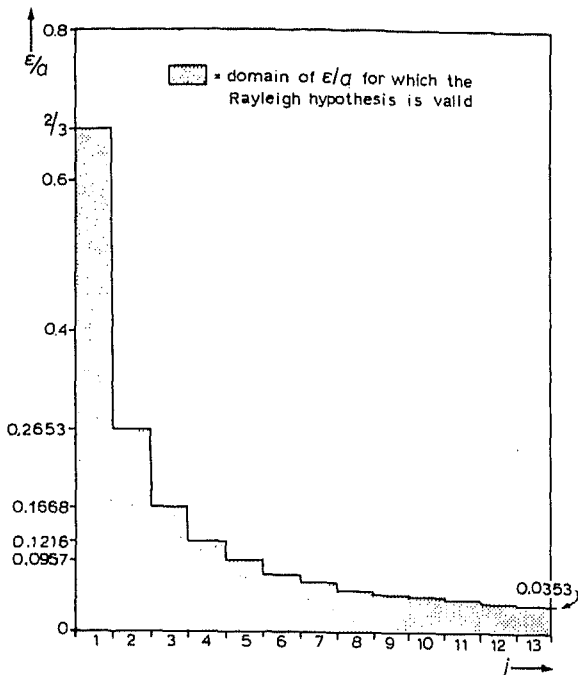


Fig. 6. Values of ϵ/a for which Rayleigh hypothesis holds, when obstacle is represented by $\rho(\theta) = a + \epsilon \sin(j\theta)$.

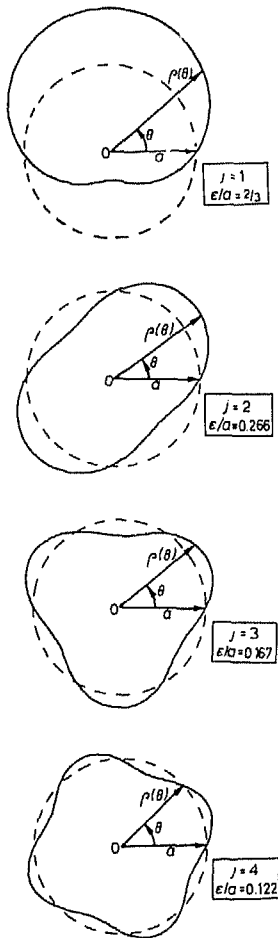


Fig. 7. Obstacles represented by $\rho(\theta) = a + \epsilon \sin(j\theta)$, for which Rayleigh hypothesis is just not valid. For smaller values of ϵ/a Rayleigh hypothesis holds.

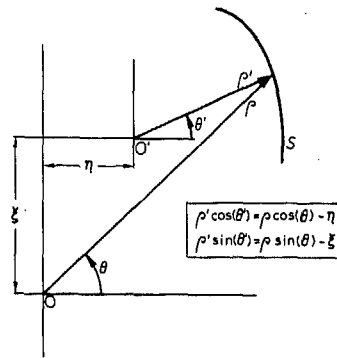


Fig. 8. Translated coordinate system.

outside and on the obstacle S . Let r_{\min}' be the radius of the circle with center O' inscribed in S and let

$$w_1'(\theta') = \{\rho'(\theta')\}^{-1} \exp(i\theta'), \tag{24}$$

then the Rayleigh hypothesis is true if (see (17))

$$|r_{\min}' w_{1,p}'| > 1, \tag{25}$$

in which $w_{1,p}'$ is a certain singularity in the w_1' plane nearest to the origin (see the analysis in Section III). We note that the translation of the coordinate system must be done in such a way that $\rho'(\theta')$ remains a single-valued function of θ' , and hence $\rho'(\theta')$ is an analytic function of θ' along $\text{Im}(\theta') = 0$. It is obvious that $\rho'(\theta')$ and $\rho(\theta)$ are related to each other through the translation formula

$$\rho'(\theta') \exp(-i\theta') = -\eta + i\xi + \rho(\theta) \exp(-i\theta). \tag{26}$$

Using the definitions of $w_1'(\theta')$ and $w_1(\theta)$ we observe that this translation formula (26) can be written as

$$w_1'(\theta') = \frac{w_1(\theta)}{(-\eta + i\xi)w_1(\theta) + 1}. \tag{27}$$

Hence, w_1' is a bilinear transformation of the complex variable w_1 through (27). To every value of w_1 corresponds just one value of w_1' in the complex w_1' plane. Hence, a singularity in the w_1' plane corresponds uniquely to a singularity in the w_1 plane, while we know from Section IV that the latter corresponds to a point $\theta = \theta_p$ ($\text{Im}(\theta_p) < 0$), being either a root of (14), a nonanalytic point of $\rho(\theta)$, or a zero of $\rho(\theta)$. Knowing these points, $w_{1,p}'$ can be determined using (27). The condition (25) then becomes

$$\left| r_{\min}' \frac{w_1(\theta_p)}{(-\eta + i\xi)w_1(\theta_p) + 1} \right| > 1. \tag{28}$$

Summarizing, we observe that after a translation of the coordinate system, condition (17) for the validity of the Rayleigh hypothesis is replaced by condition (28).

As an example we again consider the scattering by the perturbed circular cylinder with representation

$$\rho(\theta) = a + \epsilon \sin \theta, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \epsilon < a. \tag{29}$$

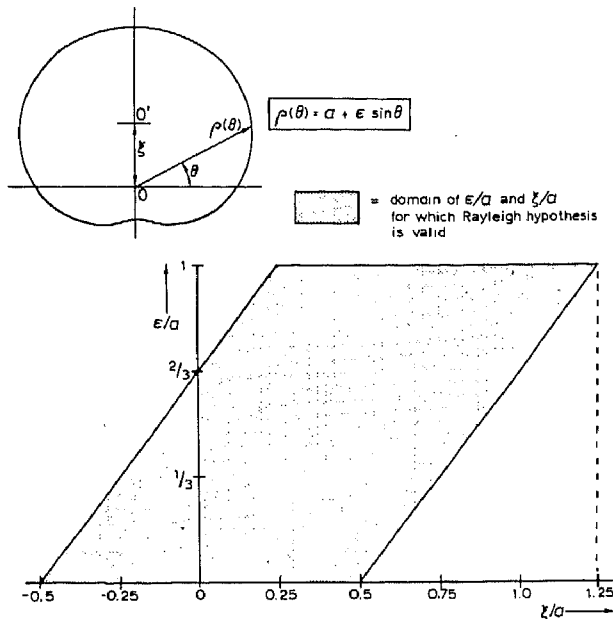


Fig. 9. Values of ϵ/a and ξ/a for which Rayleigh hypothesis applies, when origin is chosen at $O'(0, \xi)$.

In view of the symmetry of the obstacle and to be sure that $\rho'(\theta')$ is a single-valued function of θ' along $\text{Im}(\theta') = 0$, we apply a coordinate translation with $\eta = 0$ (see Fig. 8). Then, with $w_1(\theta_p) = 2i/\epsilon$ (see Section IV), condition (28) becomes

$$r_{\min}' / |\xi - \frac{1}{2}\epsilon| > 1. \quad (30)$$

Since $r_{\min}' = a + \epsilon - \xi$ when $\epsilon < \xi$, and $r_{\min}' = a - \epsilon + \xi$ when $\epsilon > \xi$, condition (30) can be rewritten as

$$\begin{aligned} a + \epsilon - \xi &> \xi - \frac{1}{2}\epsilon && \text{when } \epsilon < \xi, \\ a - \epsilon + \xi &> \xi - \frac{1}{2}\epsilon && \text{when } \xi < \epsilon < 2\xi, \\ a - \epsilon + \xi &> \frac{1}{2}\epsilon - \xi && \text{when } 2\xi < \epsilon. \end{aligned} \quad (31)$$

These conditions can be rewritten as

$$\frac{3}{4}\epsilon - \frac{1}{2}a < \xi < \frac{3}{4}\epsilon + \frac{1}{2}a, \quad \text{when } 0 \leq \epsilon < a. \quad (32)$$

In Fig. 9 we illustrate the range of ξ and ϵ for which the Rayleigh hypothesis is valid. From this figure we conclude that, provided the origin is suitably chosen, the Rayleigh hypothesis is valid for each ϵ ($0 \leq \epsilon < a$).

Obviously, the domain of analyticity of (23) depends on the choice of the origin, and hence the domain where singularities of the expansion are located varies if O' is varied. This leads to the question of whether by a proper choice of η and ξ , the maximum diameter of the latter domain can be minimized. Replacing r_{\min}' in the convergence condition by an arbitrary radius r_0' , we obtain the condition

$$r_0' > |-\eta + i\xi + \{w_1(\theta_p)\}^{-1}|. \quad (33)$$

As long as condition (33) holds, the expansion of the scattered field in (23) represents an analytic function in the domain $r' > r_0'$, even when $r_0' < r_{\min}'$. In the case of our example of the perturbed circular cylinder with representation (29), we have $w_1(\theta_p) = 2i/\epsilon$, so that condition (33) becomes

$$r_0' > |\xi - \frac{1}{2}\epsilon + i\eta|. \quad (34)$$

From (34) we observe that any $r_0' > 0$ can be taken, provided that $\xi = \frac{1}{2}\epsilon$ and $\eta = 0$. Consequently, for the obstacle under consideration, the singularity in the representation (23) is a single point $r' = 0$, provided that $\xi = \epsilon/2$ and $\eta = 0$. The location of this singular point $\eta = 0, \xi = \epsilon/2$, is in agreement with condition (32), confirming the requirement for the validity of the Rayleigh hypothesis that the source point of the exterior scattered field must lie inside the inscribed circle with center at the chosen origin. The best choice of O' is at this source point of the exterior scattered field.

A coordinate translation for the other obstacles dealt with in Section II yields no larger range for application of the Rayleigh hypothesis. The reason is that we then have two or more roots of (22). At these points the values of $|w_1(\theta_p)|$ are the same, but the values of $\arg\{w_1(\theta_p)\}$ are different. Hence, trying to minimize the right side of (33), for a certain root $\theta = \theta_p$, leads to an increase of the right side of (33) for another root of (22). This fact can also be found in Fig. 7 where, in view of the symmetry of the obstacles for $j > 1$, the best choice of the origin is at the point 0.

VI. CONCLUSION

The problem of studying the legitimacy of the Rayleigh hypothesis has been reduced to one of locating the singularities in the representation of the exterior scattered field. This can be done very easily by taking as a starting point the Rayleigh theory itself. An investigation as to the analyticity of the Rayleigh solution in the domain of interest yields a condition under which the Rayleigh hypothesis can be used. In this way, for any obstacle of arbitrary cross section, it can be investigated whether the Rayleigh hypothesis holds or not. In our treatment we have assumed that the obstacle has a smooth surface. When the surface has edges the Rayleigh hypothesis never holds. Further, we have assumed that the obstacle can be described by a radius ρ being a single-valued function of the angular variable θ . When this is not the case we can extend the analysis, employing an arbitrary parametric representation $\rho = \rho(s)$ and $\theta = \theta(s)$, in which the parameter s is extended to complex values.

APPENDIX A

Let us assume that there exists a solution of the problem (cf. Fig. 1):

- 1) $\nabla^2 G + k^2 G = -\delta(r - r')$ outside S ,
- 2) $G = 0$ on S (Dirichlet boundary condition),
- 3) G satisfies the radiation condition as $|r| \rightarrow \infty$.

Let U_a define an analytic solution of the homogeneous Helmholtz equation outside S . Further, let U_a satisfy the radiation condition as $|r| \rightarrow \infty$. Using Green's theorem, the radiation conditions and the boundary condition with respect to G , we arrive at

$$U_a(r') = \int_S U_a \frac{\partial G}{\partial n} dA, \quad r' \text{ outside } S, \quad (A1)$$

in which S is an analytic surface (without edges) and $\partial/\partial n$ is the derivative in the direction of the outward normal to S . Also, let

$$U_a \equiv U^s - U_R^s, \quad (A2)$$

being the difference between two different solutions of our

relevant scattering problem, originating from the same incident field. Then

$$U_a = 0 \text{ on } S \text{ (Dirichlet boundary condition)} \quad (A3)$$

and hence, from (A1) and (A3), $U_a \equiv 0$ everywhere outside S . From (A2) it then follows that $U^s = U_R^s$ everywhere outside S . Hence, we are dealing with a unique solution of our scattering problem.

ACKNOWLEDGMENT

The authors wish to thank Professor A. T. de Hoop for useful discussions.

REFERENCES

- [1] J. W. Strutt (Lord Rayleigh), *The Theory of Sound*, 2nd Ed., London: Macmillan, 1896 vol II, p. 89, (also available in a 1945 edition from New York Dover).
- [2] —, "On the dynamical theory of gratings," *Proc. Roy. Soc. (London)*, vol. A79, pp. 399-416, 1907.
- [3] L. N. Deriugin, "Equations for coefficients of wave reflections from a periodically uneven surface," *Dokl. Akad. Nauk SSSR (USSR)*, vol. 87, pp. 913-916, 1952.
- [4] B. A. Lippmann, "Variational formulation of a grating problem," Nuclear Development Associates, Inc., White Plains, NY, Rep. NDA 18-8, 1952 (available from NTIS, Washington, DC 20402).
- [5] —, "Note on the theory of gratings," *J. Opt. Soc. Am.*, vol. 43, p. 408, 1953.
- [6] B. A. Lippmann and A. Oppenheim, "Towards a theory of Wood's anomalies," Technical Research Group, New York, NY, Rep. (contract no. OSR-TR-54-25), June 30, 1954 (available from NTIS, Washington, DC 20402).
- [7] R. Petit, "Diffraction d'une onde plane monochromatique par un réseau périodique infiniement conducteur," *Comptes Rendus Acad. Sci. (Paris)*, vol. B257, pp. 2018-2021, 1963.
- [8] R. Petit and M. Cadilhac, "Etude théorique de la diffraction par un réseau," *Comptes Rendus Acad. Sci. (Paris)*, vol. B259, pp. 2077-2080, 1964.
- [9] —, "Sur la diffraction d'une onde plane par un réseau infiniement conducteur," *Comptes Rendus Acad. Sci. (Paris)*, vol. B262, pp. 468-471, 1966.
- [10] —, "Sur la diffraction d'une onde plane par un réseau infiniement conducteur," *Comptes Rendus Acad. Sci. (Paris)*, vol. B264, pp. 1441-1444, 1967.
- [11] R. F. Millar, "On the Rayleigh assumption in scattering by a periodic surface," *Proc. Camb. Phil. Soc.*, vol. 65, pp. 773-791, 1969.
- [12] —, "On the Rayleigh assumption in scattering by a periodic surface: Part II," *Proc. Camb. Phil. Soc.*, vol. 69, pp. 217-225, 1971.
- [13] M. Neviere and M. Cadilhac, "Sur la validité du développement de Rayleigh," *Opt. Commun.*, vol. 2, pp. 235-238, 1970.
- [14] R. F. Millar, "Singularities of two-dimensional exterior solutions of the Helmholtz equation," *Proc. Camb. Phil. Soc.*, vol. 69, pp. 175-188, 1971.
- [15] —, "The Rayleigh hypothesis and a related least-square solution to scattering problems for periodic surfaces and other scatterers," *Radio Sci.*, vol. 8, pp. 785-796, 1973.
- [16] P. M. van den Berg and J. T. Fokkema, "The Rayleigh hypothesis in the diffraction theory of gratings," *J. Opt. Soc. Am.*, vol. 69, pp. 27-31, 1979.
- [17] H. A. Kalhor, "Numerical evaluation of Rayleigh hypothesis for analyzing scattering from corrugated gratings," *IEEE Trans. Antennas Propagat.*, vol. AP-24, pp. 884-889, 1976.
- [18] P. C. Clemmow and V. H. Weston, "Diffraction of a plane wave by an almost circular cylinder," *Proc. Roy. Soc. (London)*, vol. A264, pp. 246-268, 1961.
- [19] C. Yeh, "Perturbation approach to the diffraction of electromagnetic waves by arbitrary shaped dielectric obstacles," *Phys. Rev.*, vol. A135, pp. 1193-1201, 1964.
- [20] C. R. Mullin, R. Sandburg, and C. O. Velline, "A numerical technique for the determination of scattering cross-section of infinite cylinders of arbitrary geometrical cross-section," *IEEE Trans. Antennas Propagat.*, vol. AP-13, pp. 141-149, 1965.
- [21] R. F. Harrington, "On the calculation of scattering by conducting cylinders," *IEEE Trans. Antennas Propagat.*, vol. AP-13, pp. 812-813, 1965.
- [22] R. H. T. Bates, "The point-matching method for interior and exterior two-dimensional boundary value problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-15, pp. 185-187, 1967.
- [23] A. Wirgin, "Sur la validité d'une représentation de la solution de l'équation de Laplace," *Comptes Rendus Acad. Sci. (Paris)*, vol. A267, pp. 724-728, 1968.
- [24] R. H. T. Bates, "Modal expansions for electromagnetic scattering from perfectly conducting cylinders of arbitrary cross-section," *Proc. IEE*, vol. 115, pp. 1443-1445, 1968.
- [25] —, "The theory of the point-matching method for perfectly conducting waveguides and transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 294-301, 1969.
- [26] —, "Rayleigh hypothesis, the extended boundary condition, and point matching," *Electron. Lett.*, vol. 5, pp. 654-655, 1969.
- [27] M. L. Burrows, "Equivalence of the Rayleigh solution and the extended boundary-condition-solution for scattering problems," *Electron. Lett.*, vol. 5, pp. 277-278, 1969.
- [28] R. F. Millar and M. L. Burrows, "Rayleigh hypothesis in scattering problems," *Electron. Lett.*, vol. 5, pp. 416-418, 1969.
- [29] R. F. Millar, "On the legitimacy of an assumption underlying the point matching method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 325-327, 1970 (including a reply by R. H. T. Bates and comments on this reply by R. F. Millar).
- [30] M. L. Burrows, "Examples of the generalized function validity of the Rayleigh hypothesis," *Electron. Lett.*, vol. 5, pp. 694-695, 1969.
- [31] R. F. Millar, "The location of singularities of two-dimensional harmonic functions; Part I—Theory, Part II—Applications," *Siam J. Math. Anal.*, vol. 1, pp. 333-353, 1970.
- [32] J. C. Bolomey and A. Wirgin, "Numerical comparison of the Green's function and the Waterman and Rayleigh theories of scattering from a cylinder with arbitrary cross-section," *Proc. IEE*, vol. 121, pp. 794-804, 1974.
- [33] V. H. Ahlfors, *Complex Analysis*, New York: McGraw-Hill, 1953, ch. IV, pp. 140-149.
- [34] M. A. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, New York: Dover, 1965, p. 360.
- [35] E. C. Titchmarsh, *The Theory of Functions*, 2nd Ed. New York: Oxford University Press, 1939.



Peter M. van den Berg was born in Rotterdam, The Netherlands, on November 11, 1943. He received a degree in electrical engineering from the Polytechnic School of Rotterdam in 1964. He received the B.S. and M.S. degrees in electrical engineering from the Delft University of Technology, The Netherlands, in 1966 and 1968, respectively, and the Ph.D. degree in technical sciences from the Delft University of Technology, The Netherlands, 1971.

From 1967 to 1968 he was employed as a Research Engineer by the Dutch Patent Office. From 1968 until the present he has been a member of the Scientific Staff of the Electromagnetic Research Group of the Delft University of Technology, The Netherlands, first as a Lecturer and later as a Senior Lecturer. During these years he carried out research and taught courses in the area of wave propagation and radiation problems, with particular reference to periodic structures and radiation phenomena produced by moving charges. During the academic year 1973-1974 he was a visiting Lecturer in the Department of Mathematics, University of Dundee, Scotland.



Jacob T. Fokkema was born in Leeuwarden, The Netherlands, on January 21, 1948. He received a degree in electrical engineering from the Polytechnic School of Leeuwarden in 1969. He received the B.S. and M.S. degrees in electrical engineering from the Delft University of Technology, The Netherlands, in 1973 and 1976, respectively. At present he is a doctoral student at the Delft University of Technology, The Netherlands, doing research in the area of diffraction of elastic waves by a periodic surface.