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# THE RECIPROCAL SUM OF THE AMICABLE NUMBERS

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Abstract. In this paper, we improve on several earlier attempts to show that the reciprocal sum of the amicable numbers is small, showing this sum is $< 215$.

## 1. Introduction

Let $\sigma(n)$ denote the sum-of-divisors function; that is, $\sigma(n) = \sum_{d|n} d$. A pair of distinct numbers $n, n'$ is said to form an *amicable pair* if $\sigma(n) = \sigma(n') = n + n'$, and we call an integer *amicable* if it is a member of such a pair. This concept was first noted by Pythagoras who used the function $s(n) = \sigma(n) - n$. Thus, $n$ is amicable if and only if $s(s(n)) = n$ and $s(n) \neq n$. There are more than 1.2 billion amicable pairs known (see [6]) but we do not know whether there are infinitely many of them.

Though studied by many since antiquity, the amicable numbers were not known to comprise a set of asymptotic density 0 until 1955, when this was shown by Erdős [8]. It was not known until 1981 that the amicable numbers have a finite reciprocal sum; see [16]. Roughly using the approach of [16], Bayless and Klyve [2] were able to show that the reciprocal sum of the amicable numbers is less than 656 000 000. This is in contrast to the lower bound of 0.011984 computed from the known amicable numbers, so there is clearly a huge gap between this upper bound and the lower bound!

The paper [16] on the distribution of the amicable numbers was improved in the recent paper [17], and using some ideas from this paper, the first-named author [9] was able to about halve the gap (on a logarithmic scale), showing the reciprocal sum of the amicable numbers is less than 4084. Here we make further progress.

**Theorem 1.1.** *The reciprocal sum of the amicable numbers is smaller than* 215.

One of the ideas from [9], namely using an averaging argument to show there are few abundant numbers ($s(n) > n$) among the odd numbers, is taken further here to include numbers that are 2 (mod 4) and not divisible by 5. In addition, we establish some new estimates on the reciprocal sum of numbers without large prime factors. These estimates may prove to be useful in other problems, such as in [1]. We carve out various subsets of the amicable numbers such as the odd amicables and the even pairs which do not agree (mod 4). In particular, these two subsets have a considerably smaller reciprocal sum than what we are able to prove for the complementary set.

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## 2. Lemmas

**Lemma 2.1.** *With $\gamma$ as Euler's constant, we have for $x > 0$ that*

$$\Big|\sum_{n\leqslant x}\frac{1}{n}-(\log x+\gamma)\Big|<\frac{1}{x}.$$

*Proof.* The result holds trivially when $0 < x < 1$, so assume $x \geqslant 1$. By partial summation

$$\sum_{n\leqslant x}\frac{1}{n}=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\,dt=\log x+\frac{\lfloor x\rfloor}{x}+\int_1^\infty\frac{\lfloor t\rfloor-t}{t^2}\,dt+\int_x^\infty\frac{t-\lfloor t\rfloor}{t^2}\,dt.$$

The next-to-last integral is $\gamma - 1$ so that

$$\sum_{n\leqslant x}\frac{1}{n}=\log x+\gamma-\frac{x-\lfloor x\rfloor}{x}+\int_x^\infty\frac{t-\lfloor t\rfloor}{t^2}\,dt.$$

Since this last integral is positive and smaller than $1/x$, the result follows. □

Let $\varphi$ denote Euler's function, let $\mu$ denote the Möbius function, and let $\omega$ denote the function which counts the distinct prime divisors of a positive integer.

**Corollary 2.1.** *For $x > 0$ and $u$ a positive integer,*

$$\Bigg|\sum_{\substack{n\leqslant x\\ \gcd(n,u)=1}}\frac{1}{n}-\frac{\varphi(u)}{u}(\log x+\gamma)+\sum_{d\mid u}\frac{\mu(d)\log d}{d}\Bigg|<\frac{2^{\omega(u)}}{x}.$$

*Proof.* This result follows immediately from Lemma 2.1 and the identity

$$\sum_{\substack{n\leqslant x\\ \gcd(n,u)=1}}\frac{1}{n}=\sum_{d\mid u}\mu(d)\sum_{\substack{n\leqslant x\\ d\mid n}}\frac{1}{n}=\sum_{d\mid u}\frac{\mu(d)}{d}\sum_{n\leqslant x/d}\frac{1}{n}.$$

□

We remark that in [12] it is observed that, with $p$ a prime variable,

$$\sum_{d\mid u}\frac{\mu(d)\log d}{d}=-\frac{\varphi(u)}{u}\sum_{p\,\mid\,u}\frac{\log p}{p-1},$$

thus enabling the sum on $d$ in Corollary 2.1 to be more easily computed when $\omega(u)$ is large.

As is common, we use the letter $e$ for the base of the natural logarithms.

**Lemma 2.2.** *For any $z > 0$ we have*

$$\sum_{z<n\leqslant ez}\frac{1}{n}<1+\frac{1}{z}.$$

*Let $\mathcal{S}$ be a set of positive integers. We have*

$$\sum_{\substack{z<n\leqslant ez\\ \exists s\in\mathcal{S},\, s\mid n}}\frac{1}{n}<\sum_{s\in\mathcal{S},\, s\leqslant ez}\frac{1}{s}+\frac{1}{z}\sum_{s\in\mathcal{S},\, s\leqslant ez}1.$$

*Proof.* The first estimate is trivial if $z < 1$, so assume that $z \geqslant 1$. Then

$$\sum_{z<n\leqslant ex} \frac{1}{n} \leqslant \frac{1}{\lceil z\rceil} + \sum_{\lceil z\rceil+1\leqslant n\leqslant ez} \frac{1}{n} < \frac{1}{\lceil z\rceil} + \int_{\lceil z\rceil}^{ez} \frac{dt}{t} \leqslant 1 + \frac{1}{z}.$$

For the second estimate, we have that the sum in question is at most

$$\sum_{s\in\mathcal{S},\, s\leqslant ez} \sum_{\substack{z<n\leqslant ez\\ s\mid n}} \frac{1}{n} = \sum_{s\in\mathcal{S},\, s\leqslant ez} \frac{1}{s} \sum_{z/s<m\leqslant ez/s} \frac{1}{m} < \sum_{s\in\mathcal{S},\, s\leqslant ez} \frac{1}{s}\left(1+\frac{s}{z}\right), \tag{2.1}$$

using the first estimate, and the result follows. □

**Lemma 2.3.** *Let $\mathcal{S}$ be a set of positive integers coprime to the positive integer $u$. We have*

$$\sum_{\substack{z<n\leqslant ez\\ \exists s\in\mathcal{S},\, s\mid n\\ \gcd(n,u)=1}} \frac{1}{n} < \frac{\varphi(u)}{u} \sum_{s\in\mathcal{S},\, s\leqslant ez} \frac{1}{s} + \frac{2^{\omega(u)}(1+1/e)}{z} \sum_{s\in\mathcal{S},\, s\leqslant ez} 1.$$

*Proof.* We use Corollary 2.1 for the sum on $m$ in (2.1). □

**Lemma 2.4.** *For a real number $x \geqslant e$, we have*

$$\log\log x < \sum_{n\leqslant x/e} \frac{1}{n\log(x/n)} < \log\log x + \frac{1}{\log x}.$$

*Further, for $x \geqslant 16$,*

$$\sum_{\substack{n\leqslant x/e\\ n \text{ odd}}} \frac{1}{n\log(x/n)} < \log\log x - \frac{1}{2}\log\log(x/2) + \frac{1}{\log x} < \frac{1}{2}\log\log x + \frac{7/5}{\log x},$$

$$\begin{aligned}\sum_{\substack{n\leqslant x/e\\ 2\mid n,\, 3\nmid n}} \frac{1}{n\log(x/n)} &< \frac{1}{2}\log\log(x/2) - \frac{1}{6}\log\log(x/6) + \frac{1}{2\log(x/2)}\\ &< \frac{1}{3}\log\log x + \frac{3/4}{\log x}.\end{aligned}$$

*Proof.* The function $1/(t\log(x/t))$ is decreasing in $t$ on the interval $[1, x/e]$. Since it has an antiderivative $-\log\log(x/t)$, we have

$$\sum_{n\leqslant x/e} \frac{1}{n\log(x/n)} < \frac{1}{\log x} + \int_1^{x/e} \frac{dt}{t\log(x/t)} = \frac{1}{\log x} + \log\log x.$$

For the lower bound, we use

$$\sum_{n\leqslant x/e} \frac{1}{n\log(x/n)} > \int_1^{x/e} \frac{dt}{t\log(x/t)}.$$

The last two assertions follow from the first displayed result and some simple calculations. □

**Lemma 2.5.** *For positive integers $j, n$, let $\tau_j(n)$ denote the number of ordered factorizations of $n$ into $j$ positive factors. We have for any $x \geqslant 1$ that*

$$\sum_{n\leqslant x} \frac{\tau_j(n)}{n} \leqslant \frac{1}{j!}(j+\log x)^j.$$

This result is [10, (4.9)]. There are slightly better estimates available; see [15] for the case $j = 2$ and [3] (plus a routine partial summation argument) for the cases $j \geqslant 3$.

We always use the letters $p, q, r$ to represent prime numbers.

**Lemma 2.6.** *Let*

$$H(x) = \sum_{p \leqslant x} \frac{1}{p}.$$

*With* $B = 0.2614972128\ldots$ *the Mertens constant and* $x \geqslant 286$, *we have*

$$\big|H(x) - (\log\log x + B)\big| < \frac{1}{2(\log x)^2}.$$

*Further,*

$$\sum_{x < p \leqslant ex} \frac{1}{p} < \frac{1}{\log x} + \frac{1}{2(\log x)^2}.$$

*Proof.* The first assertion is [18, Theorem 5], and the second assertion follows from this and also the inequality

$$\log\log(ex) - \log\log x + \frac{1}{2(\log x)^2} + \frac{1}{2(\log(ex))^2} < \frac{1}{\log x} + \frac{1}{2(\log x)^2}.$$

□

**Lemma 2.7.** *For* $x > 1$, *we have*

$$\sum_{p > x} \frac{1}{p^2} < \frac{1}{x \log x}.$$

*Proof.* We easily verify that the lemma holds when $x \leqslant 10^4$ (in fact, the sum is smaller than $0.92/(x \log x)$ in this range), so assume that $x > 10^4$. Let $\theta(t)$ denote the Chebyshev function $\sum_{p \leqslant t} \log p$. It follows from [5] and [7] that

(2.2)

$$t - 2\sqrt{t} < \theta(t) < t \quad (1423 \leqslant t \leqslant 10^{19}), \qquad |\theta(t) - t| < \frac{t}{(\log t)^3} \quad (t \geqslant 89\,967\,803).$$

We have

$$\sum_{p > x} \frac{1}{p^2} = \sum_{p > x} \frac{\log p}{p^2 \log p} = -\frac{\theta(x)}{x^2 \log x} + \int_x^\infty \theta(t)\Big(\frac{2}{t^3 \log t} + \frac{1}{t^3 (\log t)^2}\Big)\, dt,$$

via partial summation. Assume that $x \leqslant 10^{19}$ so that (2.2) implies that

$$\begin{aligned}\sum_{p > x} \frac{1}{p^2} &< -\frac{x - 2\sqrt{x}}{x^2 \log x} + \int_x^\infty \Big(\frac{2}{t^2 \log t} + \frac{1}{t^2 (\log t)^2}\Big)\, dt \\ &= -\frac{x - 2\sqrt{x}}{x^2 \log x} + \frac{2}{x \log x} - \int_x^\infty \frac{dt}{t^2 (\log t)^2} \\ &= \frac{1}{x \log x} + \frac{2}{x^{3/2} \log x} - \int_x^\infty \frac{dt}{t^2 (\log t)^2}.\end{aligned}$$

In addition,

$$\int_x^\infty \frac{dt}{t^2 (\log t)^2} > \frac{1}{(\log ex)^2} \int_x^{ex} \frac{dt}{t^2} = \Big(1 - \frac{1}{e}\Big) \frac{1}{x (\log ex)^2}.$$

Using this estimate in the prior one, we have the lemma in the range $10^4 \leqslant x \leqslant 10^{19}$. The range $x > 10^{19}$ follows in the same way by using the second inequality in (2.2). $\square$

If $a, m$ are coprime integers with $m > 0$, let

$$\pi(x; m, a) = \sum_{\substack{p \leqslant x \\ p \equiv a \pmod{m}}} 1.$$

**Lemma 2.8.** *For $a, m$ coprime as above and $x > m$,*

$$\pi(x; m, a) < \frac{2x}{\varphi(m) \log(x/m)}.$$

*Moreover, if $A, B$ are numbers with $m < A < B$, then*

$$\sum_{\substack{A < p \leqslant B \\ p \equiv a \pmod{m}}} \frac{1}{p} < \frac{2}{\varphi(m) \log(B/m)} + \frac{2}{\varphi(m)} (\log\log(B/m) - \log\log(A/m)).$$

*Proof.* The first assertion is the version of the Brun–Titchmarsh inequality in Montgomery–Vaughan [14]. The second assertion follows directly by partial summation. $\square$

For an integer $n > 1$, let $P(n)$ denote the largest prime factor of $n$, and let $P(1) = 0$.

**Lemma 2.9.** *For $x > y \geqslant 2$ and $0 < s < 1$, let*

$$S(x, y) = \sum_{n > x,\, P(n) \leqslant y} \frac{1}{n}, \quad \zeta(s, y) = \sum_{P(n) \leqslant y} \frac{1}{n^s} = \prod_{p \leqslant y} \Big(1 + \frac{1}{p^s - 1}\Big).$$

*Then $S(x, y) \leqslant x^{-s} \zeta(1 - s, y)$. Further, if $2 \leqslant y_0 < y$, then*

$$S(x, y) \leqslant x^{-s} \exp\Big(\frac{y_0^{1-s}}{y_0^{1-s} - 1} \sum_{y_0 < p \leqslant y} \frac{1}{p^{1-s}}\Big) \prod_{p \leqslant y_0} \Big(1 + \frac{1}{p^{1-s} - 1}\Big).$$

*Proof.* The first inequality is clear since if $n > x$ we have $1/n < x^{-s}/n^{1-s}$. The second inequality follows from $1 + \alpha < e^\alpha$ for $\alpha > 0$ and the fact that $z^{1-s}/(z^{1-s} - 1)$ is decreasing in $z$ for $z \geqslant 2$. $\square$

**Lemma 2.10.** *Let $x > y \geqslant 2$, let $u = \log x / \log y$, and assume that $u \geqslant 3$ and $\log(u \log u)/\log y \leqslant 1/3$. With $S(x, y)$ as in Lemma 2.9, we have*

$$S(x, y) < 25 e^{(1+\varepsilon)u} (u \log u)^{-u} (2^{\log(u \log u)/\log y} - 1)^{-1},$$

*where $\varepsilon = 2.3 \times 10^{-8}$.*

*Proof.* Let $s = \log(u \log u)/\log y$ and apply Lemma 2.9. Then $x^{-s} = (u \log u)^{-u}$ and we have

$$S(x, y) \leqslant (u \log u)^{-u} \exp\Big(\sum_{p \leqslant y} \log\big(1 + 1/(p^{1-s} - 1)\big)\Big). \tag{2.3}$$

We have

$$\sum_{p\leqslant y}\log\Big(1+\frac{1}{p^{1-s}-1}\Big)<\sum_{p\leqslant y}\frac{1}{p^{1-s}}+\sum_{p}\Big(\log\Big(1+\frac{1}{p^{1-s}-1}\Big)-\frac{1}{p^{1-s}}\Big)\tag{2.4}$$
$$<\sum_{p\leqslant y}\frac{1}{p^{1-s}}+0.83,$$

using $s\leqslant 1/3$. Note that from [4] and [5] (also see [13, Proposition 2.1]), we have

$$\theta(t)<(1+\varepsilon)t\quad (t>0),\tag{2.5}$$

where $\varepsilon=2.3\times 10^{-8}$. Let $f(t)=1/(t^{1-s}\log t)$. By partial summation and (2.2) and (2.5), we have

$$\sum_{p\leqslant y}\frac{1}{p^{1-s}}=\sum_{p\leqslant y}f(p)\log p$$
$$=\theta(y)f(y)-\int_2^y\theta(t)f'(t)\,dt<(1+\varepsilon)yf(y)-(1+\varepsilon)\int_2^y tf'(t)\,dt,$$

using that $f'(t)<0$ for $t\geqslant 2$. Integrating by parts, we have

$$\sum_{p\leqslant y}\frac{1}{p^{1-s}}<(1+\varepsilon)2f(2)+(1+\varepsilon)\int_2^y f(t)\,dt=(1+\varepsilon)(\mathrm{Li}(y^s)-\mathrm{Li}(2^s)+2^s/\log 2),\tag{2.6}$$

where $\mathrm{Li}(t)=\int_2^t dx/\log x$. Note that

$$-\mathrm{Li}(2^s)=\int_{2^s}^2\frac{dt}{\log t}<\int_{2^s}^2\frac{dt}{(t-1)-\frac12(t-1)^2}=-\log(2^s-1)+\log(3-2^s).$$

Using this in (2.6) and noting that $y^s=u\log u$, we have

$$\sum_{p\leqslant y}\frac{1}{p^{1-s}}<(1+\varepsilon)(\mathrm{Li}(u\log u)-\log(2^s-1)+\log(3-2^s)+2^s/\log 2).$$

Finally, using this in (2.4) and (2.3), and noting that $\mathrm{Li}(u\log u)<u$ and $\log(3-2^s)+2^s/\log 2+.83<\log 25$, we have the lemma. □

*Remark* 2.1. We can use some of the techniques in the proof of Lemma 2.10 to help numerically with the estimate in Lemma 2.9. In particular, we have

$$\sum_{y_0<p\leqslant y}\frac{1}{p^{1-s}}<(1+\varepsilon)\Big(\mathrm{Li}(y^s)-\mathrm{Li}(y_0^s)+\frac{y_0^s}{\log y_0}\Big)-\theta(y_0)\frac{y_0^{s-1}}{\log y_0}.$$

We find that in the ranges in which we are using Lemma 2.9, it is helpful to take $s=\log(e^\gamma u\log u)/\log y$. Let

$$S_{\mathrm{odd}}(x,y)=\sum_{\substack{n>x\\ P(n)\leqslant y\\ n\,\mathrm{odd}}}\frac1n,\quad S_{\mathrm{even}}(x,y)=\sum_{\substack{n>x\\ P(n)\leqslant y\\ n\,\mathrm{even}}}\frac1n,\quad S_{\mathrm{even,\,no\,3}}(x,y)=\sum_{\substack{n>x\\ P(n)\leqslant y\\ 3\nmid n,\,n\,\mathrm{even}}}\frac1n.$$

In Lemma 2.9, if we know our summand $n$ is odd, as in $S_{\mathrm{odd}}(x,y)$, we may remove the factor $(1+1/(2^s-1))$ from the product. And if we know our summand is even, as in $S_{\mathrm{even}}$, we may replace the factor $(1+1/(2^s-1))$ with $1/(2^s-1)$. In the latter case, if we also know our summand is coprime to 3, as in $S_{\mathrm{even,\,no\,3}}$, we may also remove the factor $(1+1/(3^s-1))$.

## 3. Amicable numbers of moderate size

### 3.1. Parity and number of primes.

**Proposition 3.1.** *Let $\mathcal{A}_0$ denote the set of amicable numbers $n$ such that either*

(1) $n < 10^{14}$,
(2) *$n$ belongs to a pair of opposite parity, or*
(3) $10^{14} < n < e^{300}$ *and* $4 \nmid \sigma(n)$.

*The reciprocal sum of the members of $\mathcal{A}_0$ is $< 2.826$.*

*Proof.* The amicable numbers to $10^{14}$ have been completely enumerated, and their reciprocal sum is $< 0.012$, as reported in [2]. If $n$ belongs to an amicable pair of opposite parity, then $\sigma(n)$ is odd. This implies that $n$ is either a square or the double of a square. There are no examples up to $10^{14}$. Further, as is easy to see,

$$\sum_{n^2>10^{14}} \frac{1}{n^2} + \sum_{2n^2>10^{14}} \frac{1}{2n^2} < \frac{2}{10^7}. \tag{3.1}$$

If $n$ is even and $2 \parallel \sigma(n)$, then $n = pm$, where $p \equiv 1 \pmod 4$ and $m$ is either an even square or the double of one. So, the reciprocal sum of such $n$ in $(10^{14}, e^{300})$, when $p > 10^{14}$, is at most

$$\sum_{10^{14}<p<e^{300}} \frac{1}{2} \sum_{m \text{ or } 2m=\square} \frac{1}{pm} = \frac{3}{4}\zeta(2)(H(e^{300}) - H(10^{14})) < 2.753,$$

using Lemma 2.6. For the case $p < 10^{14}$, we use that for $x > 0$,

$$\sum_{j^2>x} \frac{1}{j^2} < \frac{1}{x} + \int_{\sqrt{x}}^{\infty} \frac{1}{t^2}\,dt = \frac{1}{\sqrt{x}} + \frac{1}{x}.$$

We have

$$\sum_{p<10^{14}} \frac{1}{4} \sum_{\substack{m>10^{14}/(4p)\\ m=\square}} \frac{1}{pm} < \frac{1}{4} \sum_{p<10^{14}} \frac{1}{p}\Big(\sqrt{\frac{4p}{10^{14}}} + \frac{4p}{10^{14}}\Big) = \frac{1}{2\cdot 10^7} \sum_{p<10^{14}} \frac{1}{\sqrt{p}} + \frac{\pi(10^{14})}{10^{14}}.$$

Similarly, we have

$$\sum_{p<10^{14}} \frac{1}{8} \sum_{\substack{m>10^{14}/(8p)\\ m=\square}} \frac{1}{pm} < \frac{1}{\sqrt{8}\cdot 10^7} \sum_{p<10^{14}} \frac{1}{\sqrt{p}} + \frac{\pi(10^{14})}{10^{14}}.$$

We know that $\pi(t) < \mathrm{Li}(t)$ for $t < 10^{19}$; see [5]. Using this we compute that

$$\sum_{p<10^{14}} \frac{1}{\sqrt{p}} < 332\,460.$$

We also know the exact value of $\pi(10^{14})$; it is $3\,204\,941\,750\,802$. Adding these estimates to our prior one when $p > 10^{14}$ and to (3.1), we have less than 2.814 for the reciprocal sum of the members of $\mathcal{A}_0$. ∎

*Remark* 3.1. In the sequel we will only consider amicable pairs of the same parity. We shall also assume a simple, but useful result of Lee [11] that no amicable number in an even-even pair is divisible by 3.

We would like to extend the third property in Proposition 3.1 to all even amicable numbers, but this will require some tools, which will be of use later as well.

**Proposition 3.2.** *Let $\mathcal{A}_1$ denote the set of amicable numbers $n$ not in $\mathcal{A}_0$ with $\omega(n) > 4\log\log n$. The sum of reciprocals of those amicable numbers with at least one of the pair $> e^{100}$ and at least one of the pair in $\mathcal{A}_1$ is less than* $0.029$.

*Proof.* Note that $\tau_4(n) \geqslant 4^{\omega(n)}$, using the notation in §2. For any integer $K \geqslant 10$, we have

$$\sum_{\substack{n>e^K \\ \omega(n)>4\log\log n}} \frac{1}{n} \leqslant \sum_{k\geqslant K+1} \sum_{\substack{e^{k-1}<n<e^k \\ \omega(n)>4\log(k-1)}} \frac{1}{n} < \sum_{k\geqslant K+1} 4^{-4\log(k-1)} \sum_{n<e^k} \frac{\tau_4(n)}{n}$$
$$< \frac{1}{24} \sum_{k\geqslant K+1} \frac{(4+k)^4}{(k-1)^{4\log 4}},$$

by Lemma 2.5. We can use this inequality to capture the reciprocal sum of those amicable numbers $n > e^K$ with $\omega(n) > 4\log\log n$. We must also sum $1/n'$ for such numbers $n$. If $n' > n$,

$$\frac{1}{n} + \frac{1}{n'} < \frac{2}{n}.$$

Suppose $n' < n$ and $\omega(n') \leqslant 4\log\log n'$. If $n'$ is even, then we may assume that $n$ is even as well, so that $n' > n/2$, and

$$\frac{1}{n} + \frac{1}{n'} < \frac{3}{n}. \tag{3.2}$$

Now assume that $n, n'$ are odd. Let $\mu_k$ be the product of $p/(p-1)$ over the first $\lfloor 4\log k\rfloor$ odd primes. Since

$$\omega(n') \leqslant 4\log\log n' < 4\log\log n < 4\log k,$$

we have $n + n' = \sigma(n') < \mu_k n'$, so that

$$\frac{1}{n} + \frac{1}{n'} < \frac{\mu_k}{n}. \tag{3.3}$$

Since $\mu_k > 3$ for $k \geqslant 10$, we have in all cases that (3.3) holds.

It follows from [18, Theorem 15] that if $s(n) > e^{100}$, then $n > e^{97}$. We compute that

$$\frac{1}{24} \sum_{K+1\leqslant k\leqslant 5000} \frac{\mu_k(4+k)^4}{(k-1)^{4\log 4}} < 0.0249$$

for $K = 97$. For $k > 5000$, we use

$$\mu_k < 1.3\log(1 + 4\log k). \tag{3.4}$$

This is verified directly for $5000 < k \leqslant 20\,000$, and for larger values of $k$ we use some estimates in [18], in particular, (3.11) and (3.30), where (3.30) is adapted to odd primes. We compute that

$$\frac{1}{24} \sum_{k>5000} \frac{1.3\log(1+4\log k)(4+k)^4}{(k-1)^{4\log 4}} < 0.0036.$$

This completes the proof. □

3.2. **Multipliers.** We have seen in the proof of Proposition 3.2 that if $n, n'$ form an odd amicable pair with $n > n'$ and $e^{k-1} < n < e^k$, then (3.3) holds, while if $n, n'$ form an even amicable pair, then (3.2) holds. Here $\mu_k$ is the product of $p/(p-1)$ as $p$ runs over the first $\lfloor 4\log k \rfloor$ odd primes. As in (3.4), we have $\mu_k < 1.3\log(1+4\log k)$ for $k > 5000$. In fact, we can do better in certain cases. For example, suppose that $n > n'$ and $h(n') \leqslant 2.5$. Then $n/n' \leqslant 1.5$ and $1/n + 1/n' \leqslant 2.5/n$. We shall see shortly that there are very few odd amicables where one of the pair has $h$-value $> 2.5$, so in moderate ranges we can take the odd multiplier as 2.5.

The multiplier for even amicable numbers can be improved from the "3" in (3.2) when we know that $2^j \mid n, n'$. It can be taken as $(2^{j+1}-1)/(2^j-1)$. Indeed, if $n > n'$, then $s(n)/n > s(2^j)/2^j = 1 - 2^{-j}$. Thus, $n' > (1-2^{-j})n$, and so $1/n + 1/n' < (1 + (1-2^{-j})^{-1})/n$.

3.3. **Proper prime powers.** Let $L(x) = \exp(\sqrt{\lceil \log x \rceil}/5)$, and let $L_k = L(e^k) = e^{\sqrt{k}/5}$. We have $L(x) = L_k$ for all $x \in (e^{k-1}, e^k]$.

**Proposition 3.3.** *Let $\mathcal{A}_2$ denote the set of amicable numbers $n$ not in $\mathcal{A}_0$ or $\mathcal{A}_1$ such that either*

(1) *$n > e^{750}$, $n$ is even, and $n$ is divisible by a proper prime power $> 15L(n)$,*
(2) *$n > e^{1500}$, $n$ is odd, $s(n)/n \leqslant 1.5$ when $n < e^{5000}$, and $n$ is divisible by a proper prime power $> 15L(n)$,*
(3) *$n > e^{300}$ and $P(n)^2 \mid n$.*

*The reciprocal sum of those amicable numbers $n$ with $n$ or $n'$ in $\mathcal{A}_2$ is $< 4.500$.*

*Proof.* Let $S$ be the reciprocal sum of all odd proper prime powers so that

$$S = \sum_{p \geqslant 3} \sum_{a \geqslant 2} \frac{1}{p^a} = \sum_{p \geqslant 3} \frac{1}{p(p-1)}.$$

We compute that

$$0.1064900 < S < 0.1064901. \tag{3.5}$$

By a fairly trivial argument, for $B \geqslant 12$ we have

$$\sum_{p^a > B,\, a \geqslant 2} \frac{1}{p^a} = \sum_{p > \sqrt{B}} \frac{1}{p(p-1)} + \sum_{p \leqslant \sqrt{B},\, p^a > B} \frac{1}{p^a} < \frac{1}{\sqrt{B}-1} + \frac{\pi(\sqrt{B})}{B} < \frac{2}{\sqrt{B}}. \tag{3.6}$$

We also have that for $x \geqslant 200$,

$$\sum_{p^a \leqslant x,\, a \geqslant 2} 1 = \sum_{j \geqslant 2} \pi(x^{1/j}) < x^{1/2}. \tag{3.7}$$

Let

$$\mathcal{S} = \{p^a : p \geqslant 5,\, a \geqslant 2\}, \quad \mathcal{S}_k = \mathcal{S} \cap (15L_k, e^k).$$

We have, by Lemma 2.2, Lemma 2.3, and (3.7), that for any positive integer $k$,

$$\sum_{\substack{e^{k-1} < n < e^k \\ \exists s \in \mathcal{S}_k,\, 2s \mid n \\ \gcd(n,3)=1}} \frac{1}{n} < \frac{1}{3} \sum_{s \in \mathcal{S}_k} \frac{1}{s} + 3e^{1-k} \#\mathcal{S}_k < \frac{1}{3}\Big(S - \sum_{s \in \mathcal{S},\, s \leqslant L_k} \frac{1}{s}\Big) + 3e^{1-k/2}$$

and

$$\sum_{\substack{e^{k-1}<n<e^k\\ \exists s\in\mathcal{S}_k,\, s\mid n\\ n \text{ odd}}} \frac{1}{n} < \frac{1}{2}\sum_{s\in\mathcal{S}_k}\frac{1}{s} + 3e^{1-k}\#\mathcal{S}_k < \frac{1}{2}\Big(S - \sum_{s\in\mathcal{S},\, s\leqslant L_k}\frac{1}{s}\Big) + 3e^{1-k/2}.$$

Using that even amicable numbers are not divisible by 3 (Remark 3.1), if $e^{k-1} < n < e^k$ is an even amicable number divisible by a proper prime power $> 15L_k$, then either $n$ coprime to 3 is divisible by a power of 2 that is $> 15L_k$ or $n$ coprime to 3 is divisible by the double of a member of $\mathcal{S}_k$. We have

$$\sum_{k=750}^{10\,000}\sum_{\substack{e^{k-1}<n<e^k\\ n \text{ amicable}\\ n \text{ even}\\ \exists s\in\mathcal{S}_k,\, s\mid n}}\Big(\frac{1}{n}+\frac{1}{n'}\Big) \leqslant 3\sum_{k=750}^{10\,000}\sum_{\substack{e^{k-1}<n<e^k\\ \exists s\in\mathcal{S}_k,\, 2s\mid n\\ \gcd(n,3)=1}}\frac{1}{n} < 2.4581.$$

Since $\mathcal{S}$ leaves out powers of 2, in the even case we should also be summing $2/(15L_k)$. (The factor 2 reflects the multiplier 3 and the fact that $n$ is not divisible by 3.) This adds on $< 0.1809$ summing to infinity. For the remaining even amicables $> e^{10\,000}$ we use (3.7) and (3.6) with the above method to find the reciprocal sum is $< 0.0516$. In total, the contribution to the reciprocal sum in case (1) is $< 2.6906$.

For odd amicable numbers, using a multiplier the 2.5 below $e^{5000}$, we have

$$\sum_{k=1500}^{5000}\sum_{\substack{e^{k-1}<n<e^k\\ n \text{ amicable}\\ n \text{ odd}\\ \exists s\in\mathcal{S}_k,\, s\mid n}}\Big(\frac{1}{n}+\frac{1}{n'}\Big) \leqslant 2.5\sum_{k=1500}^{5000}\sum_{\substack{e^{k-1}<n<e^k\\ \exists s\in\mathcal{S}_k,\, s\mid n\\ n,\text{ odd}}}\frac{1}{n} < 0.9949.$$

Beyond 5000 we use multiplier $1.3\log(1+4\log k)$ from (3.4) for the odd amicables and find their contribution to $e^{10\,000}$ is $< 0.0786$. Using (3.6) beyond $e^{10\,000}$ the contribution is $< 0.1198$. Finally, since $\mathcal{S}$ leaves out powers of 3, we add on the sum from $k = 1500$ to $5000$ of $1.25/(15L_k)$ and the sum beyond $k = 5000$ of $(1/2)1.3\log(1+4\log k)/(15L_k)$, which is $< 0.0159$. In all, the contribution to the reciprocal sum in case (2) is $< 1.1306$.

If $n$ is an amicable number $> e^{300}$ and $n, n' \notin \mathcal{A}_1$, then $n' > e^{298}$. Since $\omega(n) \leqslant 4\log\log n$ it follows that the largest prime power $p^a$ (proper or not) that divides $n$ is $> n^{1/(4\log\log n)}$. If $a = 1$, then $p = P(n)$ and $n$ is not in case (3). If $a > 1$, then (3.6) and (3.7) imply that the reciprocal sum in question is at most

$$\sum_{k=299}^{5000}\mu_k\Big(\frac{2}{e^{(k-1)/(8\log(k-1))}} + e^{1-k/2}\Big) < 0.6781,$$

$$\sum_{k>5000} 1.3\log(1+4\log k)\Big(\frac{2}{e^{(k-1)/(8\log(k-1))}} + e^{1-k/2}\Big) < 10^{-29}.$$

Adding together the contributions in cases (1), (2), and (3) proves the proposition. □

For an integer $n > 1$, the largest prime power that divides $n$ is at least $n^{1/\omega(n)}$. If $\omega(n) \leqslant 4\log\log n$ and $n$ is not divisible by a proper prime power $> \frac{1}{2}L(n)$, then for $n \geqslant 20$, we have $P(n) \geqslant n^{1/4\log\log n}$ and $P(n)^2 \nmid n$. We apply this to the numbers $n, n'$ in an amicable pair with $n, n'$ not in $\mathcal{A}_j$, $j < 3$. It follows that we may write $n = pm$ where $p = P(n) \nmid m$, and similarly, $n' = p'm'$ where $p' = P(n') \nmid m'$.

We now complete the argument for $4 \mid \sigma(n)$, showing that this may be assumed for even amicable numbers, since those that do not satisfy this property have a fairly small reciprocal sum.

**Proposition 3.4.** *Let $\mathcal{A}_3$ denote the set of amicable numbers $n$ with $n, n' \notin \mathcal{A}_j$ for $j < 3$, with $4 \nmid \sigma(n)$. The reciprocal sum of those amicable numbers with at least one of the pair $> e^{300}$ and with $n, n' \in \mathcal{A}_3$ is $< 0.349$.*

*Proof.* We have just seen that we have $n = pm$, $n' = p'm'$, where $p, p'$ are the largest primes in $n, n'$, appearing to just the first power. Thus, $\sigma(n) = \sigma(n')$ are both even. If they are not divisible by 4, then both $m, m'$ are either squares or doubles of squares. It is shown in [17, step (v)] that $m, m'$ uniquely determine $n, n'$. We have

$$mm' = \frac{nn'}{pp'} < n^{1-1/4\log\log n} n'^{1-1/4\log\log n'}.$$

Suppose that $e^{k-1} < n < e^k$. Then $n' < (\mu_k - 1)n$ so that

$$mm' < (\mu_k - 1)e^{2k-0.5/\log\log((\mu_k-1)e^k)} = x_k, \text{ say.} \tag{3.8}$$

Let $\mathcal{S}$ denote the set of numbers that are either squares or the doubles of squares, with counting function $\mathcal{S}(x)$. Then $\mathcal{S}(x) < 2\sqrt{x}$ for $x \geqslant 1$. The number of pairs $m, m'$ in $\mathcal{S}$ satisfying (3.8) is at most

$$\sum_{m<x_k,\, m\in\mathcal{S}} \sum_{m'<x_k/m,\, m'\in\mathcal{S}} 1 < \sum_{m<x_k,\, m\in\mathcal{S}} 2\sqrt{\frac{x_k}{m}} < (4 + 2\log x_k)\sqrt{x_k},$$

where we have used partial summation for the last estimate. Thus, the number of $n$ is upper-bounded by this last estimate, so the reciprocal sum is at most

$$\frac{(4 + 2\log x_k)\sqrt{x_k}}{e^{k-1}}.$$

Summing this expression for $k \geqslant 299$ we get a contribution of at most 0.349. □

**Corollary 3.1.** *If $n > 10^{14}$ is an amicable number with $n, n' \notin \mathcal{A}_j$, $j \leqslant 3$, then $2 \parallel n$ if and only if $2 \parallel n'$. Further, $28 \nmid n$.*

*Proof.* We have seen the first assertion in Propositions 3.1, 3.4. For the second, assume that $28 \mid n$. By the first part, $4 \mid n'$. If $2^2 \parallel n$ or $2^2 \parallel n'$, then $7 \mid \sigma(n) = \sigma(n')$, so that $28 \mid n'$ as well. Thus, both $n, n'$ are abundant, a contradiction. We thus have $2^3 \mid n, n'$. Then

$$1 = \frac{s(n)}{n}\frac{s(n')}{n'} > \frac{s(56)}{56}\frac{s(8)}{8} = 1,$$

a contradiction. This completes the proof. □

3.4. **Odd amicables of moderate size.** For the rest of this section we have $K \geqslant 50$ an integer.

**Proposition 3.5.** *We have*

$$\sum_{\substack{n<e^K \\ n \text{ odd, amicable}}} \frac{1}{n} < 0.023773K + 0.030, \qquad \sum_{\substack{n<e^K \\ n \text{ odd, amicable} \\ h(n) \text{ or } h(n')>2.5}} \frac{1}{n} < 3.777\times 10^{-5}K + 5\times 10^{-5}.$$

*Proof.* Let $h(n)=\sigma(n)/n$. We say $n$ is abundant when $h(n)>2$ and $n$ is deficient when $h(n)<2$. For any positive integer $j$ we have

$$\sum_{\substack{n<e^K\\ n\text{ odd, abundant}}}\frac{1}{n}<2^{-j}\sum_{\substack{n<e^K\\ n\text{ odd}}}\frac{h(n)^j}{n}. \tag{3.9}$$

Let $f_j(n)$ be the multiplicative function with $f_j(p^a)=h(p^a)^j-h(p^{a-1})^j$ for prime powers $p^a$ so that

$$h(n)^j=\sum_{d\mid n}f_j(d). \tag{3.10}$$

Thus, by (3.9),

$$\sum_{\substack{n<e^K\\ n\text{ odd, abundant}}}\frac{1}{n}<2^{-j}\sum_{\substack{d<e^K\\ d\text{ odd}}}\frac{f_j(d)}{d}\sum_{\substack{m<e^K/d\\ m\text{ odd}}}\frac{1}{m}.$$

By Corollary 2.1 with $u=2$, we have

$$\sum_{\substack{m\leqslant e^K\\ m\text{ odd}}}\frac{1}{m}<\frac{1}{2}K+\frac{1}{2}\gamma+\frac{1}{2}\log 2+\frac{2}{e^K}<\frac{1}{2}K+0.64$$

using $K\geqslant 50$. Thus,

$$\sum_{\substack{d<e^K\\ d\text{ odd}}}\frac{f_j(d)}{d}\sum_{\substack{m<e^K/d\\ m\text{ odd}}}\frac{1}{m}<\frac{1}{2}(K+1.28)\sum_{d\text{ odd}}\frac{f_j(d)}{d},$$

and so

$$\sum_{\substack{n<e^K\\ n\text{ odd, amicable}}}\frac{1}{n}<2\sum_{\substack{n<e^K\\ n\text{ odd, abundant}}}\frac{1}{n}<2^{-j}(K+1.28)\sum_{d\text{ odd}}\frac{f_j(d)}{d}.$$

Note the Euler product

$$\sum_{d\text{ odd}}\frac{f_j(d)}{d}=\prod_{p>2}\left(1+\frac{f_j(p)}{p}+\frac{f_j(p^2)}{p^2}+\cdots\right), \tag{3.11}$$

which allows us, for any particular value of $j$, to compute this sum to high accuracy. We find that the optimal value of $j$ is 18, and

$$2^{-j}\sum_{d\text{ odd}}\frac{f_j(d)}{d}<0.023773.$$

This completes the proof of the first assertion.

The second assertion follows by exactly the same method, where the factor $2^{-j}$ is replaced with $2.5^{-j}$. The minimum value is $3.776\times 10^{-5}$, which occurs at $j=44$. □

We shall use $K=1500$ in the first inequality of Proposition 3.5 and $K=5000$ in the second. We have

$$\sum_{\substack{n<e^{1500}\\ n\text{ odd, amicable}}}\frac{1}{n}+\sum_{\substack{e^{1500}<n<e^{5000}\\ n\text{ odd, amicable}\\ h(n)\text{ or }h(n')>2.5}}\frac{1}{n}<35.849. \tag{3.12}$$

3.5. **Even amicables of moderate size.** We now turn to even amicable numbers $< e^K$, where as before, $K \geqslant 50$ is an integer.

**Proposition 3.6.** *We have*

$$\sum_{\substack{n<e^K,\, 2\,\|\, n\\ 5\nmid n,\, h(n)>2\\ n \text{ amicable}}} \left(\frac{1}{n}+\frac{1}{n'}\right) < 0.003559K + 0.0055.$$

*Proof.* Using that $6 \nmid n$ from Remark 3.1, the sum in question is at most

$$2 \sum_{\substack{n<e^K,\, 2\,\|\, n\\ \gcd(n,15)=1\\ h(n)>2}} \frac{1}{n}.$$

If $2 \,\|\, n$ and $\gcd(n,15)=1$, then $n=2l$ where $\gcd(l,30)=1$. Since $h(2)=3/2$, we have $h(n)>2$ if and only if $h(l)>4/3$. Thus, for any positive integer $j$, we have

$$\sum_{\substack{n<e^K\\ h(n)>2\\ 2\,\|\, n,\, \gcd(n,15)=1}} \frac{1}{n} = \frac{1}{2} \sum_{\substack{l<e^K/2\\ h(l)>4/3\\ \gcd(l,30)=1}} \frac{1}{l} < \frac{1}{2}\left(\frac{3}{4}\right)^j \sum_{\substack{l<e^K/2\\ \gcd(l,30)=1}} \frac{h(l)^j}{l}$$

$$= \frac{1}{2}\left(\frac{3}{4}\right)^j \sum_{\substack{d<e^K/2\\ \gcd(d,30)=1}} \frac{f_j(d)}{d} \sum_{\substack{m<e^K/2d\\ \gcd(m,30)=1}} \frac{1}{m},$$

using (3.10). By Corollary 2.1, the inner sum here is at most

$$\frac{4}{15}(K-\log 2+\gamma)+0.438617+\frac{8}{e^K/2} < \frac{4}{15}K+0.41,$$

using $K \geqslant 50$. Further, using the Euler product in (3.11) starting at $p=7$, we find that when $j=35$,

$$\left(\frac{3}{4}\right)^j \sum_{\gcd(d,30)=1} \frac{f_j(d)}{d} < 0.013343.$$

Thus,

$$\sum_{\substack{n<e^K,\, 2\,\|\, n\\ \gcd(n,15)=1\\ n \text{ amicable}}} \frac{1}{n} < 2 \sum_{\substack{n<e^K,\, 2\,\|\, n\\ \gcd(n,15)=1\\ n \text{ abundant}}} \frac{1}{n} < 2\cdot\frac{1}{2}\cdot 0.013343\left(\frac{4}{15}K+0.41\right) < 0.003559K+0.0055.$$

This completes the proof. □

For the remaining amicables with $2\|n$ we have two remaining cases:

(1) $5 \nmid n$, $n$ deficient, $5 \mid n'$,
(2) $5 \mid n$.

Note that in case (1) we have $1/n < 1/n'$, so the reciprocal sum in case (1) is less than the reciprocal sum in case (2). Thus,

$$\sum_{\substack{10^{14}<n<e^K\\ 2\,\|\, n,\, 5\mid n\\ h(n)>2\\ n \text{ amicable}}} \left(\frac{1}{n}+\frac{1}{n'}\right) < 2 \sum_{\substack{10^{14}<n<e^K\\ 2\,\|\, n,\, \gcd(n,15)=5}} \frac{1}{n} < \frac{1}{15}K - 2.149 \tag{3.13}$$

for $K \geqslant 50$, using Corollary 2.1 and Remark 3.1.

When $2^2 \mid n$, we can use that $3, 7 \nmid n$ from Remark 3.1 and Corollary 3.1. Thus, if $2^2 \parallel n$,

$$\sum_{\substack{10^{14}<n<e^K\\ 2^2 \parallel n\\ n \text{ amicable}}} \frac{1}{n} \leqslant \sum_{\substack{10^{14}<n<e^K\\ 2^2 \parallel n\\ \gcd(n,21)=1}} \frac{1}{n} < \frac{1}{14}K - 2.302.$$

If $2^3 \parallel n$, since $5 \mid \sigma(2^3)$ and 20 is abundant, we have that not only $3, 7 \nmid n$, but $5 \nmid n$. Thus,

$$\sum_{\substack{10^{14}<n<e^K\\ 2^3 \parallel n\\ n \text{ amicable}}} \frac{1}{n} < \frac{1}{35}K - 0.921.$$

We finally consider $2^4 \mid n$. We consider two cases: $5 \mid n$ and $5 \nmid n$. In the first case, if $n/80$ is divisible by any of the 59 primes to 277, then $h(n) > 7/3$, and so $n$ cannot belong to an amicable pair with both members divisible by 4. Thus,

$$\sum_{\substack{10^{14}<n<e^K\\ 80 \mid n\\ n \text{ amicable}}} \frac{1}{n} < 0.001232K,$$

again using $K \geqslant 50$. The remaining even amicables to $e^K$ have reciprocal sum at most

$$\sum_{\substack{10^{14}<n<e^K\\ 16 \mid n,\, \gcd(n,105)=1}} \frac{1}{n} < \frac{1}{35}K - 0.921.$$

Adding together all of the contributions in this subsection, we have

$$\sum_{\substack{10^{14}<n<e^K\\ n \text{ even, amicable}}} \frac{1}{n} < 0.20003K - 6.287.$$

In particular, taking $K = 750$,

$$\sum_{\substack{10^{14}<n<e^{750}\\ n \text{ even, amicable}}} \frac{1}{n} < 143.736. \tag{3.14}$$

## 4. Large amicable numbers

We consider odd amicable numbers in $(e^{1500}, e^{5000})$ with $h(n), h(n') \leqslant 2.5$, odd amicable numbers $> e^{5000}$, and even amicable numbers $> e^{750}$.

**Proposition 4.1.** *Let $\mathcal{A}_4$ denote the set of amicable numbers $n$ such that $n, n' \notin \mathcal{A}_j$ for $j < 4$ and $\gcd(n, s(n))$ is divisible by a prime $> 31L(n)$. The reciprocal sum of those amicable numbers with at least one of the pair $> e^{750}$ in the even case and $> e^{1500}$ in the odd case, and at least one of the pair in $\mathcal{A}_4$, is at most* 0.049.

*Proof.* Let $n$ be an amicable number in the interval $(e^{k-1}, e^k)$. Let $n'' = \min\{n, n'\}$. If $n$ is even, then $n'' > n/2$, if $n < e^{5000}$ is odd, then $n'' > n/1.5$, and if $n > e^{5000}$ is odd, then $n'' > n/(\mu_k - 1)$. In all cases, if $e^{k-1} < n < e^k$, then we have $n'' > n/(\mu_k - 1)$. Let $L'_k = \exp((\sqrt{k - \log(\mu_k - 1)}/5)$. If $n$ or $n'$ is in $\mathcal{A}_4$, since $n' = s(n)$ and $n = s(n')$, then $\gcd(n, n')$ is divisible by a prime $q > 31L'_k$. Thus, it suffices to sum the reciprocals of such numbers $n$ without the need for a multiplier.

Suppose that $e^{k-1} < n < e^k$, $q \mid \gcd(n, n')$, and $q > 31L'_k$. Since $q \mid \sigma(n)$, there is a prime power $r^a \,\|\, n$ with $q \mid \sigma(r^a)$. We have $r^a > \frac{1}{2}\sigma(r^a) > \frac{1}{2}q$ so that $r^a > 15.5L'_k > 15L_k$ for $k \geqslant 750$. Thus, since we are assuming that $n \notin \mathcal{A}_2$, we have $a = 1$, and so $r \equiv -1 \pmod{q}$. In particular, $r \geqslant 2q - 1$. It simplifies matters a little if we dispose of the case $r = 2q - 1$. In this case, $n$ is divisible by $q(2q-1)$. Using Lemma 2.7, we have that the sum of $1/(q(2q-1))$ for $q > 31L'_k$ is less than $1/(31L'_k \log(31L'_k))$, while the number of integers $q(2q-1) < e^k$ is at most $e^{k/2}$. It thus follows from Lemma 2.2 and a calculation that the reciprocal sum of such $n$ which are even and $> e^{750}$ plus the reciprocal sum for such $n$ which are odd and $> e^{1500}$ is less than 0.0026.

So, we now assume that $n$ is divisible by $qr$ where $q > 31L'_k$, $r \equiv -1 \pmod{q}$, and $r \geqslant 4q - 1$. Using Lemma 2.8, the reciprocal sum of such numbers $qr < e^k$ is at most

$$\sum_{q > 31L'_k} \frac{2\log(k - \log(31L'_k))}{q(q-1)} < \frac{2\log(k - \log(31L'_k))}{(31L'_k - 1)\log(31L'_k - 1)},$$

using Lemma 2.7. Summing one-half of this for $k \geqslant 750$ we get $< 0.0308$, using Lemma 2.7, and this contributes to the reciprocal sum of even $n \in \mathcal{A}_4$. The parallel contribution for odd $n > e^{1500}$ is $< 0.0039$. We also must count the number of integers $qr < e^k$. We could use Lemma 2.8 again, but it is simpler not to use that $r$ is prime. For a given $q$, the number of integers $r$ with $q < r < e^k/q$ and $r \equiv -1 \pmod{q}$ is at most $e^k/q^2$. Using Lemma 2.2 and summing $e^{1-k}$ times this estimate for $k \geqslant 750$ (using Lemma 2.7) adds on $< 0.0134$ to the reciprocal sum for even, and the parallel contribution for odd $n > e^{1500}$ is $< 0.0008$.

Now, totalling the various contributions, we have that the sum in the proposition is at most 0.0489. □

**Proposition 4.2.** *Let $\mathcal{A}_5$ denote the set of amicable numbers $n$ such that $n, n'$ are not in $\mathcal{A}_j$ for $j < 5$ and $mm' \leqslant n/(10L(n))$. Then the reciprocal sum of those amicable numbers such that at least one of the pair is $> e^{1500}$ in the odd case and $> e^{750}$ in the even case, and at least one of the pair is in $\mathcal{A}_5$, is at most* 3.469.

*Proof.* By Proposition 3.4, we may assume that we are in one of the 3 cases

$$m, m' \text{ odd}, \quad m \equiv m' \equiv 2 \pmod{4}, \quad m \equiv m' \equiv 0 \pmod{4}.$$

As in the proof of Proposition 3.4, the pair $\{m, m'\}$ determines the pair $\{n, n'\}$.

Suppose we are in the odd-odd case. We distinguish two ranges for $n$: $e^{1500} < n < e^{5000}$ and $n > e^{5000}$. In the first range we have multiplier 2.5, since by (3.12) we are assuming that $h(n), h(n') \leqslant 2.5$. In the second range, we have multiplier $\mu_k$, where $k = \lceil \log n \rceil$. Say $n, n'$ are an amicable pair and $n/(10L(n)) > mm'$. If $n' > n$, then $n'/(10L(n')) > mm'$. Suppose that $n' < n$. Then $n' > n/1.5$ in the first range, so if $1.5n/(10L(n)) > mm'$, then $n'/(10L(n')) > mm'$. In the second range, if $n' < n$, we have $n' > n/(\mu_k - 1)$, so, if $(\mu_k - 1)n/(10L(n)) > mm'$, then $n'/(10L(n')) > mm'$.

For $n$ or $n' > e^{1500}$, $p = P(n) > n^{1/(4\log\log n)} > 3 \times 10^{28}$. So, if $n$ is abundant, then

$$h(m) = \frac{p}{p+1}h(n) > \frac{2p}{p+1} > 2 - 10^{-28}.$$

Also note that if $n > e^{1500}$, then $n' > e^{1499}$, and if $n > e^{5000}$, then $n' > e^{4998}$. Let $\nu$ be the appropriate multiplier so that $\nu = 2.5$ in the small odd range and

$\nu = 1.3\log(1+4\log k)$ for large odd cases. Let $N_0(t)$ be the number of odd amicable numbers $n \leqslant t$ with $mm' < (\nu-1)n/(10L(n))$. By partial summation,
(4.1)

$$\sum_{\substack{n \text{ or } n' \in \mathcal{A}_5 \\ nn' \text{ odd} \\ n \text{ or } n' > e^{1500}}} \left(\frac{1}{n}+\frac{1}{n'}\right) \leqslant \sum_{k \geqslant 1500} \left(\frac{N_0(e^k)}{e^k} - \frac{N_0(e^{k-1})}{e^{k-1}} + \int_{e^{k-1}}^{e^k} \frac{N_0(t)}{t^2}\,dt\right) \leqslant \int_{e^{1499}}^{\infty} \frac{N_0(t)}{t^2}\,dt.$$

Let $t' = (\nu-1)t/(10L(t))$. If $\{m, m'\} = \{m_1, m_2\}$ where $h(m_1) < h(m_2)$, then

$$N_0(t) \leqslant \sum_{\substack{m_2 < t' \\ m_2 \text{ odd} \\ h(m_2) > 2-10^{-28}}} \sum_{\substack{1 < m_1 \leqslant t'/m_2 \\ m_1 \text{ odd}}} 1 \leqslant \frac{1}{2}t' \sum_{\substack{m_2 < t' \\ m_2 \text{ odd} \\ h(m_2) > 2-10^{-28}}} \frac{1}{m_2}.$$

(Note that $m_1 \neq 1$, since all amicable numbers are composite.) We now follow the argument in the proof of Proposition 3.5. We have for any positive integer $j$ that

$$\sum_{\substack{m_2 < t',\, m_2 \text{ odd} \\ h(m_2) > 2-10^{-28}}} \frac{1}{m_2} < (2-10^{-28})^{-j} \sum_{m_2 < t',\, m_2 \text{ odd}} \frac{h(m_2)^j}{m_2}$$
$$< \frac{1}{2}(\log(t'+1.28)(2-10^{-28})^{j} \sum_{d \text{ odd}} \frac{f_j(d)}{d}.$$

Taking $j = 18$, we get

$$\sum_{\substack{m_2 < t',\, m_2 \text{ odd} \\ h(m_2) > 2-10^{-28}}} \frac{1}{m_2} < \frac{1}{2}(\log t' + 1.28) \cdot 0.023773$$

so that

$$N_0(t) < 0.005944(t'+1)(\log t' + 1.28).$$

Let $\nu_k = \nu = 2.5$ when $k \leqslant 5000$ and $\nu_k = \mu_k$ when $k > 5000$. We conclude from (4.1) that

$$\sum_{\substack{n \text{ or } n' \in \mathcal{A}_5 \\ nn' \text{ odd} \\ n \text{ or } n' > e^{1500}}} \left(\frac{1}{n}+\frac{1}{n'}\right) < 0.005944 \int_{e^{1499}}^{\infty} \frac{1}{t^2}(t'+1)(\log t' + 1.28)\,dt$$

$$< 0.005944 \sum_{k \geqslant 1500} \int_{e^{k-1}}^{e^k} \frac{1}{t^2}\frac{(\nu_k-1)t}{10L_k}(\log t + \log(\nu_k - 1) - \log(10L_k) + 1.29)\,dt$$

$$= 0.005944 \sum_{k \geqslant 1500} \frac{(\nu_k-1)(k-1/2+\log(\nu_k-1)-\log(10L_k)+0.79)}{10L_k} < 0.3387.$$

We now turn to the 2 (mod 4) case, which has multiplier $\nu = 3$. First suppose that $5 \nmid nn'$. By Remark 3.1 we have $3 \nmid nn'$. Let $N_1(t)$ denote the number of amicable numbers $n \leqslant t$ with $n \equiv 2 \pmod 4$, $3 \nmid mm'$, $5 \nmid mm'$, and $mm' < 2n/(10L(n))$. As in the odd-odd case, we wish to give an upper bound for $\int_{e^{749}}^{\infty} N_1(t)/t^2\,dt$. Say $\{m, m'\} = \{m_1, m_2\}$, where $h(m_1) < h(m_2)$. Similarly, as in the odd-odd case, since $n, n' > e^{749}$, we have $h(m_2) > 2 - 10^{-14}$. Let $t' = 2t/(10L(t)) = t/(5L(t))$, and let $N_{1,0}(t)$ be the contribution to $N_1(t)$ when $m_2 \leqslant t'/100$ and let $N_{1,1}(t)$ be the contribution when $m_2 > t'/100$. Note that

$$N_{1,0}(t) \leqslant \sum_{\substack{m_2 \leqslant t'/100,\, 2 \,\|\, m_2 \\ \gcd(m_2,15)=1 \\ h(m_2)>2-10^{-14}}} \sum_{\substack{m_1 \leqslant t'/m_2,\, 2 \,\|\, m_1 \\ \gcd(m_1,15)=1}} 1 \leqslant \frac{2}{15} \sum_{\substack{m_2 \leqslant t'/100,\, 2 \,\|\, m_2 \\ \gcd(m_2,15)=1 \\ h(m_2)>2-10^{-14}}} \left(\frac{t'}{m_2} + 4\right)$$

$$\leqslant \frac{2(1.04)}{15} t' \sum_{\substack{m_2 \leqslant t'/100,\, 2 \,\|\, m_2 \\ \gcd(m_2,15)=1 \\ h(m_2)>2-10^{-14}}} \frac{1}{m_2}.$$

For any positive integer $j$, the inner sum is

$$< (2-10^{-14})^{-j} \sum_{\substack{m_2 \leqslant t'/100,\, 2 \,\|\, m_2 \\ \gcd(m_2,15)=1}} \frac{h(m_2)^j}{m_2} = \frac{1}{2}\left(\frac{3}{2}\right)^j \left(2-10^{-14}\right)^{-j} \sum_{\substack{m \leqslant t'/200 \\ \gcd(m,30)=1}} \frac{h(m)^j}{m}$$

$$< \frac{1}{2}\left(\frac{3}{4} + 10^{-14}\right)^j \left(\frac{4}{15}(\log(t'/200) + \gamma) + .438617\right) \sum_{\gcd(d,30)=1} \frac{f_j(d)}{d}.$$

Taking $j = 35$, this last expression is

$$< \frac{1}{2}(0.013343)\frac{4}{15}(\log t' - 0.8203) < \frac{2}{15}(0.013343)(\log t' - 3.076).$$

Thus,

$$N_{1,0}(t) < \frac{4.16}{225}(0.013343)t'(\log t' - 3.076) < 0.000247t' \log t' - 0.000758t'.$$

For $N_{1,1}(t)$ we have

$$N_{1,1}(t) \leqslant \sum_{\substack{m_1 \leqslant 100,\, 2 \,\|\, m_1 \\ \gcd(m_1,15)=1}} \sum_{\substack{m_2 \leqslant t'/m_1,\, 2 \,\|\, m_2 \\ \gcd(m_2,15)=1 \\ h(m_2)>2-10^{-14}}} 1 \leqslant \sum_{\substack{m_1 \leqslant 100,\, 2 \,\|\, m_1 \\ \gcd(m_1,15)=1}} \sum_{\substack{m \leqslant t'/2m_1 \\ \gcd(m,30)=1 \\ h(m)>(2/3)(2-10^{-14})}} 1.$$

The inner sum is

$$< \left(\frac{3}{4} + 10^{-14}\right)^j \sum_{\substack{m \leqslant t'/2m_1 \\ \gcd(m,30)=1}} h(m)^j \leqslant \frac{t'}{2m_1}\left(\frac{3}{4} + 10^{-14}\right)^j \sum_{\gcd(d,30)=1} \frac{f_j(d)}{d}.$$

Taking $j = 35$ again, we have

$$N_{1,1}(t) < \frac{t'}{2}(0.013343) \sum_{\substack{m_1 \leqslant 100,\, 2 \,\|\, m_1 \\ \gcd(m_1,15)=1}} \frac{1}{m_1} < \frac{t'}{2}(0.013343)(0.825) < 0.005504t'.$$

With the prior estimate for $N_{1,0}(t)$, we have

$$N_1(t) < 0.000247t' \log t' + 0.004746t'.$$

As in the odd-odd case, we deduce that the contribution when $2 \,\|\, m, m'$ and $5 \nmid mm'$ is

$$< \sum_{k \geqslant 750} \frac{0.000247(k - 1/2 - \log(5L_k)) + 0.004746}{5L_k} < 0.0765.$$

We now bound the contribution when $2 \parallel m, m'$ and $5 \mid mm'$. If $N_2(t)$ denotes the number of pairs, we have

$$N_2(t) \leqslant \sum_{\substack{m_1 \leqslant \sqrt{t'}/2 \\ \gcd(m_1,30)=5}} \sum_{\substack{m_2 \leqslant t'/4m_1 \\ \gcd(m_2,6)=1}} 1 + \sum_{\substack{m_1 \leqslant \sqrt{t'}/2 \\ \gcd(m_1,30)=1}} \sum_{\substack{m_2 \leqslant t'/4m_1 \\ \gcd(m_2,30)=5}} 1$$

$$\leqslant \sum_{\substack{m_1 \leqslant \sqrt{t'}/2 \\ \gcd(m_1,30)=5}} \frac{1}{3}\Big(\frac{t'}{4m_1} + 2\Big) + \sum_{\substack{m_1 \leqslant \sqrt{t'}/2 \\ \gcd(m_1,30)=1}} \frac{1}{3}\Big(\frac{t'}{20m_1} + 2\Big)$$

$$< \frac{1}{60}t'\Big(\frac{1}{3}(\log(\sqrt{t'}/10)+\gamma)+0.4142\Big) + \frac{1}{60}t'\Big(\frac{4}{15}(\log(\sqrt{t'}/2)+\gamma) + 0.4387\Big) + \sqrt{t'}.$$

Thus, for $t > e^{749}$,

$$N_2(t) < \frac{1}{200}t'\log t' + 0.004115t'.$$

As before, we have the contribution to our sum being

$$< \sum_{k \geqslant 750} \frac{(1/200)(k - 1/2 - \log(5L_k)) + 0.004115}{5L_k} < 1.5220.$$

We now consider the case when $m, m'$ are both multiples of 4. We divide this into a few subcases:

(1) $v_2(m) = 2$, $v_2(m') = 2$,
(2) $\{v_2(m), v_2(m')\} = \{2, 3\}$,
(3) $\{v_2(m), v_2(m')\} = \{2, 4\}$,
(4) $v_2(m) = 2$, $v_2(m') \geqslant 5$ or $v_2(m) \geqslant 5$, $v_2(m') = 2$,
(5) $v_2(m) \geqslant 3$, $v_2(m') \geqslant 3$.

In all of these cases we have $3, 7 \nmid mm'$. In case (2), we have $5 \nmid mm'$ since $5 \mid \sigma(n) = \sigma(n')$. We also have $5 \nmid mm'$ in cases (4) and (5) since

$$\frac{s(20)}{20} \cdot \frac{s(32)}{32} > 1, \qquad \frac{s(4)}{4} \cdot \frac{s(160)}{160} > 1, \qquad \frac{s(40)}{40} \cdot \frac{s(8)}{8} > 1.$$

Also, in the part of case (3) where $v_2(m) = 2$, we have $5 \nmid m$ since $(s(20)/20)(s(16)/16) > 1$. In cases (1)–(4), we have the multiplier $7/3$, and in case (5), we have the multiplier $15/7$. Noting that we are dealing with unordered pairs $m, m'$, and using the same method as above with $N_2(t)$, we find that

$$\sum_{\substack{n \text{ or } n' \in \mathcal{A}_5 \\ n, n' \equiv 0 \pmod 4 \\ n \text{ or } n' > e^{750}}} \Big(\frac{1}{n} + \frac{1}{n'}\Big) < 0.5175 + 0.3312 + 0.2329 + 0.1656 + 0.2839 = 1.5311.$$

Totalling the contributions in the various cases completes the proof. □

**Proposition 4.3.** *Let $\mathcal{A}_6$ denote the set of amicable numbers $n$ such that $n, n'$ are not in any $\mathcal{A}_j$ for $j < 6$ and $p > n^{3/4}L(n)$. The reciprocal sum of those even amicable numbers with at least one of the pair in $\mathcal{A}_6$ and at least one $> e^{750}$ plus the corresponding reciprocal sum of odd amicable pairs with at least one of the pair $> e^{1500}$ is $< 2.061$.*

*Proof.* Assume that $t > e^{750}$ and let $N(t)$ denote the number of $n \in \mathcal{A}_6$ with $n \leqslant t$. For $n \in \mathcal{A}_6$, we have $m < n^{1/4}/L(n)$, so since $n \notin \mathcal{A}_5$, we have $m' > \frac{1}{10}n^{3/4}$. This then implies that $p' < 10n'/n^{3/4}$. Let $\nu$ be 1 less than the appropriate multiplier so that $\nu = 1.5$ in the smaller odd case, $\nu = 2$ in the even case, and $\nu = 1.3\log(1+4\log k) - 1$ in the larger odd case. In particular, $n' < \nu n$, so we have $p' < 10\nu n^{1/4}$. Write $n' = q_1 q_2 \cdots q_l$, where the $q_i$'s are pairwise coprime prime powers (possibly first powers of primes) and $q_1 > q_2 > \cdots > q_l$. We have $q_1 = p'$, so all of the $q_i$'s are $< 10\nu n^{1/4} \leqslant 10\nu t^{1/4}$. Assume that $n > t^{0.84}$, and choose $i$ minimally so that

$$D := q_1 q_2 \cdots q_i > \sqrt{t/L(t)}.$$

Then $D < 10\nu t^{3/4}/\sqrt{L(t)}$. If $D$ is divisible by a prime $< 31L(t)$, then in fact $D$ is smaller; it is $< 31L(t)\sqrt{t/L(t)} < t^{0.51}$. Further, $(31L(t))^{4\log\log t} < t^{0.32}$. Thus, if $n > t^{0.84}$ and $n$ is counted by $N(t)$, then the fact that $n$ is not in $\mathcal{A}_1$ or $\mathcal{A}_2$ implies that all of the prime factors of $D$ are greater than $31L(n)$. Since $n \notin \mathcal{A}_4$, we have $\gcd(D, \sigma(D)) = 1$.

Write $n' = DM$. It is shown in [17] that

$$\sigma(m)DM \equiv m\sigma(m) \pmod{\sigma(D)}.$$

Thus, $N(t)$ is at most $t^{0.84}$ plus the number of solutions $M$ to these congruences with $M < \nu t/D$, as $m$ runs to $t^{1/4}/L(t)$ and $D$ runs over the interval $(\sqrt{t/L(t)}, 10\nu t^{3/4}/\sqrt{L(t)})$. For a given choice of $m, D$, the number of solutions is at most

$$1 + \frac{\nu t/D}{\sigma(D)/\gcd(\sigma(m)D, \sigma(D))} \leqslant 1 + \frac{\nu t\sigma(m)}{D^2},$$

using $\gcd(D, \sigma(D)) = 1$. We have

$$\sum_{\substack{m < t^{1/4}/L(t) \\ D < 10\nu t^{3/4}/\sqrt{L(t)}}} 1 < 5\nu t/L(t)^{3/2} + 1,$$

both in the case $m$ even and in the case $m$ odd. Further, using the inequality $\sum_{m<B} \sigma(m) < B^2$,

$$\nu t \sum_{\substack{m < t^{1/4}/L(t) \\ D > \sqrt{t/L(t)}}} \frac{\sigma(m)}{D^2} < \nu t^{3/2} L(t)^{-2} \sum_{D > \sqrt{t/L(t)}} D^{-2} < \nu t/L(t)^{3/2} + \nu t^{1/2}/L(t),$$

where we also used that $\sum_{D>B} D^{-2} < 1/B + 1/B^2$.

We have

$$\sum_{\substack{n \text{ or } n' \in \mathcal{A}_6 \\ e^{k-1} < n < e^k}} \left(\frac{1}{n} + \frac{1}{n'}\right) < \int_{e^{k-1}}^{e^k} (\nu+1)\frac{N(t)}{t^2}\,dt < \int_{e^{k-1}}^{e^k} \frac{\nu+1}{t^{1.16}} + \frac{6(\nu+1)\nu}{L_k^{3/2} t} + \frac{\nu+1}{t^2}\,dt$$

$$< e^{-0.15k} + 6(\nu+1)\nu/L_k^{3/2} + (\nu+1)/(k-1)^2.$$

For evens starting at $k = 750$, we have $\nu = 2$, and the contribution is $< 2.0020$. For odds from $k = 1500$ to $5000$, we have $\nu = 1.5$ and the contribution is $< 0.0581$, and the contribution for odds with $k > 5000$ is $< 3.1 \times 10^{-5}$. In all, the total contribution is $< 2.0602$. □

**Proposition 4.4.** *Let $\mathcal{A}_7$ denote the set of amicable numbers $n$ such that neither $n$ nor $n'$ is in $\mathcal{A}_j$ for $j < 7$ and such that $P(\sigma(m)) \leqslant 100L(n)$. Then the reciprocal sum of the amicable numbers $n$ with either $n$ or $n' > e^{750}$ in the even case and $> e^{1500}$ in the odd case, and either $n$ or $n' \in \mathcal{A}_7$ is at most* 11.399.

*Proof.* Let $M_k = e^{(k-1)/4}/L_k$. Since $n \notin \mathcal{A}_6$, if $n \in (e^{k-1}, e^k)$, then $m > M_k$. Let $u_k = k^{1/4}$ and let $q = P(m)$. We consider three cases:

(1) $q \leqslant 10^7 M_k^{1/u_k}$ and $m < e^{k/2}$,
(2) $q \leqslant 10^7 M_k^{1/u_k}$ and $m > e^{k/2}$,
(3) $q > 10^7 M_k^{1/u_k}$ and $P(q+1) \leqslant 100 L_k$.

If $n$ is not in any of these cases, then $q > 10^7 M_k^{1/u_k} > 15 L_k$, so from $n \notin \mathcal{A}_2$, we have $q \,\|\, m$. Also $P(\sigma(m)) \geqslant P(q+1) > 100 L_k$ so that $n \notin \mathcal{A}_7$, so it suffices to bound the reciprocal sums for the three cases above.

For a given value of $k$ and $e^{k-1} < n < e^k$, the reciprocal sum in case (1) is at most

$$\sum_{\substack{M_k < m < e^{k/2} \\ P(m) \leqslant 10^7 M_k^{1/u_k}}} \frac{1}{m} \sum_{e^{k-1}/m < p < e^k/m} \frac{1}{p}.$$

Since $e^{k-1}/m > e^{k/2-1}$, Lemma 2.6 implies that the inner sum over $p$ is smaller than $2/(k-2) + 2/(k-2)^2$. Thus, the reciprocal sum in case (1) in the odd and even cases, respectively, is at most

$$\Big(\frac{2}{k-2} + \frac{2}{(k-2)^2}\Big) S_{\mathrm{odd}}(M_k, 10^7 M_k^{1/u_k}),$$
$$\Big(\frac{2}{k-2} + \frac{2}{(k-2)^2}\Big) S_{\mathrm{even,\,no\,3}}(M_k, 10^7 M_k^{1/u_k}),$$

using the notation of Remark 2.1. Summing the first expression using Lemma 2.9 and Remark 2.1 with $y_0 = e^{10}$ for $1500 \leqslant k \leqslant 5000$ and using multiplier 2.5, we get an estimate of $< 0.0808$. Summing the second expression for $750 \leqslant k \leqslant 5000$ with multiplier 3, we get an estimate of $< 3.1947$. Summing for $k > 5000$ and using a multiplier of $1.3\log(1 + 4\log k)$, using Lemma 2.10, we get $< 0.0052$.

The second case is done in almost the same way. Now we must estimate

$$\sum_{\substack{m > e^{k/2} \\ P(m) \leqslant 10^7 M_k^{1/u_k}}} \frac{1}{m} \sum_{e^{k-1}/m < p < e^k/m} \frac{1}{p}.$$

We know that $p > n^{1/4\log\log n} > e^{(k-1)/(4\log(k-1))}$, so the inner sum here is 0 unless $m$ is such that $e^k/m \geqslant e^{(k-1)/4\log(k-1)}$. With $a(k) := (k-1)/(4\log(k-1)) - 1$, Lemma 2.6 then implies the inner sum above is at most $1/a(k) + 1/(2a(k)^2)$. Thus, the reciprocal sum in case (1) in the odd and even cases, respectively, is at most

$$\Big(\frac{1}{a(k)} + \frac{1}{2a(k)^2}\Big) S_{\mathrm{odd}}(e^{k/2}, 10^7 M_k^{1/u_k}),$$
$$\Big(\frac{1}{a(k)} + \frac{1}{2a(k)^2}\Big) S_{\mathrm{even,\,no\,3}}(e^{k/2}, 10^7 M_k^{1/u_k}).$$

Summing the first expression using Lemma 2.9 and Remark 2.1 with $y_0 = e^{10}$ for $1500 \leqslant k \leqslant 5000$ and using multiplier 2.5, we get an estimate of $< 4 \times 10^{-8}$. Summing the second expression for $750 \leqslant k \leqslant 5000$ with multiplier 3, we get an estimate of $< 0.0005$. Summing for $k > 5000$ and using a multiplier of $1.3\log(1 + 4\log k)$, using Lemma 2.10, we get $< 8 \times 10^{-15}$.

We now turn to case (3). Write $l = n/q$. Here the reciprocal sum for $e^{k-1} < n < e^k$ is at most

$$\sum_{\substack{q>10^7 M_k^{1/u_k}\\ P(q+1)\leqslant 100L_k}} \frac{1}{q} \sum_{e^{k-1}/q<l<e^k/q} \frac{1}{l},$$

where $l$ is odd in the odd case, and in the even case, $l$ is even and not divisible by 3. Using Corollary 2.1 for the inner sum, we have a quantity at most

$$\begin{aligned}\Big(\frac{1}{2}+\frac{4{\cdot}10^7 M_k^{1/u_k}}{e^{k-1}}\Big) &\sum_{\substack{q>10^7 M_k^{1/u_k}\\ P(q+1)\leqslant 100L_k}} \frac{1}{q}\\ &< \Big(\frac{1}{2}+\frac{4{\cdot}10^7 M_k^{1/u_k}}{e^{k-1}}\Big)\frac{10^7 M_k^{1/u_k}+1}{10^7 M_k^{1/u_k}} S_{\text{even}}(10^7 M_k^{1/u_k}, 100L_k)\end{aligned}$$

in the odd case, with the same estimate but with $\frac{1}{3}$ in place of $\frac{1}{2}$ in the even case. Here we have relaxed the condition that $q$ is prime, keeping only that it is odd, so that $q+1$ is even. Summing this using Lemma 2.9 from $k = 750$ to $k = 5000$, using $x = 10^7 M_k^{1/u_k}$, $y = 100L_k$, $s = \log(2u\log u)/\log y$, and multiplier 3, we get $< 7.1773$ in the even case. For the odd case we sum from $k = 1500$ to 5000 using multiplier 2.5, getting an estimate of $< 0.9149$. We sum for $k \geqslant 5001$ using Lemma 2.10 and multiplier $1.3\log(1+4\log k)$ getting $< 0.0254$.

Thus, the total contribution to the reciprocal sum from $\mathcal{A}_7$ is $< 11.3988$. □

## 5. Conclusion

We are now faced with summing the reciprocals of those amicable numbers $n$ such that both $n, n'$ are $> e^{750}$ in the even case and $> e^{1500}$ in the odd case, and neither is in any set $\mathcal{A}_j$. As before, we have $n = pm, n' = p'm'$, where $p = P(n) \nmid m$, $p' = P(n') \nmid m'$, and $p \neq p'$. We shall assume that $p > p'$ and sum $1/n$, using an appropriate multiplier to take into account the numbers $1/n'$.

Let $r = P(\sigma(m))$, so since $n \notin \mathcal{A}_7$, we have $r > 100L(n)$. Since $r \mid \sigma(m) \mid \sigma(n) = \sigma(n')$, there are prime powers $q^\alpha \parallel m$, $q'^{\alpha'} \parallel n'$ with $r \mid \sigma(q^\alpha)$ and $r \mid \sigma(q'^{\alpha'})$. Then $q^\alpha, q'^{\alpha'} > \frac{1}{2}r > 50L(n)$, so since $n, n' \notin \mathcal{A}_2$, we have $\alpha = \alpha' = 1$. In particular, $q \equiv q' \equiv -1 \pmod r$.

Since $q' > r > 100L(n)$ and since $n \notin \mathcal{A}_4$, we have $q' \nmid n$. Since $q' \mid n' = s(n) = ps(m) + \sigma(m)$, we have

$$ps(m) + \sigma(m) \equiv 0 \pmod{q'}.$$

This implies that if $q' \mid \sigma(m)$, then $q' \mid s(m)$, which implies that $q' \mid m$, a contradiction. So, we have $q' \nmid \sigma(m)$ and the above congruence places $p$ in a residue class $a(m, q') \pmod{q'}$ for a given choice of $m$ and $q'$. Also note that $p > p' \geqslant q'$.

Write $m = qm_1$. For a given value of $k \geqslant 750$, we have

$$S_k := \sum_{\substack{n \text{ in this case} \\ e^{k-1}<n<e^k}} \frac{1}{n}$$
$$< \sum_{r>100L_k} \sum_{\substack{q<e^{k/2} \\ q\equiv -1 \pmod r}} \frac{1}{q} \sum_{\substack{q'<e^{k+1} \\ q'\equiv -1 \pmod r}} \sum_{m_1<e^k/q} \frac{1}{m_1} \sum_{\substack{e^{k-1}/qm_1<p<e^k/qm_1 \\ p\equiv a(qm_1,q') \pmod{q'} \\ p>q'}} \frac{1}{p}.$$

We begin with the inner sum. Fix $q, m_1, q'$ and let $a$ be in the residue class $a(qm_1, q')$ (mod $q'$) with $0 < a < q'$. First suppose that $q'$ is large. If $q' > e^k/qm_1$, then the sum on $p$ is 0. (In particular, we may assume that $q' < e^k/q$.) Suppose that $q' > e^{k-2}/qm_1$. Using only that $q'$ is odd, that $p$ is an odd number in the interval $(e^{k-1}/qm_1, e^k/qm_1)$, and that $p \equiv a \pmod{q'}$ with $p > q'$, we have that the sum on $p$ is at most $1/q' < qm_1/e^{k-2}$. Let $w = e^{k-1}/qq'$ and assume that $q' \leqslant e^{k-2}/qm_1$; that is, $m_1 \leqslant w/e$. Let $z = e^{k-1}/qm_1$. By Lemma 2.8, we have that

$$\sum_{\substack{z<p<ez \\ p\equiv a \pmod{q'} \\ p>q'}} \frac{1}{p} < \frac{2}{(q'-1)\log(z/q')} + \frac{2}{q'-1}\log\left(\frac{1+\log(z/q')}{\log(z/q')}\right)$$
$$< \frac{4}{(q'-1)\log(z/q')} = \frac{4}{(q'-1)\log(w/m_1)}.$$

We now sum on $m_1$. Since $q' < ez = e^k/qm_1$, we have $m_1 < e^k/qq' = ew$ so that we have

$$\sum_{m_1<ew} \frac{1}{m_1} \cdot \frac{qm_1}{e^{k-2}} + \sum_{m_1\leqslant w/e} \frac{1}{m_1} \cdot \frac{4}{(q'-1)\log(w/m_1)}.$$

We distinguish the even and odd cases. Using Lemma 2.4 and $w = e^{k-1}/qq'$, the sum on $m_1$ is

$$< \frac{e^2}{2q'} + \frac{2}{q'-1}\log k \quad \text{(odd case)}, \qquad < \frac{e^2}{3q'} + \frac{4/3}{q'-1}\log k \quad \text{(even case)}.$$

What we have at this point is

$$S_k < c \sum_{r>100L_k} \sum_{\substack{q<e^{k/2} \\ q\equiv -1 \pmod r}} \frac{1}{q} \sum_{\substack{q'<e^k/q \\ q'\equiv -1 \pmod r}} \left(\frac{4}{q'-1}\log k + \frac{e^2}{q'}\right),$$

where $c = 1/2$ in the odd case and $c = 1/3$ in the even case. Let $\iota_k = 1/(100L_k - 1)$. By Lemma 2.8, using the fact that the least prime in the residue class $-1 \pmod r$ is at least $2r-1$ and $-\log\log((2r-1)/r) < 0.37$, the sum on $q'$ is at most

$$(1+\iota_k)^2 \frac{2(4\log k + e^2)(\log k + 0.37)}{r}.$$

Similarly, the sum on $q$ is at most

$$(1+\iota_k)\frac{2(\log(k/2) + 0.37)}{r},$$

so we are left with

$$S_k < c(1+\iota_k)^3 4(4\log k + e^2)(\log k + 0.37)(\log(k/2) + 0.37) \sum_{r>100L_k} \frac{1}{r^2}.$$

We use Lemma 2.7 for the sum over $r$. In the odd case we sum our bound for $S_k$ from $k = 1500$ to 5000 with multiplier 2.5, getting $< 1.5215$. For the remainder of the odds, using multiplier $1.3\log(1+4\log k)$ adds on $< 0.0082$. For the even case, using multiplier 3 and summing for $k \geqslant 750$, we get $< 8.3484$. In total, the contribution is $< 9.879$.

Since the contribution to the reciprocal sum from the prior cases is $< 204.267$, with the result of this section our proof is complete.

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