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THE REAL FIELD WITH CONVERGENT GENERALIZED POWER SERIES

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Abstract. We construct a model complete and o-minimal expansion of the field of real numbers in which each real function given on $[0,1]$ by a series $\sum c_n x^{\alpha_n}$ with $0 \le \alpha_n \to \infty$ and $\sum |c_n| r^{\alpha_n} < \infty$ for some $r > 1$ is definable. This expansion is polynomially bounded.

1. Introduction

We develop here a new way to prove model completeness and o-minimality of certain expansions of the real field. We apply this to a particular expansion $\mathbb{R}_{\mathrm{an}^*}$, for which previous methods, from [1], [3], [12], [17], fail. Inductive arguments using blow-up maps as in Tougeron [15], [16] are an important ingredient of our approach. Also, ideas of Gabrielov (as expounded in [1]) are crucial.

Throughout this paper we let m range over $\mathbb{N} = \{0, 1, 2, \dots\}$, and we let $X = (X_1, \dots, X_m)$ be a tuple of m distinct indeterminates. We consider formal series

$$F = F(X) = \sum_{\alpha} c_\alpha X^\alpha,$$

where the multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ ranges over $[0,\infty)^m$, the coefficients c_α are real, X^α denotes the formal monomial $X_1^{\alpha_1} \cdots X_m^{\alpha_m}$, and the set

$$\mathrm{supp}(F) := \{\alpha \in [0,\infty)^m : c_\alpha \neq 0\} \quad \text{(the support of the series)}$$

is contained in the cartesian product $S_1 \times \cdots \times S_m$ of well ordered subsets $S_1, \dots, S_m$ of $[0,\infty)$. (It follows that $\mathrm{supp}(F)$ is countable.) These series are added and multiplied in the usual way, and form an $\mathbb{R}$-algebra denoted by $\mathbb{R}[\![X^*]\!]$. For each polyradius $r = (r_1, \dots, r_m)$ (that is, $0 < r_i < \infty$ for $i = 1, \dots, m$) we put

$$\|F\|_r := \sum |c_\alpha| r^\alpha \in [0,\infty]$$

and we let $\mathbb{R}\{X^*\}_r$ be the normed subalgebra of $\mathbb{R}[\![X^*]\!]$ consisting of the F's with $\|F\|_r < \infty$, with norm given by $\|\cdot\|_r$. Each $F(X) = \sum c_\alpha X^\alpha \in \mathbb{R}\{X^*\}_r$ gives rise to a continuous function $x \mapsto F(x) := \sum c_\alpha x^\alpha : [0,r_1] \times \cdots \times [0,r_m] \longrightarrow \mathbb{R}$, analytic on the interior $(0,r_1) \times \cdots \times (0,r_m)$ of its domain. Let $\mathbb{R}_{\mathrm{an}^*}$ be the expansion of the ordered real field $(\mathbb{R}, <, 0, 1, +, -, \cdot)$ by all functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ (for all $m \in \mathbb{N}$)

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that are 0 outside $[0,1]^m$ and are given on $[0,1]^m$ by a power series $F \in \mathbb{R}\{X^*\}_r$ for some polyradius r with $r_1 > 1, \ldots, r_m > 1$. If $F(X) \in \mathbb{R}\{X^*\}_r$ and $0 < r_1' < r_1$, $\ldots, 0 < r_m' < r_m$, then the function $x \mapsto F(x) : [0, r_1'] \times \cdots \times [0, r_m'] \longrightarrow \mathbb{R}$ is clearly definable in $\mathbb{R}_{\mathrm{an}^*}$. It is also easy to see that the primitives of the structure $\mathbb{R}_{\mathrm{an}}$ as defined in [6] are definable in $\mathbb{R}_{\mathrm{an}^*}$, so the subsets of $\mathbb{R}^n$ that are definable in $\mathbb{R}_{\mathrm{an}}$ are definable in $\mathbb{R}_{\mathrm{an}^*}$ as well. On the other hand, there are many one-variable functions that are definable in $\mathbb{R}_{\mathrm{an}^*}$, but not in $\mathbb{R}_{\mathrm{an}}$. For example, the function

$$x \mapsto \zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n} : [0, e^{-2}] \longrightarrow \mathbb{R}$$

(where ζ is the Riemann zeta function) is definable in $\mathbb{R}_{\mathrm{an}^*}$, but not in $\mathbb{R}_{\mathrm{an}}$, in fact, not even in $\mathbb{R}_{\mathrm{an,exp}}$. (See corollary 5.14 in [7].) Here is our main result.

Theorem A. *The expansion $\mathbb{R}_{\mathrm{an}^*}$ is model complete and o-minimal.*

We have set up this article so that much of it will be useful also in a planned sequel, where we construct other model complete and o-minimal expansions of the real field. One such expansion, worked out in the second author's doctoral thesis, is more closely related to the material in [15].

Sections 2 and 3 are of a very general nature. In section 2 we develop a geometric test for model completeness and o-minimality of expansions of the real field. Section 3 elaborates on cell decomposition, as needed later. In sections 4, 5 and 6 we consider in detail the power series rings mentioned above, establishing, among other things, Weierstrass preparation, and study a variant of the blow-up substitutions used by Tougeron [15] in his treatment of semianalytic sets with "Gevrey condition on the boundary". In section 7 we introduce the generalized semianalytic sets described locally by equations and inequalities between the power series above. In section 8 we establish Theorem A. In its proof we use inductive arguments inspired by [15] to establish the so-called "Gabrielov property" of section 2 for our generalized semianalytic sets, which allows us to draw the desired conclusion. In section 9 we obtain, by similar inductive arguments,

Theorem B. *Let $\epsilon > 0$ and let $f : (0, \epsilon) \longrightarrow \mathbb{R}$ be definable in $\mathbb{R}_{\mathrm{an}^*}$. Then there is a series $F(X) \in \mathbb{R}\{X^*\}_\delta$ for some $\delta \in (0, \epsilon)$, where X is a single variable, and there is a (possibly negative) real number r such that $f(x) = x^r F(x)$ for $x \in (0, \delta)$.*

It follows that $\mathbb{R}_{\mathrm{an}^*}$ is polynomially bounded. The o-minimality and polynomial boundedness of an expansion of the real field carries numerous topological and analytic-geometric consequences with it, such as Łojasiewicz inequalities; see [8].

We finish this introduction with some terminological conventions, in particular concerning manifolds and dimension, that are in force throughout this paper.

Notations and Conventions. We let k, l, m, n and d range over $\mathbb{N}$, and we let $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$ and $Z = (Z_1, \ldots, Z_l)$ denote tuples of distinct indeterminates. The tuples $r = (r_1, \ldots, r_m)$ and $s = (s_1, \ldots, s_n)$ always denote polyradii (as defined above), while the tuples $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ denote elements of $[0, \infty)^m$ and $[0, \infty)^n$ respectively. For any tuple $z = (z_1, \ldots, z_k) \in \mathbb{R}^k$ we put $|z| := \sup\{|z_1|, \ldots, |z_k|\}$, and we write $z' = (z_1, \ldots, z_{k-1})$ if $k \geq 1$. For polyradii $r = (r_1, \ldots, r_m)$ and $s = (s_1, \ldots, s_m)$ we write $r < s$ to mean $r_i < s_i$ for all $i = 1, \ldots, m$, and similarly for $r \leq s$.

For any set S we write $|S|$ for the cardinality of S.

All rings are assumed to be commutative with $1 \neq 0$. A **normed ring** is a ring A equipped with a **norm** $|\cdot| : A \longrightarrow [0, \infty)$, i.e. for all $x, y \in A$:

1. $|x| = 0$ if and only if $x = 0$;
2. $|x + y| \leq |x| + |y|$;
3. $|xy| \leq |x||y|$, hence $|1| \leq 1$.

Given $m \leq n$, we denote by $\Pi^n_m : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the projection on the first m coordinates. More generally, if $\lambda \in \{1, \dots, n\}^m$ is a strictly increasing sequence, we let $\Pi^n_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the projection defined by $\Pi_\lambda(x_1, \dots, x_n) = (x_{\lambda(1)}, \dots, x_{\lambda(m)})$. If n is clear from context (as is usually the case), we just write Π_m and Π_λ respectively.

Given a subset A of a topological space S, we let $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and $\mathrm{fr}\, A := \mathrm{cl}(A) \setminus A$ denote the closure, interior and frontier of A in S respectively, if the ambient space S is clear from context. If $f, g : A \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ are two functions, we write $f < g$ if $f(x) < g(x)$ for all $x \in A$; in that case we put

$$(f, g) := \{(x, t) \in A \times \mathbb{R} : f(x) < t < g(x)\}.$$

A **manifold** M is always a nonempty embedded (not just immersed) analytic submanifold of $\mathbb{R}^k$ (for some k depending on M) everywhere of the same dimension $\dim(M)$. We identify the tangent space T_xM of M at a point $x \in M$ in the usual way with a linear subspace of the ambient space $\mathbb{R}^k$ (of dimension $\dim(M)$). Note that if M is a manifold in $\mathbb{R}^k$, then M is locally closed; hence $\mathrm{fr}\, M$ is closed. In order to facilitate arguments by "induction on dimension" it will be convenient to say that a set $S \subseteq \mathbb{R}^k$ **has dimension** if S is a countable union of manifolds; in that case we put

$$\dim(S) := \max \{\dim(M) : M \subseteq S \text{ is a manifold}\}$$

for nonempty S, and $\dim(\emptyset) := -\infty$. If S happens also to be a manifold, then this agrees with the dimension of S as a manifold. This notion of dimension is a bit ad hoc, tied as it is to the notion of manifold, but it has some useful properties:

1. if $S = \bigcup_{i \in \mathbb{N}} S_i$ and each S_i has dimension, then S also has dimension and $\dim(S) = \max\{\dim(S_i) : i \in \mathbb{N}\}$;
2. if $f : M \longrightarrow \mathbb{R}^n$ is an analytic map from the manifold M into $\mathbb{R}^n$ of constant rank r, then $f(M)$ has dimension, and $\dim(f(M)) = r$.

Property (1) follows by a Baire category argument (see [4], p. 533 for details). Property (2) follows from the rank theorem, the fact that M has a countable basis for its topology, and property (1).

We will also occasionally use the following fact.

3. If $n \geq m$ and $A \subseteq \mathbb{R}^n$ as well as $\Pi_m(A) \subseteq \mathbb{R}^m$ have dimension, then $\dim(A) \geq \dim(\Pi_m(A))$.

One way to see this is to observe that if $A \subseteq \mathbb{R}^n$ has dimension, then $\dim(A) = \mathrm{Hausdorffdim}(A)$ (with respect to the usual euclidean metric on $\mathbb{R}^n$), and that $\mathrm{Hausdorffdim}(A) \geq \mathrm{Hausdorffdim}(\Pi_m(A))$, since $\Pi_m : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a Lipschitz map. (Actually, the assumption in (3) that A has dimension implies that $\Pi_m(A)$ has dimension, but we will not need this fact.)

2. Gabrielov Property, Model Completeness and O-minimality

In this section we develop a useful geometric test for the model completeness and o-minimality of expansions of the real field. We do this by axiomatizing and generalizing the arguments in the proof of Gabrielov's "Theorem of the Complement" as exposed by Bierstone and Milman in [1].

2.1 Definition. Let a collection Λ_n of bounded subsets of $\mathbb{R}^n$ be given for each n, and let $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$. We call a set $A \subseteq \mathbb{R}^n$ a **Λ-set** if $A \in \Lambda_n$; if in addition A is a manifold, we call A a **Λ-manifold**. We also call a set $E \subseteq \mathbb{R}^m$ a **sub-Λ-set** if there are $n \geq m$ and a Λ-set $A \subseteq \mathbb{R}^n$ such that $E = \Pi_m(A)$; if in addition E is a manifold, we call E a **sub-Λ-manifold**.

We say that a set $A \subseteq \mathbb{R}^n$ has the **Λ-Gabrielov property**, if for each $m \leq n$ there are connected sub-Λ-manifolds $B_1 \subseteq \mathbb{R}^{n+q_1}, \dots, B_k \subseteq \mathbb{R}^{n+q_k}$, where $q_1, \dots, q_k \in \mathbb{N}$, such that

$$\Pi_m(A) = \Pi_m(B_1) \cup \cdots \cup \Pi_m(B_k)$$

and for each $i = 1, \dots, k$ we have:

(G1) $\operatorname{fr} B_i$ is contained in a closed sub-Λ-set $D_i \subseteq \mathbb{R}^{n+q_i}$ such that D_i has dimension with $\dim(D_i) < \dim(B_i)$;

(G2) $\dim(B_i) \leq m$, and there is a strictly increasing sequence $\lambda \in \{1, \dots, m\}^d$, with $d = \dim(B_i)$, such that $\Pi_\lambda|_{B_i} : B_i \longrightarrow \mathbb{R}^d$ is an immersion.

2.2 Remarks. (1) In (G2) the sequence λ and the natural number d may depend of course on i. That $\Pi_\lambda|_{B_i}$ in (G2) is an immersion just means that Π_λ is injective on each tangent space $T_x(B_i) \subseteq \mathbb{R}^{n+q_i}$ for $x \in B_i$; since $\dim(T_x(B_i)) = d$, it follows in particular that $\Pi_\lambda(B_i)$ is open in $\mathbb{R}^d$ and that $\Pi_\lambda|_{B_i} : B_i \longrightarrow \mathbb{R}^d$ is a local homeomorphism. Note that $\Pi_m|_{B_i} : B_i \longrightarrow \mathbb{R}^m$ is then also an immersion, since $\Pi_\lambda^{n+q_i}|_{B_i} = \Pi_\lambda^m \circ (\Pi_m|_{B_i})$.

(2) In the situation of Gabrielov's Theorem of the Complement one has

$$\Lambda_n = \{A \subseteq \mathbb{R}^n : A \text{ is bounded and semianalytic in } \mathbb{R}^n\},$$

and each $A \in \Lambda_n$ has the Λ-Gabrielov property, with $q_i = 0$, $B_i \subseteq A$ and $\operatorname{fr} B_i = D_i$ a Λ-set for all i in the definition above. Because of our later use of "blowing up" it is crucial for us to allow $q_i > 0$, and to allow D_i to be a sub-Λ-set.

2.3. Let $I = [-1, 1] \subseteq \mathbb{R}$. Write E^c for the complement $I^n \setminus E$ of a set $E \subseteq I^n$. From now on in this section we assume $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$, where each Λ_n is a collection of subsets of I^n such that for every $A, B \in \Lambda_n$:

(I) $\emptyset$ and I^n belong to Λ_n, and for each pair (i,j) with $1 \leq i < j \leq n$ the diagonal $\Delta_{ij} = \{x \in I^n : x_i = x_j\}$ belongs to Λ_n, along with its complement $(\Delta_{ij})^c$;

(II) $A \cup B, A \cap B \in \Lambda_n$;

(III) $I \times A$ and $A \times I$ belong to Λ_{n+1};

(IV) A has the Λ-Gabrielov property.

2.4 Remark. Axioms (I)-(III) imply that if $A \subseteq I^m$ and $B \subseteq I^n$ are Λ-sets, then $A \times B \subseteq I^{m+n}$ is a Λ-set. This can be used to show that if $E_1, E_2 \subseteq I^m$ are sub-Λ-sets, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are sub-Λ-sets too. One checks easily that if $\lambda \in \{1, \dots, n\}^d$ is strictly increasing and $A \in \Lambda_n$, then $\Pi_\lambda(A) \subseteq I^d$ is a sub-Λ-set.

We now have the following elementary lemma.

2.5 Lemma. *Suppose for a certain d that the complement of each sub-Λ-set in I^d is a sub-Λ-set. Let $\lambda \in \{1,\dots,m\}^d$ be a strictly increasing sequence. Let E be a sub-Λ-set in I^m and suppose there is $M \in \mathbb{N}$ such that $|E \cap \Pi_\lambda^{-1}(x)| \leq M$ for all $x \in I^d$. Then the complement E^c of E in I^m is also a sub-Λ-set.*

Proof. For simplicity of notation assume $\lambda(1) = 1, \dots, \lambda(d) = d$, and write E_x for the fiber $E \cap \Pi_\lambda^{-1}(x)$, $x \in I^d$. Clearly for each $k \in \mathbb{N}$ the set $C_k := \{x \in I^d : |E_x| \geq k\}$ is a sub-Λ-set in I^d; hence $D_k := \{x \in I^d : |E_x| = k\} = C_k \setminus C_{k+1}$ is a sub-Λ-set. Now $I^d = D_0 \cup \cdots \cup D_M$, so

$$E^c = \left(\Pi_d^{-1}(D_0) \setminus E\right) \cup \cdots \cup \left(\Pi_d^{-1}(D_M) \setminus E\right).$$

Hence it suffices to show that each set $\Pi_d^{-1}(D_k) \setminus E$ is a sub-Λ-set. With $m = d + e$ and $(x,y) = (x_1,\dots,x_d,y_1,\dots,y_e)$ ranging over I^m and i,j over $\{1,\dots,k\}$, this follows from

$$\begin{aligned}(x,y) \in \Pi_d^{-1}(D_k) \setminus E \Longleftrightarrow \exists z_1 \dots z_k \in I^e \Big[x \in D_k \wedge \left(\bigwedge_{i=1}^k y \neq z_i\right) \wedge \\ \wedge \left(\bigwedge_{1 \leq i < j \leq k} z_i \neq z_j\right) \wedge \left(\bigwedge_{i=1}^k (x, z_i) \in E\right)\Big].\end{aligned}$$

□

2.6 Remark. Note that 2.4 and 2.5 go through for I any nonempty set equipped with a collection Λ_n of subsets of I^n, for each $n \in \mathbb{N}$, such that axioms (I),(II) and (III) hold. The next result is a basic tool for proving model completeness and o-minimality theorems in this paper and its sequel. Here axiom (IV) comes into play.

2.7 Theorem of the Complement. *If $E \subseteq I^m$ is a sub-Λ-set, then E^c is a sub-Λ-set.*

Remark. In the proof of the "theorem of the complement" we will use the following easy consequences of axiom (IV) for an arbitrary sub-Λ-set $E \subseteq I^m$:

1. E has only finitely many (connected) components, and each component of E is a sub-Λ-set in I^m;
2. E has dimension.

To see this, write $E = \Pi_m(A)$ with $A \in \Lambda_n$, $n \geq m$. By axiom (IV), and using the notation of 2.1, each connected component of E is a union of sets $\Pi_m^{n+q_i}(B_i)$. Hence E has only finitely many connected components, and each component of E is a sub-Λ-set. Property (2) follows in the same way, taking into account the remarks made on dimension at the end of the introduction.

Proof of the theorem of the complement. By induction on m; the case $m = 0$ is clear.

Let $m > 0$ and assume that the theorem holds for sub-Λ-sets in I^d, for all $d < m$. Let E be a sub-Λ-set in I^m. To show that E^c is a sub-Λ-set we may reduce by axiom (IV) to the case that $E = \Pi_m(B)$ for some connected sub-Λ-manifold $B \subseteq \mathbb{R}^n$, where $m \leq n$ and B has the following properties:

1. $\operatorname{fr} B$ is contained in a closed sub-Λ-set $D \subseteq I^n$ such that D has dimension with $\dim(D) < \dim(B)$;

2. $\dim(B) = d \leq m$, and there is a strictly increasing $\lambda \in \{1, \dots, m\}^d$ such that $\Pi_\lambda|_B : B \longrightarrow \mathbb{R}^d$ is an immersion.

Put $F = \Pi_\lambda(B)$, so $\Pi_\lambda^m(E) = F$. Since $\Pi_m|_B$ and $\Pi_\lambda|_B$ have constant rank d, we have $\dim(B) = \dim(E) = \dim(F) = d$.

Case 1: $d < m$. In this case we first establish

Claim. There is $M \in \mathbb{N}$ such that $|(\Pi_\lambda^m)^{-1}(x) \cap E| \leq |\Pi_\lambda^{-1}(x) \cap B| \leq M$ for all $x \in I^d$.

The left inequality is obvious. For the right inequality, put $B_x := \Pi_\lambda^{-1}(x) \cap B$ for $x \in I^d$. Note that $\Pi_\lambda|_B : B \longrightarrow \mathbb{R}^d$ is a local homeomorphism. Put $G := \Pi_\lambda(D)$. Then G is a closed sub-Λ-set of dimension $< d$; in particular, every neighbourhood of every point in G contains points of G^c. Hence if $M \in \mathbb{N}$ is such that $|B_x| \leq M$ for all $x \in G^c$, then $|B_x| \leq M$ for $x \in G$ as well. So it suffices to show there is such a constant M for $x \in G^c$.

The map $\Pi_\lambda|_{B \cap \Pi_\lambda^{-1}(G^c)} : B \cap \Pi_\lambda^{-1}(G^c) \longrightarrow G^c$ is proper: let $K \subseteq G^c$ be compact and (u_k) a sequence of points in $B \cap \Pi_\lambda^{-1}(K)$ converging to $u \in I^n$; we have to show that $u \in B \cap \Pi_\lambda^{-1}(K)$. Clearly $u \in \Pi_\lambda^{-1}(K)$; if $u \notin B$, then $u \in \operatorname{fr} B$, so $\Pi_\lambda(u) \in G$, contradicting $\Pi_\lambda(u) \in K$. Since said map is both proper and a local homeomorphism, it is a topological covering map, and hence $|B_x|$ takes a constant finite value on each component of G^c (see for example [9], 4.22). By the inductive assumption G^c is a sub-Λ-set; hence G^c has only finitely many connected components. So there is $M \in \mathbb{N}$ such that $|B_x| \leq M$ for all $x \in G^c$. This proves the claim.

Now it follows immediately from lemma (2.5) and the claim above that E^c is a sub-Λ-set.

Case 2: $d = m$. Then $\Pi_m|_B$ is a local homeomorphism; hence $\Pi_m(B)$ is open in $\mathbb{R}^m$. Note that $\Pi_m(D)$ is a (closed) sub-Λ-set of dimension $< m$, so $(\Pi_m(D))^c$ is a sub-Λ-set by case 1. Since $(\Pi_m(B \cup D))^c = (\Pi_m(B))^c \cap (\Pi_m(D))^c$, and $\Pi_m(B)$ is open and $B \cup D$ is compact, it follows that $(\Pi_m(B \cup D))^c$ is open and closed in $(\Pi_m(D))^c$; hence $(\Pi_m(B \cup D))^c$ is a sub-Λ-set by remark (1) above. Next note that

$$E^c = (\Pi_m(B))^c = (\Pi_m(B \cup D))^c \cup (\Pi_m(D) \setminus (\Pi_m(B) \cap \Pi_m(D))).$$

Since $\Pi_m(B) \cap \Pi_m(D)$ is a sub-Λ-set of dimension $< m$, it follows from case 1 that $\Pi_m(D) \setminus (\Pi_m(B) \cap \Pi_m(D))$ is a sub-Λ-set. Hence E^c is a sub-Λ-set. □

2.8 Corollary. *The structure (I, Λ) which has an n-ary relation for each set in Λ_n, $n \in \mathbb{N}$, is model complete. Its definable sets are exactly the sub-Λ-sets. (Here "definable" means "definable without parameters".)*

Proof. Let $S\Lambda_n$ be the collection of sub-Λ-sets in I^n, for each n. Then the theorem of the complement implies that $S\Lambda_n$ is a Boolean algebra of subsets of I^n. It is also clear from the definition of sub-Λ-set that $S\Lambda_n$ contains all diagonals Δ_{ij} $(1 \leq i < j \leq n)$, that $A \in S\Lambda_n$ implies $I \times A, A \times I \in S\Lambda_{n+1}$, and that if $B \in S\Lambda_{n+1}$, then $\Pi_n(B) \in S\Lambda_n$. These facts imply that every subset of I^n definable in the structure (I, Λ) must belong to $S\Lambda_n$. Since the sub-Λ-sets are in fact existentially definable in (I, Λ), it follows that (I, Λ) is model complete. □

2.9 Corollary. *Assume in addition that $\{r\} \in \Lambda_1$ for all $r \in I$ and the sets*

$$\{(x, y, z) \in I^3 : x + y = z\} \text{ and } \{(x, y, z) \in I^3 : xy = z\}$$

belong to Λ_3. Then the expansion $\mathbb{R}_\Lambda := (\mathbb{R}, <, 0, 1, +, -, \cdot, \Lambda)$ of the ordered field by the Λ-sets in $\mathbb{R}^n$ for $n = 0, 1, 2, \ldots$ is model complete and o-minimal. A set $A \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_\Lambda$ if and only if $\tau_n(A)$ is a sub-Λ-set in I^n, where $\tau_n : \mathbb{R}^n \longrightarrow I^n$ is given by $\tau_n(x_1, \ldots, x_n) = (x_1/\sqrt{1+x_1^2}, \ldots, x_n/\sqrt{1+x_n^2})$.

Proof. Let Σ_n be the collection of all sets $A \subseteq \mathbb{R}^n$ such that $\tau_n(A)$ is a sub-Λ-set in I^n. Let $(\mathbb{R}, \Sigma)$ be the structure with underlying set $\mathbb{R}$ and an n-ary relation symbol for each set $A \in \Sigma_n$, $n \in \mathbb{N}$. The previous corollary and the fact that $\tau_n \circ \Pi_n^{n+1} = \Pi_n \circ \tau_{n+1}$ for all n implies that any set $A \subseteq \mathbb{R}^n$ that is definable in $(\mathbb{R}, \Sigma)$ actually belongs to Σ_n.

A routine argument using the hypothesis of this corollary shows that the graphs of addition and multiplication belong to Σ_3. Hence all primitives of $\mathbb{R}_\Lambda$ are definable in $(\mathbb{R}, \Sigma)$. Conversely, the sets in Σ_n are clearly existentially definable in $\mathbb{R}_\Lambda$. The model completeness of $\mathbb{R}_\Lambda$ follows. Since sub-Λ-sets have only finitely many connected components, the o-minimality of $\mathbb{R}_\Lambda$ follows as well. □

2.10 Remark. This section goes through unchanged if by "manifold" we mean "nonempty embedded C^1 submanifold of $\mathbb{R}^k$ (for some k) everywhere of the same dimension", and we correspondingly extend the notion of "dimension" to subsets of $\mathbb{R}^k$ that are countable unions of such manifolds, as in "Notations and Conventions".

3. Cell Decomposition

In this section we elaborate on a result from [11] on "relatively semialgebraic" sets. We also refer to the exposition in ch. 2 of [5].

3.1. Let S be a nonempty topological space. Let $\mathcal{E}$ be a ring of continuous functions $\phi : S \longrightarrow \mathbb{R}$, the ring operations being pointwise addition and multiplication, with the identity the function on S which takes the constant value 1. Call $A \subseteq S$ an **$\mathcal{E}$-set** if A is a finite union of sets of the form

$$\{x \in S : \phi(x) = 0, \psi_1(x) > 0, \ldots, \psi_k(x) > 0\}$$

with $\phi, \psi_1, \ldots, \psi_k \in \mathcal{E}$. The $\mathcal{E}$-sets form a Boolean algebra of subsets of S.

3.2 Cell Decomposition. *Let $f_1, \ldots, f_M \in \mathcal{E}[T]$ all be of degree at most d in T, and let $f_1, \ldots, f_N$ be the list of all partials $\partial^l f_m/\partial T^l$ with $m = 1, \ldots, M$ and $0 \le l \le d$. Then S can be partitioned into finitely many $\mathcal{E}$-sets $S_1, \ldots, S_k$ such that for each connected component C of each S_i there are continuous real valued functions $\xi_{C,1} < \cdots < \xi_{C,m(C)}$ on C such that, with $\xi_{C,0} = -\infty$ and $\xi_{C,m(C)+1} = +\infty$,*

1. *each of the sets $\Gamma(\xi_{C,j})$, $1 \le j \le m(C)$, and $(\xi_{C,j}, \xi_{C,j+1})$, $0 \le j \le m(C)$, is of the form*
$$\{(x,t) \in C \times \mathbb{R} : \operatorname{sign}(f_n(x,t)) = \epsilon(n) \text{ for } n = 1, \ldots, N\}$$
for a suitable sign condition $\epsilon : \{1, \ldots, N\} \longrightarrow \{-1, 0, 1\}$;

2. *if $f_{i_1}, \ldots, f_{i_l}$ with $1 \le i_1 < \cdots < i_l \le N$ are those members of $\{f_1, \ldots, f_N\}$ which are not identically zero on $C \times \mathbb{R}$, and if $g := f_{i_1} \cdots f_{i_l}$, then $g \ne 0$ on each $(\xi_{C,j}, \xi_{C,j+1})$, and for each $j = 1, \ldots, m(C)$ there is $e \in \{1, \ldots, \deg_T(g)\}$ such that for all $(x,t) \in \Gamma(\xi_{C,j})$ we have*
$$g(x,t) = \cdots = \partial^{e-1} g/\partial T^{e-1}(x,t) = 0 \quad \text{and} \quad \partial^e g/\partial T^e(x,t) \ne 0;$$

3. *if moreover* $f_1, \dots, f_M$ *are monic in* T*, then each function* $\xi_{C,j}$*,* $1 \le j \le m(C)$*, extends uniquely to a continuous function* $\eta_{C,j} : \mathrm{cl}(C) \longrightarrow \mathbb{R}$ *such that each of the sets* $\mathrm{cl}(\Gamma(\xi_{C,j})) = \Gamma(\eta_{C,j})$ *with* $1 \le j \le m(C)$ *and* $\mathrm{cl}((\xi_{C,j}, \xi_{C,j+1}))$ *with* $0 \le j \le m(C)$ *equals*

$$\{(x,t) \in \mathrm{cl}(C) \times \mathbb{R} : \mathrm{sign}(f_n(x,t)) \in \{\epsilon(n), 0\} \text{ for } n = 1, \dots, N\},$$

where ϵ *is the corresponding sign condition from Part 1.*

Proof. Following the proofs in [5], ch. 2, we obtain a partition $S = S_1 \cup \dots \cup S_k$ for which the statement of the theorem holds with the possible exception of property (2). Note that property (1) implies that for g as in (2) we have

$$\{(x,t) \in C \times \mathbb{R} : g(x,t) = 0\} = \Gamma(\xi_{C,1}) \cup \dots \cup \Gamma(\xi_{C,m(C)}).$$

To obtain property (2), we will refine the partition $\{S_1, \dots, S_k\}$; this will not affect (1) and (3).

To find such a refinement, we apply conclusion (1) of the theorem to the list $g_1, \dots, g_{M'}$ consisting of all products $f_{i_1} \cdots f_{i_l}$ with $1 \le i_1 < \dots < i_l \le N$. Since $f_1, \dots, f_M \in \{g_1, \dots, g_{M'}\}$, the proof in ch. 2 of [5] gives a finite partition of S into $\mathcal{E}$-sets that refines the partition $\{S_1, \dots, S_k\}$. Let C' be a connected component of some element of this refinement, and let C be the (unique) connected component of one of $S_1, \dots, S_k$ such that $C' \subseteq C$. If g is the product of all those f_i that are not identically zero on $C' \times \mathbb{R}$, and if ξ is the restriction of one of the $\xi_{C,j}$ to C', then clearly g is identically zero on $\Gamma(\xi)$; but also ξ is one of the functions $\xi_{C',j'}$ obtained from the theorem applied to $g_1, \dots, g_{M'}$, and hence every partial $\partial^\nu g / \partial T^\nu$ has constant sign on $\Gamma(\xi)$. Moreover, the number of zeros of $g(x,T)$ is constant and finite as x ranges over C'. Hence some $\partial^\nu g/\partial T^\nu$, $1 \le \nu \le \deg_T(g)$, does not vanish identically on $\Gamma(\xi)$. □

3.3 Remark. In section 8 we will use this theorem in a situation where $S \subseteq \mathbb{R}^q$ for some q, C is a component of some S_i as in the theorem, and D is a manifold contained in C such that all functions $\phi|_D$ with $\phi \in \mathcal{E}$ are analytic. Then the functions $\xi_{C,j}|_D$, $j = 1, \dots, m(C)$, are also analytic. (This follows easily from part (2) above and the implicit function theorem.)

4. Generalized Power Series

4.1. We denote by X^* the multiplicative monoid whose elements are the **monomials** $X^\alpha := X_1^{\alpha_1} \cdots X_m^{\alpha_m}$ with $\alpha = (\alpha_1, \dots, \alpha_m) \in [0,\infty)^m$, multiplied according to $X^\alpha \cdot X^\beta = X^{\alpha+\beta}$. The identity element of X^* is $X^0 = 1$, where $0 = (0, \dots, 0)$.

Let us say that a set $S \subseteq [0,\infty)^m$ is **good** if for each $i = 1, \dots, m$ the set $S_i := \{\alpha_i : \alpha \in S\}$ is a well ordered subset of $[0,\infty)$, or equivalently, if there are well ordered subsets $S_1, \dots, S_m$ of $[0,\infty)$ such that $S \subseteq S_1 \times \dots \times S_m$. We partially order $[0,\infty)^m$ by setting $\alpha \le \beta$ if and only if $\alpha_i \le \beta_i$ for $i = 1, \dots, m$. Instead of $\alpha \le \beta$ we also write $X^\alpha \mid X^\beta$. We put $|\alpha| := \alpha_1 + \dots + \alpha_m$, $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_m + \beta_m)$, $\inf(\alpha, \beta) := (\min\{\alpha_1, \beta_1\}, \dots, \min\{\alpha_m, \beta_m\})$, and $\gcd(X^\alpha, X^\beta) := X^{\inf(\alpha,\beta)}$.

4.2 Lemma. *Suppose* $S \subseteq [0,\infty)^m$ *is good.*

1. $S_{\min} := \{\alpha \in S : \alpha$ is a minimal element of $S\}$ *is finite, and each element* $\alpha \in S$ *is* $\ge$ *some element of* $S_{\min}$.
2. *The set* $\{|\alpha| : \alpha \in S\}$ *is a well ordered subset of* $[0,\infty)$*, and for every* $t \in [0,\infty)$ *the set* $S(t) := \{\alpha \in S : |\alpha| = t\}$ *is finite.*

Proof. (1) Suppose $S_{\min}$ is infinite. Take a sequence $\{\alpha^n\}_{n\in\mathbb{N}}$ in $S_{\min}$ with $\alpha^k \neq \alpha^l$ for $k \neq l$. By passing to a subsequence we may assume that $\{\alpha_1^n\}_{n\in\mathbb{N}}$ is either constant or strictly increasing (use the fact that each infinite sequence in S_1 has a subsequence that is constant or strictly increasing). By repeating this argument we reduce to the case that for each $i = 1, \dots, m$ the sequence $\{\alpha_i^n\}_{n\in\mathbb{N}}$ is either constant or strictly increasing. Hence $\alpha^0 \leq \alpha^1$, contradicting the fact that α^0 and α^1 are distinct elements in $S_{\min}$.

(2) Suppose the set in (2) is not well ordered. Take a sequence $\{\alpha^n\}_{n\in\mathbb{N}}$ in S such that $|\alpha^0| > |\alpha^1| > |\alpha^2| > \dots$. By the same argument as in (1) we may pass to a subsequence and reduce to the case that $\alpha^0 \leq \alpha^1 \leq \alpha^2 \leq \dots$, contradiction. In the same way one proves the second statement of (2). ☐

For $S \subseteq [0,\infty)^m$ put $\Sigma(S) := \{\alpha^1 + \dots + \alpha^k : k \in \mathbb{N}, \alpha^1, \dots, \alpha^k \in S\}$.

4.3 Lemma. *If $S, T \subseteq [0,\infty)^m$ are good, then so are $S \cup T$ and $\Sigma(S)$.*

Proof. This is easily reduced to the case $m = 1$, for which the lemma is well known (see e.g. [10]). ☐

4.4. Let A be a ring; then $A[\![X^*]\!]$ is by definition the set of **power series in** X^* **over** A. Its elements are the formal sums

$$f(X) = \sum f_\alpha X^\alpha,$$

where α ranges over $[0,\infty)^m$, the coefficients f_α belong to A, and

$$\operatorname{supp}(f) := \{\alpha \in [0,\infty)^m : f_\alpha \neq 0\}$$

is a good subset of $[0,\infty)^m$. If $\operatorname{supp}(f)$ is finite, we call f a **polynomial in** X^*, and we denote by $A[X^*]$ the set of all polynomials in X^* with coefficients in A.

These series are added and multiplied in the usual way, just as formal power series in $A[\![X]\!]$, and form a ring under these operations, containing $A[X^*]$ as a subring. We consider the power series ring $A[\![X]\!]$ also as a subring of $A[\![X^*]\!]$, namely as the subring of all series $f(X)$ as above for which $\operatorname{supp}(f) \subseteq \mathbb{N}^m$. (Note that $\mathbb{N}^m$ is a good subset of $[0,\infty)^m$.)

The **constant term** of a series $f(X) = \sum f_\alpha X^\alpha \in A[\![X^*]\!]$ is the element $f_0 = f(0)$ of A. Note that the map $\sum f_\alpha X^\alpha \mapsto f_0 : A[\![X^*]\!] \longrightarrow A$ is a ring homomorphism.

4.5. Let $f(X) = \sum f_\alpha X^\alpha \in A[\![X^*]\!]$. The **order** of f is the element of $[0,\infty]$ defined as follows:

$$\operatorname{ord}(f) := \begin{cases} \min\{|\alpha| : f_\alpha \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{if } f = 0. \end{cases}$$

One easily checks that for $f, g \in A[\![X^*]\!]$ we have

1. $\operatorname{ord}(f+g) \geq \min\{\operatorname{ord}(f), \operatorname{ord}(g)\}$, and
2. $\operatorname{ord}(fg) \geq \operatorname{ord}(f) + \operatorname{ord}(g)$, with equality if A is an integral domain.

Hence $A[\![X^*]\!]$ is an integral domain if A is an integral domain.

4.6. Let J be any index set and $\{f_j\}_{j\in J}$ a family in $A[\![X^*]\!]$ such that

1. for each $\alpha \in [0,\infty)^m$ there are only finitely many $j \in J$ such that $\alpha \in \operatorname{supp}(f_j)$, and
2. $\bigcup_{j\in J} \operatorname{supp}(f_j)$ is a good subset of $[0,\infty)^m$.

(Note that if J is finite these conditions are automatically satisfied.) We may then clearly consider the (potentially infinite) **sum** $\sum_{j\in J} f_j$ as a well defined element of $A[\![X^*]\!]$. In the following we shall frequently use such infinite sums, and the obvious rules for manipulating them. Note that with this notation $\sum f_\alpha X^\alpha$ has acquired a new meaning (sum of the family $f_\alpha X^\alpha$ indexed by $\alpha \in [0,\infty)^m$), but this new meaning agrees of course with the given one: $f(X) = \sum f_\alpha X^\alpha$. We can also write $f(X) = \sum f_\alpha X^\alpha$ as the sum of its homogeneous parts: $f = \sum_{r\in[0,\infty)} f_{(r)}$ with $f_{(r)} := \sum_{|\alpha|=r} f_\alpha X^\alpha$ the **homogeneous part of degree** r of f. Note that by lemma 4.2 each $f_{(r)}$ is actually a polynomial in X^*.

4.7 Lemma. *Let $f(X) = \sum f_\alpha X^\alpha \in A[\![X^*]\!]$. Then f is a unit in $A[\![X^*]\!]$ if and only if its constant term f_0 is a unit in A.*

Proof. If $f(X)g(X) = 1$ with $g(X) = \sum b_\alpha X^\alpha \in A[\![X^*]\!]$, then $a_0b_0 = 1$, so a_0 is a unit.

Conversely, if $a_0b_0 = 1$ with $b_0 \in A$, then $b_0 f = 1 - h$ with $\operatorname{ord}(h) > 0$. Hence the infinite sum $\sum_{n=0}^\infty h^n$ is well defined, and clearly $1 = (\sum_{n=0}^\infty h^n)(1-h) = (\sum_{n=0}^\infty h^n)b_0 f$, so f has inverse $b_0(\sum_{n=0}^\infty h^n)$ in $A[\![X^*]\!]$. $\square$

4.8 Lemma. *Each $f \in A[\![X^*]\!]$ with $\operatorname{ord}(f) > 0$ is of the form*

$$f = X_1^{\gamma_1} f_1 + \cdots + X_m^{\gamma_m} f_m$$

with $f_i \in A[\![(X_1,\dots,X_i)^*]\!]$ for $i = 1,\dots,m$ and real numbers $\gamma_1,\dots,\gamma_m > 0$.

Proof. By induction on m; the case $m = 0$ is trivial. So let $m > 0$. Write $f \in A[\![X^*]\!]$ with $\operatorname{ord}(f) > 0$ as $f = g + h$, where g is the sum of the terms of f not involving X_m and h is the sum of the terms of f involving X_m. Then clearly $h = X_m^{\gamma_m} f_m$ for some $f_m \in A[\![X^*]\!]$ and some $\gamma_m > 0$, while the inductive hypothesis implies that $g = X_1^{\gamma_1} f_1 + \cdots + X_{m-1}^{\gamma_{m-1}} f_{m-1}$ with $f_i \in A[\![(X_1,\dots,X_i)^*]\!]$ for $i = 1,\dots,m-1$ and real numbers $\gamma_1,\dots,\gamma_{m-1} > 0$. $\square$

4.9. *Blow-up height.* Assume $m \geq 2$. Given distinct $i,j \in \{1,\dots,m\}$ and $\gamma > 0$, we define an injective monoid homomorphism $s_{ij}^\gamma : X^* \longrightarrow X^*$ such that $s_{ij}^\gamma(X_k) = X_k$ for $k \neq i$ and $s_{ij}^\gamma(X_i) = X_i X_j^\gamma$, as follows:

$$s_{ij}^\gamma(X^\alpha) := X_1^{\alpha_1}\cdots X_{j-1}^{\alpha_{j-1}} X_j^{\gamma\alpha_i+\alpha_j} X_{j+1}^{\alpha_{j+1}}\cdots X_m^{\alpha_m} = X^\alpha X_j^{\gamma\alpha_i}.$$

We call s_{ij}^γ a **singular blow-up substitution on** X.

We now assign to every pair of monomials X^α, X^β a number $b_X(X^\alpha, X^\beta) \in \mathbb{N}$ called the **blow-up height** of the pair (X^α, X^β), also denoted by $b(X^\alpha, X^\beta)$ if $X = (X_1,\dots,X_m)$ is clear from context, as follows:

Special case: $\gcd(X^\alpha, X^\beta) = 1$. We let $a := |\{i \in \{1,\dots,m\} : \alpha_i \neq 0\}|$ and $b := |\{j \in \{1,\dots,m\} : \beta_j \neq 0\}|$, and we put

$$b(X^\alpha, X^\beta) := \begin{cases} 0 & \text{if } X^\alpha = 1 \text{ or } X^\beta = 1, \\ a+b & \text{otherwise.} \end{cases}$$

General case. This is reduced to the special case by setting $b(X^\alpha, X^\beta) := b(X^{\alpha-\omega}, X^{\beta-\omega})$, where $X^\omega = \gcd(X^\alpha, X^\beta)$.

4.10 Lemma. 1. *$b(X^\alpha, X^\beta) = 0$ if and only if $X^\alpha | X^\beta$ or $X^\beta | X^\alpha$.*

2. *If $b(X^\alpha, X^\beta) = 0$ then $b(s_{ij}^\gamma(X^\alpha), s_{ij}^\gamma(X^\beta)) = 0$.*
3. *$b(X^\alpha, X^\beta) = b(X^\beta, X^\alpha)$.*

4. *If $b(X^\alpha, X^\beta) \neq 0$, then there are $\gamma > 0$ and distinct $i, j \in \{1, \dots, m\}$ such that*

$$b(s_{ij}^\gamma(X^\alpha), s_{ij}^\gamma(X^\beta)) < b(X^\alpha, X^\beta)$$

and

$$b(s_{ji}^{1/\gamma}(X^\alpha), s_{ji}^{1/\gamma}(X^\beta)) < b(X^\alpha, X^\beta).$$

Proof. (1),(2) and (3) are easy, so we prove (4).

Let $b(X^\alpha, X^\beta) \neq (0,0)$. Using the notation of 4.9 above, we assume first that $\gcd(X^\alpha, X^\beta) = 1$ with $X^\alpha \neq 1$ and $X^\beta \neq 1$. Take $i, j \in \{1, \dots, m\}$ with $\alpha_i \neq 0$ and $\beta_j \neq 0$ (so $i \neq j$), and let $\gamma := \beta_j/\alpha_i$. Then $s_{ij}^\gamma(X^\alpha) = X^\alpha X_j^{\beta_j}$ and $s_{ij}^\gamma(X^\beta) = X^\beta$. Dividing $X^\alpha X_j^{\beta_j}$ and X^β by their gcd $X_j^{\beta_j}$ we see that $b(s_{ij}^\gamma(X^\alpha), s_{ij}^\gamma(X^\beta)) < b(X^\alpha, X^\beta)$; similarly for $s_{ji}^{1/\gamma}$.

In the general case, take distinct $i, j \in \{1, \dots, m\}$ and $\gamma > 0$ such that

$$b(s_{ij}^\gamma(X^{\alpha-\omega}), s_{ij}^\gamma(X^{\beta-\omega})) < b(X^\alpha, X^\beta)$$

and

$$b(s_{ji}^{1/\gamma}(X^{\alpha-\omega}), s_{ji}^{1/\gamma}(X^{\beta-\omega})) < b(X^\alpha, X^\beta),$$

where $X^\omega = \gcd(X^\alpha, X^\beta)$. The identity $s_{ij}^\gamma(X^\alpha) = s_{ij}^\gamma(X^\omega) s_{ij}^\gamma(X^{\alpha-\omega})$ then implies $b(s_{ij}^\gamma(X^\alpha), s_{ij}^\gamma(X^\beta)) = b(s_{ij}^\gamma(X^{\alpha-\omega}), s_{ij}^\gamma(X^{\beta-\omega}))$; hence $b(s_{ij}^\gamma(X^\alpha), s_{ij}^\gamma(X^\beta)) < b(X^\alpha, X^\beta)$. The case of $s_{ji}^{1/\gamma}$ is again similar. $\square$

4.11. Next we consider a finite collection $\mathcal{G} = \{X^{\alpha(1)}, \dots, X^{\alpha(k)}\}$ of k distinct monomials in X^*, and define

$$s_{ij}^\gamma(\mathcal{G}) := \{s_{ij}^\gamma(X^{\alpha(1)}), \dots, s_{ij}^\gamma(X^{\alpha(k)})\}.$$

We associate to $\mathcal{G}$ the pair $b_X(\mathcal{G}) = (p,q) \in \mathbb{N}^2$ defined as follows: if there are pairs (l, l') with $1 \leq l < l' \leq k$ and $b(X^{\alpha(l)}, X^{\alpha(l')}) \neq 0$, then $p :=$ number of such pairs and $q :=$ minimum of the blow-up heights of all such pairs; if no such pairs exist, then $(p,q) := (0,0)$. Again, if $X = (X_1, \dots, X_m)$ is clear from the context we just write $b(\mathcal{G})$ for $b_X(\mathcal{G})$. We also order $\mathbb{N}^2$ lexicographically in what follows.

Note that $b(\mathcal{G}) = (0,0)$ means that $\mathcal{G}$ is linearly ordered by divisibility.

4.12 Lemma. 1. *If $\mathcal{G}' \subseteq \mathcal{G}$ then $b(\mathcal{G}') \leq b(\mathcal{G})$.*

2. *If $b(\mathcal{G}) \neq (0,0)$, then there are $\gamma > 0$ and distinct $i, j \in \{1, \dots, m\}$ such that*

$$b(s_{ij}^\gamma(\mathcal{G})) < b(\mathcal{G}) \text{ and } b(s_{ji}^{1/\gamma}(\mathcal{G})) < b(\mathcal{G}).$$

Proof. (1) is easy.

For (2), let $b(\mathcal{G}) = (p,q)$ with $p \in \mathbb{N} - \{0\}$, and consider monomials $X^\alpha, X^\beta \in \mathcal{G}$ for which $b(X^\alpha, X^\beta) = q$. By (4) of the previous lemma, we get $\gamma > 0$ and distinct $i, j \in \{1, \dots, m\}$ such that

$$b(s_{ij}^\gamma(X^\alpha), s_{ij}^\gamma(X^\beta)) < b(X^\alpha, X^\beta) \text{ and } b(s_{ji}^{1/\gamma}(X^\alpha), s_{ji}^{1/\gamma}(X^\beta)) < b(X^\alpha, X^\beta).$$

Then it follows from (2) of the previous lemma that

$$b(s_{ij}^\gamma(\mathcal{G})) < b(\mathcal{G}) \text{ and } b(s_{ji}^{1/\gamma}(\mathcal{G})) < b(\mathcal{G}).$$

$\square$

4.13. We now extend s_{ij}^{γ} to an injective A-algebra endomorphism of $A[\![X^*]\!]$ by putting $s_{ij}^{\gamma}(\sum f_\alpha X^\alpha) := \sum f_\alpha s_{ij}^{\gamma}(X^\alpha)$. To avoid too many nested parentheses, we will write $s_{ij}^{\gamma} f$ instead of $s_{ij}^{\gamma}(f)$.

Consider a finite collection $\mathcal{F} \subseteq A[\![X^*]\!]$ of generalized power series. For distinct $i, j \in \{1, \ldots, m\}$ we put $s_{ij}^{\gamma}(\mathcal{F}) = \{s_{ij}^{\gamma} f : f \in \mathcal{F}\}$, and let $b_X(\mathcal{F}) := b_X(\mathcal{G})$, where $\mathcal{G} := \left\{X^\alpha : \alpha \in \bigcup_{f \in \mathcal{F}} \left(\operatorname{supp}(f)\right)_{\min}\right\}$ is the (by lemma 4.2 finite) set of "minimal monomials" of members of $\mathcal{F}$. The elements of $\mathcal{G}$ are called the **minimal monomials of** $\mathcal{F}$, and $b_X(\mathcal{F})$ is the **blow-up height of** $\mathcal{F}$. (As before we write $b(\mathcal{F})$ if X is clear from the context.)

Note that each $f \in \mathcal{F}$ can be written as $f = \sum X^\omega g_\omega$, where the sum is over $\operatorname{supp}(f)_{\min}$ and each $g_\omega \in A[\![X^*]\!]$ satisfies $g_\omega(0) \neq 0$.

4.14 Proposition. 1. *If $b(\mathcal{F}) \neq (0,0)$, then there are $\gamma > 0$ and distinct $i, j \in \{1, \ldots, m\}$ such that $b(s_{ij}^{\gamma}(\mathcal{F})) < b(\mathcal{F})$ and $b(s_{ji}^{1/\gamma}(\mathcal{F})) < b(\mathcal{F})$.*

2. *If $b(\mathcal{F}) = (0,0)$, then each nonzero $f \in \mathcal{F}$ is of the form $f = X^\omega g$ with $g \in A[\![X^*]\!]$, $g(0) \neq 0$.*

Proof. For (1), using the previous lemma we get $\gamma > 0$ and distinct $i, j \in \{1, \ldots, m\}$ such that $b(s_{ij}^{\gamma}(\mathcal{G})) < b(\mathcal{G})$ and $b(s_{ji}^{1/\gamma}(\mathcal{G})) < b(\mathcal{G})$. Note that each monomial in $s_{ij}^{\gamma}(\mathcal{G})$ has a nonzero coefficient in some member of $s_{ij}^{\gamma}(\mathcal{F})$, and that each monomial with a nonzero coefficient in some member of $s_{ij}^{\gamma}(\mathcal{F})$ is divisible by a monomial in $s_{ij}^{\gamma}(\mathcal{G})$. Hence the minimal monomials of $s_{ij}^{\gamma}(\mathcal{F})$ belong to $s_{ij}^{\gamma}(\mathcal{G})$. Therefore by lemma 4.10, part (1), we get $b(s_{ij}^{\gamma}(\mathcal{F})) \leq b(s_{ij}^{\gamma}(\mathcal{G})) < b(\mathcal{F})$. Similarly we obtain $b(s_{ji}^{1/\gamma}(\mathcal{F})) < b(\mathcal{F})$.

For (2), if $b(\mathcal{F}) = (0,0)$ then $\mathcal{G}$ is linearly ordered by divisibility; hence the desired result. ∎

4.15. *Mixed series.* Let $(X, Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ be a tuple of $m + n$ distinct indeterminates. According to 4.6 a series $\sum a_{\alpha,\beta} X^\alpha Y^\beta$ in $A[\![(X,Y)^*]\!]$ can also be written as $\sum_\beta (\sum_\alpha a_{\alpha,\beta} X^\alpha) Y^\beta$. But $\sum_\beta (\sum_\alpha a_{\alpha,\beta} X^\alpha) Y^\beta$ is also (the notation for) a power series in $A[\![X^*]\!][\![Y^*]\!]$. These two ways of reading $\sum(\sum a_{\alpha,\beta} X^\alpha) Y^\beta$ agree, provided we identify the ring $A[\![(X,Y)^*]\!]$ with a subring of $A[\![X^*]\!][\![Y^*]\!]$ via the injective ring homomorphism $A[\![(X,Y)^*]\!] \longrightarrow A[\![X^*]\!][\![Y^*]\!]$ given by $\sum a_{\alpha,\beta} X^\alpha Y^\beta \mapsto \sum(\sum a_{\alpha,\beta} X^\alpha) Y^\beta$. This identification will often be made without further comment. Note that this homomorphism is not surjective in general: with $m, n > 0$, the series $\sum_{k=1}^{\infty} X_1^{1/k} Y_1^k$ is in $A[\![X^*]\!][\![Y^*]\!]$, but not in (the image of) $A[\![(X,Y)^*]\!]$. On the other hand, $A[\![X^*]\!][Y^*] \subseteq A[\![(X,Y)^*]\!]$.

We shall also be working with the subring $A[\![X^*, Y]\!]$ of $A[\![(X,Y)^*]\!]$, consisting of those $f \in A[\![(X,Y)^*]\!]$ in which the Y-indeterminates have only natural numbers as exponents. Similarly to the above, we identify $A[\![X^*, Y]\!]$ with the corresponding subring of $A[\![X^*]\!][\![Y]\!]$; note that by the example above $A[\![X^*, Y]\!] \neq A[\![X^*]\!][\![Y]\!]$, for $m, n > 0$.

4.16 Definition. Let $n > 0$. A power series $f \in A[\![X^*, Y]\!]$ is called **regular in Y_n** if $f(0, 0, Y_n) = u Y_n^d +$ terms of higher degree in Y_n, with u a unit in A; with this d we call f **regular in Y_n of order** d. We put $Y' := (Y_1, \ldots, Y_{n-1})$.

4.17 Weierstrass Division and Preparation. *Let $n > 0$ and let $f \in A[\![X^*, Y]\!]$ be regular in Y_n of order d.*

1. *There is for each* $g \in A[\![X^*, Y]\!]$ *a unique pair* (Q, R) *with* $Q \in A[\![X^*, Y]\!]$ *and* $R \in A[\![X^*, Y']\!][Y_n]$, *such that*

$$g = Qf + R \text{ and } \deg_{Y_n}(R) < d.$$

2. f *factors uniquely as* $f = UW$, *where* $U \in A[\![X^*, Y]\!]$ *is a unit and* $W \in A[\![X^*, Y']\!][Y_n]$ *is monic of degree* d *in* Y_n.

Proof. (1) The proof below is adapted from [2]. Writing $f = \sum_{k\in\mathbb{N}} f_k Y_n^k$ with each $f_k \in A[\![X^*, Y']\!]$, the coefficients $f_0, \ldots, f_{d-1}$ have order $\geq \delta$ for some $\delta > 0$, while f_d is a unit in $A[\![X^*, Y']\!]$. Thus, taking

$$u := \sum_{k\geq d} f_k Y_n^{k-d},$$

u is a unit in $A[\![X^*, Y]\!]$. Then

$$\begin{aligned} u^{-1} f &= u^{-1}\left(\sum_{k<d} f_k Y_n^k + \sum_{k\geq d} f_k Y_n^k\right) \\ &= u^{-1}\left(\sum_{k<d} f_k Y_n^k + u Y_n^d\right) \\ &= u^{-1}\left(\sum_{k<d} f_k Y_n^k\right) + Y_n^d. \end{aligned}$$

So, replacing f by $u^{-1}f$, we may as well assume that $f = Y_n^d - F$ with $F \in \mathcal{M}[\![Y_n]\!]$, where $\mathcal{M} \subseteq A[\![X^*, Y']\!]$ is the ideal of power series of order $\geq \delta$.

Claim 1. For each $G \in \mathcal{M}^l[\![Y_n]\!]$, there are $Q \in \mathcal{M}^l[\![Y_n]\!]$, $R \in \mathcal{M}^l[Y_n]$ of degree $< d$ in Y_n, and $L(G) \in \mathcal{M}^{l+1}[\![Y_n]\!]$, such that $G = Qf + R + L(G)$.

To see this, write $G = \sum_{k\in\mathbb{N}} G_k Y_n^k$ with $G_k \in \mathcal{M}^l$, so that $G = \sum_{k<d} G_k Y_n^k + Y_n^d \sum_{k\geq d} G_k Y_n^{k-d}$; hence with $R := \sum_{k<d} G_k Y_n^k$ and $Q := \sum_{k\geq d} G_k Y_n^{k-d}$ we have $G = Q(Y_n^d - F) + R + L(G)$, where $L(G) := FQ$. Clearly Claim 1 holds for this choice of Q, R and $L(G)$.

We now proceed with the proof of the existence part. Given g, we apply the claim successively to g, $L(g)$, $L(L(g)) = L^2(g)$, $\ldots$:

$$\begin{aligned} g &= Q_0 f + R_0 + L(g), \\ L(g) &= Q_1 f + R_1 + L^2(g), \\ &\vdots \\ L^l(g) &= Q_l f + R_l + L^{l+1}(g), \\ &\vdots \end{aligned}$$

with $Q_l \in \mathcal{M}^l[\![Y_n]\!]$, $R_l \in \mathcal{M}^l[Y_n]$, $\deg_{Y_n}(R_l) < d$ and $L^l(g) \in \mathcal{M}^{l+1}[\![Y_n]\!]$. Thus the power series $Q := \sum_{l\in\mathbb{N}} Q_l$ and $L := \sum_{l\in\mathbb{N}} L^l(g)$ and the polynomial $R := \sum_{l\in\mathbb{N}} R_l$ are well defined elements of $A[\![X^*]\!][\![Y]\!]$ and $A[\![X^*]\!][\![Y']\!][Y_n]$, respectively, and adding up the rows above gives $g = Qf + R$ in $A[\![X^*]\!][\![Y]\!]$; it remains to verify that $\mathrm{supp}(Q)$ and $\mathrm{supp}(R)$ are good subsets of $[0, \infty)^{m+n}$.

For $H \in A[\![X^*]\!][\![Y]\!]$ we put

$$\operatorname{supp}'(H) := \{(\alpha, \beta') \in [0,\infty)^m \times \mathbb{N}^{n-1} : (\alpha, \beta', \beta_n) \in \operatorname{supp}(H) \text{ for some } \beta_n \in \mathbb{N}\};$$

it is clearly enough to show that $\operatorname{supp}'(Q)$ and $\operatorname{supp}'(R)$ are good subsets of $[0,\infty)^{m+n-1}$.

Claim 2. $\operatorname{supp}'(L^l(g)), \operatorname{supp}'(Q_l) \subseteq \Sigma(\operatorname{supp}'(F) \cup \operatorname{supp}'(g))$ for all $l \in \mathbb{N}$.

This is trivial for $l = 0$. If $l > 0$ and the claim holds for $l-1$ in place of l, then

$$\begin{aligned}\operatorname{supp}'(L^l(g)) &= \operatorname{supp}'(FQ_{l-1}) \\ &\subseteq \operatorname{supp}'(F) + \operatorname{supp}'(Q_{l-1}) \\ &\subseteq \operatorname{supp}'(F) + \Sigma(\operatorname{supp}'(F) \cup \operatorname{supp}'(g)) \\ &\subseteq \Sigma(\operatorname{supp}'(F) \cup \operatorname{supp}'(g)),\end{aligned}$$

and hence $\operatorname{supp}'(Q_l) \subseteq \operatorname{supp}'(L^l(g)) \subseteq \Sigma(\operatorname{supp}'(F) \cup \operatorname{supp}'(g))$, which establishes Claim 2.

Therefore also $\operatorname{supp}'(R_l) \subseteq \Sigma(\operatorname{supp}'(F) \cup \operatorname{supp}'(g))$ for all l, so

$$\operatorname{supp}'(Q), \operatorname{supp}'(R) \subseteq \Sigma(\operatorname{supp}'(F) \cup \operatorname{supp}'(g)),$$

which together with lemma 4.3 implies that $\operatorname{supp}'(Q)$ and $\operatorname{supp}'(R)$ are good subsets of $[0,\infty)^{m+n-1}$, as desired.

For the uniqueness, suppose $g = Q_1 f + R_1 = Q_2 f + R_2$ with each (Q_i, R_i) satisfying the conclusions of the theorem. Then $Qf = R$ where $Q = Q_1 - Q_2$ and $R = R_1 - R_2$, so $\deg_{Y_n}(R) < d$. It suffices to derive $Q = 0$. Suppose $Q = \sum_{k\in\mathbb{N}} q_k Y_n^k \in \mathcal{M}^l[\![Y_n]\!]$ for some $l \in \mathbb{N}$. For any k the coefficient of Y_n^{d+k} in Qf is 0, so

$$0 = q_k f_d + \sum_{h<k} q_h f_{d+k-h} + \sum_{k<h\leq k+d} q_h f_{d+k-h}.$$

Since $f = Y_n^d - F$ with $F \in \mathcal{M}[\![Y_n]\!]$ and f_d is a unit, it follows that $q_k \in \mathcal{M}^{l+1}$. The index k was arbitrary, so we have shown that $Q \in \mathcal{M}^l[\![Y_n]\!]$ implies $Q \in \mathcal{M}^{l+1}[\![Y_n]\!]$, i.e. $Q = 0$.

(2) Writing again $f = \sum_{k\in\mathbb{N}} f_k Y_n^k$ with $f_k \in A[\![X^*, Y']\!]$, we get from Weierstrass division that $Y_n^d = qf + r$, with $q = \sum_{k\in\mathbb{N}} q_k Y_n^k$, $q_k \in A[\![X^*, Y']\!]$, and $r = r_0 + r_1 Y_n + \cdots + r_{d-1} Y_n^{d-1}$ with $r_h \in A[\![X^*, Y']\!]$ for $h < d$. Substituting $(0, 0, Y_n)$ for (X, Y', Y_n) gives the following equation in $A[\![Y_n]\!]$:

$$\begin{aligned}Y_n^d = {}&\left(\sum_{k\in\mathbb{N}} q_k(0) Y_n^k\right)\big(f_d(0) Y_n^d + \text{ higher degree terms}\big) \\ &+ r_0(0) + \cdots + r_{d-1}(0) Y_n^{d-1}.\end{aligned}$$

Comparing coefficients of Y_n^d gives $q_0(0) f_d(0) = 1$; hence q_0 is a unit in $A[\![X^*, Y']\!]$, and therefore q is a unit in $A[\![X^*, Y]\!]$. Thus $f = UW$ with $U = q^{-1}$ and $W = Y_n^d - R$, which proves existence. Uniqueness follows similarly by arguing backwards, and using the uniqueness in the Weierstrass division formula $Y_n^d = qf + r$. $\square$

5. Convergent Generalized Power Series

5.1. We let r and s denote polyradii (with m components unless indicated otherwise), and we write $r \leq s$ if $r_i \leq s_i$ for all i, and $r < s$ if $r_i < s_i$ for all i. Also $r^\alpha = r_1^{\alpha_1} \cdots r_m^{\alpha_m}$.

5.2. In this section A is a normed ring with norm $|\cdot|$. For $f(X) = \sum f_\alpha X^\alpha \in A[\![X^*]\!]$ and a polyradius r we define

$$\|f\|_r := \sum |f_\alpha| r^\alpha \in [0, \infty].$$

We then have, for $f, g \in A[\![X^*]\!]$ and polyradii r, s:

1. $\|f\|_r = 0$ if and only if $f = 0$;
2. $\|f + g\|_r \leq \|f\|_r + \|g\|_r$;
3. $\|fg\|_r \leq \|f\|_r \|g\|_r$;
4. if $r \leq s$, then $\|f\|_r \leq \|f\|_s$.

We only prove (3), the other rules being obvious. Let $f(X) = \sum f_\alpha X^\alpha$ and $g(X) = \sum g_\alpha X^\alpha$. Then

$$\|fg\|_r = \sum_\alpha \left| \sum_{\beta+\gamma=\alpha} f_\beta g_\gamma \right| r^\alpha \leq \sum_{\beta,\gamma} |f_\beta||g_\gamma| r^\beta r^\gamma = \|f\|_r \|g\|_r.$$

5.3. We now define $A\{X^*\}_r := \{f \in A[\![X^*]\!] : \|f\|_r < \infty\}$. Note that $A\{X^*\}_r$ is a normed ring with norm $\|\cdot\|_r$. It is clearly a subring of $A[\![X^*]\!]$ containing $A[X^*]$. We put $A\{X^*\} := \bigcup_r A\{X^*\}_r$. Since $A\{X^*\}_r \supseteq A\{X^*\}_s$ if $r \leq s$, $A\{X^*\}$ is also a subring of $A[\![X^*]\!]$. Put $A\{X^*, Y\} := A[\![X^*, Y]\!] \cap A\{(X, Y)^*\}$, and $A\{X^*, Y\}_{r,s} := A[\![X^*, Y]\!] \cap A\{(X, Y)^*\}_{(r,s)}$ for polyradii $r = (r_1, \ldots, r_m)$, $s = (s_1, \ldots, s_n)$.

Note that if $f(X) = \sum f_\alpha X^\alpha \in A\{X^*\}_r$, then $|f_\alpha| \leq \|f\|_r / r^\alpha$.

It follows that if $\{f_k(X) = \sum f_{k,\alpha} X^\alpha\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $A\{X^*\}_r$, then $\{f_{k,\alpha}\}_{k \in \mathbb{N}}$ is a Cauchy sequence in A for each α. If moreover $\lim_{k \to \infty} f_{k,\alpha} = f_\alpha \in A$ for every α, we say that the sequence $\{f_k\}$ **has formal limit** $f(X) = \sum f_\alpha X^\alpha$. The trouble is that $\mathrm{supp}(f)$ need not be a good subset of $[0, \infty)^m$ any more: take for instance $A = \mathbb{R}$, $m = 1$ and $f_k(X) = \sum_{l=1}^k \frac{1}{l^2} X^{1/l}$; then $f(X) = \sum_{l=1}^\infty \frac{1}{l^2} X^{1/l}$. But we can still say the following:

5.4 Lemma. *Let $\{f_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence in $A\{X^*\}_r$ with formal limit f such that* $\mathrm{supp}(f)$ *is a good subset of* $[0, \infty)^m$. *Then $f \in A\{X^*\}_r$.*

Proof. Writing $f(X) = \sum f_\alpha X^\alpha$, we have to show that $f \in A\{X^*\}_r$ and that $f_k \to f$ in the normed ring $A\{X^*\}_r$.

Let $\epsilon > 0$ and take $M = M(\epsilon)$ so large that $\|f_k - f_l\|_r < \epsilon$ for all $k, l > M$. Then we have, for $k, l > M$ and any finite subset $I \subseteq \mathrm{supp}(f)$,

$$\begin{aligned}\sum_{\alpha \in I} |f_\alpha - f_{k,\alpha}| r^\alpha &\leq \sum_{\alpha \in I} |f_\alpha - f_{l,\alpha}| r^\alpha + \sum_{\alpha \in I} |f_{l,\alpha} - f_{k,\alpha}| r^\alpha \\ &\leq \sum_{\alpha \in I} |f_\alpha - f_{l,\alpha}| r^\alpha + \epsilon.\end{aligned}$$

Fixing I and k and letting $l \to \infty$ in this inequality gives $\sum_{\alpha \in I} |f_\alpha - f_{k,\alpha}| r^\alpha \leq \epsilon$, and fixing k and increasing I gives $\|f - f_k\|_r \leq \epsilon$, for each $k > M$. Hence $\|f\|_r \leq \|f - f_k\|_r + \|f_k\|_r < \infty$, so $f \in A\{X^*\}_r$ and $f_k \to f$ in the normed ring $A\{X^*\}_r$. $\square$

5.5 Lemma. *If $f = \sum f_\alpha X^\alpha \in A\{X^*\}$, then* $\lim_{r \to 0} \|f\|_r = |f(0)|$.

Proof. It suffices to show that $\lim_{r\to 0}\|f-f(0)\|_r = 0$, so replacing f by $f-f(0)$ we may as well assume that $f(0)=0$. Take s such that $\|f\|_s < \infty$, and fix $\epsilon > 0$. Let $I \subseteq \operatorname{supp}(f)$ be finite such that $\sum_{\alpha\notin I} |f_\alpha| s^\alpha < \epsilon/2$, and let $\rho \le s$ be a polyradius such that $\sum_{\alpha\in I} |f_\alpha|\rho^\alpha < \epsilon/2$. Then for every $r \le \rho$,

$$\|f\|_r \le \sum_{\alpha\in I} |f_\alpha| r^\alpha + \sum_{\alpha\notin I} |f_\alpha| s^\alpha \le \epsilon.$$

Since ϵ was arbitrary, this proves the lemma. □

5.6 Corollary. *Let $f \in A\{X^*\}$. Then*

1. *f is a unit in $A\{X^*\}$ if and only if $f(0)$ is a unit in A, and*
2. *each $f \in A\{X^*\}$ with $\operatorname{ord}(f) > 0$ is of the form $f = X_1^{\gamma_1} f_1 + \cdots + X_m^{\gamma_m} f_m$ with real numbers $\gamma_1, \ldots, \gamma_m > 0$ and $f_i \in A\{(X_1,\ldots,X_i)^*\}$ for all i.*

Also, if $m \ge 1$, then $A\{X^*\} \cap A[\![(X_1,\ldots,X_{m-1})^*]\!] = A\{(X_1,\ldots,X_{m-1})^*\}$.

Proof. (1) The necessity is clear. Suppose then $f(0) \ne 0$ and write $f = f(0)(1-g)$ for some $g \in A\{X^*\}$ with $g(0)=0$. Then $1-g$ has inverse $1+g+g^2+\ldots$ in $A[\![X^*]\!]$. Take r small enough so that $\|g\|_r < 1$ (possible by lemma 5.5). Then for every $n \in \mathbb{N}$,

$$\|1+g+g^2+\cdots+g^n\|_r \le 1 + \|g\|_r + \|g\|_r^2 + \cdots + \|g\|_r^n = \frac{1-\|g\|_r^{n+1}}{1-\|g\|_r},$$

so by lemma 5.4 the inverse $1+g+g^2+\ldots$ belongs to $A\{X^*\}_r$.

(2) follows from 4.8, since $\|f\|_r = r_1^{\gamma_1}\|f_1\|_r + \cdots + r_m^{\gamma_m}\|f_m\|_r$.

The last statement is obvious. □

5.7. Given any family $\{a_j\}_{j\in J}$ of elements of A, there is at most one element $a \in A$ such that for each $\epsilon > 0$ there is a finite subset $I(\epsilon) \subseteq J$ with $|\sum_{j\in I} a_j - a| < \epsilon$ for all finite sets $I \subseteq J$ that contain $I(\epsilon)$. If $a \in A$ has this property, we say that $\sum_{j\in J} a_j$ **exists in** A and define $\sum_{j\in J} a_j := a$. Note that $\sum_{j\in J} a_j$ certainly exists in A if A is complete and $\sum_{j\in J} |a_j| < \infty$. (One checks easily that in that case $a_j \ne 0$ for only countably many $j \in J$.)

We now modify 4.6 as follows: let J be any index set and assume that $\{f_j = \sum_\alpha f_{j,\alpha} X^\alpha\}_{j\in J}$ is a family in $A[\![X^*]\!]$ such that

1. for each $\alpha \in [0,\infty)^m$ we have $\sum_{j\in J} |f_{j,\alpha}| < \infty$ and $\sum_{j\in J} f_{j,\alpha}$ exists in A, and
2. $\bigcup_{j\in J} \operatorname{supp}(f_j)$ is a good subset of $[0,\infty)^m$.

Then $\sum f_j := \sum_\alpha \left(\sum_{j\in J} f_{j,\alpha}\right) X^\alpha \in A[\![X^*]\!]$, and one easily checks that $\|\sum f_j\|_r \le \sum \|f_j\|_r$.

Suppose now that $\sum \|f_j\|_r < \infty$; then our formal power series $\sum f_j$ actually belongs to $A\{X^*\}_r$. One checks easily that then $\sum f_j$ is also the sum of the family $\{f_j\}_{j\in J}$ in the sense of the normed ring $A\{X^*\}_r$.

5.8 Substitutions. A permutation σ of the set $\{1,\ldots,m\}$ induces a monoid isomorphism $\sigma : X^* \longrightarrow X^*$ defined by $\sigma(X^\alpha) := X_{\sigma(1)}^{\alpha_1} \cdots X_{\sigma(m)}^{\alpha_m}$, which in turn extends to an A-algebra automorphism of $A[\![X^*, Y]\!]$ by putting

$$\sigma\left(\sum f_{\alpha,\beta} X^\alpha Y^\beta\right) := \sum f_{\alpha,\beta}\,\sigma(X^\alpha) Y^\beta.$$

We usually write σf for $\sigma(f)$, where $f \in A[\![X^*, Y]\!]$. Also corresponding to σ we define a map $\sigma : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{m+n}$ by $\sigma(x, y) = (x_{\sigma(1)}, \dots, x_{\sigma(m)}, y)$. (For a polyradius $r = (r_1, \dots, r_m)$ the case $n = 0$ applies, so that $\sigma(r) = (r_{\sigma(1)}, \dots, r_{\sigma(m)})$.)

In a similar way, if $s^\gamma_{ij} : X^* \longrightarrow X^*$ is a singular blow-up substitution, then s^γ_{ij} extends to an A-algebra endomorphism s^γ_{ij} of $A[\![X^*, Y]\!]$ by setting

$$s^\gamma_{ij}\left(\sum f_{\alpha,\beta} X^\alpha Y^\beta\right) := \sum f_{\alpha,\beta} s^\gamma_{ij}(X^\alpha) Y^\beta.$$

We define the corresponding map $s^\gamma_{ij} : [0, \infty)^m \times \mathbb{R}^n \longrightarrow [0, \infty)^m \times \mathbb{R}^n$ by $s^\gamma_{ij}(x, y) = (x_1, \dots, x_{i-1}, x^\gamma_j x_i, x_{i+1}, \dots, x_m, y)$. (For a polyradius $r = (r_1, \dots, r_m)$ the case $n = 0$ applies, so that $s^\gamma_{ij}(r) = (r_1, \dots, r_{i-1}, r^\gamma_j r_i, r_{i+1}, \dots, r_m)$.)

Suppose now that $f = f(X, Y) \in A[\![X^*, Y]\!]$, and let $g = (g_1, \dots, g_n) \in A[\![Z]\!]^n$ with $g_1(0) = \cdots = g_n(0) = 0$. Since $A[\![X^*, Y]\!] \subseteq A[\![X^*]\!][\![Y]\!]$, we may substitute g for Y in f and obtain an element $f(X, g(Z)) \in A[\![X^*]\!][\![Z]\!]$. One easily checks that in fact $f(X, g(Z)) \in A[\![X^*, Z]\!]$.

Partial derivatives. The operation $f \mapsto \frac{\partial f}{\partial X_i}$ on $A[\![X]\!]$ does not extend naturally to $A[\![X^*]\!]$, but the modified operation $f \mapsto X_i \frac{\partial f}{\partial X_i}$ on $A[\![X]\!]$ does have a good extension ∂_i to $A[\![X^*]\!]$: given $f(X) = \sum f_\alpha X^\alpha \in A[\![X^*]\!]$, we define

$$\partial_i f(X) := \sum \alpha_i f_\alpha X^\alpha \in A[\![X^*]\!].$$

On the other hand, considering $f(X, Y) \in A[\![X^*, Y]\!]$ as an element of $A[\![X^*]\!][\![Y]\!]$, the partial derivatives $\partial f/\partial Y_j$ defined as usual belong to $A[\![X^*, Y]\!]$, and in fact $Y_j \partial f/\partial Y_j = \partial_{m+j} f$.

5.9 Lemma. *Let $f \in A\{X^*, Y\}_{r,s}$. Then*

1. *if ϕ is either a permutation of $\{1, \dots, m\}$ or a singular blow-up substitution on X^* with $m \geq 2$, and $\tilde{r}$ is a polyradius with $\phi(\tilde{r}) \leq r$, then $\phi f \in A\{X^*, Y\}_{\tilde{r},s}$;*
2. *if $g = (g_1, \dots, g_n) \in A\{Z\}^n_t$, where $g_1(0) = \cdots = g_n(0) = 0$ and $t = (t_1, \dots, t_l)$ is a polyradius with $\|g_j\|_t \leq s_j$ for each j, then $f(X, g(Z)) \in A\{X^*, Z\}_{r,t}$;*
3. *if $i \in \{1, \dots, m\}$, then $\partial_i f \in A\{X^*, Y\}_{\tilde{r},s}$ for each $\tilde{r} < r$, and if $j \in \{1, \dots, n\}$, then $\partial f/\partial Y_j \in A\{X^*, Y\}_{r,\tilde{s}}$ for each $\tilde{s} < s$.*

Proof. (1) Assume $f = \sum f_{\alpha,\beta} X^\alpha Y^\beta$. If ϕ is the singular blow-up substitution $s^\rho_{m,m-1}$ (with $m \geq 2$) and $\phi(t) \leq r$, then $\|\phi f\|_{t,s} = \sum |f_{\alpha,\beta}| t^\alpha t^{\rho\alpha_m}_{m-1} s^\beta \leq \|f\|_{r,s}$. The other case of ϕ and (2) are similar.

(3) Let $i \in \{1, \dots, m\}$. To simplify notation we assume that $n = 0$; the case $n > 0$ is similar. Write $f(X) = \sum f_\alpha X^\alpha$; then, with $\tilde{r} < r$,

$$\begin{aligned}
\sum \alpha_i |f_\alpha| \tilde{r}^\alpha &\leq \sum_{k=1}^\infty k \left(\sum_{k-1 \leq |\alpha| < k} |f_\alpha| \tilde{r}^\alpha \right) \\
&= \sum_{k=1}^\infty k \left(\sum_{k-1 \leq |\alpha| < k} |f_\alpha| \left(\frac{\tilde{r}}{r}\right)^\alpha r^\alpha \right) \\
&\leq \sum_{k=1}^\infty k \left|\frac{\tilde{r}}{r}\right|^{k-1} \left(\sum_{k-1 \leq |\alpha| < k} |f_\alpha| r^\alpha \right),
\end{aligned}$$

where $(\tilde{r}/r)^\alpha := (\tilde{r}_1/r_1)^{\alpha_1} \cdots (\tilde{r}_m/r_m)^{\alpha_m}$ and $|\tilde{r}/r| := \max\{\frac{\tilde{r}_1}{r_1}, \dots, \frac{\tilde{r}_m}{r_m}\} < 1$. Since $\lim_{k\to\infty} k|\tilde{r}/r|^{k-1} = 0$, there is a constant $C = C(\tilde{r}) > 0$ such that $k|\tilde{r}/r|^{k-1} \le C$ for all positive $k \in \mathbb{N}$, and so

$$\sum \alpha_i |f_\alpha| \tilde{r}^\alpha \le C \sum |f_\alpha| r^\alpha < \infty.$$

The assertion about $\partial f/\partial Y_j$ is proved in the same way. □

Remark. For later use in section 9 we consider here more closely the case $m = 1$, $n = 0$. Let $\partial := \partial_1$. The proof of (3) above shows that then $\|\partial f\|_{\tilde{r}} \le C\|f\|_r$, where we can take $C := |s \log s|^{-1}$ with $s := |\tilde{r}/r|$, since

$$\max_{x\ge 0} \left(x s^{x-1}\right) = \frac{-1}{\log s} s^{\frac{-1}{\log s} - 1} \le \frac{-1}{s \log s} = C.$$

Now let $r_1 := \tilde{r} s^{-1/2}$; then $\frac{\tilde{r}}{r_1} = \frac{r_1}{r} = s^{1/2}$ and the calculation above with r_1 in place of $\tilde{r}$ gives $\|\partial f\|_{r_1} \le C \cdot 2\|f\|_r$ (since $\frac{-1}{s^{1/2}\log(s^{1/2})} \le \frac{-2}{s \log s}$), and taking r_1, ∂f and $\partial^2 f$ in place of r, f and ∂f respectively, we get $\|\partial^2 f\|_{\tilde{r}} \le C \cdot 2 \|\partial f\|_{r_1} \le C^2 \cdot 2^2 \|f\|_r$. A similar argument with any $k \in \mathbb{N}$ gives

$$\|\partial^k f\|_{\tilde{r}} \le C^k k^k \|f\|_r.$$

5.10 Weierstrass Preparation. *Let $n > 0$, and let $f \in A\{X^*, Y\}$ be regular in Y_n of order d.*

1. *There is for each $g \in A\{X^*, Y\}$ a unique pair (Q, R) with $Q \in A\{X^*, Y\}$ and $R \in A\{X^*, Y'\}[Y_n]$, such that*

$$g = Qf + R \text{ and } \deg_{Y_n}(R) < d.$$

2. *f factors uniquely as $f = UW$, where $U \in A\{X^*, Y\}$ is a unit and $W \in A\{X^*, Y'\}[Y_n]$ is monic of degree d in Y_n.*

Proof. (1) Let $g \in A\{X^*, Y\}$. We use the same notations as in the proof of (1) in Theorem 4.17. Choose $s > 0$ so that

$$\|F\|_s \le \|u^{-1}\|_s \sum_{k<d} \|F_k\|_{s'} s_n^k < s_n^d,$$

and put $\epsilon := \|F\|_s s_n^{-d} < 1$. Writing $\mathcal{N} = \mathcal{M} \cap A\{X^*, Y'\}$ and making s smaller if necessary, we may assume that G in the claim of the proof of (1) of Theorem 4.17 is in $\mathcal{N}^l\{Y_n\}_s$, so

$$\|Q\|_s \le \sum_{k\ge d} \|G_k\|_{s'} s_n^{k-d} \le \|G\|_s s_n^{-d} \text{ and } \|R\|_s \le \|G\|_s,$$

while

$$\|L(G)\|_s \le \|Q\|_s \|F\|_s \le \epsilon \|G\|_s$$

by the definition of ϵ and the estimate on $\|Q\|_s$. Applying these norm estimates successively, we get

$$\|R_l\|_s \le \|L^l(g)\|_s \le \epsilon^l \|g\|_s$$

and

$$\|Q_l\|_s \le \|L^l(g)\|_s s_n^{-d} \le \epsilon^l \|g\|_s s_n^{-d},$$

so that

$$\|R\|_s \le \frac{\|g\|_s}{1-\epsilon} \text{ and } \|Q\|_s \le \frac{\|g\|_s s_n^{-d}}{1-\epsilon}.$$

(2) follows from the proof of (2) of theorem 4.17 and from (1) above. □

6. The Real Case

From now on we are only interested in the case $A = \mathbb{R}$, with the norm on $\mathbb{R}$ given by the usual absolute value. Note that Corollary 5.6 implies that $\mathbb{R}\{X^*\}$ is a local ring with maximal ideal $\{f \in \mathbb{R}\{X^*\} : f(0) = 0\}$, and if $m = 1$, then $\mathbb{R}\{X^*\}$ is a valuation ring.

6.1. Let $f = \sum f_\beta(X)Y^\beta \in \mathbb{R}\{X^*, Y\}$, $f \neq 0$, $n > 0$. Assume there is a monomial $X^\rho \in X^*$ such that $f(X,Y) = X^\rho F(X,Y)$ with $F = \sum F_\beta(X)Y^\beta \in \mathbb{R}\{X^*, Y\}$ and $F_\beta(0) \neq 0$ for at least one β. (Note that this always holds if $m = 1$.) Take $d \in \mathbb{N}$ minimal such that there is $\beta \in \mathbb{N}^n$ with $d = |\beta|$ and $F_\beta(0) \neq 0$.

Consider a linear substitution $\theta(Y) = (Y_1 + c_1Y_n, \ldots, Y_{n-1} + c_{n-1}Y_n, Y_n)$ with $c_1, \ldots, c_{n-1} \in \mathbb{R}$, and put $\theta g := g(X, \theta(Y))$ for $g \in \mathbb{R}\{X^*, Y\}$. Then

$$\begin{aligned}\theta F(0,0,Y_n) &= F(0, c_1Y_n, \ldots, c_{n-1}Y_n, Y_n)\\ &= P(c_1, \ldots, c_{n-1})Y_n^d \ + \ \text{terms of higher degree in } Y_n,\end{aligned}$$

where P is a nonzero polynomial in $c_1, \ldots, c_{n-1}$ depending only on f (not on $c_1, \ldots, c_{n-1}$). In summary we get

Lemma. *Let $f_1, \ldots, f_l \in \mathbb{R}\{X^*, Y\}\setminus\{0\}$ be such that each $f_i(X,Y) = X^{\rho_i}F_i(X,Y)$ for some suitable $\rho_i \in [0,\infty)^m$ and $F_i \in \mathbb{R}\{X^*, Y\}$ satisfying $F_i(0,Y) \neq 0$. Then there are infinitely many linear transformations $\theta(Y) = (Y_1 + c_1Y_n, \ldots, Y_{n-1} + c_{n-1}Y_n, Y_n)$ with $(c_1, \ldots, c_{n-1}) \in \mathbb{R}^{n-1}$ such that*

$$\theta f_i(X,Y) = X^{\rho_i}G_i(X,Y)$$

with each $G_i \in \mathbb{R}\{X^*, Y\}$ regular in Y_n for $i = 1, \ldots, l$.

6.2. Given a polyradius $\rho = (\rho_1, \ldots, \rho_{m+n})$, we put

$$I_{m,n,\rho} := [0,\rho_1] \times \cdots \times [0,\rho_m] \times [-\rho_{m+1}, \rho_{m+1}] \times \cdots \times [-\rho_{m+n}, \rho_{m+n}];$$

we will denote $[0,\infty)^m \times \mathbb{R}^n$ by $I_{m,n,\infty}$. We also write $\mathbb{R}\{X^*, Y\}_\rho$ instead of $\mathbb{R}\{X^*, Y\}_{r,s}$, where $r = (\rho_1, \ldots, \rho_m)$ and $s = (\rho_{m+1}, \ldots, \rho_{m+n})$. For $n = 0$ we write $I_{m,r}$ instead of $I_{m,0,\rho}$.

To an element $f(X,Y) = \sum f_{\alpha,\beta}X^\alpha Y^\beta \in \mathbb{R}\{X^*, Y\}_\rho$ we associate a function on $I_{m,n,\rho}$ as follows. Given $(x,y) \in I_{m,n,\rho}$, the series $\sum f_{\alpha,\beta}x^\alpha y^\beta$ converges absolutely to a real number which we denote by $f(x,y)$. The function $(x,y) \mapsto f(x,y) : I_{m,n,\rho} \longrightarrow \mathbb{R}$ is continuous, since by 5.7 f is the limit in the sense of the normed ring $\mathbb{R}\{X^*, Y\}_\rho$ of its partial sums $f_J := \sum_{(\alpha,\beta)\in J} f_{\alpha,\beta}X^\alpha Y^\beta$ with J finite, which implies that the corresponding continuous functions $(x,y) \mapsto f_J(x,y) : I_{m,n,\rho} \longrightarrow \mathbb{R}$ converge uniformly on $I_{m,n,\rho}$ to the function $(x,y) \mapsto f(x,y)$.

We shall denote the function $(x,y) \mapsto f(x,y) : I_{m,n,\rho} \longrightarrow \mathbb{R}$ by f_ρ. Note that the argument above shows that $\|f_\rho\|_{\sup} \le \|f\|_\rho$ for all $f \in \mathbb{R}\{X^*, Y\}_\rho$. Let $C(I_{m,n,\rho})$ be the ring of all real valued continuous functions on $I_{m,n,\rho}$. Part (1) of the following lemma shows that the map $f \mapsto f_\rho : \mathbb{R}\{X^*, Y\}_\rho \longrightarrow C(I_{m,n,\rho})$ is a ring homomorphism.

6.3 Lemma. *Let $f, g \in \mathbb{R}\{X^*, Y\}_\rho$. Then*

1. $(f+g)_\rho(x,y) = f_\rho(x,y) + g_\rho(x,y)$ *and* $(f \cdot g)_\rho(x,y) = f_\rho(x,y) \cdot g_\rho(x,y)$ *for all* $(x,y) \in I_{m,n,\rho}$*;*
2. *if* ϕ *is either a permutation of* $\{1,\dots,m\}$ *or a singular blow-up substitution on* X^* *with* $m \geq 2$, *and* $\tilde\rho$ *is a polyradius with* $\phi(\tilde\rho) \leq \rho$, *then* $(\phi f)_{\tilde\rho}(x,y) = f_\rho(\phi(x),y)$ *for all* $(x,y) \in I_{m,n,\tilde\rho}$*;*
3. *if* $g = (g_1,\dots,g_n) \in \mathbb{R}\{Z\}_t^n$, *where* $g_1(0) = \cdots = g_n(0) = 0$ *and* $t = (t_1,\dots,t_l)$ *is a polyradius with* $\|g_j\|_t \leq \rho_{m+j}$ *for* $j = 1,\dots,n$, *then with* $h(X,Z) := f(X, g(Z)) \in \mathbb{R}\{X^*, Z\}_\tau$ *where* $\tau = (\rho_1,\dots,\rho_m,t_1,\dots,t_l)$, *we have* $h_\tau(x,z) = f_\rho(x, g_t(z))$ *for all* $(x,z) \in I_{m,l,\tau}$*;*
4. *if* $j \in \{1,\dots,n\}$ *and* $\tilde\rho < \rho$, *then for each* $(x,y) \in I_{m,n,\tilde\rho}$ *the partial derivative* $(\partial f_{\tilde\rho}/\partial y_j)(x,y)$ *exists and* $(\partial f/\partial Y_j)_{\tilde\rho}(x,y) = \partial f_{\tilde\rho}/\partial y_j(x,y)$*;*
5. *if* $i \in \{1,\dots,m\}$ *and* $\tilde\rho < \rho$, *then for all* $(x,y) \in \operatorname{int}(I_{m,n,\tilde\rho})$ *the partial derivative* $(\partial f_{\tilde\rho}/\partial x_j)(x,y)$ *exists and* $x_j(\partial f_{\tilde\rho}/\partial x_j)(x,y) = (\partial_j f)_{\tilde\rho}(x,y)$.

Proof. These statements are obvious if f and g have finite support; hence by 6.2 they follow for general f and g. □

6.4 Lemma. *The map* $f \mapsto f_r : \mathbb{R}\{X^*\}_r \longrightarrow C(I_{m,r})$ *is injective.*

Proof. Let $f(X) = \sum f_\alpha X^\alpha \in \mathbb{R}\{X^*\}_r$ and assume $f \neq 0$; we will show that f_r cannot vanish identically on any $I_{m,\tilde r}$ with $\tilde r < r$ (which is more than what we need).

By induction on m: if $m = 1$ then $X = X_1$ and, assuming f has order δ, we can write $f(X) = X^\delta(f_\delta + \sum_{\alpha>\delta} X^{\alpha-\delta})$ with $f_\delta \neq 0$. By 6.2 the series $f_\delta + \sum_{\alpha>\delta} X^{\alpha-\delta}$ also gives rise to a function on $[0,r]$. It follows from Lemma 5.5 that $f_r(x) \neq 0$ for all $x \in (0,\tilde r]$, where $\tilde r > 0$ is small enough.

Let $m > 1$; assume our claim holds for $\mathbb{R}\{(X')^*\}_{r'}$. Write a nonzero $f \in \mathbb{R}\{X^*\}_r$ as $f(X) = \sum_{\alpha_m \geq 0} f_{\alpha_m}(X') X_m^{\alpha_m}$ with $f_{\alpha_m} \in \mathbb{R}\{(X')^*\}_{r'}$, and note that $\{\alpha_m : f_{\alpha_m} \neq 0\}$ is a well ordered subset of $[0,\infty)$. Hence $\|f\|_r = \sum \|f_{\alpha_m}\|_{r'} r_m^{\alpha_m}$ and $f_r(x) = \sum (f_{\alpha_m})_{r'}(x') x_m^{\alpha_m}$ for all $x = (x', x_m) \in I_{m,r}$. Fix some $\alpha_m \in [0,\infty)$ with $f_{\alpha_m}(X') \neq 0$; by the inductive assumption there are $x' \in I_{m-1,r'}$ arbitrarily close to the origin such that $(f_{\alpha_m})_{r'}(x') \neq 0$. For such x' we have shown above (case $m = 1$) that $f_r(x', x_m) = \sum (f_{\alpha_m})_{r'}(x') x_m^{\alpha_m}$ is nonzero for all sufficiently small $x_m \in (0, r_m]$. □

Remark. It follows from Lemma 6.4 that the map $f \mapsto f_\rho : \mathbb{R}\{X^*, Y\}_\rho \longrightarrow C(I_{m,n,\rho})$ is an injective ring homomorphism.

6.5 Lemma. *Let* $f \in \mathbb{R}\{X^*, Y\}_\rho$ *with* $m \geq 2$, *and let* $\gamma, \lambda > 0$. *Suppose* $\tau \leq \rho$ *is such that* $\tau_m < \lambda$ *and* $\tau_{m-1}^\gamma(\lambda + \tau_m) < \rho_m$. *Then there is a power series* $r(f) \in \mathbb{R}\{(X')^*, (X_m, Y)\}_\tau$ *such that*

$$r(f)_\tau(x', x_m, y) = f_\rho(x', x_{m-1}^\gamma(\lambda + x_m), y)$$

for every $(x', x_m, y) \in I_{m-1,n+1,\tau}$. *(Note that here we allow negative values for* x_m*.)*

Proof. Write $f(X,Y) = \sum_{t\geq 0} f_t(X',Y) X_m^t$. Formally substituting $X_{m-1}^\gamma(\lambda + X_m)$ for X_m and using the binomial expansion $(\lambda + X_m)^t := \sum_{k\in\mathbb{N}} \binom{t}{k} \lambda^{t-k} X_m^k$, we obtain the power series

$$\begin{aligned} r(f) :&= \sum_{t\geq 0}\left(f_t(X',Y)\sum_{k\in\mathbb{N}} X_{m-1}^{\gamma t}\binom{t}{k}\lambda^{t-k}X_m^k\right) \\ &= \sum_{k\in\mathbb{N}}\left(\sum_{t\geq 0}\binom{t}{k}\lambda^{t-k}f_t(X',Y)X_{m-1}^{\gamma t}\right)X_m^k \in \mathbb{R}[\![(X')^*,(X_m,Y)]\!]. \end{aligned}$$

Next we note that

$$(\dagger)\qquad \left\|(\lambda+X_m)^t\right\|_{\tau_m} := \sum_{k\in\mathbb{N}}\left|\binom{t}{k}\right|\lambda^{t-k}\tau_m^k \leq C(\lambda+\tau_m)^t$$

for some positive $C = C(\lambda,\tau_m)$ that is independent of $t \geq 0$.

(To see this, factor out λ^t and put $x := \tau_m/\lambda$, so that the problem is reduced to estimating $\sum_{k\in\mathbb{N}}\left|\binom{t}{k}\right|x^k$ for $0 \leq x < 1$. Using that $\binom{t}{k} \geq 0$ if $k \leq t+1$ and $\left|\binom{t}{k}\right| \leq 1$ if $k > t+1$, we obtain

$$\begin{aligned} \left|\sum_{k\in\mathbb{N}}\left|\binom{t}{k}\right|x^k-(1+x)^t\right| &= \left|\sum_{k>t+1}\left|\binom{t}{k}\right|x^k-\sum_{k>t+1}\binom{t}{k}x^k\right| \\ &\leq 2\sum_{k>t+1}\left|\binom{t}{k}\right|x^k \\ &\leq 2\frac{x^t}{1-x} \\ &\leq \frac{2}{1-x}(1+x)^t. \end{aligned}$$

Hence $\sum_{k\in\mathbb{N}}\left|\binom{t}{k}\right|x^k \leq \frac{3}{1-x}(1+x)^t$, which will do.)

From (†) we obtain easily that $\|r(f)\|_\tau \leq C\|f\|_\rho < \infty$; in particular, $r(f) \in \mathbb{R}\{(X')^*,(X_m,Y)\}_\tau$. One now easily checks that the power series $r(f)$ has the desired properties. ☐

Remark. The power series $r(f)$ with $\gamma,\lambda > 0$ and $m \geq 2$ obtained in lemma 6.5 is clearly independent of τ and is called a **regular blow-up** of f. (If we want to indicate the dependence on γ,λ, we write $r_\lambda^\gamma(f)$ instead of $r(f)$.) We also denote by $r : I_{m-1,n+1,\infty} \longrightarrow \mathbb{R}^{m+n}$ the corresponding map defined by $r(x,y) := (x', x_{m-1}^\gamma(\lambda+x_m), y)$.

The proof of the previous lemma with $\gamma = 0$ gives the following.

6.6 Lemma. 1. *Let $f \in \mathbb{R}\{X^*,Y\}_\rho$, $m \geq 1$, and let $\lambda \in (0,\rho_m)$. Suppose $\tau \leq \rho$ is such that $\tau_m < \min(\lambda,\rho_m-\lambda)$. Then there is a power series $t(f) \in \mathbb{R}\{(X')^*,(X_m,Y)\}_\tau$ such that*

$$t(f)_\tau(x',x_m,y) = f_\rho(x',\lambda+x_m,y)$$

for every $(x',x_m,y) \in I_{m-1,n+1,\tau}$.

2. *Let $f \in \mathbb{R}\{X^*,Y\}_\rho$, $n \geq 1$, and let $\lambda \in (-\rho_{m+n},\rho_{m+n})$. Suppose $\tau \leq \rho$ is such that $\tau_m < \rho_{m+n}-|\lambda|$. Then there is a power series $t(f) \in \mathbb{R}\{X^*,Y\}_\tau$*

such that

$$t(f)_\tau(x,y) = f_\rho(x, y', \lambda + y_n)$$

for every $(x,y) \in I_{m,n,\tau}$.

Clearly the series $t(f)$ (in both (1) and (2) above) is independent of the choice of τ. Applying this lemma repeatedly and permuting some variables if necessary, we obtain:

6.7 Corollary ("Taylor expansion"). *Let* $f \in \mathbb{R}\{X^*, Y\}_\rho$, *and let* $a \in I_{m,n,\rho}$ *be such that* $a_i < \rho_i$ *for* $1 \le i \le m$. *Put* $m' = |\{i : 1 \le i \le m, a_i = 0\}|$, *and choose any permutation* σ *of* $\{1,\dots,m\}$ *with* $\sigma(\{i : 1 \le i \le m, a_i = 0\}) = \{1,\dots,m'\}$. *Let* $n' := m + n - m'$ *and let* $\tau = (\tau_1, \dots, \tau_{m'+n'})$ *be a polyradius such that*

$$\tau_i < \begin{cases} \rho_{\sigma^{-1}(i)} & \text{if} \quad 1 \le i \le m', \\ \min(a_{\sigma^{-1}(i)}, \rho_{\sigma^{-1}(i)} - a_{\sigma^{-1}(i)}) & \text{if} \quad m' < i \le m, \\ \rho_i - |a_i| & \text{if} \quad m < i \le m' + n' = m + n. \end{cases}$$

(Hence $a + \sigma(z) \in I_{m,n,\rho}$ *for* $z \in I_{m',n',\tau}$.*) Then there is a unique power series* $T_a f \in \mathbb{R}\{U^*, V\}_\tau$, *where* $U = (U_1, \dots, U_{m'}), V = (V_1, \dots, V_{n'})$, *such that*

$$(T_a f)_\tau(z) = f_\rho(a + \sigma(z))$$

for every $z \in I_{m',n',\tau}$. *In particular,* f_ρ *is analytic on* $\operatorname{int}(I_{m,n,\rho})$.

7. Generalized Semianalytic Sets

Given a polyradius $\rho = (\rho_1, \dots, \rho_{m+n})$, recall that

$$I_{m,n,\rho} = [0,\rho_1] \times \cdots \times [0,\rho_m] \times [-\rho_{m+1}, \rho_{m+1}] \times \cdots \times [-\rho_{m+n}, \rho_{m+n}] \subseteq \mathbb{R}^{m+n}.$$

We also write $I_{m,n,\epsilon}$ for $I_{m,n,(\epsilon,\dots,\epsilon)}$, for positive real ϵ.

7.1 Definition. We let $\mathcal{R}_{m,n,\rho}$ be the set of all functions

$$(x,y) \mapsto f(x,y) \colon I_{m,n,\rho} \to \mathbb{R}$$

with $f \in \mathbb{R}\{x^*, Y\}_{\tilde\rho}$ for some $\tilde\rho > \rho$. Then $\mathcal{R}_{m,n,\rho}$ is an $\mathbb{R}$-algebra of real valued continuous functions on $I_{m,n,\rho}$.

A set $A \subseteq I_{m,n,\rho}$ is called a **basic $\mathcal{R}_{m,n,\rho}$-set** if

$$A = \{z \in I_{m,n,\rho} : f(z) = 0, g_1(z) > 0, \dots, g_k(z) > 0\}$$

for some $f, g_1, \dots, g_k \in \mathcal{R}_{m,n,\rho}$. A finite union of basic $\mathcal{R}_{m,n,\rho}$-sets is called an **$\mathcal{R}_{m,n,\rho}$-set**. Note that the $\mathcal{R}_{m,n,\rho}$-sets form a Boolean algebra of subsets of $I_{m,n,\rho}$.

7.2 Definition. Given a point $a = (a_1, \dots, a_{m+n}) \in \mathbb{R}^{m+n}$ and a choice of signs $\sigma \in \{-1,1\}^m$, we let $h_{a,\sigma} : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{m+n}$ be the bijection given by

$$h_{a,\sigma}(z) := (a_1 + \sigma_1 z_1, \dots, a_m + \sigma_m z_m, a_{m+1} + z_{m+1}, \dots, a_{m+n} + z_{m+n}).$$

A set $A \subseteq \mathbb{R}^{m+n}$ is called **$\mathcal{R}_{m,n}$-semianalytic at** the point $a \in \mathbb{R}^{m+n}$ if there is $\epsilon > 0$ such that for each $\sigma \in \{-1,1\}^m$ we have $A \cap h_{a,\sigma}(I_{m,n,\epsilon}) = h_{a,\sigma}(A_\sigma)$ for some $\mathcal{R}_{m,n,\epsilon}$-set $A_\sigma \subseteq I_{m,n,\epsilon}$. A set $A \subseteq \mathbb{R}^{m+n}$ is **$\mathcal{R}_{m,n}$-semianalytic** if it is $\mathcal{R}_{m,n}$-semianalytic at every point $a \in \mathbb{R}^{m+n}$. For convenience, if $A \subseteq \mathbb{R}^m$ is $\mathcal{R}_{m,0}$-semianalytic we also simply say that A is **$\mathcal{R}_m$-semianalytic**.

Note that if $A, B \subseteq \mathbb{R}^{m+n}$ are $\mathcal{R}_{m,n}$-semianalytic at a, then so are $A \cup B$, $A \cap B$ and $A \setminus B$. The maps $h_{a,\sigma}$ ($a \in \mathbb{R}^{m+n}, \sigma \in \{-1,1\}^m$) form a group of permutations of $\mathbb{R}^{m+n}$. Using this fact, it is easy to check that if $A \subseteq \mathbb{R}^{m+n}$ is $\mathcal{R}_{m,n}$-semianalytic, then each set $h_{a,\sigma}(A)$ is also $\mathcal{R}_{m,n}$-semianalytic, and that for each $\lambda \in (\mathbb{R} \setminus \{0\})^{m+n}$ the set $E_\lambda(A)$ is $\mathcal{R}_{m,n}$-semianalytic, where $E_\lambda : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{m+n}$ is given by $E_\lambda(z) = (\lambda_1 z_1, \dots, \lambda_{m+n} z_{m+n})$. Furthermore, if $A \subseteq \mathbb{R}^{m+n}$ is semianalytic at a, then A is $\mathcal{R}_{m,n}$-semianalytic at a. Finally, it follows from the definition above that each bounded $\mathcal{R}_{m,n}$-semianalytic set is quantifier-free definable in $\mathbb{R}_{\text{an}^*}$. Below we write 0 for the point $(0, \dots, 0) \in \mathbb{R}^{m+n}$.

7.3 Lemma. 1. *If $A \subseteq I_{m,n,\rho}$ is an $\mathcal{R}_{m,n,\rho}$-set, then A is $\mathcal{R}_{m,n}$-semianalytic at 0.*

2. *Let $n > 0$ and $A \subseteq I_{m,n,\rho}$ be $\mathcal{R}_{m,n}$-semianalytic at 0. Then A is also $\mathcal{R}_{m+1,n-1}$-semianalytic at 0.*
3. *Each $\mathcal{R}_{m,n}$-semianalytic subset of $\mathbb{R}^{m+n}$ is $\mathcal{R}_{m+n}$-semianalytic.*
4. *Let $A \subseteq \mathbb{R}^{m+n}$ be $\mathcal{R}_{m,n}$-semianalytic at 0 and let σ be a permutation of $\{1, \dots, m\}$. Then $\sigma(A)$ is $\mathcal{R}_{m,n}$-semianalytic at 0.*

Proof. (1) Clearly we may assume that A is a basic $\mathcal{R}_{m,n,\rho}$-set. Let $\epsilon > 0$ be such that $\epsilon < \rho_i$ for $i = 1, \dots, m+n$. Let $f, g_1, \dots, g_k \in \mathcal{R}_{m,n,\rho}$ be such that

$$A = \{z \in I_{m,n,\rho} : f(z) = 0, g_1(z) > 0, \dots, g_k(z) > 0\}.$$

For $\sigma \in \{-1, 1\}^m$ we define

$$A_\sigma := A \cap \{z \in I_{m,n,\epsilon} : z_i = 0 \text{ if } \sigma_i = -1, i = 1, \dots, m\}.$$

Then each A_σ is a (basic) $\mathcal{R}_{m,n,\epsilon}$-set, and since $A \cap h_{0,\sigma}(I_{m,n,\epsilon}) = h_{0,\sigma}(A_\sigma)$ for each σ, the first statement is proved.

(2) Let $\sigma \in \{-1, 1\}^{m+1}$ and write $\sigma' = (\sigma_1, \dots, \sigma_m)$. Then there is an $\mathcal{R}_{m,n,\epsilon}$-set $A_{\sigma'}$ for some $\epsilon > 0$, such that $A \cap h_{0,\sigma'}(I_{m,n,\epsilon}) = h_{0,\sigma'}(A_{\sigma'})$. Let the variables x, t, z range over $\mathbb{R}^m$, $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively. Now note that $I_{m,n,\epsilon} \supseteq I_{m+1,n-1,\epsilon}$ and $\{f|_{I_{m+1,n-1,\epsilon}} : f \in \mathcal{R}_{m,n,\epsilon}\} \subseteq \mathcal{R}_{m+1,n-1,\epsilon}$, so the set

$$A_\sigma := \left\{(x, t, z) \in \mathbb{R}^{m+1+(n-1)} : (x, \sigma_{m+1} t, z) \in A_{\sigma'} \cap h_{0,(1,\dots,1,\sigma_{m+1})}(I_{m+1,n-1,\epsilon})\right\}$$

is an $\mathcal{R}_{m+1,n-1,\epsilon}$-set if $\sigma_{m+1} = 1$. A similar argument shows that A_σ is an $\mathcal{R}_{m+1,n-1,\epsilon}$-set if $\sigma_{m+1} = -1$. But obviously

$$A \cap h_{0,\sigma}(I_{m+1,n-1,\epsilon}) = h_{0,\sigma}(A_\sigma).$$

(3) This is an easy consequence of (2).

(4) follows from Lemma 5.9, part (1), and Lemma 6.3, part (2). □

7.4 Lemma. *Every $\mathcal{R}_{m,n,\rho}$-set $A \subseteq I_{m,n,\rho}$ is $\mathcal{R}_{m,n}$-semianalytic.*

Proof. We may assume that A is a basic $\mathcal{R}_{m,n,\rho}$-set, so there are $f, g_1, \dots, g_k \in \mathbb{R}\{X^*, Y\}_{\tilde{\rho}}$ for some polyradius $\tilde{\rho} > \rho$ such that

$$A = \{z \in I_{m,n,\rho} : f(z) = 0, g_1(z) > 0, \dots, g_k(z) > 0\}.$$

Fix $a \in \mathbb{R}^{m+n}$. We will show that A is $\mathcal{R}_{m,n}$-semianalytic at a. If $a \notin I_{m,n,\rho}$ this is clear. Suppose that $a \in I_{m,n,\rho}$. By adding suitable equalities $z_i = \pm\rho_i$ and inequalities $-\rho_i < z_i < \rho_i$ to the description of A, and then increasing ρ (which is possible because $\tilde{\rho} > \rho$), we reduce to the case where $|a_i| < \rho_i$ for $i = 1, \dots, m+n$.

Let $\tilde{A} := A - a$, the translate of A by $-a$. It is clear from Definition 7.2 that A is $\mathcal{R}_{m,n}$-semianalytic at a if and only if $\tilde{A}$ is $\mathcal{R}_{m,n}$-semianalytic at 0.

We now apply Corollary 6.7 to the functions describing A. Let σ be the permutation of $\{1, \dots, m\}$ obtained from 6.7; by Lemma 7.3, part (4), it is enough to show that $\sigma^{-1}(\tilde{A})$ is $\mathcal{R}_{m,n}$-semianalytic at 0. By 6.7 there are natural numbers $m' \leq m$ and n' with $m'+n' = m+n$ and power series $T_a f, T_a g_1, \dots, T_a g_k$ defining functions in $\mathcal{R}_{m',n',\tau}$ for some polyradius $\tau = (\tau_1, \dots, \tau_{m'+n'})$, such that

$$\sigma^{-1}(\tilde{A}) \cap I_{m',n',\tau} = \{z \in I_{m',n',\tau} : T_a f(z) = 0, T_a g_1(z) > 0, \dots, T_a g_k(z) > 0\}.$$

Hence $\sigma^{-1}(\tilde{A}) \cap I_{m',n',\tau}$ is a basic $\mathcal{R}_{m',n',\tau}$-set. Together with Lemma 7.3, parts (1) and (2), and the fact that

$$\sigma^{-1}(\tilde{A}) \cap ([-\tau_1, \tau_1] \times \cdots \times [-\tau_{m'+n'}, \tau_{m'+n'}]) = \sigma^{-1}(\tilde{A}) \cap I_{m',n',\tau},$$

this implies that $\sigma^{-1}(\tilde{A})$ is $\mathcal{R}_{m,n}$-semianalytic at 0. $\square$

8. The Main Theorem

8.1. For $p \in \mathbb{N}$ we put, with $I = [-1,1]$,

$$\Lambda_p := \{X \subseteq I^p : X \text{ is } \mathcal{R}_p\text{-semianalytic}\}.$$

Note that if $X \subseteq I^p$ is $\mathcal{R}_{m,n}$-semianalytic with $m+n = p$, then X is also $\mathcal{R}_p$-semianalytic by 7.3, part (3), so $X \in \Lambda_p$.

The system (Λ_p) is easily seen to satisfy axioms (I)-(III) of section 2; in the following we verify axiom (IV) (see Corollary 8.15): *every* Λ*-set has the* Λ*-Gabrielov property.*

8.2. In this section it is convenient to work with a more general notion of dimension than the one given in the introduction. We call $M \subseteq \mathbb{R}^n$ a C^0**-manifold of dimension** d if $M \neq \emptyset$ and each point of M has an open neighbourhood in M homeomorphic to $\mathbb{R}^d$; in this case d is uniquely determined (by a theorem of Brouwer), and we write $d = \dim(M)$. Correspondingly, we say that a set $S \subseteq \mathbb{R}^n$ **has dimension** if S is a countable union of C^0-manifolds, and in that case we put

$$\dim(S) := \begin{cases} \max\{\dim(M) : M \subseteq S \text{ is a } C^0\text{-manifold}\} & \text{if } S \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

We then have (by a Baire category argument as in [4]): if $S = \bigcup_{i \in \mathbb{N}} S_i$ and each S_i has dimension, then S has dimension and $\dim(S) = \max\{\dim(S_i) : i \in \mathbb{N}\}$. It follows easily that if S has dimension in the sense of the introduction, then S has dimension in the present sense, and the two dimensions of S agree.

This extended notion of dimension is only a temporary convenience; once we have shown in 8.9 that the sets we are dealing with are finite unions of manifolds, these sets, whose dimension was up to then taken in the extended sense, have dimension in the original sense.

8.3 Definitions. Let $m, n \in \mathbb{N}$ and let $\rho = (\rho_1, \dots, \rho_{m+n})$ be a polyradius. We call $M \subseteq \mathbb{R}^{m+n}$ an $\mathcal{R}_{m,n,\rho}$**-manifold** if

(i) M is a basic $\mathcal{R}_{m,n,\rho}$-set contained in $\operatorname{int}(I_{m,n,\rho})$, and
(ii) there are $k \leq m+n$ and $f_1, \dots, f_k \in \mathcal{R}_{m,n,\rho}$ such that M is an $(m+n-k)$-dimensional manifold on which $f_1, \dots, f_k$ vanish identically and the gradients $\nabla f_1(z), \dots, \nabla f_k(z)$ are linearly independent at each $z \in M$.

For positive real ϵ we write $\rho \leq \epsilon$ if $\rho_i \leq \epsilon$ for $i = 1, \dots, m+n$.

Given $m' \geq m$ and $n' \geq n$, we let $\Pi^{m',n'}_{m,n} : \mathbb{R}^{m'+n'} \longrightarrow \mathbb{R}^{m+n}$ be the projection map given by $\Pi^{m',n'}_{m,n}(x_1, \dots, x_{m'}, y_1, \dots, y_{n'}) = (x_1, \dots, x_m, y_1, \dots, y_n)$; we will simply write $\Pi_{m,n}$ for $\Pi^{m',n'}_{m,n}$ if m' and n' are clear from the context.

A set $U \subseteq \operatorname{int}(I_{m,n,\infty})$ is an (m,n)**-corner** if there is $\delta > 0$ with $\operatorname{int}(I_{m,n,\delta}) \subseteq U$.

Let $f = (f_1, \dots, f_\mu) \in \mathbb{R}\{X^*, Y\}^\mu$. We say that $\epsilon > 0$ is f**-admissible** if $f \in \mathbb{R}\{X^*, Y\}_\delta^\mu$ for some $\delta > \epsilon$. For f-admissible $\epsilon > 0$, $S \subseteq I_{m,n,\epsilon}$ and a sign condition $\sigma \in \{-1, 0, 1\}^\mu$ we let

$$B_S(f, \sigma) := \{(x,y) \in S : \operatorname{sign} f_1(x,y) = \sigma_1, \dots, \operatorname{sign} f_\mu(x,y) = \sigma_\mu\}.$$

Finally, we put

$$b(f) := \begin{cases} (0,0) & \text{if } m = 0, 1, \\ b_X(\{f_1, \dots, f_\mu\}) & \text{if } m > 1, \end{cases}$$

with each f_i considered as an element of $A[\![X^*]\!]$ with $A = \mathbb{R}[\![Y]\!]$.

We can now state a key result.

8.4 Proposition. *Let $f \in \mathbb{R}\{X^*, Y\}^\mu$ and let $\epsilon > 0$ be f-admissible. Then there is an (m,n)-corner $U \subseteq \operatorname{int}(I_{m,n,\epsilon})$ with the following property:*

($*$) *for every sign condition $\sigma \in \{-1,0,1\}^\mu$ there are $m_i \geq m$ and $n_i \geq n$ and connected $\mathcal{R}_{m_i,n_i,\rho^{(i)}}$-manifolds $M_i \subseteq \mathbb{R}^{m_i+n_i}$ with each polyradius $\rho^{(i)} = (\rho_1^{(i)}, \dots, \rho_{m_i+n_i}^{(i)}) \leq \epsilon$ for $i = 1, \dots, k = k(\sigma)$, such that*

$$B_U(f, \sigma) = \Pi_{m,n}(M_1) \cup \dots \cup \Pi_{m,n}(M_k),$$

and for each $M = M_i, m' = m_i$, $n' = n_i$ and $\rho' = \rho^{(i)}$ the set $\Pi_{m,n}(M)$ is a manifold and $\Pi_{m,n}|_M : M \longrightarrow \Pi_{m,n}(M)$ is an analytic isomorphism, and $\operatorname{fr} M$ *is an $\mathcal{R}_{m',n',\rho'}$-set that has dimension with* $\dim(\operatorname{fr} M) < \dim(M)$.

Remark. Suppose that $\tilde{f} = (f_1, \dots, f_\mu, f_{\mu+1}) \in \mathbb{R}\{X^*, Y\}^{\mu+1}$, $\epsilon > 0$ is $\tilde{f}$-admissible, $U \subseteq I_{m,n,\epsilon}$ and $\sigma \in \{-1,0,1\}^\mu$. Then $B_U(f,\sigma)$ is the disjoint union of the sets $B_U(\tilde{f}, (\sigma, -1))$, $B_U(\tilde{f}, (\sigma, 0))$ and $B_U(\tilde{f}, (\sigma, 1))$. Therefore, in the attempt to establish 8.4, there is no harm in replacing f by a suitable longer list, and below we will tacitly use this device.

We first establish two lemmas needed in the inductive proof of 8.4.

8.5 Lemma. *Let $m \geq 0$, $n \geq 1$ be fixed and assume 8.4 holds for all $m' \leq m$ and $n' < n$ in place of m and n. Let $f = (f_1, \dots, f_\mu) \in \mathbb{R}\{X^*, Y'\}[Y_n]^\mu$ be such that each f_i is monic in Y_n. Then for each f-admissible $\epsilon > 0$ there is an (m,n)-corner $U \subseteq \operatorname{int}(I_{m,n,\epsilon})$ for which ($*$) holds.*

Proof. Let $\epsilon > 0$ be f-admissible. By extending the list f we may as well assume that $Y_n - \epsilon, Y_n + \epsilon \in \{f_1, \dots, f_\mu\}$.

We apply Theorem 3.2 with $S = I_{m,n-1,\epsilon}$ and $\mathcal{E} = \mathcal{R}_{m,n-1,\epsilon}$ to the list $f_1, \dots, f_\mu$, where each f_i is considered as a polynomial in Y_n with coefficients in $\mathcal{E}$. Let $\phi = (\phi_1, \dots, \phi_\nu) \in \mathcal{E}^\nu$ be the tuple of all functions involved in a description of the sets $S_1, \dots, S_k$ that are obtained from 3.2. Assume $\phi_1, \dots, \phi_\nu$ are given by power series $\hat{\phi}_1, \dots, \hat{\phi}_\nu \in \mathbb{R}\{X^*, Y'\}_\delta$, where $\delta > \epsilon$, and let $\hat{\phi} := (\hat{\phi}_1, \dots, \hat{\phi}_\nu)$. By hypothesis, Proposition 8.4 applies to $\hat{\phi}$. So there is a $(m, n-1)$-corner $V \subseteq \operatorname{int}(I_{m,n-1,\epsilon})$ such that for each S_j there are $m_i \geq m$ and $n_i \geq n-1$ and connected $\mathcal{R}_{m_i,n_i,\rho^{(i)}}$-manifolds $M_i \subseteq \mathbb{R}^{m_i+n_i}$ with each polyradius $\rho^{(i)} = (\rho_1^{(i)}, \dots, \rho_{m_i+n_i}^{(i)}) \leq \epsilon$ for $i = 1, \dots, l = l(j)$, such that

$$S_j \cap V = \Pi_{m,n-1}(M_1) \cup \dots \cup \Pi_{m,n-1}(M_l),$$

and for each $M = M_i, m' = m_i$, $n' = n_i$ and $\rho' = \rho^{(i)}$ the set $\Pi_{m,n-1}(M)$ is a manifold and $\Pi_{m,n-1}|_M : M \longrightarrow \Pi_{m,n-1}(M)$ is an analytic isomorphism, and $\operatorname{fr} M$ is an $\mathcal{R}_{m',n',\rho'}$-set that has dimension with $\dim(\operatorname{fr} M) < \dim(M)$.

We will show that the (m,n)-corner $U := V \times (-\epsilon, \epsilon)$ has property $(*)$. Note that it is enough to prove $(*)$ with $(S_j \cap V) \times (-\epsilon, \epsilon)$ in place of U for each S_j as above, so from now on we fix such an S_j. Similarly, it is enough to prove $(*)$ with $(\Pi_{m,n-1}(M_i) \cap V) \times (-\epsilon, \epsilon)$ in place of U for each M_i corresponding as above to S_j. Fix such an $M = M_i$ and put $m' := m_i, n' := n_i, \rho' = (\rho_1, \dots, \rho_{m'+n'}) := (\rho_1^{(i)}, \dots, \rho_{m'+n'}^{(i)})$ and $D := \Pi_{m,n-1}(M)$ (hence D is a connected manifold). Let C be the connected component of S_j that contains D. Simplifying the notation of Theorem 3.2 correspondingly, from now on we write $d = m(C)$ and $\xi_1, \dots, \xi_d$ for the restrictions $\xi_{C,1}|_D, \dots, \xi_{C,d}|_D$. Since $Y_n + \epsilon, Y_n - \epsilon \in \{f_1, \dots, f_\mu\}$, it follows that the constant functions $-\epsilon$ and $+\epsilon$ on D are among $\xi_1, \dots, \xi_d$.

Let $h_1, \dots, h_p \in \mathcal{R}_{m',n',\rho'}$ with $p \le m'+n'$ be such that $\dim(M) = m'+n'-p$ and $h_1, \dots, h_p$ vanish identically on M, with $\nabla h_1(z), \dots, \nabla h_p(z)$ linearly independent at each point $z \in M$. Below we let x range over $\mathbb{R}^m$, u over $\mathbb{R}^{m'-m}$, y over $\mathbb{R}^n$ with $y' = (y_1, \dots, y_{n-1})$, and v over $\mathbb{R}^{n'-(n-1)}$. For $\kappa = 1, \dots, d$, we now define the connected subsets of $\mathbb{R}^{m'+n'+1}$

$$N_\kappa := \{(x,u,y,v) : (x,u,y',v) \in M, y_n = \xi_\kappa(x,y')\},$$

and for $\kappa = 1, \dots, d-1$ the connected subsets of $\mathbb{R}^{m'+n'+1}$

$$(N_\kappa, N_{\kappa+1}) := \{(x,u,y,v) : (x,u,y',v) \in M, \xi_\kappa(x,y') < y_n < \xi_{\kappa+1}(x,y')\}.$$

Note that $\Pi_{m,n}^{m',n'+1}(N_\kappa) = \Gamma(\xi_\kappa)$ and $\Pi_{m,n}^{m',n'+1}((N_\kappa, N_{\kappa+1})) = (\xi_\kappa, \xi_{\kappa+1})$. Let N be any one of the N_κ's with $-\epsilon < \xi_\kappa < \epsilon$ or any one of the $(N_\kappa, N_{\kappa+1})$'s with $-\epsilon \le \xi_\kappa < \xi_{\kappa+1} \le \epsilon$, put $\xi = \xi_\kappa, \tilde{\xi} = \xi_{\kappa+1}$, and write $\Pi_{m,n}$ for $\Pi_{m,n}^{m',n'+1}$. Let $\rho := (\rho_1, \dots, \rho_{m'+n-1}, \epsilon, \rho_{m'+n}, \dots, \rho_{m'+n'})$, so ρ is a polyradius with $m'+n'+1$ components and $\rho \le \epsilon$.

Claim. N is a connected $\mathcal{R}_{m',n'+1,\rho}$-manifold, $\Pi_{m,n}(N)$ is a manifold, $\Pi_{m,n}|_N : N \longrightarrow \Pi_{m,n}(N)$ is an analytic isomorphism, and $\operatorname{fr} N$ is an $\mathcal{R}_{m',n'+1,\rho}$-set that has dimension with $\dim(\operatorname{fr} N) < \dim(N)$.

Clearly the proof of this claim will finish the proof of Lemma 8.5.

Proof of the claim. We distinguish two cases.

Case 1: $N = N_\kappa$ for some $\kappa \in \{1, \dots, d\}$ with $-\epsilon < \xi_\kappa < \epsilon$. By remark 3.3, ξ is analytic, so N and $\Pi_{m,n}(N)$ are manifolds of dimension $m'+n'-p$ and $\Pi_{m,n}|_N : N \longrightarrow \Pi_{m,n}(N)$ is an analytic isomorphism. Since $f_1, \dots, f_\mu$ are monic, part (3) of 3.2 implies that ξ extends uniquely to a continuous function $\eta : \operatorname{cl}(D) \longrightarrow \mathbb{R}$. So

$$\operatorname{cl}(N) = \{(x,u,y,v) : (x,u,y',v) \in \operatorname{cl}(M), y_n = \eta(x,y')\},$$

and hence

$$\operatorname{fr} N = \{(x,u,y,v) : (x,u,y',v) \in \operatorname{fr} M, y_n = \eta(x,y')\};$$

in particular, $\operatorname{fr} N$ is homeomorphic to $\operatorname{fr} M$. Moreover, by part (3) of 3.2 the set $\Gamma(\eta)$ is described inside $\operatorname{cl}(D) \times \mathbb{R}$ by equations and weak inequalities involving $f_1, \dots, f_\mu$ and their derivatives $\partial^\nu f_i / \partial Y_n^\nu$. It follows from the inductive hypothesis on $\operatorname{fr} M$ that $\operatorname{fr} N$ is an $\mathcal{R}_{m',n'+1,\rho}$-set, and that $\operatorname{fr} N$ has dimension with $\dim(\operatorname{fr} N) < \dim(N)$. (Up to this point the argument also works if $\xi_\kappa = -\epsilon$ or $\xi_\kappa = \epsilon$.)

It remains to show that N is an $\mathcal{R}_{m',n'+1,\rho}$-manifold. Using part (1) of 3.2 and the inductive hypothesis on M, it follows easily that N is a basic $\mathcal{R}_{m',n'+1,\rho}$-set. Note also that $N \subseteq \operatorname{int}(I_{m',n'+1,\rho})$. Next, let $g \in \mathcal{E}[Y_n]$ be the polynomial in part (2) of 3.2 (with Y_n in place of T) and let $e \in \{1, \dots, \deg_{Y_n}(g)\}$ be such that $h_0 := \partial^{e-1} g/\partial Y_n^{e-1}$ vanishes identically on $\Gamma(\xi)$, while $\partial h_0/\partial Y_n$ vanishes nowhere on $\Gamma(\xi)$. For simplicity, denote the functions

$$(x,u,y,v) \mapsto h_0(x,y) : I_{m',n'+1,\rho} \longrightarrow \mathbb{R}$$

and

$$(x,u,y,v) \mapsto (\partial h_0/\partial Y_n)(x,y) : I_{m',n'+1,\rho} \longrightarrow \mathbb{R}$$

also by h_0 and $\partial h_0/\partial Y_n$ respectively. Clearly these two functions belong to $\mathcal{R}_{m',n'+1,\rho}$. Similarly, for each $i \in \{1, \dots, p\}$ denote the function $(x,u,y,v) \mapsto h_i(x,u,y',v) : I_{m',n'+1,\rho} \longrightarrow \mathbb{R}$ also by h_i, so that $h_0, h_1, \dots, h_p \in \mathcal{R}_{m',n'+1,\rho}$ vanish identically on N, while they have linearly independent gradients at each point of N, since $h_1, \dots, h_p$ do not depend on y_n.

Case 2: $N = (N_\kappa, N_{\kappa+1})$ for some $\kappa \in \{1, \dots, d-1\}$ with $-\epsilon \le \xi_\kappa < \xi_{\kappa+1} \le \epsilon$. Clearly $(\xi, \tilde{\xi})$ and N are manifolds of dimension $m'+n'+1-p$ and $\Pi_{m,n}|_N : N \longrightarrow (\xi, \tilde{\xi})$ is an analytic isomorphism. As in case 1 we see that ξ and $\tilde{\xi}$ extend uniquely to continuous functions $\eta, \tilde{\eta} : \operatorname{cl}(D) \longrightarrow \mathbb{R}$ respectively. To see that $\operatorname{fr} N$ is an $\mathcal{R}_{m',n'+1,\rho}$-set and has dimension, we first observe that $\operatorname{fr} N = \operatorname{cl}(N_\kappa) \cup \operatorname{cl}(N_{\kappa+1}) \cup G$, where

$$G := \{(x,u,y,v) \in \mathbb{R}^{m'+n'+1} : (x,u,y',v) \in \operatorname{fr} M, \eta(x,y') < y_n < \tilde{\eta}(x,y')\}.$$

Putting $H := \{(x,u,y',v) \in \operatorname{fr} M : \eta(x,y') < \tilde{\eta}(x,y')\}$, we see that H is open in $\operatorname{fr} M$ and hence H has dimension. It follows from the continuity of η and $\tilde{\eta}$ that G has dimension with $\dim(G) = \dim(H) + 1 < \dim(M) + 1 = m'+n'+1-p$. On the other hand, $\operatorname{cl}(N_\kappa)$ and $\operatorname{cl}(N_{\kappa+1})$ have dimension $m'+n'-p$ by case 1. Hence $\operatorname{fr} N$ has dimension with $\dim(\operatorname{fr} N) < \dim(N)$; the fact that $\operatorname{fr} N$ is an $\mathcal{R}_{m',n'+1,\rho}$-set is established as in case 1.

It remains to show that N is an $\mathcal{R}_{m',n'+1,\rho}$-manifold. Using part (1) of 3.2 and the inductive hypothesis on M, it follows easily that N is a basic $\mathcal{R}_{m',n'+1,\rho}$-set. Note also that $N \subseteq \operatorname{int}(I_{m',n'+1,\rho})$. Similarly to case 1, for each $i \in \{1, \dots, p\}$ denote the function $(x,u,y,v) \mapsto h_i(x,u,y',v) : I_{m',n'+1,\rho} \longrightarrow \mathbb{R}$ also by h_i, so that $h_1, \dots, h_p \in \mathcal{R}_{m',n'+1,\rho}$ vanish identically on N, while they have linearly independent gradients at each point of N. $\square$

8.6 Lemma. *Let $f \in \mathbb{R}\{X^*, Y\}^\mu$, and let $\epsilon > 0$ be f-admissible. Let $S \subseteq \mathbb{R}^{m+n}$, $\phi : S \longrightarrow \mathbb{R}^{m+n}$, $\tilde{m}, \tilde{n} \in \mathbb{N}$ and $\delta > 0$, and suppose we are in one of the following three situations:*

(i) *$S = \mathbb{R}^{m+n}$, $\tilde{m} = m$, $\tilde{n} = n > 0$ and there are $c_1, \dots, c_{n-1} \in \mathbb{R}$ with*

$$(1 + |c_1| + \cdots + |c_{n-1}|)\delta \le \epsilon$$

and

$$\phi(x,y) = (x, y_1 + c_1 y_n, \dots, y_{n-1} + c_{n-1} y_n, y_n) \text{ for all } (x,y) \in S$$

(then we put $\phi f := f(X, Y_1 + c_1 Y_n, \dots, Y_{n-1} + c_{n-1} Y_n, Y_n) \in \mathbb{R}\{X^*, Y\}^\mu$);

(ii) *$S = I_{m,n,\infty}$, $\tilde{m} = m > 1$, $\tilde{n} = n$ and there is $\gamma > 0$ with $\max(\delta, \delta^{\gamma+1}) \le \epsilon$ and*

$$\phi(x,y) = (x', x_{m-1}^{\gamma} x_m, y) \text{ for all } (x,y) \in S$$

(then we put $\phi f := s^{\gamma}_{m,m-1}(f) \in \mathbb{R}\{X^*, Y\}^{\mu}$ as defined in 5.8);

(iii) *$S = I_{\tilde{m},\tilde{n},\infty}$, $\tilde{m} = m-1$, $\tilde{n} = n+1$, and there are $\gamma, \lambda > 0$ such that $\max(\delta, \delta^{\gamma}(\lambda+\delta)) \le \epsilon$ and*

$$\phi(x,y) = (x', x_{m-1}^{\gamma}(\lambda + x_m), y) \text{ for all } (x,y) \in S$$

(then we put $\phi f := r^{\gamma}_{\lambda}(f) \in \mathbb{R}\{(X')^*, (X_m, Y)\}^{\mu}$ as defined in 6.5 and the remark thereafter).

Assume that δ is ϕf-admissible and that $(*)$ holds with ϕf in place of f, δ in place of ϵ and some $(\tilde{m},\tilde{n})$-corner $V \subseteq \operatorname{int}(I_{\tilde{m},\tilde{n},\delta})$ in place of U. Then $\phi(V) \subseteq \operatorname{int}(I_{m,n,\epsilon})$ and $(*)$ holds for f with $U = \phi(V)$.

Remark. The set $\phi(V)$ is an (m,n)-corner in case (i), but not necessarily in cases (ii) or (iii).

Proof. Put $U := \phi(V)$. It is easy to check that $U \subseteq \operatorname{int}(I_{m,n,\epsilon})$ and that $\phi f(x,y) = f(\phi(x,y))$ for all $(x,y) \in V$. Hence

$$B_U(f,\sigma) = \phi(B_V(\phi f, \sigma)) \tag{$\diamond$}$$

for each sign condition $\sigma \in \{-1,0,1\}^{\mu}$. In the rest of the proof we treat only case (ii) in detail (so $\tilde{m} = m > 1, \tilde{n} = n$); the other cases are handled similarly. Let M be one of the $\mathcal{R}_{m',n',\rho'}$-manifolds in $(*)$ for ϕf with δ in place of ϵ and V in place of U, $m' \ge m$ and $n' \ge n$, and polyradius $\rho' = (\rho_1, \ldots, \rho_{m'+n'}) \le \delta$. Put

$$N := \left\{(x', t, x_m, u, y, v) \in \mathbb{R}^{m'+n'+1} : (x,u,y,v) \in M, t = x_{m-1}^{\gamma} x_m\right\},$$

where t ranges over $\mathbb{R}$. Note that $\Pi^{m'+1,n'}_{m,n}(N) = \phi(\Pi_{m,n}(M))$; below we write $\Pi_{m,n}$ for $\Pi^{m'+1,n'}_{m,n}$. Let $\rho := (\rho_1, \ldots, \rho_{m-1}, \epsilon, \rho_m, \ldots, \rho_{m'+n'})$, so ρ is a polyradius with $m'+n'+1$ components and $\rho \le \epsilon$. Clearly N is a basic $\mathcal{R}_{m'+1,n',\rho}$-set and $N \subseteq \operatorname{int}(I_{m'+1,n',\rho})$.

Claim. N is a connected $\mathcal{R}_{m'+1,n',\rho}$-manifold, $\Pi_{m,n}(N)$ is a manifold, $\Pi_{m,n}|_N : N \longrightarrow \Pi_{m,n}(N)$ is an analytic isomorphism, and $\operatorname{fr} N$ is an $\mathcal{R}_{m'+1,n',\rho}$-set that has dimension with $\dim(\operatorname{fr} N) < \dim(N)$.

In view of $(\diamond)$ and $\Pi_{m,n}(N) = \phi(\Pi_{m,n}(M))$ the proof of this claim will finish the proof of case (ii) of Lemma 8.6.

Proof of the claim. It is easy to see that N is a manifold and that the map $\theta : (x', t, x_m, u, y, v) \mapsto (x,u,y,v) : N \longrightarrow M$ is an analytic isomorphism onto M. Since M is connected it follows that N is connected. Now $\phi|_{\operatorname{int}(S)} : \operatorname{int}(S) \longrightarrow \operatorname{int}(S)$ is an analytic isomorphism, $\Pi^{m',n'}_{m,n}(M)$ is contained in $\operatorname{int}(S)$ and

$$\Pi_{m,n}|_N = \phi \circ \Pi^{m',n'}_{m,n}|_M \circ \theta,$$

and hence $\Pi_{m,n}(N)$ is a manifold and $\Pi_{m,n}|_N : N \longrightarrow \Pi_{m,n}(N)$ is an analytic isomorphism. As in the proof of the previous lemma we obtain that

$$\operatorname{fr} N = \{(x', t, x_m, u, y, v) : (x,u,y,v) \in \operatorname{fr} M, t = x_{m-1}^{\gamma} x_m\},$$

from which it follows that $\operatorname{fr} N$ is an $\mathcal{R}_{m'+1,n',\rho}$-set, and homeomorphic to $\operatorname{fr} M$; hence $\operatorname{fr} N$ has dimension and $\dim(\operatorname{fr} N) = \dim(\operatorname{fr} M) < \dim(M) = \dim(N)$.

It remains to show that N is an $\mathcal{R}_{m'+1,n',\rho}$-manifold. Let $h_1,\dots,h_p \in \mathcal{R}_{m',n',\rho'}$ with $p \le m'+n'$ be such that M is a basic $\mathcal{R}_{m',n',\rho'}$-set and an open subset of $\{z \in \operatorname{int}(I_{m',n'\rho'}) : h_1(z) = \cdots = h_p(z) = 0\}$, with $\nabla h_1(z),\dots,\nabla h_p(z)$ linearly independent at each point $z \in M$. For simplicity, for each $i \in \{1,\dots,p\}$ denote the function

$$(x',t,x_m,u,y,v) \mapsto h_i(x,u,y,v) : I_{m'+1,n',\rho} \longrightarrow \mathbb{R}$$

also by h_i, so that $h_1,\dots,h_p \in \mathcal{R}_{m'+1,n',\rho}$ vanish identically on N. Also denote the function

$$(x',t,x_m,u,y,v) \mapsto t - x_{m-1}^{\gamma} x_m : I_{m'+1,n',\rho} \longrightarrow \mathbb{R}$$

by h_0, so $h_0 \in \mathcal{R}_{m'+1,n',\rho}$ and vanishes identically on N as well. But $h_0,h_1,\dots,h_p$ have linearly independent gradients at each point of N, since $h_1,\dots,h_p$ do not depend on t. $\square$

8.7. *Proof of Proposition 8.4.* Fix a tuple $f \in \mathbb{R}\{X^*,Y\}^{\mu}$, and write $b = b(f)$. We proceed by induction on the quadruples $(m,n,b) \in \mathbb{N}^4$, ordered lexicographically. The case $(m,n,b) = (0,0,0,0)$ is trivial; so we assume that $(m,n,b) > (0,0,0,0)$ and that the proposition holds for all lower values of (m,n,b). We may and shall also assume that $f_i \ne 0$ for all i. Let $\epsilon > 0$ be f-admissible. We have to find an (m,n)-corner $U \subseteq \operatorname{int}(I_{m,n,\epsilon})$ for which $(*)$ holds.

First we assume that $b = (0,0)$, and we distinguish two cases depending on the value of n. Recall that $b = (0,0)$ means that there are $\delta_i \in [0,\infty)^m$ for $i = 1,\dots,\mu$, such that

$$f_i(X,Y) = X^{\delta_i} F_i(X,Y)$$

with $F_i \in \mathbb{R}\{X^*,Y\}$ satisfying $F_i(0,Y) \ne 0$; so we may as well assume that

$$(\diamond) \qquad f_i(0,Y) \ne 0 \text{ for each } i.$$

Case 1: $n = 0$. By $(\diamond)$ and corollary 5.6 (1) we can choose $\delta \in (0,\epsilon)$ such that $f_i(x) \ne 0$ for all $x \in [0,\delta]^m$ and $i = 1,\dots,\mu$. Then with $U = (0,\delta)^m$ each set $B_U(f,\sigma)$ (where $\sigma \in \{-1,0,1\}^{\mu}$ is a sign condition) is either empty or equal to U, so it obviously has the desired properties.

Case 2: $n > 0$. By $(\diamond)$ and 6.1 there is a linear transformation $\theta(X,Y) = (X, Y_1 + c_1Y_n, \dots, Y_{n-1} + c_{n-1}Y_n, Y_n)$ with $c_1,\dots,c_{n-1} \in \mathbb{R}$ such that each $\theta f_i := f_i(\theta(X,Y))$ is regular in Y_n.

Assume for the moment that 8.4 holds with θf in place of f. Take some θf-admissible $\delta > 0$ with $(1 + |c_1| + \cdots + |c_{n-1}|)\delta \le \epsilon$ and an (m,n)-corner $V \subseteq \operatorname{int}(I_{m,n,\delta})$ such that $(*)$ holds with θf in place of f and V in place of U. Then $(*)$ holds for f and the (m,n)-corner $U := \theta(V)$ by case (i) of Lemma 8.6.

We may therefore assume that each f_i is regular in Y_n. Applying Weierstrass Preparation 5.10 to each f_i and decreasing ϵ if necessary, we obtain

$$f_i(X,Y) = U_i(X,Y) \cdot W_i(X,Y)$$

with each $U_i \in \mathbb{R}\{X^*,Y\}_{\epsilon'}$ having no zeros in $I_{m,n,\epsilon}$, and each W_i a monic polynomial in Y_n with coefficients in $\mathbb{R}\{X^*,Y'\}_{\epsilon'}$, for some $\epsilon' > \epsilon$. Clearly we may even replace f_i by W_i, so that each f_i is actually a monic polynomial in Y_n with coefficients in $\mathbb{R}\{X^*,Y'\}_{\epsilon'}$. We now use the inductive hypothesis to apply Lemma 8.5 to f, thereby proving case 2.

Next we assume $b > (0,0)$ (recall that $b > (0,0)$ implies $m > 1$ by definition of $b(f)$). By Proposition 4.14, after permuting the first m coordinates if necessary, there are $\gamma > 0$ and singular blow-up substitutions $s_0 := s^{\gamma}_{m,m-1}$ and $s_\infty := s^{1/\gamma}_{m-1,m}$ such that $b(s_0 f) < b$ and $b(s_\infty f) < b$. Note that the corresponding maps $s_0, s_\infty : I_{m,n,\infty} \longrightarrow \mathbb{R}^{m+n}$ are given by

$$s_0(x,y) = (x', x^{\gamma}_{m-1} x_m, y),$$
$$s_\infty(x,y) = (x_1, \ldots, x_{m-2}, x_m^{1/\gamma} x_{m-1}, x_m, y).$$

Take $\delta > 0$ such that δ is $s_0 f$-admissible as well as $s_\infty f$-admissible and $\max(\delta, \delta^{\gamma+1}, \delta^{(1/\gamma)+1}) \leq \epsilon$.

By the inductive hypothesis $(*)$ holds for $s_0 f$ and $s_\infty f$ in place of f with an (m,n)-corner $V_0 \subseteq \mathrm{int}(I_{m,n,\delta})$ and an (m,n)-corner $V_\infty \subseteq \mathrm{int}(I_{m,n,\delta})$ in place of U respectively. Then case (ii) of Lemma 8.6 implies that $s_0(V_0) \cup s_\infty(V_\infty) \subseteq I_{m,n,\epsilon}$ and that $(*)$ holds for f with $s_0(V_0) \cup s_\infty(V_\infty)$ in place of U.

The problem now is that $s_0(V_0) \cup s_\infty(V_\infty)$ is not in general an (m,n)-corner. But we know there is a $\tau_0 > 0$ such that $\mathrm{int}(I_{m,n,\tau_0})$ is contained in V_0. The image under s_0 of $\mathrm{int}(I_{m,n,\tau_0})$ is contained in $s_0(V_0)$, i.e. $s_0(V_0)$ contains the set

$$D_0 = \left\{(x,y) \in \mathrm{int}(I_{m,n,\tau_0}) : x_m < \tau_0 x^{\gamma}_{m-1}\right\}.$$

The same argument for s_∞ gives $\tau > 0$ such that the set

$$\begin{aligned} D_\infty :&= \{(x,y) \in \mathrm{int}(I_{m,n,\tau}) : x_{m-1} < \tau x_m^{1/\gamma}\} \\ &= \{(x,y) \in \mathrm{int}(I_{m,n,\tau}) : \tau^{-1/\gamma} x^{\gamma}_{m-1} < x_m\} \end{aligned}$$

is contained in $s_\infty(V_\infty)$. Writing $\tau_\infty := \tau^{-1/\gamma}$, we see that if $\tau_0 > \tau_\infty$, then $D_0 \cup D_\infty$ is clearly an (m,n)-corner; hence $s_0(V_0) \cup s_\infty(V_\infty)$ is an (m,n)-corner, and we are done. Suppose then that $\tau_0 \leq \tau_\infty$; it remains to cover everything in the set $\mathrm{int}(I_{m,n,\infty}) \setminus (D_0 \cup D_\infty)$ close enough to the origin in $\mathbb{R}^{m+n}$.

To do this we use regular blow-ups. By lemma 6.5, for any $\lambda > 0$ the regular blow-up substitution r^{γ}_{λ} satisfies $r^{\gamma}_{\lambda} f \in \mathbb{R}\{(X')^*, (X_m, Y)\}^{\mu}$. (The corresponding map $r^{\gamma}_{\lambda} : I_{m-1,n+1,\infty} \longrightarrow \mathbb{R}$ is given by $r^{\gamma}_{\lambda}(x,y) = (x', x^{\gamma}_{m-1}(\lambda + x_m), y)$.) Take some $r^{\gamma}_{\lambda} f$-admissible $\delta > 0$ with $\max(\delta, \delta^{\gamma}(\lambda + \delta)) \leq \epsilon$.

By the inductive hypothesis, $(*)$ holds with $r^{\gamma}_{\lambda} f$ in place of f and an $(m-1, n+1)$-corner $V_\lambda \subseteq \mathrm{int}(I_{m-1,n+1,\delta})$ in place of U. Then Lemma 8.6 implies that $(*)$ holds for f with the set $r^{\gamma}_{\lambda}(V_\lambda) \subseteq \mathrm{int}(I_{m,n,\epsilon})$ in place of U. On the other hand, there is a $\tau_\lambda \in (0, \lambda)$ such that $\mathrm{int}(I_{m-1,n+1,\tau_\lambda})$ is contained in V_λ, and hence the set

$$D_\lambda := \left\{(x,y) \in \mathrm{int}(I_{m,n,\tau_\lambda}) : (\lambda - \tau_\lambda) x^{\gamma}_{m-1} < x_m < (\lambda + \tau_\lambda) x^{\gamma}_{m-1}\right\}$$

is contained in $r^{\gamma}_{\lambda}(V_\lambda)$.

Take finitely many $\lambda_1, \ldots, \lambda_K \in [\tau_0, \tau_\infty]$ such that

$$[\tau_0, \tau_\infty] \subseteq \bigcup_{i=1}^{K} (\lambda_i - \tau_{\lambda_i}, \lambda_i + \tau_{\lambda_i}).$$

Then $D_0 \cup D_\infty \cup \bigcup_{i=1}^{K} D_{\lambda_i}$ is clearly an (m,n)-corner, and hence

$$U := s_0(V_0) \cup s_\infty(V_\infty) \cup \bigcup_{i=1}^{K} r^{\gamma}_{\lambda_i}(V_{\lambda_i}) \subseteq \mathrm{int}(I_{m,n,\epsilon})$$

is an (m,n)-corner. Therefore $(*)$ holds for f with this set U. $\square$

We now extend 8.4 to **closed** (m,n)-corners; a set $U \subseteq I_{m,n,\infty}$ is called a **closed** (m,n)**-corner** if $I_{m,n,\delta} \subseteq U$ for some $\delta > 0$. A point on the boundary of $I_{m,n,\infty}$ has some of its first m coordinates equal to 0, but after a permutation of the first m coordinates it is of the form $(0_{m-m'}, u, v)$, where $0_{m-m'}$ is the origin in $\mathbb{R}^{m-m'}$ and $(u,v) \in \mathrm{int}(I_{m',n})$, for some $m' \le m$. In this way one reduces questions about sets contained in the boundary of $I_{m,n,\infty}$ to similar questions about sets contained in $\mathrm{int}(I_{m,n,\infty})$. We now formalize this observation as follows.

A set $M \subseteq \mathbb{R}^{m+n}$ is an $\mathcal{R}_{m,n}$**-manifold** if there are $m' \le m$, a polyradius $\rho = (\rho_1, \dots, \rho_{m'+n})$, an $\mathcal{R}_{m',n,\rho}$-manifold $N \subseteq \mathrm{int}(I_{m',n,\rho})$ and a permutation ϕ of $\{1,\dots,m\}$ such that $M = \phi(\{0_{m-m'}\} \times N)$. (Here ϕ acts on $\mathbb{R}^{m+n}$ as specified in 5.8.) In this situation we will say that the $\mathcal{R}_{m,n}$-manifold M **is obtained** from the $\mathcal{R}_{m',n,\rho}$-manifold N. Note that each $\mathcal{R}_{m,n}$-manifold is a bounded $\mathcal{R}_{m,n}$-semianalytic manifold.

8.8 Lemma. *Let $f \in \mathbb{R}\{X^*, Y\}^\mu$ and let $\epsilon > 0$ be f-admissible. Then there is a closed (m,n)-corner $U \subseteq I_{m,n,\epsilon}$ with the following property:*

$(**)$ *for every sign condition $\sigma \in \{-1,0,1\}^\mu$ there are $m_i \ge m$ and $n_i \ge n$ and connected $\mathcal{R}_{m_i,n_i}$-manifolds $M_i \subseteq \mathbb{R}^{m_i+n_i}$ for $i = 1,\dots,k = k(\sigma)$ such that*

$$B_U(f,\sigma) = \Pi_{m,n}(M_1) \cup \dots \cup \Pi_{m,n}(M_k),$$

and for each $M = M_i, m' = m_i$ and $n' = n_i$ the set $\Pi_{m,n}(M)$ is a manifold and $\Pi_{m,n}|_M : M \longrightarrow \Pi_{m,n}(M)$ is an analytic isomorphism, and $\mathrm{fr}\, M$ *is $\mathbb{R}_{m',n'}$-semianalytic and has dimension with* $\dim(\mathrm{fr}\, M) < \dim(M)$.

Proof. Let $P \subseteq \{1,\dots,m\}$ and define, for $\delta > 0$,

$$I^P_{m,n,\delta} := \{(x,y) \in I_{m,n,\delta} : x_i = 0 \text{ for } i \in P, x_i > 0 \text{ for } i \in \{1,\dots,m\} \setminus P\}.$$

For the purpose of this proof we call a set $U \subseteq I_{m,n,\epsilon}$ a P**-corner** if there is $\delta \in (0,\epsilon)$ such that $I^P_{m,n,\delta} \subseteq U$. It suffices to find for each $P \subseteq \{1,\dots,m\}$ a P-corner $U_P \subseteq I_{m,n,\epsilon}$ for which $(**)$ holds with U_P in place of U, because then

$$U := \bigcup_{P \subseteq \{1,\dots,m\}} U_P$$

is a closed (m,n)-corner for which $(**)$ holds.

So let us fix some $P \subseteq \{1,\dots,m\}$. To simplify notation, assume $P = \{1,\dots,p\}$, $0 \le p \le m$. Let $0_p = (0,\dots,0)$ be the origin in $\mathbb{R}^p$, let $\tilde{X} := (X_{p+1},\dots,X_m)$ and put $\tilde{f} := f(0_p, \tilde{X}, Y) \in \mathbb{R}\{\tilde{X}^*, Y\}^\mu$. By 8.4 applied to $\tilde{f}$ there is an $(m-p,n)$-corner $U \subseteq \mathrm{int}(I_{m-p,n,\epsilon})$ for which $(*)$ holds with $\tilde{f}$ in place of f (and $\tilde{X}$ in place of X, $m-p$ in place of m). Then $U_P := \{0_p\} \times U \subseteq I_{m,n,\epsilon}$ is clearly a P-corner.

We now claim that $(**)$ holds for U_P in place of U (we will be done once this claim is established). To see why this claim holds, let $\sigma \in \{-1,0,1\}^\mu$ and let $\tilde{M}_1,\dots,\tilde{M}_k$ be the manifolds for which $B_U(\tilde{f},\sigma) = \Pi_{m-p,n}(\tilde{M}_1) \cup \dots \cup \Pi_{m-p,n}(\tilde{M}_k)$, and which have the other properties required in $(*)$ for $\tilde{f}$ in place of f. In particular, each $\tilde{M}_i$ is clearly a connected $\mathcal{R}_{m_i,n_i,\rho^{(i)}}$-manifold in $\mathbb{R}^{m_i+n_i}$ with $m_i \ge m-p$ and $n_i \ge n$ and some polyradius $\rho^{(i)}$ (here we use 7.4). One checks easily that then each $M_i := \{0_p\} \times \tilde{M}_i \subseteq \mathbb{R}^{m_i+p+n_i}$ is a connected $\mathcal{R}_{m_i+p,n_i}$-manifold, that

$$B_{U_P}(f,\sigma) = \Pi_{m,n}(M_1) \cup \dots \cup \Pi_{m,n}(M_k),$$

and that the M_i's have the other properties required to make $(**)$ hold for U_P in place of U. $\square$

8.9 Corollary. *Let $A \subseteq \mathbb{R}^{m+n}$ be bounded and $\mathcal{R}_{m,n}$-semianalytic. Then there are $m_i \geq m$ and $n_i \geq n$ and connected, bounded, $\mathcal{R}_{m_i,n_i}$-semianalytic manifolds $M_i \subseteq \mathbb{R}^{m_i+n_i}$ for $i = 1, \ldots, k$ such that*

$$A = \Pi_{m,n}(M_1) \cup \cdots \cup \Pi_{m,n}(M_k),$$

and for each $M = M_i, m' = m_i$ and $n' = n_i$ we have:

1. *there are $a \in \mathbb{R}^{m+n}$, $\sigma \in \{-1,1\}^m$ and a connected $\mathcal{R}_{m',n'}$-manifold $N \subseteq \mathbb{R}^{m'+n'}$ such that $M = h_{a,\sigma}(N)$,*
2. *$\Pi_{m,n}(M)$ is a manifold and $\Pi_{m,n}|_M : M \longrightarrow \Pi_{m,n}(M)$ is an analytic isomorphism, and*
3. *$\operatorname{fr} M$ is $\mathcal{R}_{m',n'}$-semianalytic and has dimension with $\dim(\operatorname{fr} M) < \dim(M)$.*

Proof. By the definition of "$\mathcal{R}_{m,n}$-semianalytic" and the previous lemma the corollary holds locally at each point of $\mathbb{R}^{m+n}$, and hence the boundedness of A implies that it holds globally. □

8.10 Remark. Corollary 8.9 implies that every bounded $\mathcal{R}_{m,n}$-semianalytic set has dimension not only in the sense of 8.2, but even in the sense of the introduction.

8.11 Definitions and Remarks. Given $m, n \in \mathbb{N}$ and strictly increasing sequences $\iota \in \{1, \ldots, m\}^\mu$ and $\kappa \in \{1, \ldots, n\}^\nu$ with $\mu \leq m$ and $\nu \leq n$, let $\Pi^{m,n}_{\iota,\kappa} : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{\mu+\nu}$ be the projection map given by

$$\Pi^{m,n}_{\iota,\kappa}(x,y) = (x_{\iota(1)}, \ldots, x_{\iota(\mu)}, y_{\kappa(1)}, \ldots, y_{\kappa(\nu)}).$$

As before, we simply write $\Pi_{\iota,\kappa}$ for $\Pi^{m,n}_{\iota,\kappa}$ whenever m and n are clear from the context.

Let $m \geq k \geq 0, n \geq l \geq 0$, and let M be an $\mathcal{R}_{m,n,\rho}$-manifold of dimension d for some polyradius $\rho = (\rho_1, \ldots, \rho_{m+n})$. Take functions $h_1, \ldots, h_p \in \mathcal{R}_{m,n,\rho}$ with $p = m+n-d$ such that M is a basic $\mathcal{R}_{m,n,\rho}$-set and $h_1, \ldots, h_p$ vanish identically on M while the gradients $\nabla h_1(z), \ldots, \nabla h_p(z)$ are linearly independent at each $z \in M$. For strictly increasing sequences $\iota \in \{1, \ldots, m\}^\mu$ and $\kappa \in \{1, \ldots, n\}^\nu$ with $\mu \leq m$ and $\nu \leq n$ and $\mu + \nu = d$, we let $M_{\iota,\kappa} := \{z \in M : \Pi_{\iota,\kappa}(T_z M) = \mathbb{R}^d\}$. Then $M_{\iota,\kappa}$ is of the form $\{z \in M : h_{\iota,\kappa}(z) \neq 0\}$ for some $h_{\iota,\kappa} \in \mathcal{R}_{m,n,\rho}$: if $\tilde{\iota} \in \{1, \ldots, m\}^{m-\mu}$ and $\tilde{\kappa} \in \{1, \ldots, n\}^{n-\nu}$ are strictly increasing sequences such that $\operatorname{Im}(\iota) \cap \operatorname{Im}(\tilde{\iota}) = \emptyset$ and $\operatorname{Im}(\kappa) \cap \operatorname{Im}(\tilde{\kappa}) = \emptyset$, then basic linear algebra shows that

$$M_{\iota,\kappa} = \left\{ z \in M : \det \left(\frac{\partial(h_1, \ldots, h_p)}{\partial(x_{\tilde{\iota}_1}, \ldots, x_{\tilde{\iota}_{m-\mu}}, y_{\tilde{\kappa}_1}, \ldots, y_{\tilde{\kappa}_{n-\nu}})} \right)(z) \neq 0 \right\};$$

but the function

$$h_{\iota,\kappa} := \left(\prod_{j=1}^{m-\mu} x_{\tilde{\iota}_j} \right) \det \left(\frac{\partial(h_1, \ldots, h_p)}{\partial(x_{\tilde{\iota}_1}, \ldots, x_{\tilde{\iota}_{m-\mu}}, y_{\tilde{\kappa}_1}, \ldots, y_{\tilde{\kappa}_{n-\nu}})} \right)$$

clearly has the same zeros in $\operatorname{int}(I_{m,n,\rho})$ as $\det\left(\frac{\partial(h_1,\ldots,h_p)}{\partial(x_{\tilde{\iota}_1},\ldots,x_{\tilde{\iota}_{m-\mu}},y_{\tilde{\kappa}_1},\ldots,y_{\tilde{\kappa}_{n-\nu}})}\right)$, and by 6.3, parts (4) and (5), and the definition of $\mathcal{R}_{m,n,\rho}$ we have $h_{\iota,\kappa} \in \mathcal{R}_{m,n,\rho}$. Hence each $M_{\iota,\kappa}$ is either empty or an $\mathcal{R}_{m,n,\rho}$-manifold of dimension d. Moreover, M is clearly the union of all the $M_{\iota,\kappa}$'s.

For sequences ι, κ as above, put $\iota_0 := 0$, $\kappa_0 := 0$, and let $\mu' \in \{0, \ldots, \mu\}$ and $\nu' \in \{0, \ldots, \nu\}$ be maximal with $\iota_{\mu'} \leq k$ and $\kappa_{\nu'} \leq l$ respectively. (We do not explicitly indicate the dependence of μ' and ν' on k and l, as it will be clear from the context.) If we assume that $M = M_{\iota,\kappa}$ and that $\Pi_{k,l}|_M$ has constant rank $\mu'+\nu'$, then by the rank theorem (see [14], pp. 86,89) each fiber $M_a := \Pi_{k,l}^{-1}(a) \cap M$ for $a \in \mathbb{R}^{k+l}$ is either empty or a manifold of dimension $d-(\mu'+\nu')$. Moreover, writing $\iota' := (\iota_1, \ldots, \iota_{\mu'})$, $\tilde{\iota} := (\iota_{\mu'+1}, \ldots, \iota_\mu)$ and $\kappa' := (\kappa_1, \ldots, \kappa_{\nu'})$, $\tilde{\kappa} := (\kappa_{\nu'+1}, \ldots, \kappa_\nu)$, we note that $\Pi_{\tilde{\iota},\tilde{\kappa}}|_{M_a}$ is an immersion. (To see this, note that for $z \in M_a$ the tangent space T_zM_a is a subspace of T_zM of dimension $e := d-(\mu'+\nu')$ such that $\Pi_{k,l}(T_zM_a) = 0$. Let $v_1, \ldots, v_e$ be a basis of T_zM_a; then $\Pi_{\iota,\kappa}(v_1), \ldots, \Pi_{\iota,\kappa}(v_e)$ are linearly independent in $\mathbb{R}^d$, and $\Pi_{\iota',\kappa'}(v_1) = \cdots = \Pi_{\iota',\kappa'}(v_e) = 0$. Hence $\Pi_{\tilde{\iota},\tilde{\kappa}}(v_1), \ldots, \Pi_{\tilde{\iota},\tilde{\kappa}}(v_e)$ are linearly independent in $\mathbb{R}^e$.) It follows that if C is a connected component of M_a, then $\Pi_{\tilde{\iota},\tilde{\kappa}}(C)$ is open in $\mathbb{R}^e$ and hence has nonempty frontier if $e \geq 1$, which implies (since C is bounded) that $\operatorname{fr} C \neq \emptyset$ if $e \geq 1$.

8.12 Fiber Cutting Lemma. *Let $m \geq k \geq 0$ and $n \geq l \geq 0$. Assume that M is an $\mathcal{R}_{m,n,\rho}$-manifold for some polyradius ρ, and that moreover $M = M_{\iota,\kappa}$ for some fixed strictly increasing sequences $\iota \in \{1, \ldots, m\}^\mu, \kappa \in \{1, \ldots, n\}^\nu$ with $\mu > k$ or $\nu > l$, and that $\operatorname{rank}(\Pi_{k,l}|_{T_zM}) = \mu'+\nu'$ for all $z \in M$. Then there is an $\mathcal{R}_{m,n,\rho}$-set $A \subseteq M$ with $\dim(A) < d$ such that $\Pi_{k,l}(M) = \Pi_{k,l}(A)$.*

Proof. Note that $\mu > k$ or $\nu > l$ implies $\mu' + \nu' < d$.

First observe that there is $g \in \mathcal{R}_{m,n,\rho}$ such that g is strictly positive on all of M and identically zero on $\operatorname{fr} M$: choose a set of equations and strict inequalities from $\mathcal{R}_{m,n,\rho}$ describing M, and let g be the product of all functions making up the inequalities of this description, together with the functions $x_i, \rho_i - x_i$ for $i = 1, \ldots, m$ and $y_j + \rho_{m+j}, \rho_{m+j} - y_j$ for $j = 1, \ldots, n$.

Next, by the last remark preceding this lemma, for each $a \in \Pi_{k,l}(M)$ the fiber $M_a := \Pi_{k,l}^{-1}(a) \cap M$ is a manifold of dimension $d-(\mu'+\nu') > 0$. Also by that remark, $\operatorname{fr} C \neq \emptyset$ for each connected component C of M_a, and thus $g|_{M_a}$ has critical points on each connected component of M_a, since g is positive on M_a and vanishes identically on $\operatorname{fr} M_a$; since $g|_{M_a}$ is analytic, the set of its critical points has empty interior in M_a. Let A be the set of all critical points of $g|_{M_a}$ for all $a \in \Pi_{k,l}(M)$, i.e.

$$A = \{z \in M : z \text{ is a critical point of } g|_{M_a}, a = \Pi_{k,l}(z)\}.$$

Then clearly $\Pi_{k,l}(A) = \Pi_{k,l}(M)$, and A is an $\mathcal{R}_{m,n,\rho}$-set, so by 8.9 A has dimension. Since A has empty interior in M, we have $\dim(A) < \dim(M)$. This finishes the proof of the fiber cutting lemma. $\square$

If $M \subseteq \mathbb{R}^{m+n}$ is a manifold of dimension d and $k \leq m$ and $l \leq n$, we define

$$r(M) := \max\{\operatorname{rank}(\Pi_{k,l}|_{T_zM}) : z \in M\} \leq d.$$

(Again, we do not indicate explicitly the dependence of $r(M)$ on k, l, m and n.)

8.13 Lemma. *Let $M \subseteq \mathbb{R}^{m+n}$ be an $\mathcal{R}_{m,n}$-manifold of dimension d, and let $k \leq m$ and $l \leq n$. Then*

(∗) *there are bounded, $\mathcal{R}_{m_i,n_i}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{m_i+n_i}$ satisfying $\dim(N_i) \leq d$, $m_i \geq m$ and $n_i \geq n$ for $i = 1, \ldots, K$, and there are bounded, $\mathcal{R}_{p_j,q_j}$-semianalytic sets $A_j \subseteq \mathbb{R}^{p_j+q_j}$ satisfying $\dim(A_j) < d$, $p_j \geq m$ and $q_j \geq n$ for $j = 1, \ldots, L$, such that*

$$\Pi_{k,l}(M) = \Pi_{k,l}(N_1) \cup \cdots \cup \Pi_{k,l}(N_K) \cup \Pi_{k,l}(A_1) \cup \cdots \cup \Pi_{k,l}(A_L),$$

and for each $N = N_i$ there are strictly increasing sequences $\iota \in \{1, \ldots, k\}^\mu$ and $\kappa \in \{1, \ldots, l\}^\nu$ with $\mu + \nu = \dim(N)$, such that $\Pi_{\iota,\kappa}|_N$ is an immersion.

Proof. We prove this lemma by induction on $r(M)$ simultaneously for all k, l, m, n. One easily checks (as in the proof of Lemma 8.8) that if M is obtained from an $\mathcal{R}_{m',n,\rho}$-manifold N, it is enough to prove $(*)$ with N in place of M and a possibly smaller k. We will therefore assume that M is an $\mathcal{R}_{m,n,\rho}$-manifold for some polyradius ρ.

The initial case $r(M) = 0$ is trivial (since then $\Pi_{k,l}$ is constant on each component of M), so below we assume $r(M) > 0$ and that the lemma holds for lower values of $r(M)$.

Let $\iota \in \{1, \ldots, m\}^\mu, \kappa \in \{1, \ldots, n\}^\nu$ be strictly increasing sequences such that $M_{\iota,\kappa} \neq \emptyset$, $\mu + \nu = d$ and $\mu' + \nu' = r(M)$. Note that if $\mu \leq k$ and $\nu \leq l$, 8.13 holds trivially with $K = 1, L = 0$ and $M_{\iota,\kappa}$ in place of both M and N_1. So we assume that $\mu > k$ or $\nu > l$. Then since $M_{\iota,\kappa}$ is open in M, for every $z \in M_{\iota,\kappa}$,

$$r(M) = \mu' + \nu' \leq \operatorname{rank}(\Pi_{k,l}|_{T_z M_{\iota,\kappa}}) \leq r(M),$$

and hence 8.13 with $M_{\iota,\kappa}$ in place of M follows from the fiber cutting lemma. It is therefore enough to prove $(*)$ with

$$\tilde{M} := M \setminus \bigcup_{\substack{\iota,\kappa \\ \mu'+\nu'=r(M)}} M_{\iota,\kappa}$$

in place of M.

Note first that for every $z \in \tilde{M}$, $\operatorname{rank}(\Pi_{k,l}|_{T_z M}) < r(M)$. Since $\tilde{M}$ is clearly an $\mathcal{R}_{m,n,\rho}$-set, we may apply Corollary 8.9 with $\tilde{M}$ in place of A. Denote by $M_\lambda \subseteq \mathbb{R}^{m_\lambda + n_\lambda}$ the manifolds obtained from 8.9 for $\tilde{M}$. Since for each λ the projection $\Pi_{m,n}|_{M_\lambda} : M_\lambda \longrightarrow \Pi_{m,n}(M_\lambda) \subseteq \tilde{M}$ is an analytic isomorphism, it follows that for each $w \in M_\lambda$, $z = \Pi_{m,n}(w)$, we have $\operatorname{rank}(\Pi_{k,l}^{m_\lambda,n_\lambda}|_{T_w M_\lambda}) \leq \operatorname{rank}(\Pi_{k,l}|_{T_z M}) < r(M)$, i.e. $r(M_\lambda) < r(M)$. By 8.9 again each M_λ is equal to $h_{a,\sigma}(H_\lambda)$ for some $a \in \mathbb{R}^{m_\lambda + n_\lambda}$, $\sigma \in \{-1, 1\}^{m_\lambda}$ and some $\mathcal{R}_{m_\lambda,n_\lambda}$-manifold H_λ, and clearly $r(H_\lambda) = r(M_\lambda)$. Therefore by the inductive hypothesis $(*)$ holds with each H_λ in place of M, and one easily verifies that then $(*)$ holds with each M_λ in place of M. This finishes the proof of the lemma. ☐

8.14 Proposition. *Let $A \subseteq \mathbb{R}^{m+n}$ be a bounded, $\mathcal{R}_{m,n}$-semianalytic set, and let $k \leq m$ and $l \leq n$. Then there are connected, bounded $\mathcal{R}_{m_i,n_i}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{m_i+n_i}$ with $m_i \geq m$ and $n_i \geq n$ for $i = 1, \ldots, J$, such that*

$$\Pi_{k,l}(A) = \Pi_{k,l}(N_1) \cup \cdots \cup \Pi_{k,l}(N_J)$$

and for each $N = N_i$, $m' = m_i$ and $n' = n_i$ we have:

1. $\operatorname{fr} N$ *is $\mathcal{R}_{m',n'}$-semianalytic and has dimension with* $\dim(\operatorname{fr} N) < \dim(N)$*;*
2. $\dim(N) \leq k + l$*, and there are strictly increasing sequences $\iota \in \{1, \ldots, k\}^\mu$ and $\kappa \in \{1, \ldots, l\}^\nu$ with $\mu + \nu = d := \dim(N)$ such that $\Pi_{\iota,\kappa}|_N : N \longrightarrow \mathbb{R}^d$ is an immersion.*

Proof. By induction on $e := \dim(A)$; if $e = 0$ then A is finite by 8.9, so the theorem is trivial in this case. So we assume $e > 0$ and that the theorem holds for lower values of e.

Note first that if there is a bounded $\mathcal{R}_{\tilde{m},\tilde{n}}$-semianalytic set $E \subseteq I^{\tilde{m}+\tilde{n}}$ for some $\tilde{m} \geq m$ and $\tilde{n} \geq n$ such that $A = \Pi_{m,n}(E)$ and 8.14 holds with E, $\tilde{m}$ and $\tilde{n}$ in place of A, m and n respectively, then 8.14 also holds for A, m and n; and if A is a finite union of $\mathcal{R}_{m,n}$-semianalytic sets each satisfying 8.14 in place of A, then again 8.14

also holds for A. By 8.9 and the inductive hypothesis, reasoning as at the end of the previous proof, and increasing m and n if necessary, we may therefore reduce to the case that A is a bounded, connected, $\mathcal{R}_{m,n}$-manifold M of dimension d.

Applying Lemma 8.13 to M (with m, n, k, l), let $N_1, \dots, N_K$ and $A_1, \dots, A_L$ be as in $(*)$ for M. Since for each $j = 1, \dots, L$ we have $\dim(A_j) < e$, the inductive hypothesis together with the above implies that we may even reduce to the case where $M = N_i$ for some $i \in \{1, \dots, K\}$ (again increasing m and n if necessary), i.e. condition (2) of 8.14 holds with M in place of N.

Now we again apply 8.9 with M in place of A, and we let N (with corresponding $m' \geq m$ and $n' \geq n$) be one of the M_i's thus obtained from 8.9. We now claim that conditions (1) and (2) of 8.14 hold for this N, which together with the fact that N is a connected, bounded $\mathcal{R}_{m',n'}$-manifold then finishes the proof of 8.14.

Since $\Pi_{m,n}|_N : N \longrightarrow \Pi_{m,n}(N)$ is an analytic isomorphism, $\Pi_{m,n}(N) \subseteq M$ and $\Pi_{\iota,\kappa}|_M$ is an immersion, we see that $\Pi^{m',n'}_{\iota,\kappa}|_N$ is an immersion, which establishes (2). Condition (1) follows from condition (3) of 8.9 with N in place of M. □

8.15 Corollary. *Every Λ-set $A \subseteq I^p$ has the Λ-Gabrielov property.*

Proof. Note first that if $A \subseteq I^{m+n}$ in Corollary 8.9 (resp. Proposition 8.14), then each M_i (resp. N_i) can be taken to be a subset of $I^{m_i+n_i}$ (multiply the coordinates $x_{m+1}, \dots, x_{m'}$, $y_{n+1}, \dots, y_{n'}$ by some small enough $\delta > 0$ and use the remarks in 7.2). Therefore Corollary 8.15 follows from 8.14 with $m = p$ and $n = 0$. □

Theorem A. *The expansion $\mathbb{R}_{\mathrm{an}^*}$ is model complete and o-minimal.*

Proof. Since any Λ-set $A \subseteq I^p$ is a bounded $\mathcal{R}_p$-semianalytic set, A is quantifier-free definable in $\mathbb{R}_{\mathrm{an}^*}$ by a remark in 7.2. The theorem then follows in view of Corollaries 8.15 and 2.9. □

As a consequence of 2.9 and the way we proved Theorem A we have

8.16 Proposition. *If $A \subseteq \mathbb{R}^m$ is bounded and definable in $\mathbb{R}_{\mathrm{an}^*}$, then there are $n \geq m$ and a bounded $\mathcal{R}_n$-semianalytic set $B \subseteq \mathbb{R}^n$ with $A = \Pi_m(B)$.*

9. Polynomial Boundedness

From now on we work in the structure $\mathbb{R}_{\mathrm{an}^*}$; in particular, "definable" means "definable in $\mathbb{R}_{\mathrm{an}^*}$". In this section we prove Theorem B, which characterizes definable 1-variable functions. The main step towards this goal is the curve selection result 9.6, whose proof is along the lines of Tougeron's treatment of curve selection in [15] and [16]. To deduce Theorem B from this curve selection we also need to construct the "compositional inverse" of certain elements of $\mathbb{R}\{T^*\}$; see 9.9. Here T is a single indeterminate. Note that $\mathbb{R}\{T^*\}$ is a valuation ring with residue field $\mathbb{R}$ and value group $\mathbb{R}$. Let $\mathrm{Frac}(\mathbb{R}\{T^*\})$ denote the fraction field of $\mathbb{R}\{T^*\}$; we make it into an ordered field as follows: for $0 \neq g \in \mathbb{R}\{T^*\}$, put $g > 0$ if $g(T) = \sum b_\gamma T^\gamma$ with $b_{\mathrm{ord}(g)} > 0$.

9.1 Lemma. *The local ring $\mathbb{R}\{T^*\}$ is henselian, i.e. given any*

$$f(T, W) = W^n + a_1(T)W^{n-1} + \cdots + a_n(T) \in \mathbb{R}\{T^*\}[W]$$

with $f(0,0) = 0$ and $(\partial f/\partial W)(0,0) \neq 0$, there is $\alpha(T) \in \mathbb{R}\{T^*\}$ such that $\alpha(0) = 0$ and $f(T, \alpha(T)) = 0$.

Proof. Let $f(T,W)$ be as in the lemma. Considering $f(T,W)$ as an element of $\mathbb{R}\{T^*,W\}$, this means that f is regular in W of order 1. Hence by 5.10, $f(T,W) = u(T,W)(W-\alpha(T))$ for some unit $u \in \mathbb{R}\{T^*,W\}$ and some $\alpha \in \mathbb{R}\{T^*\}$, and the lemma follows with this α. $\square$

9.2 Corollary. *The field* $\operatorname{Frac}(\mathbb{R}\{T^*\})$ *is real closed. Every* $f \in \operatorname{Frac}(\mathbb{R}\{T^*\}) \setminus \{0\}$ *is of the form* $T^r g(T)$ *for some* $r \in \mathbb{R}$ *and* $g \in \mathbb{R}\{T^*\}$ *with* $g(0) \neq 0$.

Proof. By 9.1 and the remarks preceding it, using [13]. $\square$

Before we can proceed to curve selection, we need to make sense of substituting a positive generalized power series in one variable in another generalized power series.

9.3 Definition and remarks. Let $h \in \mathbb{R}\{T^*\}$ with $h(0)=0$, and let $r>0$. Then we define

$$(1+h)^r := \sum_{k=0}^{\infty} \binom{r}{k} h^k;$$

note that $(1+h)^r$ is a well defined element of $\mathbb{R}\{T^*\}$ by 5.7.

Now let $0 < g = \sum b_\gamma T^\gamma \in \mathbb{R}\{T^*\}$, and write $g = b_{\gamma_0} T^{\gamma_0}(1+h)$ with $\gamma_0 = \operatorname{ord}(g) \geq 0$, $b_{\gamma_0} > 0$, and $h \in \mathbb{R}\{T^*\}$ with $h(0)=0$. Then we define, for any $r>0$,

$$g^r := b_{\gamma_0}^r T^{r\gamma_0}(1+h)^r.$$

More explicitly, $h = b_{\gamma_0}^{-1} \sum_{\gamma>\gamma_0} b_\gamma T^{\gamma-\gamma_0} = b_{\gamma_0}^{-1} \sum_{\theta>0} b_{\gamma_0+\theta} T^\theta$, so

$$h^k = b_{\gamma_0}^{-k} \left(\sum_{\substack{\theta_1+\cdots+\theta_k=\gamma \\ \theta_1,\ldots,\theta_k>0}} (b_{\gamma_0+\theta_1} \cdots b_{\gamma_0+\theta_k}) \right) T^\gamma.$$

Hence $g^r = \sum b_{r,\gamma} T^\gamma$ with

$$(*) \qquad b_{r,\gamma} = \sum_k \binom{r}{k} b_{\gamma_0}^{r-k} \left(\sum_{\substack{\theta_1+\cdots+\theta_k=\gamma-r\gamma_0 \\ \theta_1,\ldots,\theta_k>0}} b_{\gamma_0+\theta_1} \cdots b_{\gamma_0+\theta_k} \right).$$

(Note that since $\operatorname{supp}(g)$ is well ordered, the right-hand side of equality $(*)$ is actually a finite sum, and that it equals 0 if $\gamma < r\gamma_0$.)

For any small enough $\tau > 0$ we have by 5.5 that $\|h\|_\tau < 1$; let us fix such a number τ. Then by 5.7 and 5.2

$$\|(1+h)^r\|_\tau \leq \sum_{k=0}^{\infty} \left| \binom{r}{k} \right| \|h\|_\tau^k.$$

By (†) in the proof of 6.5 there is a constant $C>0$ depending only on $\|h\|_\tau$ (not on r), such that $\|g^r\|_\tau \leq C\|g\|_\tau^r$; indeed, it follows from (†) in the proof of 6.5 that for any $D \in (0,1)$ the constant $C := \frac{3}{1-D}$ works whenever $\|h\|_\tau \leq D$. By the binomial formula we also get for $t \in (0,\tau)$ that $g(t)>0$ and $g^r(t) = (g(t))^r$.

9.4 Lemma. *Let* $f \in \mathbb{R}\{X^*,Y\}_{\rho,\sigma}$ *for some polyradii* $\rho = (\rho_1,\ldots,\rho_m)$ *and* $\sigma = (\sigma_1,\ldots,\sigma_n)$.

1. *Let* $n \geq 1$, $g \in \mathbb{R}\{T^*\}$ *and suppose* $\|g\|_\tau < \sigma_n$, *where* $\tau > 0$. *Then there are* $\tau' \in (0, \tau]$ *and a series* $h(X, T, Y') \in \mathbb{R}\{(X, T)^*, Y'\}_{\rho,\tau',\sigma'}$, $Y' = (Y_1, \ldots, Y_{n-1})$, *such that*
$$h(x, t, y') = f(x, y', g(t))$$
for every $(x, t, y') \in \operatorname{int}(I_{m+1,n-1,(\rho,\tau',\sigma')})$.
2. *Let* $m \geq 1$, $0 < g \in \mathbb{R}\{T^*\}$ *and suppose* $\|g\|_\tau < \rho_m$, *where* $\tau > 0$. *Then there are* $\tau' \in (0, \tau]$ *and a series* $h(X', T, Y) \in \mathbb{R}\{(X', T)^*, Y\}_{\rho',\tau',\sigma}$, $X' = (X_1, \ldots, X_{m-1})$, *such that*
$$h(x', t, y) = f(x', g(t), y)$$
for every $(x', t, y) \in \operatorname{int}(I_{m,n,(\rho',\tau',\sigma)})$.

Remark. (Here we assume the lemma is true.) We note that by 6.4 the series $h(X, T, Y') \in \mathbb{R}\{(X, T)^*, Y'\}$ (respectively $h(X', T, Y) \in \mathbb{R}\{(X', T)^*, Y\}$) is unique in the sense that it depends only on $f \in \mathbb{R}\{X^*, Y\}$ and $g \in \mathbb{R}\{T^*\}$, but not on choices of ρ, σ, τ with $f \in \mathbb{R}\{X^*, Y\}_{\rho,\sigma}$ and $\|g\|_\tau < \sigma_n$ (resp. $\|g\|_\tau < \rho_m$). We will therefore simply denote $h(X, T, Y')$ by $f(X, Y', g(T))$ (resp. $h(X', T, Y)$ by $f(X', g(T), Y)$). In particular, for any $f \in \mathbb{R}\{X^*, Y\}$ with $n \geq 1$ (resp. $m \geq 1$) and any $g \in \mathbb{R}\{T^*\}$ with $g(0) = 0$ the power series $f(X, Y', g(T))$ (resp. $f(X', g(T), Y)$ with $g > 0$) is well defined.

These substitutions behave as expected. For example, let $f, g \in \mathbb{R}\{T^*\}$, $f(0) \neq 0$, $g(0) = 0$, $g > 0$; then $\frac{1}{f(g)} = \frac{1}{f}(g)$ in $\mathbb{R}\{T^*\}$, as is clear from 6.4. Below we shall freely use facts of this nature.

Proof of 9.4. We distinguish two cases.

Case 1: $g(0) = 0$.

(1) Writing $f(X, Y) = \sum_{k=0}^\infty f_k(X, Y')Y_n^k$ with $f_k \in \mathbb{R}\{X^*, Y'\}$ for $k \in \mathbb{N}$, we define
$$h(X, T, Y') := \sum_{k=0}^\infty f_k(X, Y')g(T)^k;$$
note that $h \in \mathbb{R}[\![(X, T)^*, Y']\!]$ since $\operatorname{ord}(g) > 0$. Convergence of h follows easily from the assumptions on g, and the equation of part (1) holds obviously if f has finite support, and hence by 6.2 for general f.

(2) To simplify notation, we assume throughout the rest of case 1 that $m = 1$ and $n = 0$; the general case is treated similarly. Write $f(X) = \sum a_r X^r$ and $g(T) = \sum b_\gamma T^\gamma$. Let $\gamma_0 := \operatorname{ord}(g) > 0$, and define
$$h(T) := f(0) + \sum_{r>0} a_r g(T)^r = f(0) + \sum_{\gamma>0} c_\gamma T^\gamma,$$
where, for $\gamma > 0$,
$$c_\gamma := \sum_{r\geq 0} a_r b_{r,\gamma},$$
with $b_{r,\gamma}$ as in $(*)$ of 9.3. For these definitions of $h(T)$ and c_γ to make sense, we first need to show that the last sum is actually a finite sum, and that $c_\gamma \neq 0$ only on a well ordered set of γ's. Since the proofs for these two statements are almost the same, we only prove the first one.

Note that $b_{r,\gamma} = 0$ for $\gamma < r\gamma_0$. Assume for a contradiction that $\gamma > 0$ and that there is a sequence $\{r_i\}_{i\in\mathbb{N}}$ of distinct real numbers such that $a_{r_i} b_{r_i,\gamma} \neq 0$ (hence

$r_i \le \gamma/\gamma_0$ for all i). By passing to a subsequence, we may as well assume (using the fact that f has good support) that the sequence $\{r_i\}$ is strictly increasing. Next, by $(*)$ there are for each $i \in \mathbb{N}$ a natural number $k(i) \ge 0$ and real numbers $\theta_{i,1}, \dots, \theta_{i,k(i)} > 0$ such that $\theta_{i,1} + \cdots + \theta_{i,k(i)} = \gamma - r_i\gamma_0$ and $b_{\gamma_0+\theta_{i,j}} \neq 0$ for $j = 1, \dots, k(i)$. Since the sequence $\gamma - r_i\gamma_0$ is strictly decreasing, one easily checks that then there is a strictly decreasing sequence $\{\theta_i\}_{i\in\mathbb{N}}$ such that $b_{\theta_i} \neq 0$, which contradicts the fact that $\operatorname{supp}(g)$ is well ordered.

Next we show that h converges: by 5.5 and the last remark in 9.3 there are $\tau' \in (0, \tau]$ and $C > 0$ such that $\|g^r\|_{\tau'} \le C\|g\|^r_{\tau'}$ (with C depending only on $\|g\|_{\tau'}$, not on r), and hence by 5.7

$$\|h\|_{\tau'} \le |f(0)| + \sum_{r>0} |a_r| \|g^r\|_{\tau'} \le C\|f\|_\rho.$$

The remaining equation of part (2) follows from the last remark of 9.3 if f has finite support, and hence by 6.2 it holds for general f.

Case 2: $g(0) \neq 0$. We only give a proof of part (1) in this case, since the proof of part (2) is similar.

Write $g = b_0 + \tilde{g}$ with $b_0 \in (0, \sigma_n)$ and $\tilde{g} \in \mathbb{R}\{T^*\}$ with $\tilde{g}(0) = 0$. By 6.6, part (2), there is a series $\tilde{h} \in \mathbb{R}\{X^*, Y\}$ such that for every $\sigma'_n \in (0, \sigma_n - |b_0|)$ we have $\tilde{h} \in \mathbb{R}\{X^*, Y\}_{(\rho,\sigma',\sigma'_n)}$ and

$$\tilde{h}(x, y) = f(x, y', b_0 + y_n)$$

for every $(x, y) \in I_{m,n,(\rho,\sigma',\sigma'_n)}$. Now apply part (1) with $\tilde{h}$, $\tilde{g}$ and (σ', σ'_n) in place of f, g and σ respectively. $\square$

Let $f \in \mathbb{R}\{X^*, Y\}^\mu$ with $\mu \in \mathbb{N}$, let $\epsilon > 0$ be f-admissible, and let $U \subseteq I_{m,n,\epsilon}$. We then denote by $(**)$ the statement $(*)$ of 8.4 together with the following statement: for every $M = M_i$, $m' = m_i$ and $n' = n_i$ (with M_i, m_i and n_i as in $(*)$), and every $z \in \operatorname{fr} M$,

(†) there are $\delta > 0$ and $g = (g_1, \dots, g_{m'+n'}) \in \mathbb{R}\{T^*\}^{m'+n'}_\delta$ such that $g(t) \in M$ for every $t \in (0, \delta)$ and $g(0) = z$.

We can now strengthen Proposition 8.4 as follows.

9.5 Proposition. *Let $f \in \mathbb{R}\{X^*, Y\}^\mu$ with $\mu \in \mathbb{N}$, and let $\epsilon > 0$ be f-admissible. Then there is an (m, n)-corner $U \subseteq \operatorname{int}(I_{m,n,\epsilon})$ for which $(**)$ holds.*

We proceed as in the proof of Proposition 8.4; in particular, we first need to establish the following two facts.

Sublemma 1. *Let $m \ge 0, n \ge 1$ be fixed and assume 9.5 holds for all $m' \le m$ and $n' < n$ in place of m and n. Let $f = (f_1, \dots, f_\mu) \in \mathbb{R}\{X^*, Y'\}[Y_n]^\mu$ be such that each f_i is monic in Y_n. Then there is for each f-admissible $\epsilon > 0$ an (m, n)-corner $U \subseteq \operatorname{int}(I_{m,n,\epsilon})$ for which $(**)$ holds.*

Proof. We follow the proof of 8.5 with $(**)$ in place of $(*)$ and work with the notation established in that proof. To finish the proof of Sublemma 1, we assume that (†) holds for the manifold M that we fixed in the proof of 8.5 and every $z \in \operatorname{fr} M$, and we show that then (†) also holds with N in place of M for each $N = N_\kappa$ with $-\epsilon < \xi_\kappa < \epsilon$ and each $N = (N_\kappa, N_{\kappa+1})$ with $-\epsilon \le \xi_\kappa < \xi_{\kappa+1} \le \epsilon$, and for every $z \in \operatorname{fr} N$.

Let $z \in \operatorname{fr} N$, and let w be the image of z under the projection $(x,u,y,v) \mapsto (x,u,y',v) : \mathbb{R}^{m'+n'+1} \longrightarrow \mathbb{R}^{m'+n'}$.

Case 1: $N = N_\kappa$ with $-\epsilon < \xi_\kappa < \epsilon$. By case 1 of the proof of 8.5 we have $w \in \operatorname{fr} M$. By hypothesis there are $\tau > 0$ and $h = (h_1, \dots, h_{m'+n'}) \in \mathbb{R}\{T^*\}_\tau^{m'+n'}$ such that $h(t) \in M$ for $t \in (0,\tau)$ and $h(0) = w$; below we write $\tilde{h} = (h_1, \dots, h_m, h_{m'+1}, \dots, h_{m'+n-1})$. Define the auxiliary set

$$\tilde{N} := \{(s,t) \in \mathbb{R}^2 : 0 < s < \tau, t = \xi(\tilde{h}(s))\} \subseteq \mathbb{R}^2,$$

and for simplicity of notation assume that there is $i \in \{1, \dots, \mu\}$ such that, after shrinking τ if necessary, $\phi(S,T) := f_i(\tilde{h}(S), T)$ vanishes identically on $\tilde{N}$ (in general, this is true for some $\partial^\nu f_i / \partial Y_n^\nu(\tilde{h}(S), T)$ with $\nu < \deg_{Y_n} f$, and the proof is then similar). Note that $\phi \in \mathbb{R}\{S^*\}_\tau[T]$ is monic in T. Hence by 9.2, and after decreasing τ if necessary, ϕ factors as

$$\phi(S,T) = (T - \alpha_1(S)) \cdots (T - \alpha_l(S)) \psi(S,T)$$

with $\alpha_i \in \mathbb{R}\{S^*\}_\tau$, $\psi \in \mathbb{R}\{S^*\}_\tau[T]$, such that $\psi(s,t) > 0$ for all $(s,t) \in (0,\tau) \times \mathbb{R}$. It follows from the o-minimality of $\mathbb{R}_{\mathrm{an}^*}$ that, after decreasing τ once more, there is $j \in \{1, \dots, l\}$ such that $\phi(s, \alpha_j(s)) \in \tilde{N}$ for all $s \in (0,\tau)$ and $\alpha_j(0) = \xi(\tilde{h}(0))$. It is now easy to check that (†) holds with N in place of M, with $\delta := \tau$ and

$$g := (h_1, \dots, h_{m'+n-1}, \alpha_j, h_{m'+n}, \dots, h_{m'+n'}).$$

Note that $h_1, \dots, h_{m'+n'}$ do not depend on κ.

Case 2. $N = (N_\kappa, N_{\kappa+1})$ with $-\epsilon \le \xi_\kappa < \xi_{\kappa+1} \le \epsilon$. If $z \in N_\kappa \cup N_{\kappa+1}$ then (†) holds trivially with N in place of M, so by case 2 of the proof of 8.5 we may assume that $z \in \operatorname{fr} N_\kappa \cup \operatorname{fr} N_{\kappa+1} \cup G$, and hence again $w \in \operatorname{fr} M$. Write $z = (x,u,y',t,v)$, so $w = (x,u,y',v)$. Let $t_1 < t_2$ be such that $(x,u,y',t_1,v) \in \operatorname{fr} N_\kappa$ and $(x,u,y',t_2,v) \in \operatorname{fr} N_{\kappa+1}$. By case 1 above we have $\tau > 0$ and

$$h = (h_1, \dots, h_{m'+n-1}, \alpha_1, h_{m'+n}, \dots, h_{m'+n'}),$$

$$h' = (h_1, \dots, h_{m'+n-1}, \alpha_2, h_{m'+n} \dots, h_{m'+n'})$$

in $\mathbb{R}\{T^*\}_\tau^{m'+n'+1}$ such that $h(t) \in N_\kappa$ and $h'(t) \in N_{\kappa+1}$ for $t \in (0,\tau)$ and $h(0) = (x,u,y',t_1,v)$, $h'(0) = (x,u,y',t_2,v)$. Then (†) holds with N in place of M, where $\delta := \tau$ and

$$g := (h_1, \dots, h_{m'+n-1}, \alpha_1 + c(\alpha_2 - \alpha_1), h_{m'+n}, \dots, h_{m'+n'}),$$

where $c := \frac{t - t_1}{t_2 - t_1}$. ☐

Sublemma 2. *Let $f \in \mathbb{R}\{X^*, Y\}^\mu$, and let $\epsilon > 0$ be f-admissible. Let $S, \phi, \tilde{m}, \tilde{n}$ and $\delta > 0$ be as in Lemma 8.6. Assume that δ is ϕf-admissible and that $(**)$ holds with ϕf in place of f, δ in place of ϵ and some $(\tilde{m}, \tilde{n})$-corner $V \subseteq \operatorname{int}(I_{\tilde{m},\tilde{n},\delta})$ in place of U. Then $\phi(V) \subseteq \operatorname{int}(I_{m,n,\epsilon})$, and $(**)$ holds for f with $U = \phi(V)$.*

Proof. As in the previous sublemma, we follow the proof of 8.6 with $(**)$ in place of $(*)$, and again we use the notation established in the proof of 8.6. So we assume in addition that (†) holds for M and every $z \in \operatorname{fr} M$, and we show that then (†) holds with N in place of M for every $z \in \operatorname{fr} N$. But this follows readily from the definition of N and from 9.4. ☐

Proof of 9.5. The proof of 8.4 (section 8.7) goes now through almost literally for 9.5, with some obvious adaptations: replace $(*)$ by $(**)$ and the references to 8.5 and 8.6 by references to Sublemma 1 and Sublemma 2 respectively. $\square$

9.6 Curve selection. *Let A be a definable subset of $\mathbb{R}^n$, and let $0 \in \operatorname{fr} A$. Then there are $\epsilon > 0$ and $g = (g_1, \dots, g_n) \in \mathbb{R}\{T^*\}_\epsilon^n$ such that $g(t) \in A$ for every $t \in (0, \epsilon)$ and $g(0) = 0$.*

Proof. We may of course assume that A is bounded. Note first that if 9.6 holds with A, then 9.6 also holds with $\Pi_m(A)$ in place of A and m in place of n, for any $m \le n$. Hence by 8.16 we may assume that A is $\mathcal{R}_n$-semianalytic, and by the definition of "$\mathcal{R}_n$-semianalytic", we may even assume that A is a basic $\mathcal{R}_{n,\tau}$-set for some $\tau > 0$.

Since $A = B_{I_{n,\tau}}(f, \sigma)$ for some $f \in \mathbb{R}\{X^*\}_{\tau'}^\mu$ with $\tau' > \tau$ and some $\sigma \in \{-1, 0, 1\}^\mu$, there is by 9.4 an $\mathcal{R}_{n',\rho}$-manifold $M \subseteq \mathbb{R}^{n'}$ for some $n' \ge n$ and $\rho > 0$, such that $0 \in \operatorname{fr} \Pi_n(M)$. But M is bounded, so there is $z \in \operatorname{fr} M$ with $\Pi_n(z) = 0$, and again by 9.4 there are $\epsilon > 0$ and $h = (h_1, \dots, h_{n'}) \in \mathbb{R}\{T^*\}_\epsilon^{n'}$ such that $h(t) \in M$ for all $t \in (0, \epsilon)$ and $h(0) = z$. Now take $g := (h_1, \dots, h_n)$. $\square$

Before we can deduce Theorem B from the curve selection, we need to show that the "compositional inverse" of $f \in \mathbb{R}\{T^*\}$ with $f(0) = 0$ and $f > 0$ exists in $\mathbb{R}\{T^*\}$.

9.7 Remark and definition. Let X be a single indeterminate and write ∂ for ∂_1. Let $\tilde{\rho} > \rho > \tau > 0$, and let $f \in \mathbb{R}\{X^*\}_{\tilde{\rho}}$. By 5.9 and 6.7, each derivative $(f_\rho)^{(k)}$ exists and is analytic in $(0, \rho)$, and for any $|t| < \min(\tau, \rho - \tau)$,

$$f_\rho(\tau + t) = f_\rho(\tau) + (f_\rho)'(\tau)t + \frac{1}{2!}(f_\rho)''(\tau)t^2 + \dots,$$

where the right hand side is an absolutely convergent series. Thus by 5.9 and 6.3,

$$f_\rho(\tau + t) = f_\rho(\tau) + (\partial f)_\rho(\tau)\left(\frac{t}{\tau}\right) + \frac{1}{2!}(\partial^2 f)_\rho(\tau)\left(\frac{t}{\tau}\right)^2 + \dots.$$

We define

$$\tilde{T}f(X, Y) := f(X) + \partial f(X) \cdot Y + \frac{1}{2!}\partial^2 f(X) \cdot Y^2 + \dots \in \mathbb{R}[\![X^*, Y]\!].$$

By the remark after 5.9 we have, with $s := \rho/\tilde{\rho}$ and $C := |s \log s|^{-1} > 1$,

$$\|\partial^k f\|_\rho \le C^k k^k \|f\|_{\tilde{\rho}} \le (3C)^k k! \|f\|_{\tilde{\rho}},$$

so for every $\sigma \in (0, \frac{1}{3C})$ we have $\|\tilde{T}f\|_{\rho,\sigma} \le \frac{1}{1 - 3C\sigma}\|f\|_{\tilde{\rho}} < \infty$. Hence with $|t| < \min(\frac{\tau}{3C}, \rho - \tau)$ we have

$$f(\tau + t) = \tilde{T}f\left(\tau, \frac{t}{\tau}\right).$$

We now want to prove a similar equation with τ and t replaced by suitable series in $\mathbb{R}\{T^*\}$. Note that if $g, h \in \mathbb{R}\{T^*\}$ with $\operatorname{ord}(h) \ge \operatorname{ord}(g)$, $g \ne 0$, then $h/g \in \mathbb{R}\{T^*\}$ also.

9.8 Lemma. *Let X be a single indeterminate, and let $f \in \mathbb{R}\{X^*\}$. Assume $g, h \in \mathbb{R}\{T^*\}$ with $g > 0$ and $\operatorname{ord}(h) > \operatorname{ord}(g) > 0$. Let $\tilde{T}f$ be defined for f as in 9.7.*

Then $f(g+h)$ *and* $\tilde{T}f(g,\frac{h}{g})$ *are in* $\mathbb{R}\{T^*\}$*, and*

$$f(g+h) = \tilde{T}f\left(g, \frac{h}{g}\right).$$

Proof. Use 9.4, 9.7 and 6.4. □

9.9 Lemma. *Let* $0 < f \in \mathbb{R}\{T^*\}$ *with* $f(0) = 0$*. Then there is* $g \in \mathbb{R}\{T^*\}$ *such that* $g > 0$*,* $g(0) = 0$ *and* $f(g(T)) = T$*.*

Proof. Write $f(T) = a_\gamma T^\gamma + h(T)$ with $\gamma > 0$, $a_\gamma > 0$ and $h \in \mathbb{R}\{T^*\}$ with $\eta := \operatorname{ord}(h) > \gamma$. Note first that if $\frac{1}{a_\gamma} f(g(T)) = T$ with $0 < g \in \mathbb{R}\{T^*\}$, $g(0) = 0$, then by 9.4 and 6.4 we have $f(g(T/a_\gamma)) = T$ as well, so we may assume that $a_\gamma = 1$; and similarly, if $f^{1/\gamma}(g(T)) = T$, then $f(g(T^{1/\gamma})) = T$, so we may even assume that $\gamma = 1$. We may also assume that $h \neq 0$, so $1 < \eta < \infty$. Put $\alpha := \frac{1}{2}(\eta - 1) > 0$.

Claim. There are $\rho, \tau > 0$ with $\tau < 1$, $\frac{\tau}{1-\tau^\alpha} < \rho$ and $\|f\|_{2\rho} < \infty$, and there are $\epsilon_n, \delta_n \in \mathbb{R}\{T^*\}_\tau$ for $n \in \mathbb{N}$, such that $\epsilon_0(T) = T$, $\delta_0(T) = f(T) - T$, and

($\diamond$) $\operatorname{ord}(\epsilon_n) > 1 + 2^n\alpha$ if $n > 0$, $\operatorname{ord}(\delta_n) > 1 + 2^{n+1}\alpha$, $\|\epsilon_n\|_\tau \leq \tau^{1+n\alpha}$, and $\|\delta_n\|_\tau \leq \frac{1}{12}\tau^{1+(n+1)\alpha}$, and with $g_n := \sum_{i=0}^n \epsilon_i$ we have $g_n > 0$, $\operatorname{ord}(g_n) = 1$ and

$$f(g_n) = T + \delta_n.$$

Assume for the moment that the claim holds. Let $g := \sum_{n=0}^\infty \epsilon_n \in \mathbb{R}\{T^*\}_\tau$; then $\|g\|_\tau < \rho$, so $f(g) \in \mathbb{R}\{T^*\}_{\tau'}$ for some $\tau' \in (0, \tau]$ by 9.4. Hence for any $t \in [0, \tau']$ we have $\lim_{n\to\infty} g_n(t) = g(t)$ and $\lim_{n\to\infty} \delta_n(t) = 0$, so by the continuity of f,

$$f(g(t)) = \lim_{n\to\infty} f(g_n(t)) = \lim_{n\to\infty}(t + \delta_n(t)) = t,$$

which together with 6.4 finishes the proof of the lemma.

Before we proceed to prove the claim, we note that $f'(T) := \partial f(T)/T \in \mathbb{R}\{T^*\}$ by 5.9 and that $f'(0) = 1$, so the multiplicative inverse $\frac{1}{f'}$ is in $\mathbb{R}\{T^*\}$ as well.

Proof of the claim. Put $C := |\frac{1}{2}\log\frac{1}{2}|^{-1} > 1$, $A := 72 \cdot (6C)^2 > 12$, and choose $\rho > 0$ such that $\|f\|_\sigma \leq 2\sigma$ for every $\sigma \in (0, 2\rho]$ and $\|\frac{1}{f'}\|_\rho \leq 2$. Let $\tau := \frac{2}{3}\rho$ and assume (shrinking ρ if necessary) that $\tau^\alpha \leq \frac{1}{12A}$ and $\frac{1}{1-\tau^\alpha} \leq \frac{3}{2}$. Note that further decreasing ρ does not affect the above inequalities.

We now proceed by induction on n.

Initial step. We put $\epsilon_0(T) := T$ and $\delta_0(T) := f(T) - T$; then $\operatorname{ord}(\epsilon_0) = 1$, $\operatorname{ord}(\delta_0) = \eta > 1 + \alpha$, $\|\epsilon_0\|_\tau = \tau$, and decreasing ρ if necessary we may assume that $\|\delta_0\|_\tau \leq \frac{1}{A}\tau^{1+\alpha}$. Note that now ($\diamond$) holds for $n = 0$.

Inductive step. Let $n > 0$ and assume that we are given $\delta_i, \epsilon_i \in \mathbb{R}\{T^*\}_\tau$ for $i = 0, \ldots, n-1$, such that ($\diamond$) holds with each i in place of n. Note first that

$$\|g_{n-1}\|_\tau \leq \frac{\tau}{1-\tau^\alpha} \leq \frac{3}{2}\tau = \rho;$$

if $g_{n-1} = T(1 + h_{n-1})$ with $h_{n-1}(0) = 0$, then $\|g_{n-1}\|_\tau = \tau(1 + \|h_{n-1}\|_\tau)$, so

$$\|h_{n-1}\|_\tau = \frac{\|g_{n-1}\|_\tau}{\tau} - 1 \leq \frac{1}{2}.$$

Hence from the last remark in 9.3 (with $D = 1/2$) we get for any $r \geq 0$ that $\|g_{n-1}^r\|_\tau \leq 6\|g_{n-1}\|_\tau^r \leq 6\rho^r$, and hence for any $F \in \mathbb{R}\{T^*\}_{\tilde{\rho}}$ with $\tilde{\rho} > \rho$ that

$$\|F(g_{n-1})\|_\tau \leq 6\|F\|_\rho. \tag{I}$$

By 9.8 and the inductive hypothesis we can write, for $h \in \mathbb{R}\{T^*\}$ with $\operatorname{ord}(h) > \operatorname{ord}(g_{n-1}) = 1$,

$$\begin{aligned} f(g_{n-1}+h) &= f(g_{n-1}) + f'(g_{n-1})h + \sum_{k\geq 2} \frac{1}{k!}\partial^k f(g_{n-1}) \left(\frac{h}{g_{n-1}}\right)^k \\ &= T + \delta_{n-1} + f'(g_{n-1})h + \sum_{k\geq 2} \frac{1}{k!}\partial^k f(g_{n-1}) \left(\frac{h}{g_{n-1}}\right)^k. \end{aligned}$$

Put $\epsilon_n := -\delta_{n-1}(f'(g_{n-1}))^{-1}$. Then $\operatorname{ord}(\epsilon_n) = \operatorname{ord}(\delta_{n-1}) > \operatorname{ord}(g_{n-1})$, and by the remark after 9.4, the assumptions on ρ, and (I) we have $\|\frac{1}{f'(g_{n-1})}\|_\tau \leq 6\|\frac{1}{f'}\|_\rho \leq 12$, i.e.

$$\|\epsilon_n\|_\tau \leq 12\|\delta_{n-1}\|_\tau \leq \begin{cases} \frac{1}{A}\tau^{1+\alpha} & \text{if } n=1, \\ \tau^{1+n\alpha} & \text{if } n>1. \end{cases}$$

Replacing h above by ϵ_n, we get

$$f(g_{n-1}+\epsilon_n) = T + \delta_n,$$

where $\delta_n := \sum_{k\geq 2} \frac{1}{k!}\partial^k f(g_{n-1}) \left(\frac{\epsilon_n}{g_{n-1}}\right)^k$. By the inductive hypothesis, for $k \geq 2$ we have

$$\begin{aligned} \operatorname{ord}\left(\partial^k f(g_{n-1}) \left(\frac{\epsilon_n}{g_{n-1}}\right)^k\right) &\geq 1 + k(\operatorname{ord}(\epsilon_n) - \operatorname{ord}(g_{n-1})) \\ &> 1 + 2(1 + 2^n\alpha - 1) \\ &= 1 + 2^{n+1}\alpha; \end{aligned}$$

hence $\operatorname{ord}(\delta_n) > 1 + 2^{n+1}\alpha$. Next note that, by the inductive hypothesis and the assumptions on ρ,

$$\text{(II)} \qquad \begin{aligned} \left\|\frac{\epsilon_n}{g_{n-1}}\right\|_\tau = \frac{1}{\tau}\left\|\frac{\epsilon_n}{1+h_{n-1}}\right\|_\tau &\leq \|\epsilon_n\|_\tau \frac{1}{\tau}\left(1 + \|h_{n-1}\|_\tau + \|h_{n-1}\|_\tau^2 + \dots\right) \\ &\leq \|\epsilon_n\|_\tau \frac{1}{\tau - \tau\|h_{n-1}\|_\tau} \\ &\leq \frac{2}{\tau}\|\epsilon_n\|_\tau \\ &\leq \begin{cases} \frac{2}{A}\tau^{\alpha} & \text{if } n=1, \\ 2\tau^{n\alpha} & \text{if } n>1, \end{cases} \end{aligned}$$

and by the remark after 5.9 (with $s = 1/2$), (I), and the assumptions on ρ,

$$\text{(III)} \qquad \|\partial^k f(g_{n-1})\|_\tau \leq 6\|\partial^k f\|_\rho \leq 6C^k k^k \|f\|_{2\rho} \leq 24(3C)^k k!\rho.$$

Thus using (II) and (III) gives, for $n = 1$,

$$\begin{aligned}
\|\delta_1\|_\tau &\le \sum_{k\ge 2} \frac{1}{k!} \|\partial^k f(g_0)\|_\tau \left\| \frac{\epsilon_1}{g_0} \right\|_\tau \\
&\le 24\rho \sum_{k\ge 2} (3C)^k (\frac{2}{A}\tau^\alpha)^k \\
&\le 24\rho \cdot (6C)^2 \frac{\tau^{2\alpha}}{A^2} \cdot \frac{1}{1 - \frac{6C}{A}\tau^\alpha} \\
&\le 36\tau \cdot (6C)^2 \frac{\tau^{2\alpha}}{A^2} \cdot 2 \\
&\le \frac{1}{12}\tau^{1+2\alpha},
\end{aligned}$$

and similarly, for $n > 1$,

$$\begin{aligned}
\|\delta_n\|_\tau &\le \sum_{k\ge 2} \frac{1}{k!} \|\partial^k f(g_{n-1})\|_\tau \left\| \frac{\epsilon_n}{g_{n-1}} \right\|_\tau \\
&\le 24\rho \sum_{k\ge 2} (3C)^k (2\tau^{n\alpha})^k \\
&\le 36\tau(6C)^2\tau^{2n\alpha} \cdot \frac{1}{1 - 6C\tau^{n\alpha}} \\
&\le A\tau \cdot \tau^\alpha \tau^{(n+1)\alpha} \\
&\le \frac{1}{12}\tau^{1+(n+1)\alpha},
\end{aligned}$$

so $(\diamond)$ holds for n with ϵ_n and δ_n. □

Theorem B. *Let $\epsilon > 0$, and let $f : (0, \epsilon) \longrightarrow \mathbb{R}$ be definable in $\mathbb{R}_{\mathrm{an}^*}$. Then there are a series $F(T) \in \mathbb{R}\{T^*\}$ and an $r \in \mathbb{R}$ such that $f(t) = t^r F(t)$ for all sufficiently small $t > 0$.*

Proof. Assume first that $\lim_{t\to 0} f(t) = 0$. Then $(0, 0) \in \operatorname{fr}\Gamma(f)$, so by 9.6 there are $\tau \in (0, \epsilon)$ and $g_1, g_2 \in \mathbb{R}\{T^*\}_\tau$ such that $(g_1(t), g_2(t)) \in \Gamma(f)$ for all $t \in (0, \tau)$ and $g_1(0) = g_2(0) = 0$. By 9.9 there is $h \in \mathbb{R}\{T^*\}$ such that $h > 0$, $h(0) = 0$ and $g_1(h(T)) = T$. Then it is clear that the desired result holds with $F(T) := g_2(h(T))$ and $r = 0$.

If $\lim_{t\to 0} f(t) = c < \infty$, then the theorem follows easily from the case above by considering $f - c$. If $\lim_{t\to 0} |f(t)| = \infty$, then the theorem follows similarly from the first case by considering $\frac{1}{f}$. □

9.10 Corollary. *The expansion $\mathbb{R}_{\mathrm{an}^*}$ of the real field is polynomially bounded.*

It is easy to see that for any definable set $A \subseteq \mathbb{R}^n$, the dimension $\dim(A)$ agrees with the dimension of A in the sense of o-minimal structures. Using this observation and "cell decomposition" for o-minimal structures (see for example [8]), we obtain the following consequence of Theorem B:

9.11 Corollary. *If $A \subseteq \mathbb{R}^n$ is definable (in $\mathbb{R}_{\mathrm{an}^*}$) and* $\dim(A) \le 1$, *then A is $\mathcal{R}_n$-semianalytic.*

For subsets of $\mathbb{R}^2$ the condition "$\dim(A) \le 1$" can be omitted, and the conclusion strengthened:

9.12 Corollary. *If $A \subseteq \mathbb{R}^2$ is definable, then A is $\mathcal{R}_{1,1}$-semianalytic.*

10. Concluding Remarks

1. Let $0 < \delta < \epsilon$ and let $f(T) \in \mathbb{R}\{T^*\}_\epsilon$. Then the function $f_\delta : [0, \delta] \longrightarrow \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an}^*}$, but in general not in $\mathbb{R}_{\mathrm{an,exp}}$. This is because a necessary condition for f_δ to be definable in $\mathbb{R}_{\mathrm{an,exp}}$ is for $\mathrm{supp}(f)$ to be contained in a finitely generated additive subgroup of $\mathbb{R}$, by Proposition 4.13 and the idea of the proof of Corollary 4.14 in [7]. Clearly, many well ordered subsets of $[0, \infty)$ are not contained in any finitely generated additive subgroup of $\mathbb{R}$, and for each well ordered subset S of $[0, \infty)$ there is a power series $f \in \mathbb{R}\{T^*\}_2$ with $\mathrm{supp}(f) = S$: for example, if $S = \{\gamma_n : n \in \mathbb{N}\}$, we can take $f(T) = \sum_{n=0}^{\infty} 2^{-n-\gamma_n} T^{\gamma_n}$.

2. Theorems A and B of this paper go through (with the same proofs) if the requirement of "good support" for the series $F(X)$ considered in the introduction is strengthened to "$\mathrm{supp}(F) \subseteq S_1 \times \cdots \times S_m$ with $S_i \subseteq [0, \infty)$ such that $|S_i \cap [0, R]| < \infty$ for all positive real R and $i = 1, \ldots, m$". One might wonder if this variant of our results cannot be achieved more directly as in [3] via a suitable preparation theorem for the power series rings involved. We are not aware of any useful preparation theorem of this nature. In any case, the non-noetherianity of these power series rings would seem to be another obstacle in applying this method.

In [3] it is shown that $\mathbb{R}_{\mathrm{an}}$ admits elimination of quantifiers in its natural language augmented by a symbol for the reciprocal function. We have no reason to believe that the analogous statement for $\mathbb{R}_{\mathrm{an}^*}$ is true.

3. A natural next step would be to show that the expansion $\mathbb{R}_{\mathrm{an}^*,\exp}$ of $\mathbb{R}_{\mathrm{an}^*}$ is model complete and o-minimal. (Note that in this expansion the Riemann zeta function on $(1, \infty)$ is definable.) One way to attempt this is as follows.

Let Γ be an ordered vector space over $\mathbb{R}$. There is a natural way to expand the generalized formal power series field $\mathbb{R}((t^\Gamma))$ into a structure $\mathbb{R}((t^\Gamma))_{\mathrm{an}^*}$ for the natural language of $\mathbb{R}_{\mathrm{an}^*}$, so that $\mathbb{R}_{\mathrm{an}^*}$ is a substructure of $\mathbb{R}((t^\Gamma))_{\mathrm{an}^*}$. If one could show that $\mathbb{R}_{\mathrm{an}^*} \preceq \mathbb{R}((t^\Gamma))_{\mathrm{an}^*}$ for all Γ, then the same arguments as in [6] would give us that $\mathbb{R}_{\mathrm{an}^*,\exp}$ is model complete and o-minimal. However, we have not been able to prove that $\mathbb{R}_{\mathrm{an}^*} \preceq \mathbb{R}((t^\Gamma))_{\mathrm{an}^*}$ for all Γ, though it seems quite plausible to us. The second author has obtained a complete axiomatization of the (model complete) theory $\mathrm{Th}(\mathbb{R}_{\mathrm{an}^*})$ and has proved that $\mathbb{R}_{\mathrm{an}^*}$ is existentially closed in its extension $\mathbb{R}((t^\Gamma))_{\mathrm{an}^*}$, which implies in particular that $\mathbb{R}((t^\Gamma))_{\mathrm{an}^*}$ is a substructure of a model of $\mathrm{Th}(\mathbb{R}_{\mathrm{an}^*})$.

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