The reciprocity theorem for the scattered field is the progenitor of the generalized optical theorem

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By analyzing correlation-type reciprocity theorems for wavefields in perturbed media, it is shown that the correlation-type reciprocity theorem for the scattered field is the progenitor of the generalized optical theorem. This reciprocity theorem, in contrast to the generalized optical theorem, allows for inhomogeneous background properties and does not make use of a far-field condition. This theorem specializes to the generalized optical theorem when considering a finite-size scatterer embedded in a homogeneous background medium and when utilizing the far-field condition. Moreover, it is shown that the reciprocity theorem for the scattered field is responsible for the cancellation of non-physical (spurious) arrivals in seismic interferometry, and as such provides the mathematical description of such arrivals. Even though here only acoustic waves are treated, the presented treatment is not limited to such wavefields and can be generalized to general wavefields. Therefore, this work provides the framework for deriving equivalents of the generalized optical theorem for general wavefields. © *2011 Acoustical Society of America*.
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I. INTRODUCTION

The retrieval of the Green function from the cross-correlations between observed wavefields at different receivers is in seismology commonly referred to as seismic interferometry. Due to its many potential applications in both passive and active seismology, this field of study has recently received a lot of attention and both review and tutorial articles as well as a book have been recently published on the subject.[1–4]

Recently, it was observed that there is a close connection between seismic interferometry and the generalized optical theorem.[5] This observation in essence provides an alternative way to derive the generalized optical theorem and the subsequent analysis is based on a stationary-phase analysis and a far-field condition. This inspired Halliday and Curtis[6] to derive a generalized optical theorem for surface waves using a similar derivation. Furthermore, Snieder *et al*.[7] draw connections between Green function extraction from field fluctuations, energy principles, imaging with backscattered waves, and the generalized optical theorem, while Wapenaar *et al*.[8] further elucidate the connection between seismic interferometry, the generalized optical theorem, and the scattering matrix of a point scatterer by emphasizing the role of the non-linearity of the scattering matrix of a point scatterer.

With the exception of Wapenaar *et al*.,[8] the aforementioned studies are all based on a stationary-phase analysis and a far-field condition. Here, we start with the general framework of reciprocity theorems for wavefields in perturbed media.[9,10] By analyzing the connection of these theorems to the work of Snieder *et al*.[5] and the field of seismic interferometry, we then recognize the reciprocity theorem for the scattered field as the progenitor of the generalized optical theorem and show that indeed this reciprocity theorem reduces to the generalized optical theorem when subjected to a far-field condition. The main difference in our approach versus the approach taken by Wapenaar *et al*.[8] is that here we use reciprocity theorems for wavefields in perturbed media, while Wapenaar *et al*.[8] use such theorems for unperturbed media. The advantage of our approach is that the reciprocity theorem for the scattered field, assuming the receiver locations of both wavefields in the reciprocity theorem are the same, leads immediately to the recently presented volume integral representation of the scattering cross-section.[11] Similarly the reciprocity theorem for the total wavefield leads directly to the volume integral representation of the absorption cross-section.[11] Hence the presented treatment connects the generalization of the optical theorem using volume integral representations of the extinction and absorption cross-section[11,12] with the general framework of reciprocity theorems for perturbed media and the scattered field.

The organization of this paper is as follows: First, we review the correlation-type reciprocity theorem for acoustic wavefields in non-flowing attenuative perturbed media and show its connection to seismic interferometry. Second, we

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derive the correlation-type reciprocity theorem for the scattered field and show that it is responsible for the absence of spurious arrivals in seismic interferometry and provides the mathematical description of these arrivals in terms of volume integrals. Third, using the far-field condition, we show that this reciprocity theorem reduces to the generalized optical theorem and that as such this reciprocity theorem is the progenitor of the generalized optical theorem. We proceed with some energy considerations that allow us to connect our work to previously published volume integral representations for the scattered and absorbed power in electromagnetics. We conclude with a short discussion on the generalization of the presented approach to general linear systems and the possibility of deriving a unified generalized optical theorem for general linear systems.

II. RECIPROCITY THEOREM OF THE CORRELATION-TYPE FOR ACOUSTIC WAVEFIELDS IN PERTURBED MEDIA

A reciprocity theorem describes the relation between two physical fields in a common domain. Here we treat acoustic waves in non-flowing lossy media but emphasize that a similar treatment applies to the general case of wave and diffusive fields in lossy media.[13,14] We only summarize the main points of the derivation of the correlation-type reciprocity theorem for acoustic wavefields, as there are many existing references that contain more detailed treatments.[15,16] Throughout the remainder, we shall always mean a reciprocity theorem of the correlation-type when we omit the particular type of reciprocity theorem. Moreover, we adopt the following Fourier transform convention throughout this work

$$f(x,\omega)=\int_{-\infty}^{\infty}\hat{f}(x,t)e^{-i\omega t}\mathrm{d}t,\quad \hat{f}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x,\omega)e^{i\omega t}\mathrm{d}\omega. \tag{1}$$

Since the whole analysis in this work is done in the Fourier domain and thus to avoid almost all quantities to have to be denoted with a ˆ, we have abandoned the conventional notation of using a ˆ to denote the Fourier transformed quantity, and instead have reversed the notation such that the absence of ˆ denotes the Fourier transformed quantity.

For acoustic media the particle velocity $\boldsymbol{v}$ and the acoustic pressure p obey, in the space-frequency domain, the equation of motion

$$i\omega\rho\boldsymbol{v}+\nabla p=0, \tag{2}$$

and the stress–strain relationship

$$i\omega\kappa p+\nabla\cdot\boldsymbol{v}=q, \tag{3}$$

where ω is the angular frequency, κ is the compressibility, ρ is the density, and q is the distribution of volume injection-rate density sources. Both equations are formulated in the frequency domain. Attenuation losses are included when κ is complex-valued. To simplify notation, we have here omitted the explicit dependence of q, p, and $\boldsymbol{v}$ on the spatial variable $\boldsymbol{x}$ and angular frequency ω.

To find the correlation-type reciprocity theorem, we consider the interaction quantity

$$\mathcal{I}_{A,1,B,2}:=p^*_{A,1}\boldsymbol{v}_{B,2}+\boldsymbol{v}^*_{A,1}p_{B,2}, \tag{4}$$

where the subscripts A and B denote different states of the wavefield and/or the medium, 1 and 2 denote different source locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and the asterisk indicates complex conjugation. To obtain the reciprocity theorem, we take the divergence of $\mathcal{I}_{A,1,B,2}$, apply the product rule, integrate over a volume $\mathcal{V}$, use Gauss' theorem, and use Eqs. (2) and (3) to eliminate the terms $\nabla p_{A,B}$ and $\nabla\cdot\boldsymbol{v}_{A,B}$, respectively. Doing this, we find (see also Refs. 15 or 9)

$$\begin{aligned}\mathcal{L}(p_{A,1},p_{B,2})=&\int_{x\in\mathcal{V}}\Big\{p^*_{A,1}q_{B,2}+p_{B,2}q^*_{A,1}-i\omega\boldsymbol{v}^*_{A,1}\cdot\boldsymbol{v}_{B,2}(\rho_B-\rho_A)\\&-i\omega p^*_{A,1}p_{B,2}(\kappa_B-\kappa_A)\Big\}\mathrm{d}x\\&+2\omega\int_{x\in\mathcal{V}}p^*_{A,1}p_{B,2}\mathrm{Im}\kappa_A\mathrm{d}x,\end{aligned} \tag{5}$$

where we have defined the operator

$$\begin{aligned}\mathcal{L}(p_{A,1},p_{B,2}):=&\frac{1}{i\omega}\int_{x\in\partial\mathcal{V}}\hat{\boldsymbol{n}}\cdot\left(\frac{1}{\rho_A}p_{B,2}\nabla p^*_{A,1}\right.\\&\left.-\frac{1}{\rho_B}p^*_{A,1}\nabla p_{B,2}\right)\mathrm{d}x,\end{aligned} \tag{6}$$

to denote the surface integral.

We are free to choose the states A and B. Thus we can choose state B to be a perturbed state of A and define $\rho_B:=\rho_A+\Delta\rho$, $\kappa_B:=\kappa_A+\Delta\kappa$, and thus also $p_B:=p_A+\Delta p$ and $\boldsymbol{v}_B:=\boldsymbol{v}_A+\Delta\boldsymbol{v}$. In relation to the scattering of waves, state A then gives the background wavefield p_A while state B is the total wavefield, i.e., the sum of the scattered and background wavefield. To make this explicit, we change notation and substitute $(p,\boldsymbol{v},q,\kappa,\rho)_B\rightarrow(p,\boldsymbol{v},q,\kappa,\rho)_t$ and $(p,\boldsymbol{v},q,\kappa,\rho)_A\rightarrow(p,\boldsymbol{v},q,\kappa,\rho)_0$, where the subscript t is the short form for "total." If we choose the source terms q in Eq. (5) to be δ-functions, then the pressure fields become the Green functions. Doing this, and using the sifting property of the δ-function, while introducing the new notation, we find that Eq. (5) reduces to

$$\begin{aligned}\mathcal{L}(G_{0,1},G_{t,2})=&G^*_0(\boldsymbol{x}_1,\boldsymbol{x}_2)+G_t(\boldsymbol{x}_2,\boldsymbol{x}_1)\\&+2\omega\int_{x\in\mathcal{V}}G^*_0(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x})\mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\&+\frac{1}{i\omega}\left\{\int_{x\in\mathcal{V}}\nabla G^*_0(\boldsymbol{x}_1,\boldsymbol{x})\cdot\nabla G_t(\boldsymbol{x}_2,\boldsymbol{x})\frac{\Delta\rho(\boldsymbol{x})}{\rho_0(\boldsymbol{x})\rho_t(\boldsymbol{x})}\right.\\&\left.+\omega^2G^*_0(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x})\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right\},\end{aligned} \tag{7}$$

where we defined the notation $G_{0,1}:=G_0(\boldsymbol{x},\boldsymbol{x}_1)$ and $G_{t,2}:=G_t(\boldsymbol{x},\boldsymbol{x}_2)$ and used source-receiver reciprocity on the right-hand side. This expression is the reciprocity theorem that describes the relationship between the Green function in the background medium and the total Green function.

We note that if we reorder the terms in Eq. (7), we find

$$G_t(\boldsymbol{x}_2,\boldsymbol{x}_1) = -\Bigg[G_0^*(\boldsymbol{x}_1,\boldsymbol{x}_2) + 2\omega\int_{x\in\mathcal{V}} G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) \times \mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{1}{i\omega}\Bigg\{\int_{x\in\mathcal{V}} \nabla G_0^*(\boldsymbol{x}_1,\boldsymbol{x})\cdot\nabla G_t(\boldsymbol{x}_2,\boldsymbol{x}) \times \frac{\Delta\rho(\boldsymbol{x})}{\rho_0(\boldsymbol{x})\rho_t(\boldsymbol{x})} + \omega^2 G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x})\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Bigg\}\Bigg] + \mathcal{L}(G_{0,1},G_{t,2}). \quad (8)$$

This equation is similar to the Lippmann–Schwinger equation[17] except that it is expressed in terms of the time-advanced Green function in the background medium instead of the usual time-retarded one [see also Eq. 38 of Douma[9]].

III. SEISMIC INTERFEROMETRY

The connection between seismic interferometry and reciprocity theorems was recently established by Wapenaar.[18] From the reciprocity theorem in Eq. (7), it is straightforward to find the fundamental equation underlying seismic interferometry. Replacing G_0 with G_t and thus calculating $\mathcal{L}(G_{t,1}, G_{t,2})$, while realizing that in this case the relevant medium parameters are ρ_t and κ_t and there are no medium contrasts, gives

$$\mathcal{L}(G_{t,1},G_{t,2}) = G_t^*(\boldsymbol{x}_1,\boldsymbol{x}_2) + G_t(\boldsymbol{x}_2,\boldsymbol{x}_1) + 2\omega\int_{x\in\mathcal{V}} G_t^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x})\mathrm{Im}\kappa_t(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \quad (9)$$

This is the fundamental equation underlying seismic interferometry in the presence of attenuation.[19] In case the medium is non-attenuative ($\mathrm{Im}\,\kappa_t = 0$), the volume integral on the right-hand side vanishes. In that case Eq. (9) becomes the familiar equation describing seismic interferometry [e.g., Eq. (18) in Wapenaar and Fokkema[16]]. Of course, the equivalent expression can be obtained by replacing G_t with G_0 in Eq. (7), which gives

$$\mathcal{L}(G_{0,1},G_{0,2}) = G_0^*(\boldsymbol{x}_1,\boldsymbol{x}_2) + G_0(\boldsymbol{x}_2,\boldsymbol{x}_1) + 2\omega\int_{x\in\mathcal{V}} G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_0(\boldsymbol{x}_2,\boldsymbol{x})\mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \quad (10)$$

The only difference between Eqs. (9) and (10) is that Eq. (9) describes seismic interferometry in the perturbed medium, whereas Eq. (10) does so for the background medium.

IV. RECIPROCITY THEOREM FOR THE SCATTERED FIELD

The total wavefield is the sum of the background wavefield and the scattered wavefield. In terms of Green functions, we thus have

$$G_t(\boldsymbol{x}',\boldsymbol{x}'') = G_0(\boldsymbol{x}',\boldsymbol{x}'') + G_s(\boldsymbol{x}',\boldsymbol{x}''), \quad (11)$$

where G_s denotes the Green function for the scattered field. Using this expression for the total Green function in Eq. (6), we find

$$\mathcal{L}(G_{t,1},G_{t,2}) = \mathcal{L}(G_{0,1},G_{0,2}) + \mathcal{L}(G_{0,1},G_{s,2}) + \mathcal{L}(G_{s,1},G_{0,2}) + \mathcal{L}(G_{s,1},G_{s,2}), \quad (12)$$

provided we assume there are no density contrasts on the boundary $\partial\mathcal{V}$. To obtain the reciprocity theorem for the scattered field, we want to find an analogous equation as Eqs. (9) and (10) but then for the term $\mathcal{L}(G_{s,1}, G_{s,2})$. To do this, we proceed by analyzing the terms in the right-hand side of Eq. (12) and comparing them to the right-hand side of Eq. (9).

Consider the first three terms on the right-hand side of Eq. (12). The first term is given by Eq. (10). The second term can be obtained by subtracting Eq. (10) from Eq. (7). Doing this we find

$$\mathcal{L}(G_{0,1},G_{s,2}) = G_s(\boldsymbol{x}_2,\boldsymbol{x}_1) + 2\omega\int_{x\in\mathcal{V}} G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_s(\boldsymbol{x}_2,\boldsymbol{x}) \times \mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{1}{i\omega}\Bigg\{\int_{x\in\mathcal{V}} \nabla G_0^*(\boldsymbol{x}_1,\boldsymbol{x})\cdot\nabla G_t(\boldsymbol{x}_2,\boldsymbol{x}) \times \frac{\Delta\rho(\boldsymbol{x})}{\rho_0(\boldsymbol{x})\rho_t(\boldsymbol{x})} + \omega^2 G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) \times \Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Bigg\}. \quad (13)$$

Observing from the definition of $\mathcal{L}$ in Eq. (6) that we can get the term $\mathcal{L}(G_{s,1}, G_{0,2})$ from $\mathcal{L}(G_{0,1}, G_{s,2})$ simply by interchanging the subscripts 1 and 2 and taking the complex conjugate, we find

$$\mathcal{L}(G_{s,1},G_{0,2}) = G_s^*(\boldsymbol{x}_1,\boldsymbol{x}_2) + 2\omega\int_{x\in\mathcal{V}} G_0(\boldsymbol{x}_2,\boldsymbol{x})G_s^*(\boldsymbol{x}_1,\boldsymbol{x}) \times \mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \frac{1}{i\omega}\Bigg\{\int_{x\in\mathcal{V}} \nabla G_0(\boldsymbol{x}_2,\boldsymbol{x})\cdot\nabla G_t^*(\boldsymbol{x}_1,\boldsymbol{x}) \times \frac{\Delta\rho(\boldsymbol{x})}{\rho_0(\boldsymbol{x})\rho_t(\boldsymbol{x})} + \omega^2 G_0(\boldsymbol{x}_2,\boldsymbol{x})G_t^*(\boldsymbol{x}_1,\boldsymbol{x}) \times \Delta\kappa^*(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Bigg\}. \quad (14)$$

Adding Eqs. (10), (13), and (14), we then find after some algebraic massaging

$$\begin{aligned}
&\mathcal{L}(G_{0,1},G_{0,2}) + \mathcal{L}(G_{0,1},G_{s,2}) + \mathcal{L}(G_{s,1},G_{0,2}) \\
&= \mathcal{L}(G_{t,1},G_{t,2}) - \mathcal{L}(G_{s,1},G_{s,2}) \\
&= \Bigg\{G_t^*(\boldsymbol{x}_1,\boldsymbol{x}_2) + G_t(\boldsymbol{x}_2,\boldsymbol{x}_1) + 2\omega\int_{x\in\mathcal{V}} G_t^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x})\mathrm{Im}\kappa_t(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Bigg\} \\
&\quad - i\omega\int_{x\in\mathcal{V}} \big[G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) - G_0(\boldsymbol{x}_2,\boldsymbol{x})G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big]\mathrm{Re}\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\
&\quad - \frac{1}{i\omega}\int_{x\in\mathcal{V}} \big[\nabla G_0^*(\boldsymbol{x}_1,\boldsymbol{x})\cdot\nabla G_t(\boldsymbol{x}_2,\boldsymbol{x}) - \nabla G_0(\boldsymbol{x}_2,\boldsymbol{x})\cdot\nabla G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big]\frac{\Delta\rho(\boldsymbol{x})}{\rho_0(\boldsymbol{x})\rho_t(\boldsymbol{x})}\mathrm{d}\boldsymbol{x} \\
&\quad - 2\omega\int_{x\in\mathcal{V}} G_s^*(\boldsymbol{x}_1,\boldsymbol{x})G_s(\boldsymbol{x}_2,\boldsymbol{x})\mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\
&\quad - \omega\int_{x\in\mathcal{V}} \big[2G_t^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) - G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) - G_0(\boldsymbol{x}_2,\boldsymbol{x})G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big] \times \mathrm{Im}\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \quad (15)
\end{aligned}$$

Comparing the right-hand side of Eq. (15) with Eq. (9), we see that the first term in between curly braces on the right-hand side. of Eq. (15) equals $\mathcal{L}(G_{t,1}, G_{t,2})$. It thus follows that the remaining volume integrals equal $-\mathcal{L}(G_{s,1}, G_{s,2})$. Therefore we have

$$\begin{aligned}\mathcal{L}(G_{s,1}, G_{s,2}) = i\omega \int_{x\in\mathcal{V}} \big[&G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) \\ &- G_0(\boldsymbol{x}_2,\boldsymbol{x})G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big]\mathrm{Re}\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &+ \frac{1}{i\omega}\int_{x\in\mathcal{V}} \big[\nabla G_0^*(\boldsymbol{x}_1,\boldsymbol{x})\cdot\nabla G_t(\boldsymbol{x}_2,\boldsymbol{x}) \\ &- \nabla G_0(\boldsymbol{x}_2,\boldsymbol{x})\cdot\nabla G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big]\frac{\Delta\rho(\boldsymbol{x})}{\rho_0(\boldsymbol{x})\rho_t(\boldsymbol{x})}\mathrm{d}\boldsymbol{x} \\ &+ 2\omega\int_{x\in\mathcal{V}} G_s^*(\boldsymbol{x}_1,\boldsymbol{x})G_s(\boldsymbol{x}_2,\boldsymbol{x})\mathrm{Im}\kappa_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &+ \omega\int_{x\in\mathcal{V}} \big[2G_t^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) \\ &- G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) - G_0(\boldsymbol{x}_2,\boldsymbol{x})G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big] \\ &\times \mathrm{Im}\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \end{aligned} \quad (16)$$

This is the reciprocity theorem for the scattered field.

Recently, Snieder *et al.*[5] described a, at first, puzzling experiment related to Green function extraction (or seismic interferometry). Consider the situation sketched in Fig. 1 where a scattering object is embedded in a homogeneous medium and where the wavefields of all the sources on the surrounding surface $\partial\mathcal{V}$ are recorded at receivers at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. According to Eq. (10) (assuming there is no attenuation and under a radiating boundary condition such that we can replace $\nabla G_0 = -i\boldsymbol{k}_0 G_0$) cross-correlating the total wavefields due to each source on the closed surface and integrating the result over this surface should provide the sum of the time-advanced and time-retarded Green function. It is known[20] that not all sources on this closed surface contribute equally to the Green function. The dominant contributions come from sources that have associated wavepaths that are common (or stationary) between both receivers. However, the cross-correlation of the scattered waves will produce a contribution at the travel time equal to the difference between the travel time (t_1) from the scatterer to receiver location 1 and the travel time (t_2) from the scatterer to receiver location 2. Note that this contribution is the same for all sources and thus stationary, but clearly non-physical. Since Eq. (10) is exact, there should be no such non-physical contributions. The answer to this at first puzzling thought experiment lies hidden in Eqs. (15) and (16).

We observed already that the first term in between curly braces on the right-hand side of Eq. (15) equals $\mathcal{L}(G_{t,1}, G_{t,2})$ [see Eq. (9)]. Knowing that Eq. (9) is exact and thus contains all the physics of the problem, it must follow that the remaining four volume integrals on the right-hand side of Eq. (15) are related to non-physical arrivals. These volume integrals thus account for the non-physical (or spurious) arrivals described by Snieder *et al.*[5] Since we have just shown that these four volume integrals equal $-\mathcal{L}(G_{s,1}, G_{s,2})$, it follows that the correlation-type reciprocity theorem for the scattered field is responsible for making sure there are no spurious arrivals in seismic interferometry. Snieder *et al.*[5] show by using a stationary-phase and far-field condition that the generalized optical theorem is responsible for the cancellation of the spurious arrivals. Therefore, it must follow that Eq. (16) specializes to the generalized optical theorem if we subject it to a far-field condition.

V. GENERALIZED OPTICAL THEOREM

The generalized optical theorem is a statement of energy conservation in the absence of attenuation. Therefore, for the purpose of deriving the generalized optical theorem from the reciprocity theorem for the scattered field, we assume that any imaginary parts of κ_0 and $\Delta\kappa$ in the reciprocity theorem for the scattered field [Eq. (16)] are zero. Thus, throughout this section it is understood that $\Delta\kappa$ is real.

Consider a finite (non-absorbing) scattering volume $\mathcal{V}_s \subset \mathcal{V}$ embedded in a homogeneous non-attenuating

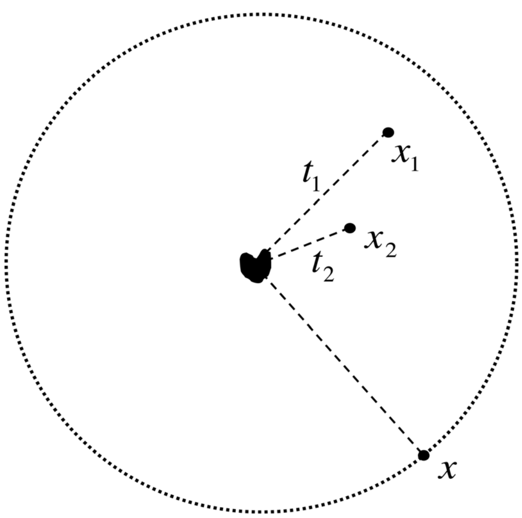

FIG. 1. Wavepaths from the scattered waves traveling from the source location to the scatterer at the origin and then to the receiver locations $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$. Correlating these scattered waves produces a non-physical arrival at the travel time t_1–t_2 that is stationary for all sources on the closed surface $\partial\mathcal{V}$.

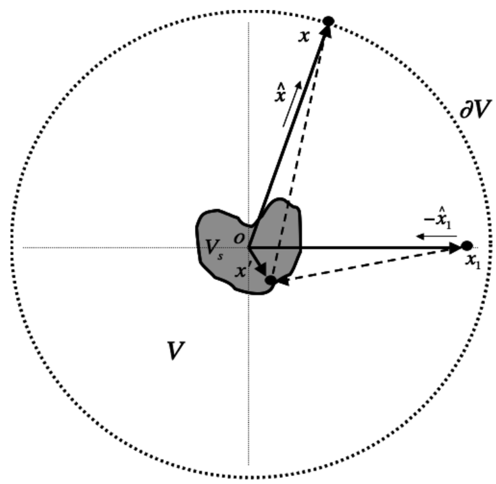

FIG. 2. Geometry of a finite-size scatterer centered at the origin embedded in a homogeneous background medium. The source is located at $\boldsymbol{x}_1$ while the receiver is located at $x \in \partial\mathcal{V}$. In the far-field we have $|\boldsymbol{x}| \gg |\boldsymbol{x}'|$ and $|\boldsymbol{x}_1| \gg |\boldsymbol{x}'|$.

background medium and centered at the origin (Fig. 2). The integral solution to the scattered field G_s is in this case given by (assuming radiating boundary conditions)

$$\begin{aligned} G_{s,1} &= G_s(\boldsymbol{x},\boldsymbol{x}_1) \\ &= -\omega^2 \int_{x'\in\mathcal{V}_s} G_0(\boldsymbol{x},\boldsymbol{x}')\Delta\kappa(\boldsymbol{x}')G_t(\boldsymbol{x}',\boldsymbol{x}_1)\mathrm{d}\boldsymbol{x}' \\ &\quad + \int_{x'\in\mathcal{V}_s} \nabla' G_0(\boldsymbol{x},\boldsymbol{x}')\cdot\nabla' G_t(\boldsymbol{x}',\boldsymbol{x}_1)\frac{\Delta\rho(\boldsymbol{x}')}{\rho_0\rho_t(\boldsymbol{x}')}\mathrm{d}\boldsymbol{x}', \end{aligned} \tag{17}$$

where ∇' means taking the gradient with respect to the $\boldsymbol{x}'$ variable, and where the background density ρ_0 does not depend on $\boldsymbol{x}$ due to the assumed homogeneous background medium. Equation (17) provides the same expression for the scattered field as in the familiar Lippmann–Schwinger equation, and thus incorporates all the non-linear scattering inside the scatterer. To derive the optical theorem, we must first define the scattering amplitude f for the scatterer. It is customary to define this scattering amplitude using a far-field condition. In that way the finite scatterer can mathematically be treated as a point scatterer by simply multiplying the background wavefield with the scattering amplitude at the central location of the scatterer.

In the far-field (i.e., if $|\boldsymbol{x}| \gg |\boldsymbol{x}'|$), the Green function G_0 is given by[21]

$$G_0(\boldsymbol{x},\boldsymbol{x}') \approx \frac{\rho_0}{4\pi}\frac{e^{-ik_0|\boldsymbol{x}|}}{|\boldsymbol{x}|}e^{ik_0\boldsymbol{x}'\cdot\hat{\boldsymbol{x}}}, \tag{18}$$

where k_0 is the magnitude of the wave-vector in the background medium, and $\hat{\boldsymbol{x}}$ denotes a unit vector in the direction of $\boldsymbol{x}$. Note that here the sign of the exponent is chosen to be consistent with the choice of Fourier transformation in Eq. (1). Using this, we can write the scattered wavefield $G_{s,1}$ due to a (finite) scatterer centered around the origin as

$$G_{s,1} = G_s(\boldsymbol{x},\boldsymbol{x}_1) \approx \frac{\rho_0}{4\pi}\frac{e^{-ik_0|\boldsymbol{x}|}}{|\boldsymbol{x}|}f(\hat{\boldsymbol{x}},-\hat{\boldsymbol{x}}_1)\frac{e^{-ik_0|\boldsymbol{x}_1|}}{|\boldsymbol{x}_1|}, \tag{19}$$

with $f(\hat{\boldsymbol{x}},-\hat{\boldsymbol{x}}_1)$ denoting the scattering amplitude due to an incident wave in the $-\hat{\boldsymbol{x}}_1$ direction that is scattered in the $\hat{\boldsymbol{x}}$ direction. To find an expression for f, we use Eq. (19) in Eq. (17). Recognizing that in the far-field $|\boldsymbol{x}_1| \gg |\boldsymbol{x}'|$, we can also use Eq. (18) to get the far-field expression for $G_0(\boldsymbol{x}',\boldsymbol{x}_1)$ in $G_t(\boldsymbol{x}',\boldsymbol{x}_1) = G_0(\boldsymbol{x}',\boldsymbol{x}_1) + G_s(\boldsymbol{x}',\boldsymbol{x}_1)$. Note that, however, we cannot use Eq. (19) for $G_s(\boldsymbol{x}',\boldsymbol{x}_1)$ since $\boldsymbol{x}'$ is inside the scattering volume and can thus not be considered large. Doing this, we find that the scattering amplitude f can be written as

$$\begin{aligned} f(\hat{\boldsymbol{x}}_1,-\hat{\boldsymbol{x}}_2) := -\frac{\rho_0}{4\pi}\Bigg[&\omega^2\int_{x'\in\mathcal{V}_s} e^{ik_0\boldsymbol{x}'\cdot\hat{\boldsymbol{x}}_1}\Delta\kappa(\boldsymbol{x}') \\ &\times\Big\{e^{ik_0\boldsymbol{x}'\cdot\hat{\boldsymbol{x}}_2} + \frac{4\pi}{\rho_0}|\boldsymbol{x}_2|e^{ik_0|\boldsymbol{x}_2|}G_s(\boldsymbol{x}',\boldsymbol{x}_2)\Big\}\mathrm{d}\boldsymbol{x}' \\ &-\int_{x'\in\mathcal{V}_s}\nabla' e^{ik_0\boldsymbol{x}'\cdot\boldsymbol{x}_1}\cdot\nabla'\Big\{e^{ik_0\boldsymbol{x}'\cdot\hat{\boldsymbol{x}}_2} \\ &+\frac{4\pi}{\rho_0}|\boldsymbol{x}_2|e^{ik_0|\boldsymbol{x}_2|}G_s(\boldsymbol{x}',\boldsymbol{x}_2)\Big\}\frac{\Delta\rho(\boldsymbol{x}')}{\rho_0\rho_t(\boldsymbol{x}')}\mathrm{d}\boldsymbol{x}'\Bigg]. \end{aligned} \tag{20}$$

Note that this definition of f incorporates all the non-linear scattering inside the scatterer, since the right-hand side of Eq. (20) includes the scattered field G_s. Therefore, even though at first sight it appears that Eq. (20) contains only monopole and dipole terms, the higher-order multi-pole terms are included through the presence of the scattered field G_s on the right-hand side. Hence, no Born approximation is used here. By construction it follows from using reciprocity relation $G_s(\boldsymbol{x}_1,\boldsymbol{x}_2) = G_s(\boldsymbol{x}_2,\boldsymbol{x}_1)$ together with Eq. (19), that f satisfies the necessary symmetry relation $f(\hat{\boldsymbol{x}}_1,-\hat{\boldsymbol{x}}_2) = f(\hat{\boldsymbol{x}}_2,-\hat{\boldsymbol{x}}_1)$.

We aim to show that using the far-field condition and the above definition of the scattering amplitude, the reciprocity theorem for the scattered field in Eq. (16) specializes to the generalized optical theorem. Ignoring attenuation (i.e., $\mathrm{Im}\kappa_0 = \mathrm{Im}\Delta\kappa = 0$) this reciprocity theorem [Eq. (16)] can be written as

$$\begin{aligned} i\omega\mathcal{L}(G_{s,1},G_{s,2}) = &-\omega^2\int_{x\in\mathcal{V}_s}\big[G_0^*(\boldsymbol{x}_1,\boldsymbol{x})G_t(\boldsymbol{x}_2,\boldsymbol{x}) \\ &- G_0(\boldsymbol{x}_2,\boldsymbol{x})G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big]\Delta\kappa(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &+\int_{x\in\mathcal{V}_s}\big[\nabla G_0^*(\boldsymbol{x}_1,\boldsymbol{x})\cdot\nabla G_t(\boldsymbol{x}_2,\boldsymbol{x}) \\ &-\nabla G_0(\boldsymbol{x}_2,\boldsymbol{x})\cdot\nabla G_t^*(\boldsymbol{x}_1,\boldsymbol{x})\big]\frac{\Delta\rho(\boldsymbol{x})}{\rho_0\rho_t(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{21}$$

with

$$\begin{aligned} i\omega\mathcal{L}(G_{s,1},G_{s,2}) = \frac{1}{\rho_0}\int_{x\in\partial\mathcal{V}}\hat{\boldsymbol{n}}\cdot\big[&G_s(\boldsymbol{x},\boldsymbol{x}_2)\nabla G_s^*(\boldsymbol{x},\boldsymbol{x}_1) \\ &-G_s^*(\boldsymbol{x},\boldsymbol{x}_1)\nabla G_s(\boldsymbol{x},\boldsymbol{x}_2)\big]\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{22}$$

where we multiplied on both sides by $i\omega$ and used that there are no density contrasts on $\partial\mathcal{V}$. The background is homogeneous and the density ρ_0 therefore does not depend on $\boldsymbol{x}$. Using Eq. (19) in Eq. (22) and approximating $\hat{\boldsymbol{n}}\cdot\nabla(f(\hat{\boldsymbol{x}},-\hat{\boldsymbol{x}}_1)e^{-ik_0|x|}/|x|) \approx -ik_0 f(\hat{\boldsymbol{x}},-\hat{\boldsymbol{x}}_1)e^{-ik_0|x|}/|\boldsymbol{x}|$ (consistent with the far-field condition), it follows that

$$\begin{aligned} \text{left-hand side of Eq. (21)} = &\frac{2ik_0\rho_0}{(4\pi)^2}\frac{e^{-ik_0(|\boldsymbol{x}_2|-|\boldsymbol{x}_1|)}}{|\boldsymbol{x}_1||\boldsymbol{x}_2|} \\ &\times\oint_{4\pi}f^*(\hat{\boldsymbol{x}},-\hat{\boldsymbol{x}}_1)f(\hat{\boldsymbol{x}},-\hat{\boldsymbol{x}}_2)\mathrm{d}\Omega, \end{aligned} \tag{23}$$

where we used that $\mathrm{d}\boldsymbol{x}/|\boldsymbol{x}|^2 = \mathrm{d}\Omega$. Using Eq. (18) in Eq. (21) to get the far-field expressions for G_0 and $G_t = G_0 + G_s$, together with the definition of the scattering amplitude f in Eq. (20), we then also have

$$\begin{aligned} \text{right-hand side of Eq. (21)} = &\frac{\rho_0}{4\pi}\frac{e^{-ik_0(|\boldsymbol{x}_2|-|\boldsymbol{x}_1|)}}{|\boldsymbol{x}_1||\boldsymbol{x}_2|} \\ &\times[f(-\hat{\boldsymbol{x}}_1,-\hat{\boldsymbol{x}}_2) - f^*(-\hat{\boldsymbol{x}}_2,-\hat{\boldsymbol{x}}_1)]. \end{aligned} \tag{24}$$

Therefore it follows from Eqs. (23) and (24) that we must have

$$\frac{k_0}{4\pi}\oint_{4\pi} f^*(\hat{x}, -\hat{x}_1) f(\hat{x}, -\hat{x}_2) \mathrm{d}\Omega = \frac{1}{2i}[f(-\hat{x}_1, -\hat{x}_2) - f^*(-\hat{x}_2, -\hat{x}_1)]. \quad (25)$$

This is the generalized optical theorem.[22–24] Hence, the correlation-type reciprocity theorem for the scattered field reduces to the generalized optical theorem when subjected to the far-field condition. This reciprocity theorem can therefore be seen as the progenitor of the generalized optical theorem.

VI. ENERGY CONSIDERATIONS

The interaction quantity $\mathcal{I}_{A,1,B,2}$ is the product of pressure, i.e., the force per unit of surface, and velocity, i.e., distance per unit of time. Force times distance equals the work done, and the work per unit time is equal to the power. Therefore, provided the source locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ coincide, the interaction quantity is the power flux. This flux is also referred to as the acoustic intensity.[25] In quantum mechanics the equivalent vector is the probability current density,[26] while in electromagnetics it is the Poynting vector.[27]

Consider now the situation where both source locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are outside of the volume $\mathcal{V}$ (see Fig. 3) and where the scatterer and the background can both be absorbing. Integrating the power flux over a surface gives the net power. Therefore the surface integral $\mathcal{L}(G_A, G_B)$ is nothing more than an estimate of the power; since this interpretation of the surface integral makes sense only when both source locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the same, we drop the subscripts 1 and 2 from here onward. This means that $\mathcal{L}(G_s, G_s)$ equals the power P_{scat} lost due to scattering. In the absence of sources inside the volume $\mathcal{V}$, the energy that is not scattered must be absorbed. Therefore, the power associated with the total field must equal negative the absorbed power. That is, we have (ignoring normalization factors)

$$P_{\mathrm{scat}} = \mathcal{L}(G_s, G_s) \quad \text{and} \quad P_{\mathrm{abs}} = -\mathcal{L}(G_t, G_t). \quad (26)$$

The total extinguished power P_{ext} equals the sum of the scattered and absorbed power. Therefore we have

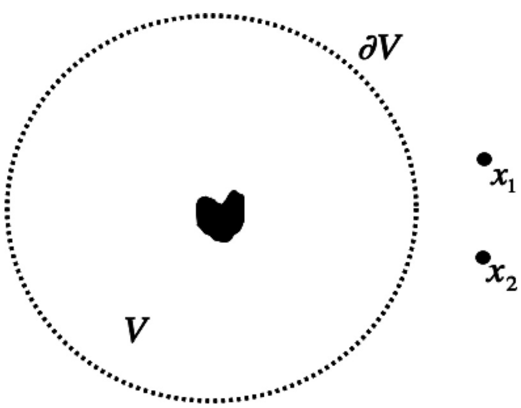

FIG. 3. Geometry for the derivation of the volume integral representations of the scattered and absorbed cross-sections. The sources $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are both assumed to be outside of the volume $\mathcal{V}$.

$$P_{\mathrm{ext}} = P_{\mathrm{ext}} + P_{\mathrm{ext}} = \mathcal{L}(G_s, G_s) - \mathcal{L}(G_t, G_t). \quad (27)$$

In the absence of sources and attenuation in the background medium, the net power for the background field must equal zero; i.e., what comes into the volume $\mathcal{V}$ must also come out. Therefore in this case we have

$$\mathcal{L}(G_0, G_0) = 0. \quad (28)$$

Using Eq. (12) in Eq. (27) together with Eq. (28), we then find the surface integral representation for the extinguished power, given by

$$P_{\mathrm{ext}} = -\{\mathcal{L}(G_0, G_s) + \mathcal{L}(G_s, G_0)\}. \quad (29)$$

This implies that Eq. (12) merely states the conservation of power (provided the source locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the same).

VII. VOLUME INTEGRAL REPRESENTATIONS FOR THE SCATTERED, ABSORBED, AND EXTINGUISHED POWER

The reciprocity theorems allow the surface integrals to be rewritten in terms of volume integrals. Using the reciprocity theorem for the scattered field in Eq. (16) (and remembering that we assume the locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to be equal, i.e., $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{x}$), it immediately follows from Eq. (26) that

$$P_{\mathrm{scat}} = 2\omega \int_{\boldsymbol{x}' \in \mathcal{V}} \mathrm{Im}\left[G_0(\boldsymbol{x}', \boldsymbol{x}) G_t^*(\boldsymbol{x}', \boldsymbol{x})\right] \mathrm{Re}\Delta\kappa(\boldsymbol{x}') \mathrm{d}\boldsymbol{x}' + \frac{2}{\omega}\int_{\boldsymbol{x}' \in \mathcal{V}} \mathrm{Im}\left[\nabla' G_0^*(\boldsymbol{x}', \boldsymbol{x}) \cdot \nabla' G_t(\boldsymbol{x}', \boldsymbol{x})\right] \frac{\Delta\rho(\boldsymbol{x}')}{\rho_0(\boldsymbol{x}')\rho_t(\boldsymbol{x}')} \mathrm{d}\boldsymbol{x}', \quad (30)$$

where we assumed the scatterer and background to be non-absorbing, such that P_{scat} is indeed the cross-section due to scattering only. Note that all the non-linear scattering is included by the presence of the total field G_t on the right-hand side of Eq. (30) and that there is thus no underlying Born approximation in this result. Similarly, we can also use the reciprocity theorem for the total field in Eq. (9) to find a volume integral representation for the absorbed power P_{abs}. Note that the first two terms in the right-hand side of Eq. (9) are due to the fact that we chose the source terms q in Eq. (5) to be δ-functions with support inside the volume $\mathcal{V}$. Therefore, in the absence of sources inside $\mathcal{V}$ those first two terms vanish. Using Eq. (26), it then thus follows that

$$P_{\mathrm{abs}} = 2\omega \int_{\boldsymbol{x}' \in \mathcal{V}} |G_t(\boldsymbol{x}', \boldsymbol{x})|^2 \mathrm{Im}\Delta\kappa(\boldsymbol{x}') \mathrm{d}\boldsymbol{x}', \quad (31)$$

where we assumed the background medium to be non-attenuative [i.e., $\mathrm{Im}\kappa_t = \mathrm{Im}(\kappa_0 + \Delta\kappa) = \mathrm{Im}\Delta\kappa$], such that we get the absorption cross-section of the scatterer only. The volume integral representation for the extinguished power follows simply by taking the sum of the volume integral representations for P_{scat} and P_{abs} in Eqs. (30) and (31), respectively. Equations (30) and (31) are the acoustic equivalents of the equations obtained by Carney *et al.*[11] for scalar electromagnetic waves [their Eqs. (7) and (13), respectively] that were later extended to vector electromagnetic waves by

Lytle *et al.*[12] They assume a contrast in the susceptibility whereas in our treatment the contrast is in the compressibility and density.

VIII. GENERALIZATION TO GENERAL LINEAR SYSTEMS

We identified the reciprocity theorem for the scattered field as the progenitor of the generalized optical theorem. In this work, we have treated acoustic waves in attenuative non-flowing media with contrasts in both compressibility and density. However, in light of the recent work of Wapenaar *et al.*[13] and Wapenaar,[14] who presented a unified theory for Green function extraction for general wave and diffusive fields in lossy media based on reciprocity theorems, it follows that one can derive a much more general form of the reciprocity theorem for the scattered field that relates to general linear systems. That in turn means that, based on the treatment presented here, one should be able to write down the generalized optical theorem in a unified form that captures general wavefields. In light of this and the connection we showed with the volume integral representations of $P_{\rm scat}$ and $P_{\rm abs}$ given by Carney *et al.*,[11] Lytle *et al.*[12] recently extended the treatment of Carney *et al.*[11] to the case of vector electromagnetic wavefields.

IX. CONCLUSIONS

Using reciprocity theorems for acoustic wavefields in perturbed media, we have shown that the correlation-type reciprocity theorem for the scattered field is the progenitor of the generalized optical theorem. In contrast to the generalized optical theorem, this reciprocity theorem allows for inhomogeneous background properties, attenuation, and does not depend on a far-field condition. By considering the special case of a finite-size scatterer embedded in a homogeneous non-absorbing background medium, we have shown that this theorem specializes to the generalized optical theorem when using the far-field condition. This leads to a (volume) integral representation of the non-linear scattering amplitude. Moreover, by analyzing seismic interferometry in the context of reciprocity theorems in perturbed media, we have shown that the correlation-type reciprocity theorem for the scattered field is responsible for the cancellation of the nonphysical (or spurious) arrivals in seismic interferometry. As such, this reciprocity theorem provides the mathematical description of the spurious arrivals.

In light of the recent work on unified Green function retrieval for general linear systems, a reciprocity theorem for the scattered field can be derived for general linear systems. Since, here, we identified the correlation-type reciprocity theorem of the scattered field as the progenitor of the generalized optical theorem, a similar treatment as the one provided here should allow for the derivation of a unified optical theorem for general wavefields.

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[1]E. Larose, L. Margerin, A. Derode, B. van Tiggelen, M. Campillo, N. Shapiro, A. Paul, L. Stehly, and M. Tanter, "Correlation of random wavefields: An interdisciplinary review," Geophysics **71**, SI11–SI21 (2006).

[2]G. T. Schuster, *Seismic Interferometry* (Cambridge University Press, Cambridge, UK, 2009), 272 p.

[3]K. Wapenaar, D. Dragonov, R. Snieder, X. Campman, and A. Verdel, "Tutorial on seismic interferometry: Part 1—Basic principles and applications," Geophysics **75**, 75A195–75A209 (2010).

[4]K. Wapenaar, E. Slob, R. Snieder, and A. Curtis, "Tutorial on seismic interferometry: Part 2—Underlying theory and new advances," Geophysics **75**, 75A211–75A227 (2010).

[5]R. Snieder, K. van Wijk, M. Haney, and R. Calvert, "Cancellation of spurious arrivals in Green's function extraction and the generalized optical theorem," Phys. Rev. E **78**, 036606 (2008).

[6]D. Halliday and A. Curtis, "Generalized optical theorem for surface waves and layered media," Phys. Rev. E **79**, 056603 (2009).

[7]R. Snieder, F. Sanchéz-Sesma, and K. Wapenaar, "Field fluctuations, imaging with backscattered waves, a generalized energy theorem, and the optical theorem," SIAM J. Imaging Sci. **2**, 763–776 (2009).

[8]K. Wapenaar, E. Slob, and R. Snieder, "On seismic interferometry, the generalized optical theorem, and the scattering matrix of a point scatterer," Geophysics **75**, SA27–SA35 (2010).

[9]H. Douma, "Generalized representation theorems for acoustic wavefields in perturbed media," Geophys. J. Int. **179**, 319–332 (2009).

[10]I. Vasconcelos, R. Snieder, and H. Douma, "Representation theorems and Green's function retrieval for scattering in acoustic media," Phys. Rev. E **80**, 036605 (2009).

[11]P. S. Carney, J. C. Schotland, and E. Wolf, "Generalized optical theorem for reflection, transmission, and extinction of power for scalar fields," Phys. Rev. E **70**, 036611 (2004).

[12]D. R. Lytle, P. S. Carney, J. C. Schotland, and E. Wolf, "Generalized optical theorem for reflection, transmission, and extinction of power for electromagnetic fields," Phys. Rev. E **71**, 056610 (2005).

[13]K. Wapenaar, E. Slob, and R. Snieder, "Unified Green's function retrieval by cross correlation," Phys. Rev. Lett. **97**, 234301 (2006).

[14]K. Wapenaar, "General representations for wavefield modeling and inversion in geophysics," Geophysics **72**, SM5–SM17 (2007).

[15]J. T. Fokkema and P. M. van den Berg, *Seismic Applications of Acoustic Reciprocity* (Elsevier, Amsterdam, The Netherlands, 1993).

[16]K. Wapenaar and J. T. Fokkema, "Green's function representations for seismic interferometry," Geophysics **71**, SI33–SI46 (2006).

[17]L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968), pp. 318–319.

[18]K. Wapenaar, "Retrieving the elastodynamic Green's function of an arbitrary inhomogeneous medium by cross correlation," Phys. Rev. Lett. **93**, 254301 (2004).

[19]R. Snieder, "Extracting the Green's function of attenuating heterogeneous acoustic media from uncorrelated waves," J. Acoust. Soc. Am. **121**, 2637–2643 (2007).

[20]R. Snieder, "Extracting the Green's function from the correlation of coda waves: A derivation based on stationary phase," Phys. Rev. E **69**, 046610 (2004).

[21]A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Institute of Electrical and Electronics Engineers, New York, 1997), p. 16.

[22]R. Glauber and V. Schomaker, "The theory of electron diffraction," Phys. Rev. **89**, 667–671 (1953).

[23]R. G. Newton, "Optical theorem and beyond," Am. J. Phys. **44**, 639–642 (1976).

[24]P. L. Marston, "Generalized optical theorem for scatterers having inversion symmetry: Applications to acoustic backscattering," J. Acoust. Soc. Am. **109**, 1291–1295 (2001).

[25]P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (Princeton University, Princeton, NJ, 1968), p. 249.

[26]E. Merzbacher, *Quantum Mechanics*, 2nd ed. (Wiley, New York, 1970), p. 37.

[27]M. Born and E. Wolf, *Principles of Optics*, 3rd ed. (Pergamon, Oxford, UK, 1970), pp. 9–10.