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The Recovery of the Chiral Symmetry in Lattice Gross-Neveu Model

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The recovery of the chiral symmetry is carefully analyzed in the lattice Gross-Neveu model with Wilson's fermion, by using the effective potential obtained in the large N limit. It turns out that we have to introduce two bare coupling constants for four-fermi interactions as well as the bare mass term in order to obtain the chiral symmetric theory in the continuum limit. A method is proposed to extract the genuine order parameter that scales in the continuum limit.

§ 1. Introduction

The chiral symmetry is one of the important properties to explain the mass spectrum of the hadrons. The π-meson is thought to be the Nambu-Goldstone boson associated with the dynamical breakdown of the chiral symmetry. The strong interaction is governed by the Quantum Chromodynamics (QCD) whose Lagrangian is chiral symmetric. It is important to show the dynamical breakdown of the chiral symmetry and to calculate π-meson mass in the framework of the QCD. For calculating such non-perturbative effects the lattice regularization is suitable.

There is a problem to define a chiral symmetric QCD on a lattice.[1] This problem is the spectral doubling of fermions and to avoid this spectral doubling we must add the Wilson term to the Lagrangian.[1] The Wilson term, however, breaks the chiral symmetry explicitly. It is known to be impossible to obtain the chiral symmetric lattice QCD without the spectral doubling.[2] Probably this property may represent the existence of the chiral anomaly. If we want to obtain the correct continuum limit we must use the QCD Lagrangian with the Wilson term. Therefore the chiral symmetry of the QCD is explicitly broken by the Wilson term which disappears in the naive (classical) continuum limit. Therefore we expect that the chiral symmetry breaking effect of the Wilson term also disappears in the true continuum limit besides the chiral anomaly.

To see whether our expectations are true or not we investigate the chiral symmetric fermion model, Gross-Neveu model on a two dimensional lattice. The recovery of the chiral symmetry is usually measured by the scaling behavior of the chiral order parameter.[3] But in this paper we investigate the effective potential instead. If the effective potential is a chiral symmetric in the continuum limit, our expectation is valid.

This paper is organized as follows. In § 2 we analyze the continuum Gross-Neveu model in the presence of the explicit breaking of chiral symmetry. In § 3 we analyze the lattice Gross-Neveu model, especially its continuum limit. It is shown that the effective potential becomes chiral symmetric in the continuum limit, if and only if we introduce two bare-coupling constants of the four-fermi interaction and adjust them. This result is contrary to our naive expectation. In § 4 we propose the method to analyze this two couplings model on a finite lattice and in § 5 results of our analysis are given. In § 6 we discuss the implication of the results. In the Appendix we discuss the recovery of chiral symmetry of the continuum Gross-Neveu model with the chiral non-invariant regularization.

§ 2. Continuum Gross-Neveu model

Let us first recapitulate the two dimensional Gross-Neveu model[4] described by the Lagrangian:

$$L=\bar{\psi}(i\gamma\cdot\partial-m_0)\psi+\frac{g^2}{2N}(\bar{\psi}\psi)^2\,, \tag{2·1}$$

where ψ denotes N Dirac fermion $\psi_k(k=1, 2, \cdots, N)$, coupled through a scalar interaction. We have used the notation

$$\bar{\psi}\psi=\sum_{k=1}^{N}\bar{\psi}_k\psi_k\,.$$

This theory is invariant under a discrete chiral symmetry: $\psi\to\gamma_5\psi$, $\bar{\psi}\to-\bar{\psi}\gamma_5$ when $m_0=0$. Later in this section we will describe a generalized model invariant under continuous chiral transformation. It is convenient to replace (2·1) by an equivalent Lagrangian

$$L_\sigma=\bar{\psi}(i\gamma\cdot\partial-\sigma)\psi-\frac{N}{2g^2}(\sigma-m_0)^2\,, \tag{2·2}$$

where, by the equation of motion

$$\sigma=m_0-\frac{g^2}{N}\bar{\psi}\psi\,. \tag{2·3}$$

To solve the model we integrate out the fermion fields and obtain an effective action describing the self-interaction of σ:

$$Z=\int[d\sigma][d\psi]\exp\left\{i\int L_\sigma dx\right\}\equiv\int[d\sigma]\exp\{iNS_{\text{eff}}(\sigma)\}\,, \tag{2·4}$$

where

$$S_{\text{eff}}(\sigma)=\int dx\left\{-\frac{1}{2g^2}(\sigma-m_0)^2\right\}+\frac{1}{i}\text{Tr ln}(\sigma-i\partial\!\!\!/)\,. \tag{2·5}$$

Since the exponent of Eq. (2·4) is of order N, integrations over $\sigma(x)$ are performed by the saddle-point method when N is sufficiently large and g^2 fixed, giving the systematic expansion of the effective potential in powers of $1/N$. By decomposing $\sigma(x)$ into a sum of the constant classical field σ_c and the fluctuating quantum field $\sigma'(x)$ with a constraint

$$\int dx\ \sigma'(x)=0\,, \tag{2·6}$$

we find

$$S_{\text{eff}}=-\Omega V(\sigma_c)+\frac{1}{2}\int dx\int dy\ \sigma'(x)iG_\sigma^{-1}(x,y)\sigma'(y)+S_{\text{int}}(\sigma_c, \sigma')\,, \tag{2·7}$$

where Ω is the space-time volume and

$$V(\sigma_c)=-m_0\cdot\sigma_c/g^2+\sigma_c{}^2/2g^2-\int\frac{d^2k}{(2\pi)^2i}\ln(\sigma_c{}^2-k^2-i\varepsilon)+\text{const}\,, \tag{2·8}$$

$$iG_\sigma{}^{-1}(p)=-1/g^2-\int\frac{d^2k}{(2\pi)^2i}\text{tr}\left[\frac{1}{\sigma_c-\not{k}}\cdot\frac{1}{\sigma_c-(\not{p}+\not{k})}\right], \tag{2·9}$$

is the propagator of σ' in momentum space, and

$$S_{\text{int}}(\sigma_c,\sigma')=-\frac{1}{i}\sum_{n=3}^{\infty}\frac{(-1)^n}{n}\text{Tr}\left[\frac{1}{\sigma_c-i\not{\partial}}\sigma'\right]^n. \tag{2·10}$$

Now, the effective potential, the energy density of the ground state in the presence of the background field σ_c, is defined by

$$Z=\int d\sigma_c\exp(-iN\Omega V_{\text{eff}}(\sigma_c))\,. \tag{2·11}$$

By comparing Eqs. (2·4) and (2·11), we find an expression similar to Jackiw's formula[5)]

$$V_{\text{eff}}(\sigma_c)=V(\sigma_c)+\frac{1}{2N}\int\frac{d^2k}{(2\pi)^2i}\ln(-iG_0{}^{-1}(k))+\frac{i}{N\Omega}\ln\langle\exp(iNS_{\text{int}})\rangle^{\text{1PI}}_{G_\sigma}\,. \tag{2·12}$$

The last term is the sum of connected one particle irreducible (1PI) vacuum graphs obtained by using the conventional Feynman rules, with $(1/N)\,G_\sigma$ as the propagator. We have to keep only 1PI graphs because of the constraint (2·6). The first term is independent of N; the second term, the one-loop determinant, is proportional to $1/N$. The remaining terms are at most of order $1/N^2$. This is seen by counting the number of $1/N$: Each propagator carries factor $1/N$. Each vertex is of order N. Then, the contribution of a vacuum graph with n_P propagators and n_V vertices is proportional to $N^{-n_P+n_V-1}$ $=N^{-n_L}$ with n_L being the number of independent loops. Thus the $1/N$ expansion for V_{eff} is nothing but the loop expansion in a theory described by $S_{\text{eff}}(\sigma)$.

Hereafter we shall confine ourselves to the large N limit, where $V_{\text{eff}}(\sigma_c)$ is simply given by $V(\sigma_c)$. If we introduce the straight cutoff M in the euclidean momentum space, we find the expression for V_{eff}, when the cutoff M tends to infinity with Λ and m kept fixed

$$V_{\text{eff}}(\sigma_c)=-m\sigma_c+\frac{1}{4\pi}\sigma_c{}^2\ln\frac{\sigma_c{}^2}{e\Lambda^2}\,, \tag{2·13}$$

where the renormalization point independent scale parameter Λ and mass parameter m which characterize the explicit breaking of chiral symmetry are defined by

$$\frac{1}{g^2}=\frac{1}{2\pi}\ln\frac{M^2}{\Lambda^2} \tag{2·14}$$

and

$$m=\frac{m_0}{g^2}\,. \tag{2·15}$$

If we had introduced a renormalization point μ and a renormalized coupling constant g_R[4)] by

$$\frac{1}{g^2}=\frac{1}{g_R{}^2}+\frac{1}{2\pi}\ln\frac{M^2}{\mu^2}\,,$$

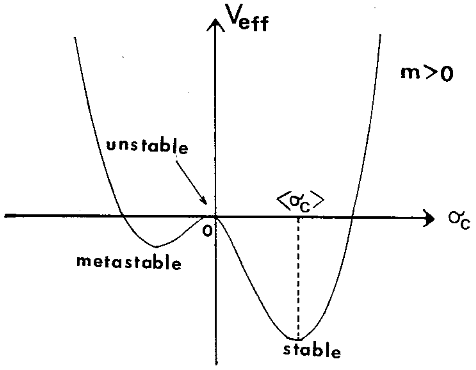


Fig. 1. Shape of the effective potential (2·13) when the current quark mass m is positive and small. Among three extrema, the stable vacuum corresponds to the absolute minimum of V_{eff}.

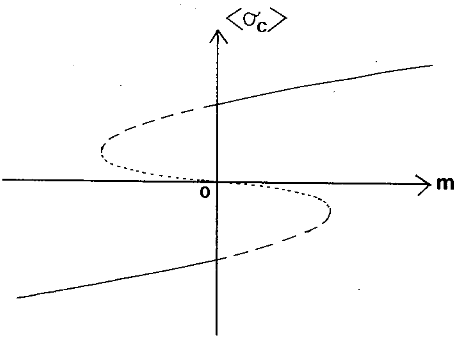


Fig. 2. Dependence of the constituent quark mass (order parameter) $\langle\sigma_c\rangle$ on the current quark mass m. Solid line represents the stable vacuum. Dashed (dotted) line indicates the metastable (unstable) vacuum. The presence of the gap when $m=0$ shows the spontaneous breakdown of the chiral symmetry.

then Λ would have been expressed as

$$\Lambda=\mu\exp(-\pi/g_R^2)\,.$$

In the large N limit, the wave function renormalization of σ is not necessary.

The vacuum expectation value of σ_c is determined by looking for the true minimum of $V_{eff}(\sigma_c)$, i.e., by solving the renormalized gap equation

$$\frac{\partial V_{eff}(\sigma_c)}{\partial\,\sigma_c}=-m+\frac{1}{2\pi}\sigma_c\ln\frac{\sigma_c^2}{\Lambda^2}=0\,. \qquad (2\cdot16)$$

When m is small, this gap equation has three solutions. In this case, the true ground state can be chosen by looking at the shape of the $V_{eff}(\sigma_c)$ as is shown in Fig. 1. Other two solutions correspond to either metastable or unstable state. Therefore, the stability of the ground state requires that the order parameter $\langle\sigma_c\rangle$ always has the same sign as the explicit breaking parameter m of chiral symmetry. Namely, when $|m|$ is small, $\langle\sigma_c\rangle$ is given by

$$\langle\sigma_c\rangle=\begin{cases}\Lambda+\pi\cdot m\,, & (m>0)\\ -\Lambda-\pi\cdot m\,. & (m<0)\end{cases}$$

Note that the order parameter $\langle\sigma_c\rangle$ has a gap, a clear evidence of dynamical breaking of chiral symmetry, when m changes sign as is shown in Fig. 2, indicating the first order phase transition as a function of m. Half of this gap determines the magnitude of the order parameter in the chiral symmetry limit $m=0$.

Now let us discuss a generalization of the Gross-Neveu model with continuous chiral symmetry, defined by the Lagrangian:

$$L=\bar{\psi}(i\partial\!\!\!/-m_0)\psi+\frac{g^2}{2N}\{(\bar{\psi}\psi)^2+(\bar{\psi}i\gamma_5\psi)^2\}\,. \qquad (2\cdot17)$$

This theory is invariant under the continuous chiral transformation: $\psi \to e^{i\theta\gamma_5}\psi$, $\bar{\psi} \to \bar{\psi} e^{i\theta\gamma_5}$, when $m_0=0$. The corresponding equivalent Lagrangian is

$$L_\sigma = \bar{\psi}(i\partial\!\!\!/ - \sigma - i\gamma_5 \Pi)\psi - \frac{N}{2g^2}\{(\sigma - m_0)^2 + \Pi^2\}, \tag{2·18}$$

where, by the equation of motion

$$\sigma = m_0 - \frac{g^2}{N}\bar{\psi}\cdot\psi, \tag{2·19}$$

$$\Pi = -\frac{g^2}{N}\bar{\psi} i\gamma_5 \psi. \tag{2·20}$$

The effective potential in the large N limit is obtained in a similar way

$$V_{\rm eff}(\sigma_c, \Pi_c) = -m\sigma_c + \frac{1}{4\pi}({\sigma_c}^2 + {\Pi_c}^2)\ln\frac{{\sigma_c}^2 + {\Pi_c}^2}{e\cdot\Lambda^2} \tag{2·21}$$

and has a rotational symmetry in the σ_c-Π_c plane in the chiral symmetry limit ($m=0$). The renormalized gap equations read

$$\frac{\partial V_{\rm eff}}{\partial \sigma_c} = -m + \frac{\sigma_c}{2\pi}\ln\frac{{\sigma_c}^2 + {\Pi_c}^2}{\Lambda^2} = 0, \tag{2·22}$$

$$\frac{\partial V_{\rm eff}}{\partial \Pi_c} = \frac{\Pi_c}{2\pi}\ln\frac{{\sigma_c}^2 + {\Pi_c}^2}{\Lambda^2} = 0. \tag{2·23}$$

From these equations, we can determine the vacuum expectation values when $m \neq 0$

$$\Pi_c = 0, \tag{2·24}$$

$$m = \frac{\sigma_c}{2\pi}\ln\frac{{\sigma_c}^2}{\Lambda^2}. \tag{2·25}$$

Again the theory shows the first order phase transition when m passes through 0. When $m=0$, the vacuum is degenerate and determined up to chiral rotations by

$${\sigma_c}^2 + {\Pi_c}^2 = \Lambda^2. \tag{2·26}$$

It was pointed out by Witten[6] some time ago that the large N limit does not commute with the large volume limit in the chiral symmetric GN model ($m=0$). Therefore, the GN model in the large N limit should be regarded as a theoretical laboratory to derive useful information in the chiral symmetric case.

§3. Lattice Gross-Neveu model and continuum limit

In this section, we work on euclidean square lattice with lattice spacing a. The lattice points are labeled by

$$x_\mu = n_\mu a, \qquad n_\mu = 0, \pm 1, \pm 2, \cdots, \mu = 1, 2. \tag{3·1}$$

The range of momenta is restricted to

$$-\pi/a < k_\mu < \pi/a. \tag{3·2}$$

The natural way to find a lattice version of the Gross-Neveu model with continuous chiral symmetry is to replace the differentials by differences:

$$S=\frac{a}{2}\sum_{x,\mu}\bar{\psi}(x)\gamma_\mu\{\psi(x+a_\mu)-\psi(x-a_\mu)\}+\sum_x a^2 m_0\bar{\psi}\psi$$
$$-a^2\frac{g^2}{2N}\sum_x\{(\bar{\psi}\psi)^2+(\bar{\psi}i\gamma_5\psi)^2\}, \tag{3·3}$$

where a_μ is a vector along the μ direction with length a and γ_μ's are hermitian and satisfy $\{\gamma_\mu, \gamma_\nu\}=2\delta_{\mu\nu}$. This naive Dirac action leads to the notorious species doubling. One of the possible ways proposed by Wilson to avoid this problem is to introduce an irrelevant operator

$$-\frac{ra}{2}\sum_{x,\mu}\{\bar{\psi}(x)\cdot\psi(x+a_\mu)+\bar{\psi}(x+a_\mu)\cdot\psi(x)-2\bar{\psi}(x)\cdot\psi(x)\} \tag{3·4}$$

with $0<r\leqq 1$. The free fermion propagator

$$\left\{\sum_\mu i\gamma_\mu\frac{\sin k_\mu a}{a}+m_0+\frac{r}{a}\sum_\mu(1-\cos k_\mu a)\right\}^{-1} \tag{3·5}$$

now describes four kinds of particles with masses m_0, m_0+2r/a and m_0+4r/a in the vicinities of $k=(0, 0)$, $(0, \pi/a)$ and $(\pi/a, 0)$, and $(\pi/a, \pi/a)$, respectively. Thus, we have just one fermion in the continuum limit $a\to 0$. An obvious disadvantage of Wilson's formulation is that chiral symmetry is explicitly broken by the additional term (3·4), even if $m_0=0$. Since chiral symmetry is restored in the continuum limit for free field theory, we may expect that it is also restored for interacting field theories in the continuum limit. In order to test this idea, we examined a continuum theory with chiral non-invariant regularization in the Appendix and found that indeed chiral symmetry can be restored in the continuum limit if we start with a bare action not invariant under chiral symmetry. In Wilson's formulation of the lattice Gross-Neveu model, therefore, it is natural to start with an action

$$S=\frac{a}{2}\sum_{x,\mu}\{\bar{\psi}(x)(\gamma_\mu-r)\psi(x+a_\mu)-\bar{\psi}(x+a_\mu)(\gamma_\mu+r)\psi(x)\}$$
$$+a\sum_x(m_0 a+2r)\bar{\psi}(x)\psi(x)-a^2\sum_x\left\{\frac{g_\sigma^2}{2N}(\bar{\psi}\psi)^2+\frac{g_\pi^2}{2N}(\bar{\psi}i\gamma_5\psi)^2\right\}. \tag{3·6}$$

The interaction term no longer has chiral symmetry, instead, g_σ^2 and g_π^2 are to be chosen so that the renormalized theory has chiral symmetry. The corresponding equivalent action with auxiliary fields is

$$S=\frac{a}{2}\sum_{x,\mu}\{\bar{\psi}(x)(\gamma_\mu-r)\psi(x+a_\mu)-\bar{\psi}(x+a_\mu)(\gamma_\mu+r)\psi(x)\}$$
$$+2ar\sum_x\bar{\psi}(x)\psi(x)+a^2\sum_x\bar{\psi}\{\sigma+i\gamma_5\Pi\}\psi+a^2\sum_x\left\{\frac{N}{2g_\sigma^2}(\sigma-m_0)^2+\frac{N}{2g_\pi^2}\Pi^2\right\}, \tag{3·7}$$

where, by the equation of motion

$$\sigma(x)=m_0-\frac{g_\sigma^2}{N}\bar{\psi}(x)\psi(x), \tag{3·8}$$

$$\Pi(x)=-\frac{g_\pi^{\,2}}{N}\bar{\psi}(x)\cdot i\gamma_5\cdot\psi(x)\,. \tag{3·9}$$

The effective potential in the large N limit is obtained as in the previous section

$$V_{\text{eff}}=\frac{1}{2g_\sigma^{\,2}}(\sigma_c-m_0)^2+\frac{1}{2g_\pi^{\,2}}\Pi_c^{\,2}-I\,, \tag{3·10}$$

where

$$I=\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\ln\left\{\sum_\mu\frac{\sin^2k_\mu a}{a^2}+\left(\sigma_c+\frac{r}{a}\sum_\mu(1-\cos k_\mu a)\right)^2+\Pi_c^{\,2}\right\}. \tag{3·11}$$

This effective potential does not have the rotational symmetry in (σ_c, Π_c) plane even if $m_0=0$ and $g_\sigma=g_\pi$, because of the Wilson term (3·4). We shall postpone the detailed analysis of the gap equation for finite lattice spacing to the next section, and discuss the continuum limit of our theory in the rest of this section.

In order to evaluate the integral (3·11) in the continuum limit ($a\to 0$), let us first rewrite it as follows:

$$I=\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\left\{\ln\Delta+\ln\left(1+\frac{\varepsilon}{\Delta}\right)\right\} \tag{3·12}$$

with

$$\Delta\equiv\sum_\mu\frac{\sin^2k_\mu a}{a^2}+r^2\left(\sum_\mu\frac{1-\cos k_\mu a}{a}\right)^2+(\sigma_c^{\,2}+\Pi_c^{\,2})\,, \tag{3·13}$$

$$\varepsilon\equiv 2r\sigma_c\sum_\mu(1-\cos k_\mu a)/a\,. \tag{3·14}$$

We then expand the integrand into a power series of ε:

$$I=I_0+I_1+I_2+\cdots\cdots\,, \tag{3·15}$$

where

$$I_0=\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\ln\Delta\,, \tag{3·16}$$

$$I_n=-\frac{(-1)^n}{n}\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\frac{\varepsilon^n}{\Delta^n}\,.\qquad (n\geqq 1) \tag{3·17}$$

Note that I_0 has rotational symmetry in the σ_c-Π_c plane, whereas I_n's ($n\geqq 1$) do not. In fact, it can be shown that $I_1(I_2)$ reduces to a linear (quadratic) term in σ_c while $I_n(n\geqq 3)$ vanishes in the continuum limit ($a\to 0$). This is seen by rewriting Eq. (3·17), using a rescaled variable $\xi_\mu=k_\mu a$, as

$$I_n=-\frac{(-1)^n}{n}(2r\sigma_c)^n a^{n-2}$$
$$\times\int_{-\pi}^{\pi}\frac{d^2\xi}{(2\pi)^2}\frac{(\sum_\mu(1-\cos\xi_\mu))^n}{[\sum_\mu\sin^2\xi_\mu+r^2(\sum_\mu(1-\cos\xi_\mu))^2+a^2(\sigma_c^{\,2}+\Pi_c^{\,2})]^n}\,. \tag{3·18}$$

These integrals are well defined in the limit $a\to 0$. Thus, retaining only divergent or finite quantities, we find

$$I_1=\frac{2r\sigma_c}{a}C_1\,, \tag{3·19}$$

$$I_2=-2r^2\sigma_c{}^2C_2\,, \tag{3·20}$$

$$I_n=0\,,\qquad (n\geqq 3) \tag{3·21}$$

where

$$C_1=\int_{-\pi}^{\pi}\frac{d^2\xi}{(2\pi)^2}\frac{\sum_\mu(1-\cos\xi_\mu)}{\sum_\mu\sin^2\xi_\mu+r^2[\sum_\mu(1-\cos\xi_\mu)]^2}\,,\qquad (=0.385,\ r^2=1) \tag{3·22}$$

$$C_2=\int_{-\pi}^{\pi}\frac{d^2\xi}{(2\pi)^2}\frac{[\sum_\mu(1-\cos\xi_\mu)]^2}{\{\sum_\mu\sin^2\xi_\mu+r^2[\sum_\mu(1-\cos\xi_\mu)]^2\}^2}\,.\qquad (=0.155,\ r^2=1) \tag{3·23}$$

Now, let us turn to the evaluation of I_0. If we introduce an integral representation of I_0:

$$I_0=\int_0^{\sigma_c{}^2+\Pi_c{}^2}d\rho F(\rho)\,, \tag{3·24}$$

it is not difficult to show that

$$F(\rho)=\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\frac{1}{\sum_\mu\sin^2k_\mu a/a^2+r^2/a^2\{\sum_\mu(1-\cos k_\mu a)\}^2+\rho} \tag{3·25}$$

$$\xrightarrow[a\to 0]{}\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\frac{1}{\sum_\mu k_\mu{}^2+\rho}+C_0\,, \tag{3·26}$$

where C_0 is a finite constant defined by

$$C_0\equiv\int_{-\pi}^{\pi}\frac{d^2\xi}{(2\pi)^2}\frac{\sum_\mu(\xi_\mu{}^2-\sin^2\xi_\mu)-r^2\{\sum_\mu(1-\cos\xi_\mu)\}^2}{[\sum_\mu\sin^2\xi_\mu+r^2[\sum_\mu(1-\cos\xi_\mu)]^2]\cdot[\sum_\mu\xi_\mu{}^2]}\,.\qquad (=0.427,\ r^2=1) \tag{3·27}$$

By comparing the first term in Eq. (3·26) with the corresponding integral in the continuum theory, we find

$$F(\rho)=\frac{1}{4\pi}\ln\frac{1}{a^2\rho}+\hat{C}_0\,, \tag{3·28}$$

where a new constant $\hat{C}_0$ is defined by

$$\hat{C}_0=C_0+C_0'=0.627 \tag{3·29}$$

with

$$\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\sum_\mu\frac{1}{k_\mu{}^2+\rho}\equiv\frac{1}{4\pi}\ln\frac{1}{a^2\rho}+C_0'\,.$$

By substituting this expression to Eq. (3·24), we obtain

$$I_0=-\frac{1}{4\pi}(\sigma_c{}^2+\Pi_c{}^2)\ln\frac{a^2(\sigma_c{}^2+\Pi_c{}^2)}{e}+\hat{C}_0(\sigma_c{}^2+\Pi_c{}^2)\,. \tag{3·30}$$

Now, we are ready to discuss the continuum limit of our theory. By retaining only those terms that give non-vanishing contributions when $a\to 0$, we conclude

$$V_{\rm eff}=-\left(\frac{m_0}{g_\sigma^2}+\frac{2r}{a}C_1\right)\sigma_c+\left(\frac{1}{2g_\pi^2}-\hat{C}_0+\frac{1}{4\pi}\ln a^2\right)\Pi_c^2$$

$$+\left(\frac{1}{2g_\sigma^2}-\hat{C}_0+2r^2C_2+\frac{1}{4\pi}\ln a^2\right)\sigma_c^2+\frac{1}{4\pi}(\sigma_c^2+\Pi_c^2)\ln\frac{\sigma_c^2+\Pi_c^2}{e}, \qquad (3\cdot31)$$

where $\hat{C}_0$, C_1 and C_2 are even functions of r. Since we are interested in *a renormalized theory with chiral symmetry,* we choose the a-dependence of g_σ^2, g_π^2 and m_0 as follows:

$$\frac{1}{2g_\sigma^2}=\hat{C}_0-2r^2C_2+\frac{1}{4\pi}\ln\frac{1}{\Lambda^2a^2}, \qquad (3\cdot32)$$

$$(\Lambda a=\exp[-4r^2\pi\cdot C_2+2\pi\cdot\hat{C}_0-\pi/g_\sigma^2])$$

$$\frac{1}{2g_\pi^2}=\hat{C}_0+\frac{1}{4\pi}\ln\frac{1}{\Lambda^2a^2}, \qquad (3\cdot33)$$

$$\frac{m_0}{g_\sigma^2}=-\frac{2r}{a}C_1+m=\frac{\delta m}{g_\sigma^2}+m, \qquad (3\cdot34)$$

where the scale parameter Λ and the mass parameter m should be kept finite in the continuum limit $a\to 0$. With this choice, the renormalized effective potential is given by

$$V_{\rm eff}(\sigma_c,\Pi_c)=-m\sigma_c+\frac{1}{4\pi}(\sigma_c^2+\Pi_c^2)\ln\frac{\sigma_c^2+\Pi_c^2}{e\Lambda^2}. \qquad (3\cdot35)$$

Since this expression is symmetric under continuous chiral transformations when $m=0$, we may interpret m as the mass parameter characterizing the explicit breaking of chiral symmetry. In fact, Eq. (3·34) coincides with the corresponding definition of m in the continuum theory if $r=0$, i.e., in the absence of the Wilson term which breaks chiral symmetry explicitly. When $m=0$, the first term on the right-hand side of Eq. (3·34) represents the term necessary to compensate the chiral symmetry breaking effect due to the Wilson term. It is now obvious why we introduced two coupling constants in our lattice action (3·6). Had we not introduced two coupling constants, the resultant renormalized effective potential would not have chiral symmetry, because of the second term on the right-hand side of Eq. (3·32) which vanishes in the absence of the Wilson term.

By minimizing the effective potential, it is possible to obtain $\langle\sigma_c\rangle$. Since the scale of the physical spectrum is given by $\langle\sigma_c\rangle$, we may call this quantity *the constituent quark mass*; on the other hand, m may be called *the current quark mass.* The relation between the current and constituent quark masses is given by the renormalized gap equation (2·25). In the previous section, we mentioned that these two masses must have the same sign on the ground of the absolute stability of the vacuum (Fig. 2). This relation is also derived from a criterion of local stability: The second derivative of the effective potential at the minimum is related to the pion mass

$$m_\pi^2\propto\frac{1}{2}\frac{\partial^2V_{\rm eff}}{\partial\Pi_c^2}\bigg|_{\Pi_c=0}=\frac{1}{4\pi}\ln\frac{\sigma_c^2}{\Lambda^2}.$$

By using the renormalized gap equation (2·25), we find

$$m_\pi{}^2 \propto \frac{m}{2\sigma_c}\,. \tag{3·36}$$

Because $m_\pi{}^2$ has to be positive, we conclude that m and $\langle\sigma_c\rangle$ must have the same sign.

§ 4. Bare gap equations on a finite lattice and a numerical method

In this section we will investigate bare gap equation on a finite lattice (i.e., lattice spacing a is non-zero). First we will propose a numerical method to obtain a-dependent quantities using bare gap equations. Secondly we will summarize properties of the numerical method which will be used in § 5.

Varying the effective potential (3·10) by σ and Π, we obtain bare gap equations for a finite lattice spacing (simply σ and Π instead of σ_c and Π_c):

$$\frac{\sigma-m_0}{2g_\sigma{}^2}=\int_{-\pi/a}^{\pi/a}\frac{d^2p}{(2\pi)^2}\frac{\{\sigma+r/a\sum_\mu(1-\cos p_\mu a)\}^2a^2}{\sum_\mu\sin^2p_\mu a+\{\sigma a+r\sum_\mu(1-\cos p_\mu a)\}^2+\Pi^2a^2}\,, \tag{4·1}$$

$$\frac{\Pi}{2g_\pi{}^2}=\int_{-\pi/a}^{\pi/a}\frac{d^2p}{(2\pi)^2}\frac{\Pi a^2}{\sum_\mu\sin^2p_\mu a+\{\sigma a+r\sum_\mu(1-\cos p_\mu a)\}^2+\Pi^2a^2}\,. \tag{4·2}$$

Here $g_0{}^2$, $g_\pi{}^2$ and m_0 are bare parameters. On the analogy of the continuum case (§ 2) we set Π equal to zero hereafter. In this case Eq. (4·2) is always satisfied, therefore we will solve only Eq. (4·1).

A numerical method to satisfy renormalization conditions and to obtain a-dependence of physical quantities is as follows.

(i) When $g_\sigma{}^2$ and m_0 are fixed, Eq. (4·1) defines σ as a function of $g_\sigma{}^2$ and m_0. If there are several solutions to Eq. (4·1) we compare the value of V_{eff} for each solution in order to choose the unique solution $\sigma(g_\sigma{}^2, m_0)$ corresponding to the absolute minimum of V_{eff}.

(ii) Varying m_0 with $g_\sigma{}^2$ fixed we plot the value of the order parameter $\langle\bar\psi\psi\rangle$ which is given by

$$-g_\sigma{}^2\langle\bar\psi\psi\rangle/N=\sigma(g_\sigma{}^2, m_0)-m_0\,.$$

At some value of m_0, $\langle\bar\psi\psi\rangle$ may have a gap which is a signal of the first order phase transition. The value of m_0 where $\langle\bar\psi\psi\rangle$ has a gap is the mass counter term necessary to cancel the effect of the Wilson term and denoted $\delta m(g_\sigma{}^2)$. Then a renormalized mass m is defined as the deviation from this transition point:

$$m_0/g_\sigma{}^2=\delta m(g_\sigma{}^2)/g_\sigma{}^2+m\,.$$

Furthermore, the half of this gap defines the value of the order parameter σ_{CL} in the chiral symmetry limit ($m=0$):

$$\sigma_{\text{CL}}\equiv\sigma(g_0{}^2, \delta m(g_\sigma{}^2))\,.$$

(iii) We determine $g_\pi{}^2$ so that π-meson mass vanishes at $m_0=\delta m$. This condition is

$$1/2g_\pi{}^2=\int_{-\pi/a}^{\pi/a}\frac{d^2p}{(2\pi)^2}\frac{1}{S(p, \sigma_{\text{CL}})}\,, \tag{4·3}$$

where $S(p, \sigma_{\text{CL}})=a^{-2}[\sum_\mu\sin^2p_\mu a+\{\sigma_{\text{CL}}a+r\sum_\mu(1-\cos p_\mu a)\}^2]$.

(iv) Varying g_σ^2 we get σ_{CL}, δm and g_π^2 as functions of g_σ^2 and compare these values to the scaling behaviors of σ_{CL}, δm and g_π^2 predicted in § 3.

The result of the numerical study will be given in § 5.

Before ending this section we summarize general properties of numerical method, which will be used in § 5. Hereafter we set $r=1$.

(1) $\int_p 1/S(p,\sigma)=\int_p 1/S(p,\sigma')$,

where $\sigma' a=-4-\sigma a$ and $\int_p$ stands for $\int_{-\pi/a}^{\pi/a}\frac{d^2p}{(2\pi)^2}$.

∵If we make a change of integration variable such that $p_\mu=p_\mu'+\pi/a$, we get

$$\int_p 1/S(p,\sigma)=\int_{p'} 1/S(p'+\pi/a,\sigma')=\int_{p'} 1/S(p',\sigma')$$

(2) $\sigma'-m_0(\sigma')=-(\sigma-m_0(\sigma))$

with g_σ^2 fixed.

∵

$$\sigma'-m_0(\sigma')=2g_\sigma^2\int_{p'}\{\sigma'-1/a\sum_\mu(1-\cos p_\mu' a)\}/S(p',\sigma')$$

$$=2g_\sigma^2\int_p\{-4/a-\sigma+1/a\sum_\mu(1+\cos p_\mu a)\}/S(p,\sigma)$$

$$=-2g_\sigma^2\int_p\{\sigma+1/a\sum_\mu(1-\cos p_\mu a)\}/S(p,\sigma)$$

$$=-(\sigma-m_0(\sigma)) .$$

(3) $m_0(\sigma')=-4/a-m_0(\sigma)$.

∵From (2)

$$-4/a-\sigma-m_0(\sigma')=-\sigma+m_0(\sigma) ,$$

$$m_0(\sigma')=-4/a-m_0(\sigma) .$$

(4) $V_{\rm eff}(\sigma,m_0(\sigma))=V_{\rm eff}(\sigma',m_0(\sigma'))$ with g_σ^2 fixed.

$$∵ V_{\rm eff}(\sigma',m_0(\sigma'))=\{\sigma'-m_0(\sigma')\}^2/2g_\sigma^2-\int_{p'}\log S(p',\sigma')$$

$$=\{\sigma-m_0(\sigma)\}^2/2g_\sigma^2-\int_p\log S(p,\sigma) \quad \text{from (1) and (2)}$$

$$=V_{\rm eff}(\sigma,m_0(\sigma)) .$$

(5) From (1)~(4) the graph of $\sigma-m_0$ vs m_0 is point symmetric at the $(\sigma-m_0, m_0)=(0,-2/a)$.

From fact (5) there is at least one phase transition point at $m_0=-2/a$ if $\sigma\neq\sigma'$ there. (For example see Fig. 3 in § 5.)

§ 5. Results of the numerical calculation and the scaling behavior

In this section we summarize results of the numerical calculation and discuss the scaling behavior of σ_{CL}, δm and g_π^2.

First we plotted $\sigma - m_0 = -g_\sigma^2 \langle \bar{\psi}\psi \rangle / N$ against m_0 by solving Eq. (4·1) numerically to find transition points. There are two cases:

(i) *Strong coupling region*

For $1/g_\sigma^2 \leqq 0.3$ there is only one transition point. A typical graph in this range is given in Fig. 3(a). In this coupling range no separation of the fermion doubling mode occurs.

(ii) *Intermediate and weak coupling regions*

For $1/g_\sigma^2 \geqq 0.4$ there are three transition points. A typical graph in this range is given in Fig. 3(b). Each transition point corresponds to each continuum limit depending on a different region in the momentum space. For example, in Fig. 3(b) point A corresponds to $p = (0, 0)$, point B corresponds to $p = (\pi/a, 0)$ or $(0, \pi/a)$ and point C corresponds to $p = (\pi/a, \pi/a)$. In this coupling range the effect of the Wilson term separates three doubling modes. This separation occurs at $1/g_\sigma^2 \fallingdotseq 0.4$ this is faster than usual case (at $1/g_\sigma^2 \fallingdotseq 1.0$. See Refs. 3) and 7).). Note that our g_σ^2 corresponds to $g_s^2 N$ in these references. The true continuum limit is given by the transition point A.

Secondly we discuss the scaling behaviors. As explained in the last section, the position of the first order phase transition determines $\delta m(g_\sigma^2)$, the mass counter term to cancel the chiral symmetry breaking effect caused by the Wilson term. The half of the gap of the order parameter at this phase transition point determines the value of the genuine order parameter $\sigma_{CL} \equiv \Lambda$ in the chiral symmetry limit ($m_0 = \delta m$, i.e., $m = 0$). Finally, g_π^2 is determined by the massless condition for the pion in this limit. Numerical results for these quantities are shown in Figs. 4(a) ~ (c). These numerical results should be compared with the scaling behaviors in the continuum limit derived in § 3. From

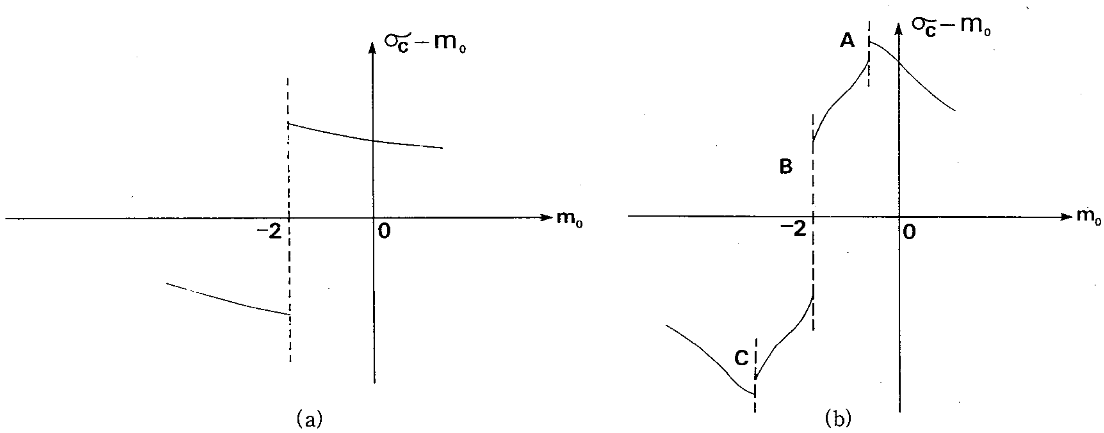
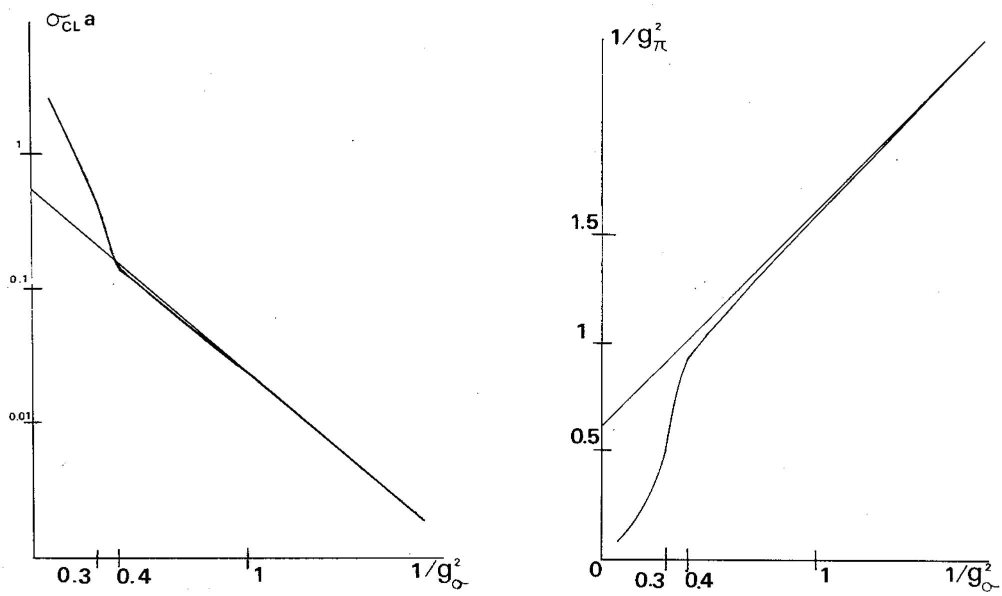
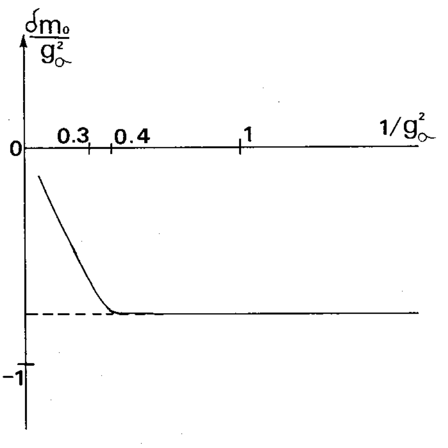


Fig. 3. Dependence of $\sigma_c - m_0$ on m_0.
(a) A typical graph for $1/g_\sigma^2 \leqq 0.3$. There is only one transition point.
(b) A typical graph for $1/g_\sigma^2 \geqq 0.4$. There are three transition points A, B and C. True continuum limit corresponds to point A.



(a) (b)



(c)

Fig. 4. (a) Dependence of $\sigma_{CL}a$ on $1/g_\sigma^2$. Straight line represents the scaling behavior of $\sigma_{CL}a$: $\sigma_{CL}a=0.57\exp[-\pi/g_\sigma^2]$.
(b) Dependence of $1/g_\pi^2$ on $1/g_\sigma^2$. Straight line represents the scaling behavior of $1/g_\pi^2$: $1/g_\pi^2=1/g_\sigma^2+0.617$.
(c) Dependence of $\delta ma/g_\sigma^2$ on $1/g_\sigma^2$. Straight line represents the scaling behavior of $\delta ma/g_\sigma^2$: $\delta ma/g_\sigma^2=-0.769$.

Eqs. (3·32)~(3·34) the scaling behavior of σ_{CL}, δm and g_π^2 is given by

$$(C_0=0.427,\ \hat{C}_0=0.627,\ C_1=0.385,\ C_2=0.155)$$

$$\Lambda a\equiv\sigma a=0.57\exp[-\pi/g_\sigma^2]\,, \qquad (\text{for } m=0)$$

$$\delta ma=-0.769\ g_\sigma^2\,,$$

$$1/g_\pi^2=1/g_\sigma^2+0.617\,.$$

From these results we see that the scaling behavior of σ_{CL}, δm and g_π^2 are good for $1/g_\sigma^2 \geqq 0.4$ and are better than usual one bare coupling case. (See Refs. 3) and 7).) It should be noted that $-g_\sigma^2\langle\bar{\psi}\psi\rangle$ itself does not follow a simple scaling law although the magnitude of the gap $\sigma_{CL}=\Lambda$, the genuine order parameter, follows the simple scaling law. The behavior of this quantity is rather complicated by the presence of the mass counter term:

$$-g_\sigma^2\langle\bar\psi\psi\rangle/N=\sigma-m_0$$

$$=\begin{cases}-\delta m+(\pi-g_\sigma^2)m+\Lambda\,, & (m>0,\ \text{small})\\ -\delta m+(\pi+g_\sigma^2)m-\Lambda\,. & (m<0,\ \text{small})\end{cases}$$

In Ref. 7) one of us proposed a new method to improve the order parameter $-g_\sigma^2\langle\bar\psi\psi\rangle/N$. Our idea is as follows. The mass counter term, which violates the scaling behavior of $-g_\sigma^2\langle\bar\psi\psi\rangle/N$ is an odd function of the Wilson parameter r, therefore if we define

$$\langle\bar\psi\psi\rangle_q=\textstyle\sum_{r=\pm1}\langle\bar\psi\psi\rangle_r/2\,,$$

the effect of δm may be dropped and the scaling behavior of $\langle\psi\psi\rangle_q$ may become simpler. ($\langle\ \cdot\ \rangle_r$ represents the expectation value with the Wilson parameter r and $\langle\ \cdot\ \rangle_q$ is called the "quenched average".) This idea was applied to the usual bare-coupling GN model and $\langle\bar\psi\psi\rangle_q$ has, indeed, better scaling behavior.[7]

Now we apply this quenched average to two-couplings GN model. From (4·1)

$$\frac{\sigma(r)a-m_0(\sigma(r))a}{2g_\sigma^2}=\int_p\frac{\sigma(r)a}{S(p,\sigma(r),r)}+r\int_p\frac{\sum_\mu(1-\cos p_\mu a)}{S(p,\sigma(r),r)}$$

$$=\int_{p'}\frac{\sigma(r)a+4r}{S(p',\sigma(r)+4r/a,-r)}-r\int_{p'}\frac{\sum_\mu(1-\cos_\mu p'a)}{S(p',\sigma(r)+4r/a,-r)}\,, \tag{5·1}$$

where

$$a^2S(p,\sigma,r)=\sum_\mu\sin^2p_\mu a+\{\sigma a+r\sum_\mu(1-\cos p_\mu a)\}^2\,,$$

$$p_\mu'=p_\mu+\pi/a$$

then we obtain

$$\tilde\sigma(-r)-m_0(\tilde\sigma(-r))=\sigma(r)-m_0(\sigma(r))\,, \tag{5·2}$$

where

$$\tilde\sigma(-r)=\sigma(r)+4r/a\,. \tag{5·3}$$

Furthermore we obtain

$$m_0(\tilde\sigma(-r))=m_0(\sigma(r))+4r/a\,. \tag{5·4}$$

(5·2) and (5·4) show that the graph of $\sigma-m_0$ vs m_0 for $r=-1$ is the same as the graph for $r=1$ if we shift $m_0\to m_0+4/a$. The true continuum limit for $r=1$ is point A in Fig. 3(b) but the true continuum limit for $r=-1$ is corresponding to point C when we shift $m_0\to m_0+4/a$. We define

$$\sigma_{\rm CL}(1,+)=\lim_{m_0\to\delta m+}\sigma(1)\,,$$

$$\sigma_{\rm CL}(1,-)=\lim_{m_0\to\delta m-}\sigma(1)\,. \tag{5·5}$$

From (5·3), (5·5) and Fig. 3(b) we obtain

$$\sigma_{\rm CL}(-1,+)=\sigma'_{\rm CL}(1,-)+4/a=-\sigma_{\rm CL}(1,-)\,. \tag{5·6}$$

(Notice that $\sigma'=-4/a-\sigma$. See § 4.) Furthermore from (5·4) and fact (3) in § 4 we obtain

$$\delta m(\sigma_{\rm CL}(-1,+))=\delta m(\sigma'_{\rm CL}(1,-))+4/a=-\delta m(\sigma_{\rm CL}(1,-))\,. \tag{5·7}$$

Finally the quenched average of $-g_\sigma{}^2\bar{\psi}\psi/N$ in the chiral limit is calculated as

$$\begin{aligned}-g_\sigma{}^2\langle\bar{\psi}\psi\rangle/N&=-g_\sigma{}^2\sum_{r=\pm1}\lim_{m_0\to\delta m(r)+}\langle\bar{\psi}\psi\rangle_r/2N\\&=\sum_{r=\pm1}\lim_{m_0\to\delta m(r)+}\{\sigma(r)-m_0(r)\}/2\\&=\{\sigma_{\rm CL}(1,+)-\delta m(1,+)+\sigma_{\rm CL}(-1,+)-\delta m(-1,+)\}/2\\&=\{\sigma_{\rm CL}(1,+)-\sigma_{\rm CL}(1,-)\}/2\,,\end{aligned} \tag{5·8}$$

where we use the fact that $\delta m(1,+)=\delta m(1,-)$. (5·8) shows that the effect of δm, which violates the simple scaling behavior of $-g_\sigma{}^2\langle\bar{\psi}\psi\rangle_r/N$, disappear in the quenched average. In Fig. 5 we plotted $-g_\sigma{}^2\langle\bar{\psi}\psi\rangle_q/N$. Figure 5 shows that the scaling begins at $1/g_\sigma{}^2\approx0.4$ where the separation of the fermion doubling occurs. This separation makes $-g_\sigma{}^2\langle\bar{\psi}\psi\rangle_q/N$ jump suddenly at this point. In this case both the separation of the doubling and the scaling of the order parameter occurs at the same value of $1/g_\sigma{}^2$. In other words in the coupling region where the doubling mode is negligible the quenched average of the chiral order parameter scales.

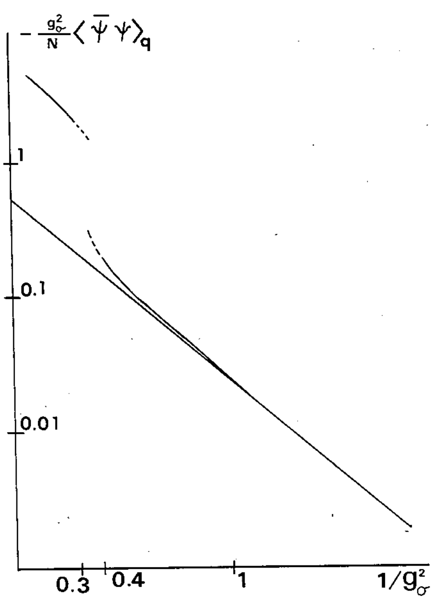


Fig. 5. Dependence of $-g_\sigma{}^2\langle\bar{\psi}\psi\rangle_q/N$ on $1/g_\sigma{}^2$. Straight line represents the scaling behavior of $-g_\sigma{}^2\langle\bar{\psi}\psi\rangle_q/N$:
$$-\frac{g_\sigma{}^2}{N}\langle\bar{\psi}\psi\rangle_q=0.57\exp[-\pi/g_\sigma{}^2].$$

§ 6. Conclusions and discussion

Contrary to our naive expectation the full chiral symmetry cannot be restored even in the continuum limit if we use the one bare-coupling GN model with the Wilson fermion. We have to introduce two bare-couplings $1/g_\sigma{}^2$ and $1/g_\pi{}^2$ and adjust them in order to obtain the chiral symmetric effective potential in the continuum limit. In other words in order to obtain the chiral symmetry as the result we must start from the chiral non-symmetric action which includes the bare mass, the Wilson term and the interaction with two bare-couplings.

The bare mass term to compensate the effects of the Wilson term is chosen so as to recover the *discrete* chiral symmetry ($\sigma\to-\sigma$), whereas the bare coupling constant g_π is chosen so as to recover the *continuous* chiral symmetry (the rotational symmetry in σ-Π space). The recovery of the discrete chiral symmetry is indicated by the existence of the first order phase transition when the bare mass m_0 varies. The genuine order parameter $\sigma_{\rm CL}$, the discontinuity of the naive order parameter $-g_\sigma^2\langle\bar\psi\psi\rangle/N$ follows a simple scaling low in the continuum limit. From the results of § 5 we can conclude that the separation of the fermion doubling in our two couplings model occurs at the value $1/g_\sigma^2=0.3\sim0.4$ which is smaller than the usual value $1/g_\sigma^2\approx1.0$ and that scalings of $\sigma_{\rm CL}$, δm and g_π^2 begin at the same value of $1/g_\sigma^2$.

Our main interest, of course, is the chiral property of the QCD. There is no room to introduce two bare-couplings of the gauge interaction in QCD. We expect that the chiral symmetry of the lattice QCD is restored in the continuum limit by simply introducing the bare mass term. In perturbative theory, this mass counter term is chosen so as to cancel the explicit breaking of chiral symmetry due to the Wilson term. In non-perturbative domain, however, there is no definite criterion for the choice of this mass counter term. Usually, this mass counter term is fixed by the massless condition for the pion. For finite lattice spacing, however, the existence of the massless pion does not mean the recovery of the chiral symmetry. Furthermore, it is difficult to extract the genuine order parameter out of the naive order parameter $\langle\bar\psi\psi\rangle$ of the chiral symmetry. On the other hand, if it is possible to find the existence of the first order phase transition when m_0 is varied, we can fix the mass counter term by its location and extract the genuine order parameter from the magnitude of the gap. Of course, this method does not guarantee the recovery of the continuous chiral symmetry although the first order phase transition is certainly related to the recovery of the discrete chiral symmetry. We leave the comparison of these two methods to fix the mass counter term to future work.

Appendix

In this appendix, we discuss the recovery of chiral symmetry in the continuum Gross-Neveu model when we adopted a chiral non-invariant regularization, a simple analog in continuum theory of Wilson's formulation of the lattice Gross-Neveu model.

Our Lagrangian is

$$L=\bar\psi(i\partial\!\!\!/-m_0)\psi-\frac{r}{M}\bar\psi\Box\psi+\frac{1}{2N}\{g_\sigma^2(\bar\psi\psi)^2+g_\pi^2(\bar\psi i\gamma_5\psi)^2\}\,, \qquad (\mathrm{A}\cdot1)$$

where the second term, a continuum analog of the Wilson term, breaks chiral symmetry even if $m_0=0$. It is proportional to $1/M$, M being the ultraviolet cutoff in momentum space. The reason we have introduced two coupling constants will become clear later. It is convenient to introduce auxilliary fields σ and Π, then (A·1) can be rewritten as

$$L=\bar\psi\left\{i\partial\!\!\!/-\left(\sigma+i\gamma_5\Pi+r\frac{\Box}{M}\right)\right\}\psi-N\left\{\frac{1}{2g_\sigma^2}(\sigma-m_0)^2+\frac{1}{2g_\pi^2}\Pi^2\right\}. \qquad (\mathrm{A}\cdot2)$$

As we have done in § 3, it is straightforward to obtain the effective potential in the large N limit:

$$V_{\rm eff}=\frac{1}{2g_\sigma^2}(\sigma_c-m_0)^2+\frac{1}{2g_\pi^2}\Pi_c^2-\int\frac{d^2k}{(2\pi)^2}\ln\left[\left(\sigma_c+\frac{rk^2}{M}\right)^2+\Pi_c^2+k^2\right], \qquad (\mathrm{A}\cdot3)$$

where the domain of integration is restricted to $k^2\leqq M^2$. Contrary to the lattice regularization, it is possible to perform the momentum integration analytically. Neglecting terms of order $1/M$ when the cutoff M tends to infinity, we find

$$V_{\rm eff}(\sigma_c, \Pi_c) = -\sigma_c\left(\frac{m_0}{g_\sigma^2} + \frac{M}{2\pi r}\ln(1+r^2)\right)$$

$$+\sigma_c^2\left(\frac{1}{2g_\sigma^2} - \frac{1}{4\pi}\ln\frac{M^2}{1+r^2} + \frac{1}{2\pi}\cdot\frac{r^2}{1+r^2}\right) + \Pi_c^2\left(\frac{1}{2g_\pi^2} - \frac{1}{4\pi}\ln\frac{M^2}{1+r^2}\right)$$

$$+\frac{1}{4\pi}(\sigma_c^2 + \Pi_c^2)\ln\frac{\sigma_c^2+\Pi_c^2}{e}. \qquad (\text{A}\cdot 4)$$

In order to have a renormalized theory with chiral symmetry, we choose the cutoff dependence of bare quantities m_0, g_σ^2 and g_π^2 as

$$m = \frac{m_0}{g_\sigma^2} + \frac{M}{2\pi r}\ln(1+r^2)\,, \qquad (\text{A}\cdot 5)$$

$$\frac{1}{2g_\sigma^2} = \frac{1}{4\pi}\ln\frac{M^2}{(1+r^2)\Lambda^2} - \frac{1}{2\pi}\frac{r^2}{1+r^2}\,, \qquad (\text{A}\cdot 6)$$

$$\frac{1}{2g_\pi^2} = \frac{1}{4\pi}\ln\frac{M^2}{(1+r^2)\Lambda^2}\,, \qquad (\text{A}\cdot 7)$$

where the renormalized scale parameter and the mass parameter m should be kept fixed in the limit $M\to\infty$. With this choice of bare quantities, we obtain

$$V_{\rm eff}(\sigma_c, \Pi_c) = -m\sigma_c + \frac{1}{4\pi}(\sigma_c^2+\Pi_c^2)\ln\frac{\sigma_c^2+\Pi_c^2}{e\Lambda^2}. \qquad (\text{A}\cdot 8)$$

It is now clear why we introduced two coupling constants in the bare Lagrangian: If we had introduced just one coupling constant $g^2 = g_\sigma^2 = g_\pi^2$ then the quadratic terms in Eq. (A·4) would not have rotational symmetry in (σ_c, Π_c) plane.

Finally, we note that bare coupling constants g_σ^2 and g_π^2 are even functions of r whereas the second term in Eq. (A·5), the bare mass term necessary to cancel the effects of chiral non-invariant reguralization, is an odd function of r.

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