THE REAL SYMPLECTIC GROUP OF A HILBERT SPACE IS ESSENTIALLY SIMPLE

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Let $K_{\mathbb{R}}$ be a separable *real* Hilbert space, possibly finite dimensional, but then of even dimension and not zero. We denote by $L(K_{\mathbb{R}})$ the real Banach algebra of bounded linear operators on $K_{\mathbb{R}}$, and for $A \in L(K_{\mathbb{R}})$ we denote by A' the adjoint of A. Let J be an orthogonal operator on $K_{\mathbb{R}}$ with $J^2 = -1$ (*orthogonal* means isometric and invertible); it is easily checked that any two such operators are conjugate in the orthogonal group of $K_{\mathbb{R}}$. We define the *real symplectic group* to be

$$\mathrm{Sp}(K_{\mathbb{R}}, J) = \{A \in L(K_{\mathbb{R}}) \mid A'JA = J = AJA'\}.$$

Each $A \in \mathrm{Sp}(K_{\mathbb{R}}, J)$ is invertible with inverse $-JA'J$.

We are interested in *normal subgroups* of $\mathrm{Sp}(K_{\mathbb{R}}, J)$. Our motivation for this is related to theoretical physics; see [**18**]. Indeed we plan to offer alternative proofs for Theorems 6.2 and 6.3 of Shale [**18**] in the same way as [**10**] reformulates part of the material in [**2**] and [**19**]; but these are not included in the present paper. If $K_{\mathbb{R}}$ is finite dimensional, the only non trivial normal subgroup of $\mathrm{Sp}(K_{\mathbb{R}}, J)$ is the centre, which has two elements, namely $+1$ and -1 [**1**; Theorem 3.26 and 5.1]. When $K_{\mathbb{R}}$ is infinite dimensional, there are other normal subgroups such as

$$\mathrm{Sp}(K_{\mathbb{R}}, J; C_0) = \{A \in \mathrm{Sp}(K_{\mathbb{R}}, J) \mid X-1 \text{ is of finite rank}\}$$

and the *extended Fredholm subgroup* of the real symplectic group, that we define to be

$$\mathrm{SpE}(K_{\mathbb{R}}, J; C) = \{A \in \mathrm{Sp}(K_{\mathbb{R}}, J) \mid \text{either } X-1 \text{ or } X+1 \text{ is compact}\}.$$

(J. Eells pointed out to me that the denomination above is imposed by tradition: see [**4**] and [**16**].) It is easy to check that $\mathrm{Sp}(K_{\mathbb{R}}, J; C_0)$ is a simple group and that it is contained in any non central normal subgroup of $\mathrm{Sp}(K_{\mathbb{R}}, J)$ (apply the argument of Section III in [**8**]).

The main result of this paper is that *any non trivial normal subgroup of the real symplectic group is contained in its extended Fredholm subgroup.* At the same time, we show an analogous result for a group of operators on a complex Hilbert space which are unitary with respect to a non degenerate and non positive hermitian form, namely for the group denoted by $U(H, H)$ below. The two theorems appear more precisely in Section 6. The main result is almost equivalent to the simplicity modulo its centre of some group of invertible operators in the Calkin algebra of the complex Hilbert space $K = K_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$; the word "essentially" is thus used in our title according to its standard meaning (see references in [**9**]).

The proofs which follow are very geometrical. We define various groups of interest in the first section, and then actions of these on the relevant "classical

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bounded domains". This makes it then possible to define "geodesic symmetries" in Section 3 (even though we do not define explicitly any Riemannian or Finslerian structure on the domains). The next step is to analyze the situation modulo compact operators. We prove then in this setting the essential simplicity of the groups of interest. The last section completes the proof of the main result itself. We use previous results on the essential simplicity of the unitary group of a Hilbert space; in this sense, the symplectic group seems a good deal harder to handle than the orthogonal group.

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1. *The groups*

As any vector in $K_{\mathbb{R}}$ is orthogonal to its image by J, we may (and we will) assume without loss of generality that there exists a real Hilbert space $H_{\mathbb{R}}$ with $K_{\mathbb{R}} = H_{\mathbb{R}} \oplus H_{\mathbb{R}}$ such that the (2×2)-matrix (with operator entries) associated to J is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let H be the complex Hilbert space $H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, let $K = H \oplus H = K_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and let

$$J_{\mathbb{R}}: \begin{cases} K = K_{\mathbb{R}} \oplus iK_{\mathbb{R}} \longrightarrow K \\ \qquad \xi \oplus i\eta \quad \mapsto \xi \oplus (-i\eta) \end{cases}$$

be the conjugation with fixed points $K_{\mathbb{R}}$ (a *conjugation* is an isometric semi-linear operator with square equal to the identity).

Any operator $A \in L(K_{\mathbb{R}})$ extends uniquely to a $\mathbb{C}$-linear operator on K that we denote again by A. For A in the complex C*-algebra $L(K)$ of bounded operators on K, we write A^* for its adjoint in the usual sense, $\bar{A}$ for the conjugate operator $J_{\mathbb{R}} A J_{\mathbb{R}}$ and A' for the transposed operator $J_{\mathbb{R}} A^* J_{\mathbb{R}}$. (If $K_{\mathbb{R}} = \mathbb{R}^{2n}$, then A may be identified with a $(2n \times 2n)$-matrix with complex entries; then $\bar{A}$ is the conjugate matrix and A' is the transposed matrix.) The map $J_{\mathbb{Q}} = J J_{\mathbb{R}} = J_{\mathbb{R}} J$ is an *anticonjugation*, that is an isometric and semi-linear operator with square equal to minus the identity.

Having identified $L(K_{\mathbb{R}})$ with a subring of $L(K)$, we may now write

$$\mathrm{Sp}(K_{\mathbb{R}}, J) = \{A \in L(K) \mid A'JA = J = AJA' \text{ and } AJ_{\mathbb{R}} = J_{\mathbb{R}}A\}.$$

We define the *complex symplectic group* to be

$$\begin{aligned} \mathrm{Sp}(K, J_{\mathbb{Q}}) &= \{A \in L(K) \mid A^* J_{\mathbb{Q}} A = J_{\mathbb{Q}} = A J_{\mathbb{Q}} A^*\} \\ &= \{A \in L(K) \mid A'JA = J = AJA'\}, \end{aligned}$$

so that the real symplectic group is just one of its real forms:

$$\mathrm{Sp}(K_{\mathbb{R}}, J) = \{A \in \mathrm{Sp}(K, J_{\mathbb{Q}}) \mid \bar{A} = A\}.$$

If the dimension of $H_{\mathbb{R}}$ is finite, say n, the standard notations for the real and complex symplectic groups are respectively $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{Sp}(n, \mathbb{C})$; see [**12**; p. 340].

Let V be the unitary operator on K defined by

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}.$$

As $\exp(-i\pi/4)V$ is in $\mathrm{Sp}(K, J_{\mathbb{Q}})$, the map $A \mapsto V^{-1}AV$ is an inner automorphism of $\mathrm{Sp}(K, J_{\mathbb{Q}})$. The element $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in \mathrm{Sp}(K, J_{\mathbb{Q}})$ is in the image of $\mathrm{Sp}(K_{\mathbb{R}}, J)$ under this map if and only if $E = \bar{B}$ and $D = \bar{C}$; indeed, A should commute with

$$V^{-1}J_{\mathbb{R}}V = V^{-1}\bar{V}J_{\mathbb{R}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} J_{\mathbb{R}} = J_{\mathbb{R}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and the claim follows easily. Hereafter, we shall always denote this group by G; more explicitly

$$G = \left\{ A = \begin{pmatrix} B & C \\ \bar{C} & \bar{B} \end{pmatrix} \in L(K) \;\middle|\; \begin{array}{l} B'\bar{C} \text{ and } BC' \text{ are symmetric and} \\ B'\bar{B} - C^*C = 1 = BB^* - CC^* \end{array} \right\}.$$

("*Symmetric*" means "equal to its transpose", and not "equal to its adjoint"; as $J_{\mathbb{R}}$ maps H onto itself, the maps $B \mapsto \bar{B}$ and $B \mapsto B'$ of $L(H)$ into itself are of course defined as above.)

Let F be the unitary operator on K defined by

$$F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The group

$$U(H, H) = \{A \in L(K) \mid A^*FA = F = AFA^*\}$$

is sometimes called the general symplectic group [17], and also the *group of Q-unitary elements* [13], with Q the sesquilinear form defined on $K \times K$ by $Q(\xi, \eta) = \langle F\xi \mid \eta \rangle$. A straightforward computation shows that $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in L(K)$ is in $U(H, H)$ if and only if

$$B^*B - D^*D = 1 = E^*E - C^*C, \qquad B^*C = D^*E,$$

$$BB^* - CC^* = 1 = EE^* - DD^*, \qquad BD^* = CE^*.$$

If one also has $E = \bar{B}$ and $D = \bar{C}$, these reduce to the relations defining G. If $\dim_{\mathbb{C}} H = n < \infty$, the standard notation for $U(H, H)$ is $U(n, n)$.

2. *Actions on the relevant classical bounded domains*

Consider now the *open unit ball* in $L(H)$,

$$\mathscr{B}_1 = \{Z \in L(H) \mid \|Z\| < 1\},$$

and its *symmetric part*

$$\mathscr{B}'_1 = \{Z \in \mathscr{B}_1 \mid Z' = Z\}.$$

We include the following well-known lemma for completeness.

LEMMA 1. *Let $Z \in \mathscr{B}_1$ and $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in U(H, H)$; write $P = BZ + C$ and $Q = DZ + E$. Then*

(i) *Q is invertible;*

(ii) *$PQ^{-1} \in \mathscr{B}_1$;*

(iii) *if $Z \in \mathscr{B}'_1$ and if $A \in G$, then $PQ^{-1} \in \mathscr{B}'_1$.*

Proof. For $X \in L(H)$, the relation $X \geqslant 0$ means that X is positive while $X > 0$ means that X is both invertible and positive.

The formulas defining $U(H, H)$ imply that

$$E^*E = 1 + C^*C \geqslant 1, \qquad EE^* = 1 + DD^* \geqslant 1,$$

so that E is invertible. The second of these may be written as

$$1 = E^{-1}E^{*-1} + E^{-1}D(E^{-1}D)^*,$$

so that $E^{-1}D < 1$ and $\|E^{-1}DZ\| \leqslant \|E^{-1}D\| \, \|Z\| < 1$. It follows that $Q = E(1 + E^{-1}DZ)$ is invertible. Next

$$\begin{aligned} Q^*Q - P^*P &= Z^*(D^*D - B^*B)Z + Z^*(D^*E - B^*C) + (E^*D - C^*B)Z + E^*E - C^*C \\ &= -Z^*Z + 1 > 0, \end{aligned}$$

so that

$$1 - (PQ^{-1})^*PQ^{-1} = Q^{-1*}(Q^*Q - P^*P)Q^{-1} > 0;$$

this implies that $\|PQ^{-1}\| < 1$. And finally if A is in G, then

$$P'Q - Q'P = Z'(B'\bar{C} - C^*B)Z + Z'(B'\bar{B} - C^*C) + (C'\bar{C} - B^*B)Z + C'\bar{B} - B^*C = Z' - Z.$$

If $Z' = Z$, then

$$PQ^{-1} = (Q'^{-1}P'Q)Q^{-1} = Q'^{-1}P' = (PQ^{-1})'$$

is also in $\mathscr{B}'_1$.

We identify $\mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ with the centre

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \in U(H, H) \,\middle|\, \lambda \in \mathbb{S}^1 \right\}$$

of $U(H, H)$ and we observe that $\{\pm 1\}$ is the centre of G.

PROPOSITION 2. *The mapping*

$$\begin{cases} U(H,H)\times\mathscr{B}_1 \longrightarrow \mathscr{B}_1 \\ \left(\begin{pmatrix} B & C \\ D & E\end{pmatrix}, Z\right) \longmapsto (BZ+C)(DZ+E)^{-1} \end{cases}$$

is an action of $U(H,H)$ by biholomorphic transformations of $\mathscr{B}_1$ which is transitive. The centre $\mathbb{S}^1$ acts trivially and the induced action of $U(H,H)/\mathbb{S}^1$ is effective. The isotropy group of $U(H,H)$ at the origin of $\mathscr{B}_1$ is

$$U(H)\times U(H) \approx \left\{\begin{pmatrix} B & 0 \\ 0 & E\end{pmatrix} \in L(K) \,\middle|\, B \text{ and } E \text{ unitary}\right\}.$$

Similarly the restricted mapping $G\times\mathscr{B}'_1 \to \mathscr{B}'_1$ is an action of G by biholomorphic transformations of $\mathscr{B}'_1$ which is transitive, and effective up to the centre $\{\pm 1\}$. The isotropy group of G at the origin of $\mathscr{B}'_1$ is

$$U(H) \approx \left\{\begin{pmatrix} B & 0 \\ 0 & \bar{B}\end{pmatrix} \in L(K) \,\middle|\, B \text{ is unitary}\right\}.$$

Proof. All the claims are straightforward to check, except perhaps for the transitivity. For $Z \in \mathscr{B}_1$, define

$$B_Z = (1-ZZ^*)^{-1/2}, \qquad C_Z = ZE_Z, \qquad D_Z = Z^*B_Z, \qquad E_Z = (1-Z^*Z)^{-1/2},$$

$$A_Z = \begin{pmatrix} B_Z & C_Z \\ D_Z & E_Z\end{pmatrix}.$$

It is routine to check that A_Z is in $U(H,H)$ (and also selfadjoint), that the transformation it defines maps the origin to Z, and that $A_Z \in G$ in case $Z' = Z \in \mathscr{B}'_1$.

Let us give any subset of $L(K)$ the topology induced by the norm. In particular, $U(H,H)$ and G are topological groups, with the metric for which the distance between A_1 and A_2 is $\|A_2 - A_1\|$. Let

$$\begin{aligned} \mathbf{p} &= \{A \in L(K) \mid A^* = -FAF = A\} \\ &= \left\{A \in L(K) \,\middle|\, A = \begin{pmatrix} 0 & C \\ C^* & 0\end{pmatrix} \text{ with } C \in L(H)\right\}. \end{aligned}$$

Then

$$\exp : \mathbf{p} \to \{A \in U(H,H) \mid A \text{ is positive}\}$$

is a homeomorphism. Indeed, for any $A \in \mathbf{p}$ one has $\exp(A^*) = F(\exp A)^{-1}F$, so that $\exp A$ is in $U(H,H)$, and $\exp A$ is clearly positive. Conversely, for A positive in $U(H,H)$, the expression $\log A$ is defined according to the rules of functional calculus and is thus hermitian; moreover

$$\log(A^*) = \log(FA^{-1}F) = -F(\log A)F$$

and A is in $\mathbf{p}$. As exp and log are inverse to each other, exp is a homeomorphism.

PROPOSITION 3. *Let $U(H)\times U(H)$ be identified with a subgroup of $U(H,H)$ as in Proposition 2 and let $\mathbf{p}$ be as above. Then*

$$\begin{cases} (U(H)\times U(H))\times \mathbf{p} \longrightarrow U(H,H) \\ \qquad (V,A) \qquad \longmapsto V\exp(A) \end{cases}$$

is a homeomorphism. Similarly

$$\begin{cases} U(H)\times\left\{A = \begin{pmatrix} 0 & C \\ C' & 0\end{pmatrix} \in L(K) \,\middle|\, \bar{C} = C\right\} \longrightarrow G \\ \qquad\qquad (V,A) \qquad\qquad\qquad \longmapsto V\exp(A) \end{cases}$$

is a homeomorphism.

Proof. The inverse map of $(U(H)\times U(H))\times\mathbf{p} \to U(H,H)$ can be described as follows. Let $A\in U(H,H)$ and let $A = VP$ be its polar decomposition, with $P = (A^*A)^{1/2}$. As A is in $U(H,H)$, so is A^*, and hence also A^*A; from the facts established just before the proposition, it follows that $P = \exp\left(\frac{1}{2}\log(A^*A)\right)$ is also in $U(H,H)$. Therefore $A\mapsto (V,P)$ is the desired inverse map. This completes the argument for $U(H,H)$, and that for G is similar.

Let us first point out that Proposition 3 is the integral form of a Cartan decomposition for the Lie algebras of $U(H,H)$ and of G; see [**12**; §III.7]. Secondly, Proposition 3 together with Kuiper's Theorem [**15**] shows that $U(H,H)$ and G are contractible; this is a strengthening of Lemma 1.2 in [**17**].

3. *Symmetries and involutions*

The operator $S_0 = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}$ is both in $U(H,H)$ and in G, and the associated transformation in $\mathscr{B}_1$ or in $\mathscr{B}'_1$ is $Z\mapsto -Z$; we define S_0 to be the *geodesic symmetry of* $\mathscr{B}_1$ (or $\mathscr{B}'_1$) *at the origin.* The isotropy groups at the origin characterized in Proposition 2 are then

$$U(H)\times U(H) = \{A\in U(H,H) \mid S_0AS_0^{-1} = A\},$$

$$U(H) = \{A\in G \mid S_0AS_0^{-1} = A\}.$$

Then let $Z\in\mathscr{B}_1$ and let A in $U(H,H)$ be any operator mapping the origin to Z (for example that denoted by A_Z in the proof of Proposition 2). The *geodesic symmetry of* $\mathscr{B}_1$ (or of $\mathscr{B}'_1$ if $Z\in\mathscr{B}'_1$) *at* Z is defined to be $S_Z = A_ZS_0A_Z^{-1}$.

LEMMA 4. *For any $W \in \mathscr{B}_1$, there exists $Z \in \mathscr{B}_1$ with $W = 2Z(1+Z^*Z)^{-1}$. The geodesic symmetry S_Z interchanges the origin and W. If $W \in \mathscr{B}'_1$, then Z is also in $\mathscr{B}'_1$.*

Proof. Consider the functions ϕ and ψ of $[0,1[$ into itself defined by $\phi(x) = 2x(1+x^2)^{-1}$ and $\psi(y) = \big(1-(1-y^2)^{1/2}\big)y^{-1}$ (with $\psi(0) = 0$), which are inverse to each other. The function ψ is given by its Taylor series at the origin in the whole of $[0,1[$; we shall write

$$\psi(y) = y \sum_{n=0}^{\infty} c_n y^{2n}$$

and define

$$Z = W \sum_{n=0}^{\infty} c_n (W^*W)^n .$$

If $W = VP$ is the polar decomposition of W (with $P^2 = W^*W$), then $Z = V\psi(P)$ and

$$2Z(1+Z^*Z)^{-1} = V2\psi(P)\big(1+\psi(P)^*\psi(P)\big)^{-1} = V\phi(\psi(P)) = VP = W .$$

Let A_Z be as in the proof of Proposition 2, and write $A_Z = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ for simplicity. Then

$$S_Z = A_Z S_0 A_Z^{-1} = \begin{pmatrix} B & C \\ D & E \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} B^* & -D^* \\ -C^* & E^* \end{pmatrix}$$

$$= \begin{pmatrix} i(BB^*+CC^*) & i(-BD^*-CE^*) \\ i(DB^*+EC^*) & i(-DD^*-EE^*) \end{pmatrix} = \begin{pmatrix} i(2BB^*-1) & -2iCE^* \\ 2iDB^* & i(1-2EE^*) \end{pmatrix}$$

maps the origin to

$$\begin{aligned} -2iCE^*[i(1-2EE^*)]^{-1} &= -2ZEE^*[1-2EE^*]^{-1} \\ &= -2Z(1-Z^*Z)^{-1}[1-2(1-Z^*Z)^{-1}]^{-1} \\ &= -2Z[(1-2(1-Z^*Z)^{-1})(1-Z^*Z)]^{-1} \\ &= 2Z(1+Z^*Z)^{-1} = W. \end{aligned}$$

As $S_Z^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on the ball, S_Z maps W back to the origin.

Observe that $Z = \sum_{n=0}^{\infty} c_n (WW^*)^n W$. As $X \mapsto X'$ is an antiautomorphism of the algebra of operators on H, one has

$$Z' = W' \sum_{n=0}^{\infty} c_n (W'^*W')^n ,$$

so that $Z' = Z$ in case $W' = W$.

An *involution* in a group is an element of period two.

PROPOSITION 5. *If $n > 1$, then any element in $U(H,H)$ or in G (and with determinant $+1$ in case $n < \infty$) is a product of involutions in the group.*

Proof. The symmetry S_0 is a product of involutions. If H is infinite dimensional, this is because multiplication by i is a product of involutions in $U(H)$; the latter fact is completely elementary, but is also a particular case of the theorem in [**6**]. If H is finite dimensional, the claim above follows for example from the simplicity of $SU(H,H)$ and of G modulo their centres (where $SU(H,H)$ is the kernel of the determinant from $U(H,H)$ to the group of non zero complex numbers). The argument fails when $n = 1$ because the only involution in $G \approx \mathrm{SL}(2,\mathbb{R})$ is central. If $n > 1$, any geodesic symmetry in $U(H,H)$ or in G is also a product of involutions, being conjugate to S_0.

Let $A \in U(H,H)$ (with determinant $+1$ in case $n < \infty$) and let W be the image of the origin by A. If Z is as in Lemma 4, the mapping $S_Z A$ has the origin as a fixed point: $S_Z A \in U(H)\times U(H)$. It is a product of involutions by [**6**] if $n = \infty$, and because $SU(H,H)$ modulo its centre is simple if $n < \infty$. The same argument applies for the group G.

PROPOSITION 6. *Suppose that $A \in U(H,H)$ has some finite power A^n which is central. Then A is conjugate to an element in the isotropy subgroup $U(H)\times U(H)$ at the origin. The analogous statement for G is also true.*

Proof. Theorem II in [**14**] shows that A has (at least) one fixed point Z in $\mathscr{B}_1$. If A_Z is as in the proof of Proposition 2, then $A_Z^{-1} A A_Z$ fixes the origin, and is thus in $U(H)\times U(H)$.

The proof can be adapted to the case of G.

COROLLARY 7. *Suppose that H is infinite dimensional.*

(i) *Let Γ be a normal subgroup of $U(H,H)$ (respectively G) containing the isotropy group $U(H)\times U(H)$ (respectively $U(H)$); then Γ is trivial.*

(ii) *The groups $U(H,H)$ and G are perfect.*

Proof. Assertion (i) follows from Propositions 5 and 6 (with $n = 2$). Then (ii) follows from the perfection of $U(H)$; see for example [**5**; Problem 191].

4. *Descent to the Calkin level*

We assume from now on that H is infinite dimensional.

We consider the Calkin algebras $\check{L}(H)$ of H and $\check{L}(K)$ of K, the images $\check{\mathscr{B}}_1$ and $\check{\mathscr{B}}'_1$ of $\mathscr{B}_1$ and $\mathscr{B}'_1$ by the canonical projection $\pi : L(H) \to \check{L}(H)$, and the images $\check{U}(H,H)_0$ and $\check{G}_0$ of $U(H,H)$ and G by the canonical projection $L(K) \to \check{L}(K)$ (the latter being again denoted by π).

In order to avoid cumbersome repetitions, we shall write down hereafter statements and proofs for G only, and leave the analogous ones for $U(H,H)$ to the reader.

PROPOSITION 8. *The action of G on $\mathscr{B}'_1$ described in Proposition 2 induces a transitive action of $\check{G}_0$ on $\check{\mathscr{B}}'_1$.*

The proof is in [13].

Objects at the Calkin level will often be denoted by small letters corresponding to similar objects in $L(H)$ and $L(K)$. For example $s_0 = \pi(S_0) \in \check{G}_0$ is the geodesic symmetry of $\check{\mathscr{B}}'_1$ at the origin, and $s_z = \pi(S_z)$ (where Z is any point in $\mathscr{B}'_1 \cap \pi^{-1}(z)$), is an element of square -1 which fixes $z \in \check{\mathscr{B}}'_1$. The isotropy group at the origin of $\check{\mathscr{B}}'_1$, which is also the set of those a in $\check{G}_0$ such that $s_0 a s_0 = a$, will be identified with $\check{U}(H)_0 = \pi(U(H))$. (Zero subscripts indicate that one could define larger groups $\check{G}$ and $\check{U}(H)$ of which $\check{G}_0$ and $\check{U}(H)_0$ would be the connected components for the topology defined by the norm.)

We write $f = \pi(F) \in \check{L}(K)$. We may consider both $\pi(J_{\mathbb{R}})$ and $\check{L}(K)$ as inside the "real Calkin algebra" of K, namely the quotient of the ring of $\mathbb{R}$-linear continuous operators on K by the ideal of $\mathbb{R}$-linear compact operators on K. In this way, it makes sense to write that some element $a \in \check{L}(K)$ commutes with $\pi(J_{\mathbb{R}})$. Similarly, if $t = \pi(T)$, where

$$T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} J_{\mathbb{R}} = J_{\mathbb{R}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

it makes sense to write that $at = ta$.

LEMMA 9. *Let $a \in \check{L}(K)$; suppose that $a^n = \lambda$ for some integer $n \geqslant 2$ and for some $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Let $\mu_1, \ldots, \mu_n$ be the n-th roots of λ and define*

$$p_k = n^{-1}(1 + \mu_k^{-1} a + \ldots + \mu_k^{-n+2} a^{n-2} + \mu_k^{-n+1} a^{n-1})$$

for each $k = 1, 2, \ldots, n$. Then the following hold.

(i) *$p_k^2 = p_k$ and $p_k p_l = 0$ for $k, l = 1, 2, \ldots, n$ with $k \neq l$.*

(ii) *$\sum_{k=1}^{n} p_k = 1$ and $\sum_{k=1}^{n} \mu_k p_k = a$.*

(iii) *$a \in \check{G}_0$ if and only if the three following conditions hold:*

(1) *$\lambda = \pm 1$;*

(2) *$p_k f$ is selfadjoint ($k = 1, 2, \ldots, n$);*

(3) *$p_k t = t p_k$ ($k = 1, 2, \ldots, n$).*

Proof. Each claim follows from straightforward computations. Let us check for example that (1), (2) and (3) hold whenever $a \in \check{G}_0$. First $a^n = \lambda \in \check{G}_0$, so that $\lambda = \pm 1$. Then as $|\mu_k| = 1$ and $f^* = f$, we have

$$(p_k f)^* = f p_k^* = n^{-1} f (1 + \mu_k a^* + \ldots + \mu_k^{n-2} (a^*)^{n-2} + \mu_k^{n-1} (a^*)^{n-1}),$$

and as $afa^* = f$, so that $f(a^*)^j = \lambda^{-1} a^{n-j} f$, one has

$$\begin{aligned} f p_k^* &= n^{-1}(1 + \lambda^{-1} \mu_k a^{n-1} + \ldots + \lambda^{-1} \mu_k^{n-2} a^2 + \lambda^{-1} \mu_k^{n-1} a) f \\ &= n^{-1}(1 + \mu_k^{-1} a + \mu_k^{-2} a^2 + \ldots + \mu_k^{-n+1} a^{n-1}) f = p_k f. \end{aligned}$$

Finally, as $at = ta$, it follows that $p_k t = tp_k$ $(k = 1, 2, ..., n)$.

LEMMA 10. *Let $p \in \check{L}(K)$ be an idempotent with pf selfadjoint. Then there exists an idempotent $P \in L(K)$ with PF selfadjoint and $\pi(P) = p$. If moreover $pt = tp$ then one may choose P such that $PT = TP$.*

Proof. Choose $Q \in L(K)$ with $\pi(Q) = p$. One may assume that QF is selfadjoint (if not, pass to $\frac{1}{2}(Q+FQ^*F)$). As Q^2-Q is compact, its spectrum is countable; by the spectral mapping theorem, the spectrum of Q is also countable; hence there is a real number r with $0 < r < 1$ such that the open set Ω of those complex numbers z satisfying $\mathrm{Re}(z) \neq r$ contains the spectrum of Q. Let ϕ be the locally constant function on Ω such that $\phi(0) = 0$ and $\phi(1) = 1$. Using holomorphic functional calculus (see for example [3; Chapitre I, §4, No. 8]), we define

$$P = \phi(Q) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z-Q},$$

where γ is a simple closed path in Ω surrounding those z in the spectrum of Q with $\mathrm{Re}(z) > r$. Then P has all desired properties.

LEMMA 11. *Let $p_1, ..., p_n \in \check{L}(K)$ be such that*

(i) *$p_k^2 = p_k$ and $p_k p_l = 0$ for $k, l = 1, ..., n$ with $k \neq l$,*

(ii) $\sum_{k=1}^{n} p_k = 1$,

(iii) *$p_k f$ is selfadjoint and $p_k t = tp_k$ for $k = 1, ..., n$.*

Then there exist $P_1, ..., P_n \in L(K)$ such that

(iv) *$P_k^2 = P_k$ and $P_k P_l = 0$ for $k, l = 1, ..., n$ with $k \neq l$,*

(v) $\sum_{k=1}^{n} P_k = 1$,

(vi) *$P_k F$ is selfadjoint and $P_k T = TP_k$ for $k = 1, ..., n$,*

(vii) *$\pi(P_k) = p_k$ for $k = 1, ..., n$.*

Proof. Let P_1 in $L(K)$ be an idempotent with $P_1 F$ selfadjoint and $P_1 T = TP_1$ projecting to p_1 in the Calkin algebra (this is possible by Lemma 10). Chose any $R_2 \in L(K)$ with $\pi(R_2) = p_2$ and $P_1 R_2 = R_2 P_1 = 0$. Define $Q_2 = \frac{1}{2}(R_2 + FR_2^* F)$; then $\pi(Q_2) = p_2$ and (using that $P_1 F$ is selfadjoint)

$$P_1 Q_2 = \tfrac{1}{2}(P_1 R_2 + F(R_2 P_1)^* F) = 0, \qquad Q_2 P_1 = \tfrac{1}{2}(R_2 P_1 + F(P_1 R_2)^* F) = 0.$$

One may now repeat the argument of Lemma 10 for the subspace $\mathrm{Ker}(P_1)$ of K and thus construct $P_2 \in L(K)$ having all the desired properties. Similarly, one constructs $P_3, ..., P_{n-1}$; these with $P_n = 1 - \sum_{j=0}^{n-1} P_j$ gives a set $(P_1, ..., P_n)$ satisfying (iv) to (vii).

PROPOSITION 12. *Let $a \in \check{G}_0$; suppose that a^n is central in $\check{G}_0$ for some integer $n \geqslant 2$. Then there exists $A \in G$ with A^n central in G and with $\pi(A) = a$.*

Proof. With the above notation, $A = \sum_{k=1}^{n} \mu_k P_k$ will do.

PROPOSITION 13. (i) *Any element of $\check{G}_0$ is a product of involutions.*

(ii) *Any element a in $\check{G}_0$ for which some finite power a^n is in the centre is conjugate to an element in $\check{U}(H)_0$.*

(iii) *Let $a \in \check{U}(H)_0$; suppose that a is non central and a^2 is central in $\check{G}_0$; if $\| \ \|$ denotes the norm in $\check{L}(K)$, then $\|a-\beta\| \geqslant \sqrt{2}$ for each complex number β.*

Proof. The first two claims follow from Propositions 5, 6 and 12.

Let now a be as in (iii). There exist a complex number λ of modulus one and a projection $p \in \check{L}(K)$ which is neither zero nor the identity such that $a = \lambda(1-p)-\lambda p$. Then

$$\|a-\beta\| = \max(|\lambda-\beta|, |\lambda+\beta|) \geqslant \sqrt{2}.$$

5. *Simplicity of $\check{G}_0/\{\pm 1\}$*

We keep the above notation, as well as the conventions of Section 4: the dimension of H is infinite, and any statement about G has an implicit counterpart for $U(H, H)$. Let Γ be a normal subgroup of G which is *not* contained in its extended Fredholm subgroup

$$\{A \in G \mid A-1 \text{ or } A+1 \text{ is compact}\}.$$

The goal for this section is to show that $\pi(\Gamma) = \check{G}_0$.

The group $\check{G}_0$ has a natural topology, which can be described either as the quotient topology on G (topologized as for Proposition 3) modulo the Fredholm subgroup (which is closed), or as the topology induced by the inclusion of $\check{G}_0$ in the Banach space $\check{L}(K)$. Let Γ_0 be the connected component of the identity in Γ (with the topology induced by G) and let $\check{\Gamma}_0 = \pi(\Gamma_0)$. It is straightforward to check that the quotient of G by its extended Fredholm subgroup has trivial centre. Hence there exist $A \in \Gamma$ and $B \in G$ with BAB^{-1} and A *not* congruent modulo the extended Fredholm subgroup. As G is arc-connected by Proposition 3, there exists a continuous path $t \mapsto B_t$ from $[0, 1]$ to G connecting 1 to B. Define $C_t = B_t A B_t^{-1} A^{-1}$; then $C_t \in \Gamma$ for each $t \in [0, 1]$ and C_1 is not in the extended Fredholm subgroup of G. The group $\check{\Gamma}_0$ is therefore an arc-connected non central normal subgroup of $\check{G}_0$.

Let $a \in \check{\Gamma}_0$ with a outside the centre of $\check{G}_0$, and let $w = a^{-1}(0) \in \check{\mathscr{B}}'_1$. By Lemma 4, there exists a geodesic symmetry s_z which interchanges w and the origin in $\check{\mathscr{B}}'_1$; it follows that as_z fixes the origin and is consequently in $\check{U}(H)_0$. Observe that

$$a(s_z a s_z^{-1}) = -(as_z)^2 \in \check{U}(H)_0 \cap \check{\Gamma}_0.$$

The next claim is that $\check{U}(H)_0 \cap \check{\Gamma}_0$ is not contained in the centre of $\check{G}_0$. There are three cases to consider.

If $(as_z)^2$ is not central in $\check{G}_0$, the claim is obviously correct; if as_z is central in G_0, then a is the product of s_z by a central element in $\check{G}_0$ and the claim is correct by Proposition 13 (ii). We may therefore suppose that $k = -as_z$ is not central in $\check{G}_0$ and that k^2 is central; this third case is longer to deal with than the first two.

Suppose first that $\check{\Gamma}_0$ contains some non central element b which is *not* like $a = ks_z$; then arguing with b instead of a as above establishes the claim. Hence we may suppose that *any* element in $\check{\Gamma}_0$ is "like" a. More specifically, let

$$\begin{cases} [0,1] \to \check{\Gamma}_0 \\ \quad t \mapsto a_t \end{cases}$$

be a continuous path connecting some point in the centre of $\check{G}_0$ to $a = a_1$, with a_t not central for $t > 0$; then $w_t = (a_t)^{-1}(0)$ depends continuously on t, and so does the geodesic symmetry s_t constructed as above which interchanges the origin with w_t; if $k_t = -a_t s_t$, then $a_t = k_t s_t$. When we say that any element is like a, we mean in particular that k_t is not central and that k_t^2 is central in $\check{G}_0$ for each $t \in]0,1]$. Now observe that

$$\begin{cases} [0,1] \to U(H)_0 \\ \quad t \mapsto k_t \end{cases}$$

is a continuous path with k_t central in $\check{G}_0$ if and only if $t = 0$; this is absurd by Proposition 13 (iii).

We have shown that $\check{\Gamma}_0 \cap \check{U}(H)_0$ is not contained in the centre of $\check{G}_0$. Suppose first that it is not contained in the centre of $\check{U}(H)_0$; then $\check{\Gamma}_0$ contains $\check{U}(H)_0$ by Section VI of [**8**] (where $\check{U}(H)_0$ is denoted by $\mathrm{Cal}(H)_0^u$). This is actually always the case. Suppose indeed that $\check{\Gamma}_0 \cap \check{U}(H)_0$ contains some element b in the centre of $\check{U}(H)_0$ but not in that of $\check{G}_0$. Then there exists a complex number β with $|\beta| = 1$ and $\beta \neq \pm 1$ such that

$$b = \begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix}.$$

Let $H = \tilde{H} \oplus \tilde{H}$ be an orthogonal decomposition of H into two isomorphic $J_{\mathbb{R}}$-invariant subspaces, and write accordingly operators on K as (4×4)-matrices with entries in $L(\tilde{H})$. The group of those operators in G associated with (4×4)-matrices with scalar entries is a subgroup S of G isomorphic to $\mathrm{Sp}(2, \mathbb{R})$. The image $\check{S} = \pi(S)$ is also isomorphic to $\mathrm{Sp}(2, \mathbb{R})$ and contains b. As $\mathrm{Sp}(2, \mathbb{R})$ is simple modulo its centre, the normal subgroup $\check{S} \cap \check{\Gamma}_0$ of $\check{S}$ is $\check{S}$ itself. It follows that $\check{\Gamma}_0$ contains an element which is not central in $\check{U}(H)_0$, so that the previous argument applies, and $\check{\Gamma}_0$ contains $\check{U}(H)_0$ in any case. From Proposition 13 it is now clear that $\check{\Gamma}_0$ is $\check{G}_0$ itself. We have reached the assigned goal and reformulate it for reference.

PROPOSITION 14. (i) *The quotient of the real symplectic group by its extended Fredholm subgroup, which is $\check{G}_0/\{\pm 1\}$, is a simple group.*

(ii) *The quotient $\check{U}(H, H)_0/\mathbb{S}^1$ is a simple group.*

6. *End of proof and final remarks*

Let Γ be as at the beginning of Section 5; we know that $\pi(\Gamma) = \pi(G)$ but we still have to show that $\Gamma = G$.

Write $H = \tilde{H} \oplus \tilde{H}$ as at the end of Section 5, and identify any number which the corresponding multiple of the relevant identity operator. For each real number t define

$$A_t = \begin{pmatrix} \operatorname{ch} 2t & 0 & -\operatorname{sh} 2t & 0 \\ 0 & i & 0 & 0 \\ -\operatorname{sh} 2t & 0 & \operatorname{ch} 2t & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Then $A_t^{-1}(0) = \begin{pmatrix} \operatorname{th} 2t & 0 \\ 0 & 0 \end{pmatrix}$. Define

$$S_t = \begin{pmatrix} i\operatorname{ch} 2t & 0 & -i\operatorname{sh} 2t & 0 \\ 0 & i & 0 & 0 \\ i\operatorname{sh} 2t & 0 & -i\operatorname{ch} 2t & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

which is the geodesic symmetry having $\begin{pmatrix} \operatorname{th} t & 0 \\ 0 & 0 \end{pmatrix}$ as fixed point and which interchanges the origin and $\begin{pmatrix} \operatorname{th} 2t & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathscr{B}'_1$. A trivial computation shows that

$$A_t S_t = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

so that

$$(A_t S_t)^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

Now there is $B \in \Gamma$ with $\pi(B) = \pi(A_1)$. Let $W = B^{-1}(0)$ and define S_Z as in Lemma 4. Then $BS_Z BS_Z^{-1} = -(BS_Z)^2$ is in $A(H) \cap \Gamma$ because it fixes the origin, and is not the sum of a scalar operator and a compact operator because $\pi(-BS_Z)^2 = -\pi(A_1 S_1)^2$ is not central. It follows from **[8]** that Γ contains $U(H)$, and then from Corollary 7 that $\Gamma = G$. Expressed then in terms of Sp rather than of G, our main theorem reads as follows.

THEOREM I. *Let Γ be a non trivial normal subgroup of the real symplectic group* $\mathrm{Sp}(H_{\mathbb{R}}, J)$. *Then either Γ is the centre $\{\pm 1\}$, or*

$$\mathrm{Sp}(K_{\mathbb{R}}, J; C_0) \subset \Gamma \subset \mathrm{SpE}(K_{\mathbb{R}}, J; C).$$

Similarly, we have the following.

THEOREM II. *Let Γ be a non trivial normal subgroup of $U(H, H)$. Then either $\Gamma \subset \mathbb{S}^1$ is central, or Γ contains*

$$\{X \in U(H, H) \mid X - 1 \textit{ is of finite rank and } \det(X) = 1\}$$

and is contained in the extended Fredholm subgroup of $U(H, H)$, that is in

$$\{X \in U(H, H) \mid \textit{there exists } \lambda \in \mathbb{S}^1 \textit{ with } \lambda - X \textit{ compact}\}.$$

It is natural to ask whether the above proof could be adapted to any "classical Banach–Lie group of bounded operators" (see [7]). One of the difficulties would be the handling of the analogues of all noncompact irreducible Riemannian symmetric spaces, and not only those of the hermitian ones as above.

More generally, let G be a connected Banach–Lie group which is an algebraic subgroup of the group of invertible elements in some real Banach algebra with unit [11]. Assume that the Lie algebra $\mathbf{g}$ of G contains a closed ideal $\mathbf{n}$ with $\mathbf{g}/\mathbf{n}$ algebraically simple. Let N be the normal subgroup of G generated by the exponential of $\mathbf{n}$. Is G/N algebraically simple?

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