THE REAL AND THE SYMMETRIC NONNEGATIVE INVERSE EIGENVALUE PROBLEMS ARE DIFFERENT

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ABSTRACT. We show that there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ that occur as the eigenvalues of an entry-wise nonnegative n-by-n matrix but do not occur as the eigenvalues of a symmetric nonnegative n-by-n matrix. This solves a problem posed by Boyle and Handelman, Hershkowitz, and others. In the process, recent work by Boyle and Handelman that solves the nonnegative inverse eigenvalue problem by appending 0's to given spectral data is refined.

1.

Let $M_{m,n}$ ($M_{m,n}(\mathbb{R})$) denote the set of all m-by-n complex (real) matrices, and let $M_m = M_{m,m}$ ($M_m(\mathbb{R}) = M_{m,m}(\mathbb{R})$). The *nonnegative inverse eigenvalue problem* (NIEP) asks which sets of n complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ occur as the eigenvalues (spectrum) of some entry-wise nonnegative matrix $A \in M_n$. In case the data: $\lambda_1, \ldots, \lambda_n$ are real, two natural variations suggest themselves:

(1) The *real nonnegative inverse eigenvalue problem* (RNIEP) asks which sets of n real numbers occur as the spectrum of a nonnegative $A \in M_n$; and

(2) the *symmetric nonnegative inverse eigenvalue problem* (SNIEP) asks which sets of n real numbers occur as the spectrum of a symmetric nonnegative matrix $A \in M_n$.

Each problem remains open in general. Several necessary conditions for NIEP are known. Suppose that $\lambda_1, \ldots, \lambda_n$ are complex numbers. For every positive integer k define the moments

$$
(1) \qquad S_k(\lambda) = \sum_{i=1}^{n} \lambda_i^k.
$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of an n-by-n nonnegative (positive) matrix A, we must have $S_k(\lambda) \geq 0$ ($S_k(\lambda) > 0$) for every positive integer k, because $S_k(\lambda)$ is just the trace of A^k. The Perron-Frobenius theorem implies that $\max_{1 \leq i \leq n} |\lambda_i|$ is an eigenvalue of A. (The Perron-Frobenius condition and the nonnegativity of the moments are not independent; see a remark after the statement of Theorem 2.) We also have:

Theorem 1 ([7, 9]). *Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of an n-by-n nonnegative matrix. Then, for any positive integers k and t we have*

$$(2) \qquad n^{k-1} S_{kt}(\lambda) \geq S_t(\lambda)^k.$$

Note that the nonnegativity of the moments is independent of the order of the matrix in contrast to the inequalities (2).

If the SNIEP has a solution for given real data, then, of course, the RNIEP also has a solution. In particular, in case $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$, then both problems trivially have a solution. When at least one of the λ's is negative, there are several construction techniques to solve the RNIEP under additional conditions on the λ's ([13, 3, 4, 8, 10, 11, 12], etc.). Interestingly, these construction techniques actually solve the SNIEP (or may be modified to do so) under the assumed circumstance. No example of data has previously been presented for which the RNIEP has a solution while the SNIEP does not, and in low dimensions ($n \leq 4$) the two are actually equivalent. Thus, it is natural to ask whether RNIEP and SNIEP are generally equivalent, and various people have raised this question, cf. Question 3.4 in [2], and [6]. Using recent work of [1] on the general NIEP, older results in [7, 9] and a cone-theoretic dimension argument, we show that the two problems are different. In the process, we refine the aforementioned work in the case of the RNIEP and, especially in the case of the SNIEP. Briefly put, according to [1], if $\lambda_1, \ldots, \lambda_n$ meet certain necessary conditions, it is possible to append sufficiently many 0's: $\lambda_{n+1} = \lambda_{n+2} = \cdots = \lambda_m = 0$ so that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of a positive $A \in M_m$. However, m (and, implicitly, rank A) may have to be large relative to n. (No explicit bounds are given.) Using [7] or [9], it is possible to give real data that meet the criteria of [1] but for which m must be very large. If A were symmetric (or, more generally, diagonalizable), rank A would then be very low relative to m. Using a cone-theoretic dimension argument, we show that if $A \in M_m$ is a sufficiently low rank nonnegative matrix, then there is another nonnegative matrix $A' \in M_{m'}$, with $m' < m$, such that the nonzero part of the spectrum of A' agrees with that of A. This rules out the possibility of the SNIEP having a solution for certain data for which [1] guarantees a solution to the RNIEP.

2.

Nonzero numbers $\lambda_1, \ldots, \lambda_n$ are said to be the *nonzero spectrum* of $A \in M_m$, $m \geq n$, if the nonzero eigenvalues of A coincide with $\lambda_1, \ldots, \lambda_n$, counting multiplicities. We define an equivalence relation on square complex matrices of all dimensions as follows: $A \in M_m$ and $A' \in M_{m'}$ are said to *have the same nonzero spectrum*, written

$$A \sim A'$$

if, for each $0 \neq \lambda \in \mathbb{C}$, λ occurs as an eigenvalue of A with the same algebraic multiplicity (possibly 0) as it does in A'. Recall that, if $B \in M_{m,m_1}$ and $C \in M_{m_1,m}$, then $BC \sim CB$.

The following is a specialization of one of the results in [1, p. 313], but is sufficient for our purposes.

Theorem 2 ([1]). *Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero and that $\lambda_1 = \max_{1 \leq i \leq n} |\lambda_i|$. Then $\{\lambda_1, \ldots, \lambda_n\}$ is the nonzero spectrum of an entry-wise pos-*

itive matrix in M_m, for some $m \geq n$, if and only if

- (a) $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$;
- (b) $S_k(\lambda) > 0$, $k = 1, 2, \ldots$; *and*
- (c) *all coefficients of the polynomial $\prod_{i=1}^{n}(x - \lambda_i)$ are real.*

Note that the conditions (a)–(c) are not independent. It is clear from Newton's identities that (b) implies (c). It was shown in [5] that if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ satisfy $S_k(\lambda) \geq 0$ for $k = 1, 2, \ldots$ then there exists $i \in \{1, 2, \ldots, n\}$ such that $\lambda_i = |\lambda_i| \geq |\lambda_j|$ for all $j = 1, 2, \ldots, n$.

Finally, we also make use of the following fact.

Theorem 3 (Carathéodory). *Let E be an l-dimensional real vector space, and let $v_i \in E$, $i = 1, 2, \ldots, p$. Let K be the convex cone generated by $v_1, v_2, \ldots, v_p$. Then each point in K can be expressed as a linear combination, with nonnegative coefficients, of l or fewer of the v_i's.*

3.

Our two main observations are the following.

Theorem 4. *If $A \in M_m$ is an entry-wise nonnegative matrix of rank r, then there is a nonnegative $A' \in M_{r^2}$ such that $A \sim A'$.*

Proof. If $m \leq r^2$, there is nothing to show. Thus, we assume $r^2 < m$. Considering A as an element of $M_m(\mathbb{R})$, we may write

$$A = B^{\mathrm{T}}C,$$

in which $B, C \in M_{r,m}(\mathbb{R})$. Partition B and C by columns as

$$B = [b_1 b_2 \cdots b_m], \qquad C = [c_1 c_2 \cdots c_m].$$

Now, $A \sim CB^{\mathrm{T}} = \sum_{i=1}^{m} c_i b_i^{\mathrm{T}}$, but, as the $c_i b_i^{\mathrm{T}}$ lie in the r^2-dimensional vector space $M_r(\mathbb{R})$, Theorem 3 shows that

$$CB^{\mathrm{T}} = \sum_{j=1}^{r^2} \alpha_j c_{i_j} b_{i_j}^{\mathrm{T}},$$

for some nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{r^2}$. Let $B' = [b_{i_1} b_{i_2} \cdots b_{i_{r^2}}]$, $C' = [c_{i_1} c_{i_2} \cdots c_{i_{r^2}}]$, and $D_\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_{r^2})$. Then we have $CB^{\mathrm{T}} = C'D_\alpha B'^{\mathrm{T}} \sim B'^{\mathrm{T}}C'D_\alpha \in M_{r^2}(\mathbb{R})$. The j, k entry of $B'^{\mathrm{T}}C'D_\alpha$ is $\alpha_k b_{i_j}^{\mathrm{T}} c_{i_k} \geq 0$, because $\alpha_k \geq 0$ and $b_{i_j}^{\mathrm{T}} c_{i_k}$ is the i_j, i_k entry of $A \geq 0$. This completes the proof. □

A modification of the above argument yields a stronger statement in the symmetric case.

Theorem 5. *Let $A \in M_m$ be a symmetric, nonnegative matrix of rank r. Then, there exists a symmetric, nonnegative matrix $A' \in M_{r(r+1)/2}$ such that $A \sim A'$.*

Proof. Let $q = r(r + 1)/2$. If $m \leq q$, there is nothing to show, so we may assume $q < m$. Via Sylvester's law, A may be written in inertial form as

$$A = B^{\mathrm{T}}EB,$$

in which $B \in M_{r,m}(\mathbb{R})$ and $E = I_{r_1} \oplus -I_{r_2}$, with $r_1 + r_2 = r$. Partition B by columns as $B = [b_1 b_2 \cdots b_m]$.

Now, $A \sim BB^\mathrm{T}E = \sum_{i=1}^m b_i(Eb_i)^\mathrm{T}$, and notice that the $b_i(EB_i)^\mathrm{T}$ lie in a q-dimensional vector space, namely the subspace of matrices $X \in M_r(\mathbb{R})$ such that XE is symmetric. Theorem 3 then yields $BB^\mathrm{T}E = \sum_{j=1}^q \alpha_j b_{i_j}(Eb_{i_j})^\mathrm{T}$ in which $\alpha_1, \ldots, \alpha_q \geq 0$. Setting $B' = [b_{i_1} b_{i_2} \cdots b_{i_q}]$ and $D_\alpha = \mathrm{diag}(\alpha_1^{1/2}, \ldots, \alpha_q^{1/2})$, we have $BB^\mathrm{T}E = B'D_\alpha(B'D_\alpha)^\mathrm{T}E \sim D_\alpha B'^\mathrm{T}EB'D_\alpha \in M_q(\mathbb{R})$, which, as before, is entry-wise nonnegative. $\qquad\square$

4.

Suppose $\lambda_1, \ldots, \lambda_n$ are nonzero complex numbers that meet the conditions of Theorem 2. Then, Theorem 1 has the effect of placing a lower bound on the number $m - n$ of 0's that need be appended to $\lambda_1, \ldots, \lambda_n$ in order to achieve the spectrum of a nonnegative m-by-m matrix (as Theorem 2 qualitatively guarantees is possible). The larger m is, the more likely the necessary conditions of Theorem 1 are to be met. Arbitrarily large numbers of 0's may need to be added to meet the conditions of Theorem 1. Theorem 4 shows that if sufficiently many 0's need be added, they must have a nonsimple Jordan structure in any realizing nonnegative matrix A (as the rank cannot be too low). In the event that $\lambda_1, \ldots, \lambda_n$ are real and nonzero and we wish to have the realizing nonnegative A symmetric, the Jordan structure of the added 0's would have to be simple. Thus, if Theorem 1 says that too many zeros need be added, there will be no symmetric nonnegative matrix with nonzero spectrum $\lambda_1, \ldots, \lambda_n$, even though there will be a nonsymmetric (even positive) one if the conditions of Theorem 2 are met. Theorem 5 further quantifies the limitations on symmetric realizability. As a specific example consider $n = 6$ and

$$\lambda_1 = \sqrt[3]{51} + \varepsilon, \quad \lambda_2 = \lambda_3 = \lambda_4 = 1, \quad \text{and} \quad \lambda_5 = \lambda_6 = -3$$

for "small" $\varepsilon > 0$. It is easy to verify that for any $\varepsilon > 0$, these λ's meet the conditions of Theorem 2. However, as $S_3(\lambda) = 0$ when $\varepsilon = 0$, for any integer $m > 6$ there is an ε_m such that for $\varepsilon = \varepsilon_m$, at least $m - 6$ 0's must be added to meet the conditions of Theorem 1, and therefore, at least $m - 6$ 0's must be added to obtain the realization guaranteed by Theorem 2. Consider now $m = 22$. For $\varepsilon = \varepsilon_{22}$, Theorem 5 shows there is no *symmetric* nonnegative matrix (of any dimension) whose nonzero spectrum is $\lambda_1, \lambda_2, \ldots, \lambda_6$. If there were one, Theorem 5 guarantees one in M_{21}, a contradiction to Theorem 1. According to Theorem 2, there is a nonsymmetric nonnegative (even positive) matrix whose nonzero spectrum is $\lambda_1, \ldots, \lambda_6$; suppose it lies in M_m. Then $\lambda_1, \ldots, \lambda_m$, with $\lambda_7 = \cdots = \lambda_m = 0$, provides an example of the difference between the RNIEP and the SNIEP.

5.

The above remarks raise a number of natural further questions. For example, more needs to be known about the minimum number of 0's that need be added to obtain the realization guaranteed by Theorem 2 (and other results in [1]). How "nonsimple" must the Jordan structure of the added 0's be? To what extend is Theorem 1 (or other results in [9]) indicative of the true number of 0's needed? Is there a [1]-type result for the SNIEP? What are the additional necessary conditions? More specifically, can the estimates r^2 and $r(r + 1)/2$, respectively, in Theorems 4 and 5 be sharpened? We do not know examples to show that either is sharp. Is there any example of n nonzero real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are not the spectrum

of a symmetric, nonnegative matrix in M_n and such that there exist $m \geq n$ and a symmetric, nonnegative matrix in M_m whose nonzero spectrum is $\lambda_1, \lambda_2, \ldots, \lambda_n$? For a set of $n \leq 4$ real nonzero numbers simple arguments with known techniques show that adding 0's cannot help in the SNIEP, but 4 may be too small a value of n to be indicative.

Author's note

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