ONLINE APPENDIX

The real effects of monetary shocks in sticky price models: a sufficient statistic approach

F. Alvarez , H. Le Bihan, and F. Lippi

A Comparing BBP to CPI data to estimate kurtosis

We match a subset of our French CPI data with the prices from 3 French retailers taken from the Billion Price Project (BPP) dataset, see Cavallo (2015).[25] Table 2 offers two comparisons. The first three columns compare the BPP data from 2 large retailers with our CPI data for a similar type of outlet: to this end we restrict our dataset to CPI price records in "hypermarkets", excluding gasoline. The last two columns compare the BPP data from a large retailer specialized in electronics and appliances with the CPI data for goods in the category of appliances and electronic (we use the Coicop nomenclature, collected in outlets type "hypermarkets","supermarkets", and "large area specialists"). Comparing the values of kurtosis from both data sets suggests that $\Omega/\zeta \cong 2$, see equation (2). We can apply this magnitude to the full sample of CPI data, for which no "measurement error-free" counterpart like the BPP exists (and the feasible correction for heterogeneity is only partial), to obtain a corrected kurtosis. The number thus obtained for the kurtosis is near 4, so it lays in between the kurtosis of the Normal and the Laplace distribution.

B Proofs

Proof. (of Proposition 1). Let $p(0) = 0$. Define $x(t) \equiv ||p(t)||^2 - n\,\sigma^2\,t$ for $t \geq 0$. Using Ito's lemma we can verify that the drift of $||p||^2$ is $n\sigma^2$, and hence $x(t)$ is a Martingale. By the optional sampling theorem $x(\tau)$, the process stopped at τ, is also a martingale. Then

[25] We are extremely grateful to Alberto Cavallo for producing these statistics for us.

Table 2: Comparison of the CPI vs. the BPP data in France

CPI category:	Hypermarkets			Appliances and electronic	
Data source:	BPP retailer 1	BPP retailer 5	CPI Hypermarkets	BPP retailer 4	CPI Large ret. electr.
duration (months)	8.6	8.1	4.8	6.4	7.2
kurtosis	5.5	4.3	10.1	2.8	6.3

Note: The BPP data are documented in Cavallo (2015). Results were communicated by the author. For CPI data source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. The sub-sample in the third column features the CPI records for the outlet type "hypermarkets". The sub-sample in the 5th column features the CPI records in the category of "appliances and electronic", as identified using the Coicop nomenclature, collected in the following outlets type: "hypermarkets","supermarkets", and "large area specialists". Data are standardized within each subsample using Coicop categories.

$\mathbb{E}\left[x(\tau) \;\middle|\; p(0)\right] = \mathbb{E}\left[||p(\tau)||^2 \;\middle|\; p(0)\right] - n\sigma^2\mathbb{E}\left[\tau \;\middle|\; p(0)\right] = x(0) = 0$ and since $N(\Delta p_i) = 1/\mathbb{E}\left[\tau \;\middle|\; p(0)\right]$ and $Var(\Delta p_i) = \mathbb{E}\left[||p(\tau)||^2 \;\middle|\; p(0)\right]/n$ we obtain the desired result. □

Proof. (of Lemma 1). First, note that since two value functions differ by a constant, then all their derivatives are identical. Hence, if the one for the discount rate and arrival rate of free adjustment $(r + \lambda, 0)$ satisfies value matching and smooth pasting, so does the one for discount rate and arrival rate of free adjustment $(r, \lambda, 0)$, for the same boundary. Second, consider the range of inaction, subtracting the value function for the problem with parameters $(r+\lambda, 0)$ from the one with parameters (r, λ), and using that all the derivatives are identical, one verifies that if the Bellman equation holds for the problem with $(r + \lambda, 0)$, so it does for the problem with (r, λ). □

Proof. (of Proposition 2). The first part is straightforward given Lemma 1 and Proposition 3 in Alvarez and Lippi (2014). The second part is derived from the following implicit expression determining $\bar{y}$ (see the proof of Proposition 3 in Alvarez Lippi for the derivation):

$$\psi \;=\; \frac{B}{r+\lambda}\bar{y}\left[1 - \frac{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y} + \bar{y}^2 + \bar{y}^2\sum_{i=1}^{\infty}\kappa_i\;(r+\lambda)^i\;\bar{y}^i}{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y} + 2\bar{y}^2 + \bar{y}^2\sum_{i=1}^{\infty}\kappa_i\;(i+2)\;(r+\lambda)^i\;\bar{y}^i}\right] \tag{25}$$

where $\kappa_i = (r+\lambda)^{-i}\prod_{s=1}^{i}\frac{1}{\sigma^2(s+2)(n+2s+2)}$. So we can rewrite equation (25) as: $\psi = \frac{B}{r+\lambda}\bar{y}$ $[1 - \xi(\sigma^2, r+\lambda, n, \bar{y})]$. Since $\bar{y} \to \infty$ as $\psi \to \infty$ then we can define the limit:

$$\lim_{\psi\to\infty}\frac{\psi}{\bar{y}} = \frac{B}{r+\lambda}\left[1 - \lim_{\bar{y}\to\infty}\xi(\sigma^2, r+\lambda, n, \bar{y})\right]$$

Simple analysis can be used to show that $\lim_{\bar{y}\to\infty}\xi(\sigma^2, r+\lambda, n, \bar{y}) = 0$ which gives the expression in the proposition (see the technical Appendix J in Alvarez, Le Bihan, and Lippi

(2016) for a detailed derivation). $\square$

Proof. (of Proposition 3). To characterize $N(\Delta p_i)$ we write the Kolmogorov backward equation for the expected time between adjustments $\mathcal{T}(y)$ which solves: $\lambda \mathcal{T}(y) = 1 + \; n\, \sigma^2\, \mathcal{T}'(y) + \; 2\, y\, \sigma^2\, \mathcal{T}''(y)$ for $y \in (0, \bar{y})$ and $\mathcal{T}(\bar{y}) = 0$ (see the technical Appendix K in Alvarez, Le Bihan, and Lippi (2016) for details on the solution to this equation). Then the expected number of adjustments is given by $N(\Delta p_i) = 1/\mathcal{T}(0)$, subject to $\mathcal{T}(0) < \infty$.

The solution of the second order ODE for $\mathcal{T}(y)$ has a power series representation: $\mathcal{T}(y) = \sum_{i=0}^{\infty} \alpha_i\; y^i$, for $y \in [0, \bar{y}]$, with the following conditions on its coefficients $\{\alpha_i\}$: $\alpha_1 = \frac{\lambda \alpha_0 - 1}{n\sigma^2}$, $\alpha_{i+1} = \frac{\lambda}{(i+1)\; \sigma^2\; (n+2i)}\; \alpha_i$, for $i \geq 1$ and where $0 < \alpha_0 < 1/\lambda$ is chosen so that $0 \geq \alpha_i$ for $i \geq 1$, $\lim_{i \to \infty} \frac{\alpha_{i+1}}{\alpha_i} = 0$ and $0 = \sum_{i=0}^{\infty} \alpha_i\; \bar{y}^i$. Moreover, $\mathcal{T}(0) = \alpha_0$ is an increasing function of $\bar{y}$ since α_0 solves:

$$0 = \alpha_0 + \frac{(\alpha_0 - 1/\lambda)}{n} \left(\frac{\bar{y}\lambda}{\sigma^2}\right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1)\;(n+2k)}\right) \left(\frac{\bar{y}\lambda}{\sigma^2}\right)^i\right]$$

Note that for $i \geq 1$: $\alpha_i = \alpha_i / \left[i!\; (n/2 + i)\right] (\lambda/(2\,\sigma^2))$, and using the properties of the Γ function

$$\alpha_i = \Gamma(n/2) / \left(\Gamma(n/2 + i) \left(\lambda/(2\,\sigma^2)\right)^i (\alpha_0 - 1/\lambda)\right. .$$

Solving for α_0 and using $\mathcal{L} \equiv \lambda / N(\Delta p_i) = \lambda \mathcal{T}(0) = \lambda \alpha_0$. Thus

$$\ell = \left(\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{n}{2}\right)}{i!\,\Gamma\left(\frac{n}{2}+i\right)} \left(\frac{\lambda \bar{y}}{2\,\sigma^2}\right)^i\right) \Big/ \left(\sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}\right)}{i!\,\Gamma\left(\frac{n}{2}+i\right)} \left(\frac{\lambda \bar{y}}{2\,\sigma^2}\right)^i\right)$$

which is equation (6). $\square$

Proof. (of Proposition 4). We first state a lemma about the density $f(y)$.

Lemma 3 *Let $f(y; n, \frac{\lambda}{\sigma^2}, \bar{y})$ be the density of $y \in [0, \bar{y}]$ in equation (7) satisfying the boundary conditions. For any $k > 0$ we have: $f\left(y; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) = \frac{1}{k}\; f\left(\frac{y}{k}; n, \frac{\lambda k}{\sigma^2}, \frac{\bar{y}}{k}\right)$.*

Proof. (of Lemma 3). Consider the function $f(y; n, \frac{\lambda}{\sigma^2}, \bar{y})$ solving equation (7) (and boundary conditions) for given $n, \frac{\lambda}{\sigma^2}, \bar{y}$. Without loss of generality set $\sigma' = \sigma$ and consider $\bar{y}' = \bar{y}/k$ and $\lambda' = \lambda k$. Notice that by setting $C_1' = C_1 k$ and $C_2' = C_2 k$ we verify that the boundary conditions hold (because $C_1'/C_2' = C_1/C_2$) and that (7) holds (which is readily verified by a change of variable). $\square$

We now prove the proposition. Let $w\left(\Delta p_i; n, \ell, Std(\Delta p_i)\right)$ be the density function in equation (9). Next we verify equation (10). From the first term in equation (9) notice that

$$(1-\ell)\,\omega\left(\Delta p_i;\bar{y}\right) = s\ (1-\ell)\,\omega\left(s\Delta p_i; s^2\bar{y}\right)$$

where the first equality uses the homogeneity of degree -1 of $\omega(\Delta p_i; y)$ (see equation (8)). From the second term in equation (9) for $n \geq 2$

$$\ell\int_0^{\bar{y}} \omega(\Delta p_i;y) f(y;n,\frac{\lambda}{\sigma^2},\bar{y})dy = \ell\int_0^{\bar{y}} s\ \omega\left(s\Delta p_i;s^2 y\right)\ s^2\ f\left(ys^2;n,\frac{\lambda}{s^2\sigma^2},\bar{y}s^2\right)dy$$

where the first equality follows from Lemma 3 for $k = 1/s^2$, and the homogeneity of degree -1 of $\omega(\cdot,\cdot)$. Further we note

$$\ell\int_0^{\bar{y}} s\ \omega\left(s\Delta p_i;s^2 y\right) s^2\ f\left(ys^2;n,\frac{\lambda}{s^2\sigma^2},\bar{y}s^2\right)dy = s^3\ell\int_0^{\bar{y}} \omega\left(s\Delta p_i;s^2 y\right) f\left(ys^2;n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)dy$$

where $\frac{\lambda'\bar{y}'}{\sigma'^2} = \frac{\lambda\bar{y}}{\sigma^2}$, so that ℓ is the same across the two economies. Using $z = y\ s^2$

$$s^3\ell\int_0^{\bar{y}} \omega\left(s\Delta p_i;s^2 y\right) f\left(ys^2;n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)dy = s\ \ell\int_0^{\bar{y}'} \omega\left(s\Delta p_i;z\right) f\left(z;n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)dz\ .$$

where $\bar{y}' = s^2\bar{y}$, which completes the verification of equation (10). $\square$

Proof. (of Proposition 5). For any $p \in \mathbb{R}^n$ with $||p||^2 \leq \bar{y}$, we write $m(p;\bar{y},\sigma,\lambda)$ to emphasize the dependence on $(\bar{y},\sigma,\lambda)$. A guess and verify strategy can be used to show the following scaling property of the function m: Let $k > 0$, then for all $p \in \mathbb{R}^n$ with $||p||^2 \leq \bar{y}$:

$$m(kp;k^2\bar{y},k\sigma,\lambda) = k\,m(p;\bar{y},\sigma,\lambda) \quad \text{and} \quad m(p;\bar{y},\sigma\sqrt{k},\lambda k) = \frac{1}{k}\,m(p;\bar{y},\sigma,\lambda)\,.$$

It is straightforward to verify that this function satisfies the ODE and boundary conditions for $m(p)$ (see e.g. the one in the main text for the $n = 1$ case). Recall the homogeneity of $f(y)$ stated in Lemma 3. Finally, note that the density $g(p)$ can be expressed as a function of the density $f(y)$ given in equation (7) and the density of the sum of n coordinates of a random variable uniformly distributed on a n dimensional hypersphere of square radius y, as obtained in Equation 21 in Alvarez and Lippi (2014). These properties applied to equation (16) establish the scaling property stated in the proposition.

Proof. (of Proposition 6). We first notice that for some special cases a simple analytic proof is available. These cases concern $n = 1$ or $n = \infty$ with $\ell \in (0,1)$; alternatively, they concern $1 < n < \infty$ and $\ell = 0$ or $\ell = 1$. See Appendix G for details.

We now assume $1 \leq n < \infty$ and $0 < \ell < 1$ and prove that $\mathcal{M}'(0) = \frac{Kur(\Delta p_i)}{6\,N(\Delta p_i)}$. The proof is structured as follows. First we derive an analytic expressions for $\frac{Kur(\Delta p_i)}{6\,N(\Delta p_i)}$ and for $\mathcal{M}'(0)$. Each expression is a power series that involves only two parameters: n and ℓ. Verifying the equality is readily done numerically to any arbitrary degree of accuracy. A simple Matlab code for the verification, called `solveMp0.m`, is available on our websites.

We first show that $\frac{Kur(\Delta p_i)}{6\,N(\Delta p_i)}$ can be written as the right hand side of equation (19). This is done in two steps. First notice that $\frac{Kur(\Delta p_i)}{N(\Delta p_i)}$ $=(1/\lambda)\mathcal{L}(\phi,n)\,Kur\,(\Delta p_i)$ where $\mathcal{L}$ is given in Proposition 3. The second step is to derive an analytic expression for the $Kur\,(\Delta p_i)$. We notice that

$$Kur\,(\Delta p_i) = \frac{\mathbb{E}\,(\Delta p_i^4(\tau)\,|\,y(0)=0)}{Var\,(\Delta p_i)^2} = \frac{Q(0)}{\frac{\sigma^4}{N(\Delta p_i)^2}} = \frac{(\lambda/\sigma^2)^2\,Q(0)}{(\mathcal{L}(\phi,n))^2}$$

where τ is the stopping time associated with a price change, and where $Q(y)$ is the expected fourth moment at the time of adjustment τ conditional on a current squared price gap y, i.e.

$$Q(y) = \mathbb{E}\left(\Delta p_i^4(\tau)\,|\,y(0)=y\right) = \frac{3}{(n+2)n}\,\mathbb{E}\left(y^2(\tau)|y(0)=y\right)$$

where $y(\tau)$ is the value of the squared price gap at the stopping time. Notice that for $y \in [0,\bar{y}]$ the function $Q(y)$ obeys the o.d.e.:

$$\lambda Q(y) = \lambda\frac{3y^2}{(n+2)n} + Q'(y)\,n\sigma^2 + Q''(y)\,2\sigma^2 y$$

with boundary condition $Q(\bar{y}) = \frac{3\bar{y}^2}{(n+2)n}$. The solution of Q has a power series representation which is easily obtained by matching coefficients and using the boundary conditions. Using this power series in the expression for $\frac{Kur(\Delta p_i)}{N(\Delta p_i)}$ obtained in the first step gives the expression on the right hand side of equation (19). See the technical Appendix L in Alvarez, Le Bihan, and Lippi (2016) for details on the algebra.

Next we derive an expression for $\mathcal{M}'(0)$, which holds for all $1 \le n < \infty$ and $0 \le \ell \le 1$:

Lemma 4 *Let $\mathcal{M}(\cdot;n)$ be the area under the IRF of output and $f(\cdot;n)$ be the density of the invariant distribution for an economy with n products and parameters $(\bar{y},\lambda,\sigma^2)$. Let $\mathcal{T}_{n+2}(y)$ be the expected time until either $y(t)$ hits $\bar{y}$ or that until there is a free adjustment opportunity, whichever happens first, starting at $y(0)=y$, for an economy with $n+2$ products and the same parameters $(\bar{y},\lambda,\sigma^2)$. Then*

$$\mathcal{M}'(0;n) = \frac{1}{\epsilon}\int_0^{\bar{y}}\left[\mathcal{T}_{n+2}(y) + \frac{2}{n}\,\mathcal{T}'_{n+2}(y)\,y\right] f(y;n)\,dy \quad . \tag{26}$$

The function $\mathcal{T}_n(y)$ is characterized in the proof of Proposition 3 where we give an explicit power series representation for this function. The proof of Lemma 4 uses a characterization of $m(p)$ in terms of a two dimensional vector (z, y), where z is the sum of the n coordinates of p. The function $m(z, y)$ solves a PDE whose solution can be expressed in terms of $\mathcal{T}_n(y)$ (see the technical Appendix M in Alvarez, Le Bihan, and Lippi (2016) for details).

To compute the right hand side of equation (26), we separately characterize $\mathcal{T}_{n+2}(y) + \frac{2}{n}\,\mathcal{T}'_{n+2}(y)\,y$ and $f(y; n)$. Using the power series representation of $\mathcal{T}_{n+2}(y)$ (see proof of Proposition 3) it is immediate to obtain a power series representation of $\mathcal{T}_{n+2}(y) + \frac{2}{n}\,\mathcal{T}'_{n+2}(y)\,y$. This gives (see the technical Appendix N in Alvarez, Le Bihan, and Lippi (2016) for details):

$$\mathcal{T}_{n+2}(y) + \frac{2}{n}\,\mathcal{T}'_{n+2}(y)\,y = \frac{\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{i!\,\Gamma\left(\frac{n}{2}+i+1\right)} \left[\left(\frac{\lambda \bar{y}}{2\,\sigma^2}\right)^i - \left(1+\frac{2i}{n}\right)\,\left(\frac{\lambda y}{2\sigma^2}\right)^i\right]}{\sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{i!\,\Gamma\left(\frac{n}{2}+i+1\right)} \left(\frac{\lambda \bar{y}}{2\,\sigma^2}\right)^i} \tag{27}$$

For f we use the characterization in equation (7) in term of modified Bessel functions of the first and second kind (see the technical Appendix O in Alvarez, Le Bihan, and Lippi (2016) for a step-by-step derivation). These functions have a power series representation, which we use to solve for the two unknown constants C_1, C_2. This gives:

$$f(y) \;=\; \left[\frac{\left(\frac{\lambda y}{2\sigma^2}\right)^{\left(\frac{n}{2}-1\right)}\ \sum_{i=0}^{\infty} \beta_{i,\frac{n}{2}-1}\ \left(\frac{\lambda y}{2\sigma^2}\right)^i}{\left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^{\left(\frac{n}{2}-1\right)}\ \sum_{i=0}^{\infty} \beta_{i,\frac{n}{2}-1}\ \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^i} - \frac{\sum_{i=0}^{\infty} \beta_{i,1-\frac{n}{2}}\ \left(\frac{\lambda y}{2\sigma^2}\right)^i}{\sum_{i=0}^{\infty} \beta_{i,1-\frac{n}{2}}\ \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^i}\right] \Bigg/ \left[\frac{\left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^{\left(\frac{n}{2}-1\right)}\ \sum_{i=0}^{\infty} \beta_{i,\frac{n}{2}-1}\ \frac{\bar{y}}{i+n/2}\ \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^i}{\left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^{\left(\frac{n}{2}-1\right)}\ \sum_{i=0}^{\infty} \beta_{i,\frac{n}{2}-1}\ \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^i} - \frac{\sum_{i=0}^{\infty} \beta_{i,1-\frac{n}{2}}\ \frac{\bar{y}}{i+1}\ \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^i}{\sum_{i=0}^{\infty} \beta_{i,1-\frac{n}{2}}\ \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^i}\right] \tag{28}$$

where the two sequence of coefficients β are defined in term of the Γ function as

$$\beta_{i,\frac{n}{2}-1} \equiv \frac{1}{i!\ \Gamma(i+n/2)} \quad \text{and} \quad \beta_{i,1-\frac{n}{2}} \equiv \frac{1}{i!\ \Gamma(i+2-n/2)} \quad \text{for } i = 0, 1, 2, \ldots \quad .$$

The expression in equation (28) holds for all *real* numbers $n \geq 1$, except when n is an even natural number (due to a singularity of the power expansion of the modified Bessel function of the second kind). Yet the expression is continuous in n.

Finally, we establish an equivalence to verify equation (19):

Lemma 5 *The equality between equation (26) and the ratio $Kur(\Delta p_i)/(6N(\Delta p_i))$, as from* *equation (19), is equivalent to the following equality*

$$\sum_{j=1}^{\infty} \frac{\gamma_j \, \frac{1}{1+j}}{\sum_{s=0}^{\infty} \gamma_s \, \frac{1}{1+s}} \, j \quad = \quad \sum_{j=1}^{\infty} \gamma_j \left(1 + \frac{2\,j}{n}\right) \times \tag{29}$$

$$\left(\left[\frac{\sum_{i=0}^{\infty} \xi_i \, \frac{1}{\frac{n}{2}+i+j}}{\sum_{i=0}^{\infty} \xi_i} - \frac{\sum_{i=0}^{\infty} \rho_i \, \frac{1}{i+1+j}}{\sum_{i=0}^{\infty} \rho_i} \right] \Big/ \left[\frac{\sum_{i=0}^{\infty} \xi_i \, \frac{1}{\frac{n}{2}+i}}{\sum_{i=0}^{\infty} \xi_i} - \frac{\sum_{i=0}^{\infty} \rho_i \, \frac{1}{i+1}}{\sum_{i=0}^{\infty} \rho_i} \right] \right) ,$$

where the sequences $\{\gamma_j, \xi_j, \rho_j\}_{j=0}^{\infty}$ *are defined as*

$$\gamma_j \;\equiv\; \frac{\Gamma\left(\frac{n}{2}+1\right)}{j!\,\Gamma\left(\frac{n}{2}+1+j\right)} \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^{j} \,,\; \xi_j \equiv \frac{1}{j!\,\Gamma\left(j+\frac{n}{2}\right)} \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^{\left(\frac{n}{2}+j-1\right)} \quad \textit{and} \quad \rho_j \equiv \frac{1}{j!\,\Gamma\left(j+2-\frac{n}{2}\right)} \left(\frac{\lambda \bar{y}}{2\sigma^2}\right)^{j} .$$

The derivation of equation (29) uses equation (27) and equation (28) to compute equation (26). Verifying equation (29) is straightforward since both sides are simple functions of convergent power series, which are arbitrarily well approximated by a finite sum. As explained above, for even values of n this expression should be understood as the limit for $n \to 2\,k$ (or, numerically, as the sum for values of n close to $2\,k$ for $k \in \mathbb{N}$ and $k \geq 1$). □

Proof. (of Proposition 7). The idea is to show that for any n and ℓ we have

$$Kur(\Delta p_i;\mu) = Kur(\Delta p_i;-\mu) \;,\; N(\Delta p_i)(\mu) = N(\Delta p_i)(-\mu) \;,\; \mathcal{M}(\delta;\mu) = -\mathcal{M}(-\delta;-\mu) \tag{30}$$

for all (μ, δ) in a neighborhood of $(0, 0)$. Note that differentiating the last expression with respect to δ, and evaluating it at $\delta = 0$ we obtain that $\mathcal{M}'(0;\mu) = \mathcal{M}'(0;-\mu)$. Hence we have that $Kur(\Delta p_i;\cdot), N(\Delta p_i)(\cdot)$ and $\mathcal{M}'(0;\cdot)$ are symmetric functions of inflation around $\mu = 0$. Hence, if they are differentiable, they must have zero derivative with respect to inflation at zero inflation. The symmetry in equation (30) follows from the symmetry on the firm's problem with respect to positive and negative drift. To establish this symmetry we proceed in two steps. First we analyze the symmetry of the decision problem for the firm of Section 3.2. Second, we consider the approximation to the GE problem for values $\mu \neq 0$. All the arguments follow a guess and verify strategy of a simple nature but with heavy notation. The technical Appendix S in Alvarez, Le Bihan, and Lippi (2016) provides the details of the proof.

Proof. (of Proposition 8.) The proof proceeds by verification. We analyze the condition that ensures that every firm with $||p||^2 = y \leq \bar{y}$ before the shock will find that $||p - \iota\delta||^2 \geq \bar{y}$ after the shock, where ι is a vector of ones. See the technical Appendix Q in Alvarez, Le Bihan, and Lippi (2016) for a detailed derivation.

Proof. (of Lemma 2) To prove the lemma we let $z(t) = x^4(t)$ and $y(t) = x^2(t)$. We use Ito's lemma to obtain:

$$dy(t) = \sigma(t)^2\,dt \;+\; 2x(t)\,\sigma(t)dW(t) \qquad \text{or} \quad y(t) = \int_0^t \sigma(s)^2\,ds \;+\; \int_0^t 2\,x(s)\,\sigma(s)\,dW(s)$$

and taking expected values: $\mathbb{E}_0\left[y(T)\right] = \mathbb{E}_0\left[\int_0^T \sigma(t)^2\,dt\right] = \int_0^T \mathbb{E}_0\left[\sigma(t)^2\right]dt$. Likewise

$$dz(t) = 6\,x^2(t)\,\sigma(t)^2\,dt \;+\; 4x^3(t)\,\sigma(t)dW(t)$$

then $\mathbb{E}_0\left[z(T)\right] = 6\,\mathbb{E}_0\left[\int_0^T \sigma(t)^2\,y(t)\,dt\right] = 6\,\int_0^T \mathbb{E}_0\left[\sigma(t)^2\,y(t)\right]dt$. Now note that:

$$\mathbb{E}_0\left[\sigma(t)^2\,y(t)\,dt\right] = \mathbb{E}_0\left[\sigma(t)^2\,\left(\int_0^t \sigma(s)^2\,ds \;+\; \int_0^t 2\,x(s)\,\sigma(s)\,dW(s)\right)\right]$$

and using the independence of $\{W(t)\}$ and $\{\sigma(t)\}$ we have: $\mathbb{E}_0\left[\sigma(t)^2\,y(t)\right] = \mathbb{E}_0\left[\int_0^t \sigma(t)^2\,\sigma(s)^2\,ds\right]$. Then, replacing this expression and noticing that $K(T) = \mathbb{E}_0\left[z(T)\right] / \left(\mathbb{E}_0\left[y(T)\right]\right)^2$, we obtain the desired result. □

Proof. (of Proposition 9.) Define the probability that $u(t) = 1$ if $u(0) = i$ as $P_1\left(t\,|i\,\right)$, or: $P_1\left(t\,|\,i\,\right) \equiv \Pr\left\{u(t) = 1\,|\;u(0) = i\right\}$ for $i \in \{0,1\}$. These probabilities are given by:

$$P_1\left(t\,|\,0\,\right) = \frac{\theta_0}{\theta_0+\theta_1}\left[1 - e^{-(\theta_0+\theta_1)t}\right] \quad , \quad P_1\left(t\,|\,1\,\right) = \frac{\theta_0}{\theta_0+\theta_1}\left[1 + \frac{\theta_1}{\theta_0}e^{-(\theta_0+\theta_1)t}\right] .$$

We now use these probabilities to compute two expressions that appear in equation (21). (see the technical Appendix T in Alvarez, Le Bihan, and Lippi (2016) for the details on the derivation). The first expression is the expected second moment $v(t) \equiv \mathbb{E}_0\left[\sigma(t)^2\right]$ which is given by

$$v(t) = \mathbb{E}_0\left[\sigma(t)^2\right] = \sigma_0^2\frac{\theta_1}{\theta_1+\theta_0} \;+\sigma_1^2\frac{\theta_0}{\theta_0+\theta_1} \quad . \tag{31}$$

Notice that this expected variance is independent of the horizon t. The second expression is $k(t,s) \equiv \mathbb{E}_0\left[\,\sigma(t)^2\,\sigma(s)^2\,\right]$, or the expected fourth moment over an horizon t conditional on $\sigma(t)$. This is given by

$$k(t,s) = \left[\sigma_0^2\frac{\theta_1}{\theta_1+\theta_0} + \sigma_1^2\frac{\theta_0}{\theta_1+\theta_0}\right]^2 + \left(\sigma_1^2 - \sigma_0^2\right)^2\,\frac{\theta_0\theta_1}{(\theta_0+\theta_1)^2}\,e^{-(\theta_0+\theta_1)(t-s)} \tag{32}$$

Using equation (31) and equation (32) into equation (21) gives

$$K(T) = 3 + 6\,\frac{\left(\sigma_1^2 - \sigma_0^2\right)^2 \frac{\theta_0\theta_1}{(\theta_0+\theta_1)^2}}{\left[\sigma_0^2 \frac{\theta_1}{\theta_1+\theta_0} + \sigma_1^2 \frac{\theta_0}{\theta_1+\theta_0}\right]^2}\,\frac{\left[T\left(\theta_0+\theta_1\right) - 1 + e^{-(\theta_1+\theta_0)T}\right]}{(\theta_1+\theta_0)^2 T^2} \quad .$$

Without loss of generality, since the expression is homogeneous of degree zero on (σ_1, σ_0), we can set $\sigma_1 = 1$. We can also use $\theta = (1/2)(\theta_1 + \theta_0)$. Finally for any θ we can let $s = \theta_0/(\theta_0+\theta_1)$ to obtain equation (22). □

Proof. (of Proposition 10.) First we compute the frequency of adjustment. Let $T_i(p)$ denote the expected time to hit a barrier conditional on the state p. The Kolmogorov backward equation gives the following system of ODEs for the expected times:

$$\begin{cases} \theta_0(T_0 - T_1) = 1 + T_0''\,\frac{\sigma_0^2}{2} \\ \theta_1(T_1 - T_0) = 1 + T_1''\,\frac{\sigma_1^2}{2} \end{cases}$$

which is symmetric $T_i(p) = T_i(-p)$ with boundary condition $T_i(\bar{p}) = 0$. The solution is

$$\begin{cases} T_0(p) = \frac{(\theta_0+\theta_1)(\bar{p}^2-p^2)}{\sigma_0^2\theta_1+\sigma_1^2\theta_0} + \frac{\sigma_1^2\theta_0(\sigma_0^2-\sigma_1^2)}{(\sigma_0^2\theta_1+\sigma_1^2\theta_0)^2}\,\left(\frac{e^{\chi p}+e^{-\chi p}}{e^{\chi\bar{p}}+e^{-\chi\bar{p}}} - 1\right) \\ T_1(p) = \frac{(\theta_0+\theta_1)(\bar{p}^2-p^2)}{\sigma_0^2\theta_1+\sigma_1^2\theta_0} + \frac{\sigma_0^2\theta_1(\sigma_1^2-\sigma_0^2)}{(\sigma_0^2\theta_1+\sigma_1^2\theta_0)^2}\,\left(\frac{e^{\chi p}+e^{-\chi p}}{e^{\chi\bar{p}}+e^{-\chi\bar{p}}} - 1\right) \qquad \text{where} \quad \chi \equiv \sqrt{2\frac{\sigma_0^2\theta_1+\sigma_1^2\theta_0}{\sigma_0^2\sigma_1^2}} \end{cases}$$

This implies that the average time between price adjustment is given by

$$\frac{1}{N_a} = \frac{T_0(0)\theta_1 + T_1(0)\theta_0}{\theta_0+\theta_1} = \frac{(\theta_0+\theta_1)\bar{p}^2}{\sigma_0^2\theta_1+\sigma_1^2\theta_0} + \frac{\theta_0\theta_1(\sigma_1^2-\sigma_0^2)^2}{(\theta_0+\theta_1)(\sigma_0^2\theta_1+\sigma_1^2\theta_0)^2}\,\left(1 - \frac{2}{e^{\chi\bar{p}}+e^{-\chi\bar{p}}}\right)$$

or, rewriting in terms of the fundamental parameters that pin down $K(T)$, namely θ, s, ξ, and the implied parameter $\hat{\rho} = \frac{\theta_1}{\theta_0} = \frac{1-s}{s}$ we have equation (23).

Now we turn to computing the cumulative output effect. Use the approximation

$$\mathcal{M}(\delta) \quad \approx \quad \delta\,\mathcal{M}'(0) = \frac{2\delta}{\epsilon}\int_0^{\bar{p}} \left(m_0(p)g_0'(p) + m_1(p)g_1'(p)\right)\,dp$$

Next we solve for the terms in the equation. First consider the ODE that characterizes $m_i(p)$:

$$\begin{cases} \theta_0(m_0 - m_1) = -p + \frac{\sigma_0^2}{2}m_0'' \\ \theta_1(m_1 - m_0) = -p + \frac{\sigma_1^2}{2}m_1'' \end{cases}$$

The function must satisfy $m_i(p) = -m_i(-p)$ and the boundaries $m_i(\bar{p}) = 0$. The solution is

$$\begin{cases} m_0(p) = \frac{(\theta_0+\theta_1)\, p\left(p^2-\bar{p}^2\right)}{3(\sigma_0^2\theta_1+\sigma_1^2\theta_0)} + \frac{\sigma_1^2\theta_0(\sigma_1^2-\sigma_0^2)}{(\sigma_0^2\theta_1+\sigma_1^2\theta_0)^2} \left(\frac{e^{\chi p}-e^{-\chi p}}{e^{\chi \bar{p}}-e^{-\chi \bar{p}}}\,\bar{p} - p\right) \\ m_1(p) = \frac{(\theta_0+\theta_1)\, p\left(p^2-\bar{p}^2\right)}{3(\sigma_0^2\theta_1+\sigma_1^2\theta_0)} + \frac{\sigma_0^2\theta_1(\sigma_0^2-\sigma_1^2)}{(\sigma_0^2\theta_1+\sigma_1^2\theta_0)^2} \left(\frac{e^{\chi p}-e^{-\chi p}}{e^{\chi \bar{p}}-e^{-\chi \bar{p}}}\,\bar{p} - p\right) \end{cases} \quad \text{where} \quad \chi \equiv \sqrt{2\frac{\sigma_0^2\theta_1+\sigma_1^2\theta_0}{\sigma_0^2\sigma_1^2}}$$

Finally we compute the invariant distribution of price gaps. Let $g_i(p)$ be the density for price gaps in state i which must be symmetric around $p = 0$, zero at the boundary: $g_i(\bar{p}) = 0$.

$$\begin{cases} \frac{\sigma_0^2}{2} g_0''(p) = \theta_1 g_1(p) - \theta_0 g_0(p) \\ \frac{\sigma_1^2}{2} g_1''(p) = \theta_0 g_0(p) - \theta_1 g_1(p) \end{cases}$$

For $p \in [-\bar{p}, \bar{p}]$, the shape of the densities is linear triangular, with density functions

$$\begin{cases} g_0(p) = \frac{\theta_1}{\theta_0+\theta_1} \frac{\bar{p}-|p|}{\bar{p}^2} \\ g_1(p) = \frac{\theta_0}{\theta_0+\theta_1} \frac{\bar{p}-|p|}{\bar{p}^2} \end{cases}$$

□

C On the implied cost of price adjustment

In this section we give a characterization of the model implications for the size of the menu cost, i.e. a mapping between observable statistics and the value of ψ/B or ψ (we also discuss how to measure B). We consider two measures for the cost of price adjustment: the first one is the cost of a single price adjustment as a fraction of profits: ψ/n. Recall that ψ is the cost that a firm must pay if it decides to adjust all prices instantaneously (i.e. without waiting for a free adjustment). Measuring this cost as a fraction of profits transforms these magnitudes into units that have an intuitive interpretation. The second measure is the average flow cost of price adjustment given by: $N(\Delta p_i)\, \frac{\psi}{n}\, (1-\ell)$. This cost measures the average amount of resources that the firm pays to adjust prices per period. The latter measure is useful because it relates more directly to what has been measured in the data by Levy et al. (1997); Zbaracki et al. (2004), namely the "average" cost of a price adjustment. The next proposition analyzes the mapping between the scaled menu cost ψ/n , and B, ℓ, n, $N(\Delta p_i)$ and $Var(\Delta p_i)$.

Proposition 11 *Fix the number of products $n \geq 1$ and let $r \downarrow 0$. There is a unique triplet $(\sigma^2, \lambda, \psi)$ consistent with any triplet $\ell \in [0,1], Var(\Delta p_i) > 0$ and $N(\Delta p_i) > 0$. Moreover,* *fixing any value ℓ, the menu cost $\psi \geq 0$ can be written as:*

$$\frac{\psi}{n} = B \; \frac{Var(\Delta p_i)}{N(\Delta p_i)} \; \Psi\,(n\,,\,\ell) \tag{33}$$

where Ψ is only a function of (n, ℓ). For all $n \geq 1$ the function $\Psi(n, \cdot)$ satisfies:

$$\lim_{\ell \to 0} \Psi\,(n,\,\ell) = \frac{n}{2\,(n+2)}\;,\quad \lim_{\ell \to 1} \Psi\,(n,\,\ell) = \infty\;,\quad \lim_{\ell \to 1} \Psi\,(n,\,\ell)\,(1-\ell) = 0\;, \tag{34}$$

$$\lim_{\ell \to 1} \frac{\Psi(n',\ell)/n'}{\Psi(n,\ell)/n} \leq 1 \;\; \textit{for } n' \geq n, \;\; \textit{and} \;\; \lim_{n \to \infty} \frac{\Psi(n,\ell)/n}{\Psi(1,\ell)/1} \to 0 \;\; \textit{as } \ell \to 1\,. \tag{35}$$

Equation (33) shows that for any fixed $n \geq 1$ and $\ell \in [0,1]$ the menu cost ψ is proportional to the ratio $Var(\Delta p_i)/N(\Delta p_i)$. Second, equation (33) shows that the menu cost is proportional to B, which measures the benefits of closing a price gap. The parameter B is related to the constant demand elasticity faced by firms η (see Section 3), so that $B = \eta(\eta-1)/2$, which can be written in terms of the (net) markup over marginal costs $\mathfrak{m} \equiv 1/(\eta-1)$ so that $B = (1+\mathfrak{m})/(2\mathfrak{m}^2)$.[26] The last expression is useful to calibrate the model using empirical estimates of the markup such as the ones by Christopoulou and Vermeulen (2012): the estimated markups average around 28% for the US manufacturing sector, and around 36% for market services (slightly smaller values are obtained for France, see their Table 1).[27] A similar value for the US, namely a markup rate of about 33%, is used by Nakamura and Steinsson (2010).

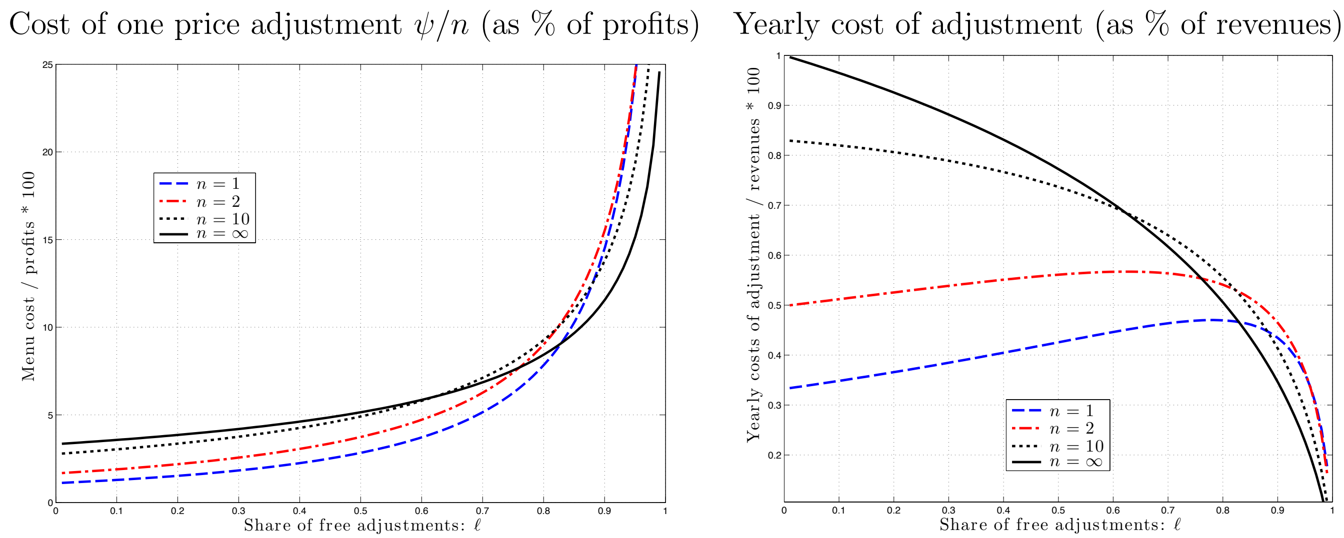
The left panel of Figure 6 illustrates the comparative static effect of ℓ and n on the implied menu cost, fixing $B\,Var(\Delta p_i)/\,N(\Delta p_i)$, i.e. it plots the function $\Psi(n,\ell)$. Fixing a value of n it can be seen that the menu cost ψ/n is increasing in ℓ. Indeed equation (34) shows that as $\ell \to 1$, the implied menu cost diverges to $+\infty$. On the other hand, for $\ell = 0$ and $n = 1$, our version of *Golosov-Lucas*'s model, the menu cost attains its smallest (strictly positive) value. Fixing ℓ and moving across lines shows that the implied fixed cost ψ/n is not monotone in the number of products n. Indeed, as stated in equation (34) for a very small share ℓ the values of ψ/n are increasing in n. On the other hand, for larger value of the share ℓ, the order of the implied fixed cost is reversed.

The model also has clear predictions about the per period (say yearly) cost of price adjustments borne by the firms: $(1-\ell)\,N(\Delta p_i)\,\psi/n$. In spite of the fact that the cost of a single deliberate price adjustment diverges as $\ell \to 1$, the total yearly cost of adjustment converge to

[26]Nakamura and Steinsson (2010) notice that lower markups (higher values of demand elasticity) η must imply higher menu costs, as shown by equation (33). Footnote 14 in their paper discusses evidence on the markup rates across several microeconomic studies and macro papers.

[27] The evidence for the US services is consistent with the gross margins, based on accounting data, reported in the Annual Retail Trade Survey by the US Census (see http://www.census.gov/retail/).

Figure 6: Implied cost of price adjustment



All economies in the figures feature $Std(\Delta p_i) = 0.10$ and a markup of 25%. For those in the left panel we set $N(\Delta p_i) = 1.5$.

zero continuously. This can be seen in the right panel of Figure 6. A simple transformation gives the yearly cost of price adjustments as a fraction of revenues: $\frac{(1-\ell)N(\Delta p_i)\ \psi/n}{\eta}$, where the scaling by η transforms the units from fraction of profits into fraction of revenues.[28] This statistic is useful because it has empirical counterparts, studied e.g. by Levy et al. (1997). Using equation (33) and the previous definition for the markup yields

$$\frac{\text{Yearly costs of price adjustment}}{\text{Yearly revenues}} = \frac{1}{2}\,\frac{Var(\Delta p_i)}{\mathfrak{m}}\,(1-\ell)\,\Psi\,(n\,,\,\ell) \tag{36}$$

Figure 6 plots the two cost measures in equation (33) and (36) as functions of ℓ, n for an economy with $N(\Delta p_i) = 1.5$, $Std(\Delta p_i) = 0.10$ and a markup $\mathfrak{m} \approx 25\%$ (i.e. $B = 10$). We see this parametrization as being consistent with the US data on price adjustments, markups, and the size distribution of price changes discussed above. The figure illustrates how observations on the costs of price adjustments can be used to parametrize the model. Levy et al. (1997) and Dutta et al. (1999) (Table IV and Table 3, respectively) document that for multi-product stores (a handful of supermarket chains and one drugstore chain) the average cost of price adjustment is around 0.7 percent of revenues. For an economy with $n = 10$ (a reasonable parametrization to fit the size-distribution of price changes) the right panel of the figure shows that the model reproduces the yearly cost of 0.7% of revenues when

[28] Since $R = \eta\,\Pi$ where R is revenues per good and Π profits per good.

the fraction of free adjustments ℓ is around 60%. The left panel in the figure indicates that at this level of ℓ the cost of one price adjustment is around 5% of profits.

Proof. (of Proposition 11). To obtain the expression in equation (33) we use the characterization of $\ell = \mathcal{L}\left(\frac{\lambda \bar{y}}{n\sigma^2}, n\right)$ of Proposition 3, it is equivalent to fix a value of $\phi \equiv \frac{\lambda \bar{y}}{n\sigma^2}$. We let the optimal decision rule be $\bar{y}\,(\psi/B, \sigma^2, r+\lambda, n)$ so that we have:

$$\bar{y}\left(\frac{\psi}{B}, \sigma^2, r+\lambda, n\right)\ \frac{\lambda}{n\,\sigma^2} = \phi$$

To be consistent with $Var(\Delta p_i)$ and $N(\Delta p_i)$ we have, using Proposition 1 and $\ell = \mathcal{L}(\phi, n)$:

$$N(\Delta p_i) = \lambda/\mathcal{L}(\phi, n) \ \text{ and } \ \frac{\lambda}{\sigma^2} = \mathcal{L}(\phi, n)/Var(\Delta p_i)\ .$$

Thus, after taking $r \downarrow 0$ and using the expression above we can write:

$$\bar{y}\left(\frac{\psi}{B}\,,\ N(\Delta p_i)Var(\Delta p_i)\,,\ \ell\, N(\Delta p_i)\,,\ n\right)\ \frac{\ell}{n\,Var(\Delta p_i)} = \mathcal{L}^{-1}(\ell; n)$$

Fixing n and ℓ and computing the total differential for this expression with respect to $(\psi/B, N(\Delta p_i), Var(\Delta p_i))$, and denoting by η_ψ, η_{σ^2}, η_λ the elasticities of $\bar{y}$ with respect to ψ/B, σ^2, λ we have:

$$\eta_\psi\,\hat{\psi} + \eta_{\sigma^2}\,(\hat{N}(\Delta p_i) + \hat{V}ar(\Delta p_i)) + \eta_\lambda\,\hat{N}(\Delta p_i) = \hat{V}ar(\Delta p_i)$$

where a *hat* denotes a proportional change. Using Proposition 3-(iv) in Alvarez and Lippi (2014) and Lemma 1 we have that these elasticities are related by: $\eta_\lambda = 2\eta_\psi - 1$ and $\eta_{\sigma^2} = 1-\eta_\psi$ .Thus $\eta_\psi\,\hat{\psi} + (1-\eta_\psi)\,(\hat{N}(\Delta p_i) + \hat{V}ar(\Delta p_i)) + (2\eta_\psi - 1)\,\hat{N}(\Delta p_i) = \hat{V}ar(\Delta p_i)$. Rearranging and canceling terms: $\eta_\psi\,\hat{\psi} + \eta_\psi\,\hat{N}(\Delta p_i) - \eta_\psi \hat{V}ar(\Delta p_i) = 0$. Dividing by η_ψ we obtain that $\hat{\psi} = \hat{V}ar(\Delta p_i) - \hat{N}(\Delta p_i)$. Additionally, since $\bar{y}$ is a function of ψ/B, then we can write $\psi/n = B\,(Var(\Delta p_i)/N(\Delta p_i))\ \Psi(n, \ell)$ for some function $\Psi(n, \ell)$.

That $\psi \to \infty$ as $\ell \to 1$ follows because $\mathcal{L}(\phi, n) \to 1$ as $\phi \to \infty$ and because, by Proposition 3-(i) in Alvarez and Lippi (2014), $\bar{y}$ is increasing in ψ and has range and domain $[0, \infty)$. For $\lambda = 0$ and $N(\Delta p_i) > 0$ we obtain: $\frac{\psi}{n} = B\ \frac{V(\Delta p)}{N(\Delta p_i)}\ \frac{n}{2\,(n+2)}$ .This follows from using the square root approximation of $\bar{y}$ for small $\psi\,(\lambda + r)^2$, the expression for $N(\Delta p_i) = n\sigma^2/\bar{y}$ and Proposition 1, i.e. $N(\Delta p_i)\,Var(\Delta p_i) = \sigma^2$. To obtain the expression for $\Psi(n, 0)$ we use Proposition 6 in Alvarez and Lippi (2014) where it is shown that for $\lambda = 0$ then $Kur\,(\Delta p_i) = 3n/(n+2)$.

D The CPI response to a monetary shock

To compute the IRF of the aggregate price level we analyze the contribution to the aggregate price level by each firm. Firms start with price gaps distributed according to g, the invariant distribution. Then the monetary shock displaces them, by subtracting the monetary shock δ to each of them. After that we divide the firms in two groups. Those that adjust immediately and those that adjust at some future time. Note that, for each firm in the cross section, it suffices to keep track only of the contribution to the aggregate price level of the first adjustment after the shock because the future contributions are all equal to zero in expected value.

Let $g\left(p; n, \lambda/\sigma^2, \bar{y}\right)$ be the density of firms with price gap vector $p = (p_1, ..., p_n)$ at time $t = 0$, just before the monetary shock, which corresponds to the invariant distribution with constant money supply. The density g equals the density f of the steady state square norms of the price gaps given by Lemma 3 evaluated at $y = p_1^2 + \cdots + p_n^2$ times a correction factor:[29]

$$g\left(p_1, ..., p_n\,; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) = f\left(p_1^2 + \cdots + p_n^2\,; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \frac{\Gamma\left(n/2\right)}{\pi^{n/2}\ \left(p_1^2 + \cdots + p_n^2\right)^{(n-2)/2}} \tag{37}$$

To define the impulse response we introduce two extra pieces of notation. First we let $\{(\bar{p}_1(t,p), \ldots, \bar{p}_n(t,p))\}$ the process for n independent BM, each one with variance per unit of time equal to σ^2, which at time $t = 0$ start at p, so $\bar{p}_i(0,p) = p_i$. We also define the stopping time $\tau(p)$, also indexed by the initial value of the price gaps p as the minimum of two stopping times, τ_1 and $\tau_2(p)$. The stopping time τ_1 denotes the first time since $t = 0$ that jump occurs for a Poisson process with arrival rate λ per unit of time. The stopping time $\tau_2(p)$ denotes the first time that $||\bar{p}(t,p)||^2 > \bar{y}$. Thus $\tau(p)$ is the first time a price change occurs for a firm that starts with price gap p at time zero. The stopped process $\bar{p}(\tau(0), p)$ is the vector of price gaps at the time of price change for such a firm.

The impulse response for the aggregate price level can be written as:

$$\mathcal{P}(t, \delta; \sigma, \lambda, \bar{y}) \;\;=\;\; \Theta(\delta; \sigma, \lambda, \bar{y}) \,+\, \int_0^t \theta(\delta, s; \sigma, \lambda, \bar{y})\, ds\,, \tag{38}$$

where $\Theta(\delta)$ gives the impact effect, the contribution of the monetary shock δ to the aggregate price level on impact, i.e. at the time of the monetary shock. The integral of the θ's gives the remaining effect of the monetary shock in the aggregate price level up to time t, i.e. $\theta(\delta, s)ds$ is the contribution to the increase in the average price level in the interval of times $(s, s + ds)$

[29] See Section 5 of Alvarez and Lippi (2014) for this result and the technical Appendix P in Alvarez, Le Bihan, and Lippi (2016) for a derivation.

from a monetary shock of size δ. Figure 3 displays several examples of impulse responses (the figures plots output, i.e. $(\delta - \mathcal{P})/\epsilon$). The functions θ and Θ are readily defined in terms of the density g, the process $\{\bar{p}\}$ and the stopping times τ:

$$\Theta(\delta; \sigma, \lambda, \bar{y}) \equiv \int_{||p(0)-\iota\delta|| \geq \bar{y}} \left(\delta - \frac{\sum_{j=0}^{n} p_j(0)}{n} \right) g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y} \right) \, dp_1(0) \cdots dp_n(0)$$

and $\theta(\delta, t; \sigma, \lambda, \bar{y})$ is the density, i.e. the derivative with respect to t of the following expression:

$$\int_{||p(0)-\iota\delta|| < \bar{y}} \mathbb{E}\left[-\frac{\sum_{j=0}^{n} \bar{p}_j\left(\tau(p), p\right)}{n} \mathbf{1}_{\{\tau(p) \leq t\}} \,\Big|\, p = p(0) - \iota\delta \right] g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y} \right) dp_1(0) \cdots dp_n(0)$$

where ι is a vector of n ones. This expression takes each firm that has not adjusted prices on impact, i.e. those with $p(0)$ satisfying $||p(0) - \iota\delta|| < \bar{y}$, weights them by the relevant density g, displaces the initial price gaps by the monetary shock, i.e. sets $p = p(0) - \iota\delta$, and then looks a the (negative) of the average price gap at the time of the first price adjustment, $\tau(p)$, provided that the price adjustment has happened before or at time t. We make 3 remarks about this expression. First, price changes equal the negative of the price gaps because price gaps are defined as prices minus the ideal price. Second, we define θ as a density because, strictly speaking, there is no effect on the price level due to price changes at *exactly* time t, since in continuous time there is a zero mass of firms adjusting at any given time. Third, we can disregard the effect of any subsequent adjustment because each of them has an expected zero contribution to the average price level. Fourth, the impulse response is based on the steady-state decision rules, i.e. adjusting only when $y \geq \bar{y}$ even after an aggregate shock occurs.

Given the results in Proposition 3 -Proposition 4 we can parametrize our model either in terms of $(n, \lambda, \sigma^2, \psi/B)$ or instead parametrize it, for each n, in terms of the implied observable statistics $(N(\Delta p_i), Std(\Delta p_i), \ell)$. These propositions show that this mapping is indeed one-to-one and onto. We refer to ℓ as an "observable" statistic, because we have shown that the "shape" of the distribution of price changes depends only on it.

Proposition 12 *Consider an economy whose firms produce n products and with steady state statistics $(N(\Delta p_i), Std(\Delta p_i), \ell)$. The cumulative proportional response of the aggregate price level $t \geq 0$ periods after a once and for all proportional monetary shock of size δ can be obtained from the one of an economy with one price change per period and with unitary standard deviation of price changes as follows:*

$$\mathcal{P}\left(t\,,\,\delta\,;\,N(\Delta p_i)\,,\,Std(\Delta p_i)\right) = Std(\Delta p_i)\;\mathcal{P}\left(t\,N(\Delta p_i)\,,\,\frac{\delta}{Std(\Delta p_i)}\,;\,1\,,\,1 \right)\;. \qquad (39)$$

This proposition extends the result of Proposition 8 in Alvarez and Lippi (2014) to the case of $\ell \equiv \lambda/N(\Delta p_i) > 0$.[30] The proof proceeds by verification. It is made of three parts. First we introduce a discrete-time, discrete-state version of the model. Second we show the scaling of time with respect to N_a , and finally the homogeneity of degree one with respect to $Std(\Delta p_i)$ and δ. The step by step passages of the proof are reported in the technical Appendix P in Alvarez, Le Bihan, and Lippi (2016).

The proposition establishes that the shape of the impulse response is completely determined by 2 parameters: n and ℓ, whose comparative static is explored in Figure 3. Economies sharing these parameters but differing in terms of $N(\Delta p_i)$ or $Std(\Delta p_i)$ are immediately analyzed by rescaling the values of the horizontal and/or vertical axis. In particular, a higher frequency of price adjustments will imply that the economy "travels faster" along the impulse response function (this is the sense of the rescaling the horizontal axis). Instead, the effect of a larger dispersion of price changes is seen by rescaling the monetary shock δ by $Std(\Delta p_i)$ and by a proportional scaling of the vertical axis. A further simplification to the last result is given by next corollary, showing that for small values of the monetary shocks one can overlook the scaling by $Std(\Delta p_i)$ so that, for a given n and ℓ determining the shape, the most important parameter is the frequency of price changes $N(\Delta p_i)$:

Corollary 1 *For small monetary shocks $\delta > 0$, the impulse response is independent of $Std(\Delta p_i)$. Differentiating equation (39) gives:*

$$\mathcal{P}\left(t\,,\,\delta\,;\,N(\Delta p_i)\,,\,Std(\Delta p_i)\,\right) = \delta\,\frac{\partial}{\partial\delta}\,\mathcal{P}\left(t\,N(\Delta p_i)\,,\,0\,;\,1\,,\,1\,\right) + o(\delta)$$

for all $t > 0$ and, since $f(\bar{y}) = 0$, then the initial jump in prices can be neglected, i.e.:

$$\mathcal{P}\left(0\,,\,\delta\,;\,N(\Delta p_i)\,,\,Std(\Delta p_i)\,\right) \equiv \Theta_{n,\ell}\left(\delta; Std\left(\Delta p_i\right)\right) = o(\delta)\ .$$

E An economy with heterogenous sectors

Assume that there are S sectors, each with an expenditure weight $e(s) > 0$, and with different parameters so that each has $N(s)$ price changes per unit of time, and a distribution of price changes with kurtosis $Kur(s)$. In this case, after repeating the arguments above for each sector and aggregating, we obtain that the area under the IRF of *aggregate* output for a small

[30] The proof in Alvarez and Lippi is constructive in nature, exploiting results from applied math on the characterization of hitting times for brownian motions in hyper-spheres, which is not longer valid for $\lambda > 0$. Here we use a different strategy which relies on limits of discrete-time, discrete state approximations.

monetary shock δ is

$$\mathcal{M}(\delta) \cong \delta \mathcal{M}'(0) = \frac{\delta}{6\,\epsilon} \sum_{s \in S} \frac{e(s)}{N(s)} Kur(s) = \frac{\delta}{6\,\epsilon}\, D \sum_{s \in S} d(s)\, Kur(s) \tag{40}$$

where D is the expenditure-weighted average duration of prices $D \equiv \sum_{s \in S} \frac{e(s)}{N(s)}$ and the $d(s) \equiv \frac{e(s)}{N(s)D}$ are weights taking into account both relative expenditures and durations. In the case in which all sectors have the same durations then $d(s) = e(s)$ and $\mathcal{M}$ is proportional to the kurtosis of the standardized data. Likewise, the same result applies if all sectors have the same kurtosis.[31] In general, if sectors are heterogenous in the durations (or expenditures), then the kurtosis of the sectors with longer duration (or expenditures) receive a higher weight in the computation of $\mathcal{M}$. For the French data, computation of the duration weighted kurtosis in equation (40) increases the estimated cumulative effect by about 15%, reflecting a correlation between the kurtosis and the duration of price changes.

F Frequency of price changes in Retail vs. Wholesale

In this appendix we document that wholesale prices are as sticky as retail prices for a broad cross section of products sold in grocery stores. For wholesale price we use PromoData, a dataset on manufacturer prices for packaged foods from grocery wholesalers (the largest wholesaler in each location). PromoData provides the price per case charged by the manufacturer to the wholesaler for a UPC in a particular day, for 48 markets, over the period 2006-2012. The data includes information on almost 900 product categories and more than 500,000 UPC×Market products, and contain information on both base prices and "trade deals" (discounts offered to the grocery wholesalers to encourage promotions). We compute the frequency of price changes using base prices (excluding trade deals) as well as including trade deals.[32]

The frequency of price adjustment at the retail level is computed using the IRI Symphony data. The dataset contains weekly scanner price and quantity data covering a panel of stores in 50 metropolitan areas from January 2001 to December 2011, with multiple chains of retailers for each market. The dataset contains around 2.4 billion transactions from over

[31] The effect of heterogeneity in $N(\Delta p_i)$ on aggregation is well known, so that D is different from the average of $N(\Delta p_i)$'s, see for example Carvalho (2006) and Nakamura and Steinsson (2010).

[32] In PromoData firms report only the dates in which their prices change. We thus assume that the price is constant between reporting dates. We discard the last price (uncompleted spell) and consider products with at least two price changes. The frequency of adjustment is computed at the weekly level for comparability with the retail data sets (even though our data may have a higher frequency). The frequency of adjustment is computed for each product (i.e. UPC x Market given that the data is not at the store level) and then aggregated using equal product weights.

Table 3: Weekly Frequency of Price Adjustment - Wholesale vs Retail Level

Data	Period	Frequency excl. Sales	Frequency
All Products			
PromoData (Wholesale)	2006-2012	0.09	0.14
IRI Symphony (Retail)	2001-2011	0.11	0.22
All Products (2006 - 2011)			
PromoData (Wholesale)	2006-2011	0.08	0.14
IRI Symphony (Retail)	2006-2011	0.12	0.23
Coffee			
PromoData (Wholesale)	2006-2012	0.17	0.20
IRI Symphony (Retail)	2001-2011	0.10	0.19
RMS	2006-2012	-	0.16

The table reports the weekly frequency of price adjustment using three datasets: Nielsen's PromoData, IRI Symphony, and Nielsen's Retail Scanner (RMS) data. The frequency of adjustment is computed at the product level and then aggregated across products using equal weights.

170,000 UPCs and around 3,000 stores. Goods are classified into 31 general product categories and a sales flag is provided when an item is on discount (thus we compute the frequency both including and excluding sales as in Section 2.2). To correct for measurement error (due to composition and time aggregation) we only retain price changes within the interval $0.1 \leq |\Delta p_i| \leq 100 \cdot \log(10/3)$. Finally, to compare with and extend Nakamura and Steinsson (2008), we compute the frequency of price changes for coffee using data on retail prices and sales from the Retail Scanner Data (RMS) by Nielsen. Our data is at the week-product-store level for the period of 2006-2013. The structure of the dataset is the same as the IRI Symphony data except that the RMS does not provide a sales flag, and covers about 200 cities.

Table 3 summarizes the main findings of this measurement exercise. The weekly frequency of price adjustment (sales excluded) for the entire wholesale data (PromoData) is 0.09 per week which compares with a mean frequency of adjustment of about 0.11 per week in the retail (IRI) data. Frequencies of comparable magnitude are detected across samples from different segments of the distribution chain, as well as for different items (coffee and beer, not reported) in the samples that exclude sales. Including sales makes the frequency of adjustment in retail somewhat higher than the frequency in wholesale.

G Simple special cases of Proposition 6

This section discusses some limiting cases in which tractable closed form expressions for the cumulative effect $\mathcal{M}$ as well as the frequency and kurtosis of price adjustments can be derived. The first two cases we illustrate assume either $n = 1$ or $n \to \infty$: we derive the implications for the cumulative output effect while considering the full range of values for $\ell \in (0, 1)$ and keeping the frequency of price changes constant. The last case restricts attention to $\ell = 0$ or $\ell = 1$ but allows for any value of $n \geq 1$.

G.1 Analytical computation of $\mathcal{M}$ in the case of $n = 1$

We give an analytical summary expression for the effect of monetary shocks in two interesting cases, those for one product, i.e. $n = 1$, and those for the large number of product, i.e. $n = \infty$. The summary expression is the area under the impulse response for output, i.e. the sum of the output above steady state after a monetary shock of size $\delta > 0$, which we denote as:

$$\mathcal{M}_n(\delta) = (1/\epsilon) \int_0^{\infty} [\delta - \mathcal{P}_n(\delta, t)] \; dt \tag{41}$$

where $1/\epsilon$ is related to the uncompensated labor supply elasticity and $\mathcal{P}_n(\delta, t)$ is the cumulative effect of monetary shock δ in the (log) of the price level after t periods. For large enough shocks, given the fixed cost of changing prices, the model display more price flexibility. Because of their prominence in the literature, and because of realism, we consider the case of small shocks δ by taking the first order approximation to equation (41), so we consider $\mathcal{M}_n(\delta) \approx \mathcal{M}_n'(0)\delta$.

For the case of $n = 1$ we obtain an analytical expression which, after normalizing by $N(\Delta p_i)$ depends only on $\lambda / N(\Delta p_i)$. Thus as $\lambda / N(\Delta p_i)$ ranges from 0 to 1 the model ranges from a version of the menu cost model of *Golosov and Lucas* to a version using *Calvo* pricing. The analytical expression is based upon the following characterization:

$$\mathcal{M}_1(\delta) = (1/\epsilon) \int_{-\bar{p}}^{\bar{p}-\delta} m(p_0) \, g(p_0 + \delta) \, dp_0 \tag{42}$$

where p_0 is the price gap *after* the monetary shocks and where $m(p)$ gives the contribution to the area under the IRF of firms that start with price gap, after the shock, equal to p_0. Since the monetary shock happens when the economy is in steady state, the distribution right after the shock has the steady state density h displaced by δ. Immediately after the shock the firms with the highest price gap have price gap $\bar{p} - \delta$. Note that the integral in equation (42) does not include the firms that adjust on impact, those that before the shock have price gaps in the interval $[-\bar{p}, \bar{p} - \delta)$, whose adjustment does not contribute to the IRF. The definition of m is:

$$m(p) = -\mathbb{E}\left[\int_0^\tau p(t)\,dt \;\middle|\; p(0) = p\right]$$

where τ is the stopping time denoting the first time that the firm adjusts its price. This function gives the integral of the negative of the price gap until the first price adjustment. This expression is based on the fact that those firms with negative price gaps, i.e. low markups, contribute positively to output being in excess of its steady state value, and those with high markups contribute negatively. Given a decision rule summarized by $\bar{p}$ we can characterize m as the solution to the following ODE and boundary conditions:

$$\lambda m(p) = -p + \frac{\sigma^2}{2} m''(p) \;\text{ for all } p \in [-\bar{p}, \bar{p}] \;\text{ and }\; m(p) = 0 \;\text{ otherwise .}$$

The solution for the function m is:

$$m(p) = -\frac{p}{\lambda} + \frac{\bar{p}}{\lambda}\left(\frac{e^{\sqrt{2\phi}\frac{p}{\bar{p}}} - e^{-\sqrt{2\phi}\frac{p}{\bar{p}}}}{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}}}\right) \quad \text{for all } \; p \in [-\bar{p}\,,\, \bar{p}] \;\; .$$

$\phi \equiv \lambda \bar{p}^2/\sigma^2$. We then have:

$$\mathcal{M}(\delta) \approx \mathcal{M}'(0)\delta = (\delta/\epsilon)\int_{\bar{p}}^{\bar{p}} m(p)\, g'(p)\, dp = (\delta/\epsilon)\, 2\int_0^{\bar{p}} m(p)\, g'(p)\, dp$$

since $m(\bar{p})g(\bar{p}) = 0$. The last equality uses that m is negative symmetric, i.e. $m(p) = -m(-p)$, and that g is symmetric around zero. Using the expression for g in Section 3.1

$$g'(p) = -\frac{2\phi}{2\bar{p}^2\left(e^{\sqrt{2\phi}} - 1\right)^2}\left(e^{\sqrt{2\phi}(2-\frac{p}{\bar{p}})} + e^{\sqrt{2\phi}\frac{p}{\bar{p}}}\right) \quad \text{for} \quad p \in [0, \bar{p}] \;.$$

we obtain:

$$\delta\mathcal{M}'(0) \;\; = \;\; \left(\frac{\delta}{\epsilon}\right)\frac{-2}{\lambda\left(e^{\sqrt{2\phi}} - 1\right)^2}\left(e^{\sqrt{2\phi}}\left(1 + \phi - \frac{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}}}{2}\right)\right) \quad .$$

Using the expression for $N(\Delta p_i)$ for the $n = 1$ and simple algebra we can rewrite it as:

$$\delta\mathcal{M}'(0) \;\; = \;\; \left(\frac{\delta}{\epsilon}\right)\frac{1}{N(\Delta p_i)}\frac{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}}}{\left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2\right)^2}\left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2 - 2\phi\right)$$

which yields the cumulative output effect of a small monetary shock of size δ.[33]

Kurtosis. We now verify that the expression can be equivalently obtained by computing the kurtosis, as stated in Proposition 6. For notation convenience let $x \equiv \sqrt{2\phi}$. Using the distribution of price changes derived in Section 3.1 and the definition of kurtosis we get

$$Kur\left(\Delta p_i\right) = \frac{2\ell\left(\frac{12}{x^4} - \frac{12+x^2}{x^2\left(e^{x/2}-e^{-x/2}\right)^2}\right) + 1 - \ell}{\left(2\ell\left(\frac{1}{x^2} + \frac{1}{2-e^{-x}-e^{x}}\right) + 1 - \ell\right)^2} = \frac{12 - \frac{12x^2+x^4}{\left(e^{x/2}-e^{-x/2}\right)^2} + x^4\frac{1-\ell}{2\ell}}{2\ell\left(1 + \frac{x^2}{2-e^{-x}-e^{x}} + x^2\frac{1-\ell}{2\ell}\right)^2}$$

Recall from Section 3.1 that $\ell = \frac{e^x+e^{-x}-2}{e^x+e^{-x}}$ so that , after some algebra

$$Kur\left(\Delta p_i\right) = 6 \, \frac{e^x + e^{-x}}{\left(e^x + e^{-x} - 2\right)^2} \left(e^x + e^{-x} - 2 - x^2\right)$$

It is immediate that the kurtosis and the cumulative output effect satisfy Proposition 6.

G.2 Analytical computation of $\mathcal{M}$ in the case of $n = \infty$

Define

$$Y_n(t,\delta) \equiv \frac{1}{n}\sum_{i=1}^{n}[p_i(t) - \delta] = Y_n(t,0) - 2\delta\frac{\sum_{i=1}^{n} p_i(t)}{n} + \delta^2 .$$

where the $p_i(t)$ are independent of each other, start at $p_i(0) = 0$ and have normal distribution with $\mathbb{E}\left[p_i(t)\right] = 0$ and $Var\left[p_i(t)\right] = \sigma^2 t$. Then, by an application of the law of large numbers, we have:

$$Y_\infty(t,\delta) = Y_\infty(t,0) + \delta^2 = t\sigma^2 + \delta^2$$

Letting $\bar{Y} \equiv \lim_{n\to\infty} \bar{y}(n)/n$ we can represent the steady state optimal decision rule as adjusting prices when t, the time elapsed since last adjustment, attains $T = \bar{Y}/\sigma^2$. We compute the density of the distribution of products indexed by the time elapsed since the last adjustment t and, abusing notation, we denote it by f. This distribution is a truncated

[33] As a check of this formula compute the case for $\phi = 0$, i.e. the cumulative output for the Golosov-Lucas model. In this case we let $\lambda = 0$ and $\bar{p} > 0$. In this case we have: $m(p) = -\frac{\bar{p}^2 p}{3\sigma^2} + \frac{p^3}{3\sigma^2}$. Also $g'(p) = -1/\bar{p}^2$ for $p \in (0, \bar{p}]$, so we have:

$$\mathcal{M}'(0)\delta = \left(\frac{\delta}{\epsilon}\right)\frac{2}{-3\sigma^2\bar{p}^2}\int_0^{\bar{p}}\left[-\bar{p}^2 p + p^3\right]dp = \left(\frac{\delta}{\epsilon}\right)\frac{-2}{3\sigma^2\bar{p}^2}\left[-\frac{\bar{p}^4}{2} + \frac{\bar{p}^4}{4}\right] = \left(\frac{\delta}{\epsilon}\right)\frac{2\bar{p}^2}{3\sigma^2}\frac{2}{8} = \left(\frac{\delta}{\epsilon}\right)\frac{1}{N(\Delta p_i)}\frac{1}{6}$$

which is the same value obtained by taking the limit for $\phi \to 0$ in the general expression above.

exponential with decay rate λ and with truncation T, thus the density is:

$$f(t) = \lambda \frac{e^{-\lambda t}}{1 - e^{-\lambda T}} \text{ for all } t \in [0, T] \ .$$

The (expected) number of price changes per unit of time is given by the sum of the free adjustments and the ones that reach T, so

$$N(\Delta p_i) = \lambda + f(T) = \lambda \left[1 + \frac{e^{-\lambda T}}{1 - e^{-\lambda T}} \right] = \frac{\lambda}{1 - e^{-\lambda T}}$$

Note that, using the definition of T given above, $\lambda T = \bar{Y} \lambda / \sigma^2$ the parameter which indexes the shape of f and of the distribution of price changes. Since this figures prominently in this expressions we define:

$$\phi \equiv \lambda T = \frac{\bar{Y} \lambda}{\sigma^2} \ .$$

which is consistent with the definition of ϕ in Proposition 3. Using this definition we get:

$$\ell = \frac{\lambda}{N(\Delta p_i)} = 1 - e^{-\phi} \text{ and thus } N(\Delta p_i) = \frac{\lambda}{1 - e^{-\phi}}$$

Impulse Response of Prices to a monetary Shock. We can now define the impulse response. Note that after the monetary shock firms that have adjusted their prices t periods ago, in average will adjust their price up by δ. This highlights that as $n \to \infty$ there is no selection.

Now we turn to the characterization of the impact effect Θ. In this case we have

$$Y_\infty(t, \delta) = Y_\infty(t, 0) + \delta^2 = t\sigma^2 + \delta^2 \geq \bar{Y} = \sigma^2 T \iff t \geq T - \delta^2/\sigma^2 \ .$$

Thus the impact effect is:

$$\Theta(\delta) = \delta \int_{T-\delta^2/\sigma^2}^{T} f(t) dt = \delta \, \frac{e^{-\lambda T + \frac{\lambda}{\sigma^2}\delta^2} - e^{-\lambda T}}{1 - e^{-\kappa}} = \delta \, \frac{e^{-\kappa + \frac{\lambda}{\sigma^2}\delta^2} - e^{-\kappa}}{1 - e^{-\kappa}}$$

Using that $N(\Delta p_i) Var(\Delta p_i) = \sigma^2$ we can write:

$$\Theta(\delta) = \delta + \delta \, \frac{e^{-\kappa + \frac{\lambda}{N(\Delta p_i)} \frac{\delta^2}{Var(\Delta p_i)}} - 1}{1 - e^{-\kappa}} = \delta + \delta \, \frac{\left(1 - \frac{\lambda}{N(\Delta p_i)}\right) e^{\frac{\lambda}{N(\Delta p_i)} \frac{\delta^2}{Var(\Delta p_i)}} - 1}{\lambda / N(\Delta p_i)}$$

Note that

$$\lim \Theta(\delta) = \begin{cases} \delta \left(\frac{\delta}{Std(\Delta p_i)}\right)^2 & \text{as } \lambda/N(\Delta p_i) \to 0 \\ 0 & \text{as } \lambda/N(\Delta p_i) \to 1 \end{cases}$$

and in general

$$\frac{\Theta(\delta)}{\partial(\lambda/N(\Delta p_i))} = \delta \, \frac{e^{\frac{\lambda}{N(\Delta p_i)} \frac{\delta^2}{Var(\Delta p_i)}} \left(\frac{\delta^2}{Var(\Delta p_i)} \frac{\lambda}{N(\Delta p_i)} \left(1 - \frac{\lambda}{N(\Delta p_i)}\right) - 1 \right) + 1}{(\lambda/N(\Delta p_i))^2} < 0$$

whenever $\delta < 2\, Std(\Delta p_i)$.

$$\theta(t) \;=\; \delta e^{-\lambda t} \left[f\left(T - \delta^2/\sigma^2 - t\right) + \lambda \int_0^{T-\delta^2/\sigma^2-t} f(s) ds \right] = \delta \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda T}} \quad .$$

We can interpret $\theta(t)dt$ as $\theta(t)$ times the number of firms that adjust its price at times (t, dt). This is the sum of two terms. The first term is the fraction that adjust because they hit the boundary between t and $t + dt$. The second term is the fraction that have not yet adjusted times the fraction that adjust, λdt due to a free opportunity. Both terms are multiplied by $e^{-\lambda t}$ to take into account those firms that have received a free adjustment opportunity before after the monetary shock but before t.

Thus we have:

$$\mathcal{P}_\infty(t, \delta) \;=\; \Theta(\delta) + \delta \int_0^t \frac{\lambda e^{-\lambda s}}{1 - e^{-\lambda T}} ds \;=\; \Theta(\delta) + \delta \frac{1 - e^{-\frac{\lambda}{N(\Delta p_i)} \, t \, N(\Delta p_i)}}{\lambda/N(\Delta p_i)}$$

Using $\mathcal{P}_\infty$ we can compute the IRF for output, and a summary measure for it, namely the area below it:

$$\mathcal{M}_\infty(\delta) \;=\; \frac{1}{\epsilon} \int_0^T \left[\delta - \mathcal{P}_\infty(\delta, t)\right] dt \;\approx\; \frac{\delta}{\epsilon\, N(\Delta p_i)} \left[\frac{1 - (1 + \phi)\, e^{-\phi}}{(1 - e^{-\phi})^2} \right]$$

where the approximation uses the expression for small δ, i.e. its first order Taylor's expansion.

Kurtosis. For completeness we also include here an expression for the kurtosis of the distribution of price changes in the case of $n = \infty$. Price changes are distributed as:

$$\mathbb{E}\left[(\Delta p_i)^2\right] \;=\; \sigma^2/N(\Delta p_i) = \frac{\sigma^2}{\lambda} \frac{\lambda}{N(\Delta p_i)} = \frac{T\sigma^2}{T\lambda} \frac{\lambda}{N(\Delta p_i)} = T\sigma^2 \frac{1}{T\lambda} \frac{\lambda}{N(\Delta p_i)}$$

$$\begin{aligned}
\mathbb{E}\left[(\Delta p_i)^4\right] &= 3\frac{\lambda}{N(\Delta p_i)}\int_0^T \frac{(\sigma^2 t)^2 \lambda e^{-\lambda t}}{1-e^{-\lambda T}} dt + \left(1-\frac{\lambda}{N(\Delta p_i)}\right) 3\left(\sigma^2 T\right)^2 \\
&= 3\sigma^4 T^2 \left[\frac{2-e^{-\lambda T}\left(\lambda T\left(\lambda T+2\right)+2\right)}{\left(T\lambda\right)^2} + \left(1-\frac{\lambda}{N(\Delta p_i)}\right)\right]
\end{aligned}$$

Some algebra shows that kurtosis is then given by:

$$\frac{\mathbb{E}\left[(\Delta p_i)^4\right]}{\left(\mathbb{E}\left[(\Delta p_i)^2\right]\right)^2} = 6\,\frac{1-e^{-\phi}\left(1+\phi\right)}{\left(1-e^{-\phi}\right)^2}$$

It is immediate to use the expressions above to verify Proposition 6.

G.3 Analytical computation for $\ell = 0$ or $\ell = 1$ (any n).

For $\ell = 0$, or equivalently $\lambda = 0$, we use the result in Alvarez and Lippi (2014) for

$$\mathcal{T}_{n+2}(y) = \frac{\bar{y}-y}{(n+2)\sigma^2}$$

gives:

$$\mathcal{M}'(0) = \frac{1}{n\epsilon}\int_0^{\bar{y}} \left[\frac{n(\bar{y}-y)-2y}{(n+2)\sigma^2}\right] f(y)\, dy$$

and using the following expression for f from Alvarez and Lippi (2014) :

$$\begin{aligned}
f(y) &= \frac{1}{\bar{y}}\left[\log(\bar{y})-\log(y)\right] \text{ if } \; n=2, \text{ and} \\
f(y) &= (\bar{y})^{-\frac{n}{2}}\left(\frac{n}{n-2}\right)\left[(\bar{y})^{\frac{n}{2}-1}-(y)^{\frac{n}{2}-1}\right] \text{ otherwise}
\end{aligned}$$

gives that:

$$\mathcal{M}'(0) = \frac{1}{n\epsilon}\frac{2\,\bar{y}\,n(n-2)}{(n^2-4)\sigma^2} = \frac{1}{\epsilon}\frac{Kurt(\Delta p_i)}{6\,N(\Delta p_i)}$$

which verifies the equality in Proposition 6.

For $1 < n < \infty$ and $\ell = 1$, with $\lambda > 0$ and $\sigma^2 > 0$, using Proposition 3 it must be the case that $\bar{y} = \infty$. In this case, $N(\Delta p_i) = \lambda\ell = \lambda$, and the distribution of price changes is independent across each of the n products, and given by a Laplace distribution, which has kurtosis 6. Likewise $\mathcal{T}_{n+2}(y) = 1/\lambda$ for all $y \geq 0$. Thus, using equation (26) we obtain the desired result.