Bull Braz Math Soc, New Series (2018) 49:369–394
https://doi.org/10.1007/s00574-017-0058-4

The Reeb Graph of a Map Germ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with Non Isolated Zeros

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Received: 24 November 2016 / Accepted: 6 November 2017 / Published online: 16 November 2017

Abstract We consider the topological classification of finitely determined map germs $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ with $f^{-1}(0) \neq \{0\}$. The case $f^{-1}(0) = \{0\}$ was treated in another recent paper by the authors. The main tool used to describe the topological type is the link of $[f]$, which is obtained by taking the intersection of its image with a small sphere S^1_δ centered at the origin. The link is a stable map $\gamma_f : N \to S^1$, where N is diffeomorphic to a sphere S^2 minus $2L$ disks. We define a complete topological invariant called the generalized Reeb graph. Finally, we apply our results to give a topological description of some map germs with Boardman symbol $\Sigma^{2,1}$.

Keywords Topological equivalence · Classification · Link · Reeb graph

Erica Boizan and Juan J. Nuño-Ballesteros have been partially supported by CAPES-PVE Grant 88881.062217/2014-01. The João Carlos Ferreira Costa has been partially supported by CAPES and FAPESP. The Juan J. Nuño-Ballesteros has been partially supported by DGICYT Grant MTM2015–64013–P.

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Mathematics Subject Classification 58K15 · 58K40 · 58K65

1 Introduction

Since our study is local we will work with the notion of map germs from $\mathbb{R}^3$ to $\mathbb{R}^2$. Map germs are equivalence classes of mappings which coincide in a neighborhood of the origin in $\mathbb{R}^3$. We will denote it by $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$. Given a map germ $[f]$ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with non isolated zeros, we can take a representative f of $[f]$, where $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$, U is a neighborhood at the origin and $f^{-1}(0) \neq \{0\}$.

This paper is devoted to the topological classification of smooth map germs from $\mathbb{R}^3$ to $\mathbb{R}^2$ with non isolated zeros which are finitely determined. The hypothesis of finite determinacy guarantees that our map germs can be assumed polynomials. Restricted to polynomial map germs from $\mathbb{R}^3$ to $\mathbb{R}^2$ of a given degree, it follows from Thom's work (Thom 1964) that the number of topological types is finite. In other words, this problem is tame in the sense that it does not have topological moduli.

The topological structure of a finitely determined map germ $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is given by the so-called link of $[f]$ (cf. Fukuda 1981, 1985). The link of $[f]$ is obtained by taking a small enough representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ and the intersection of its image with a small enough sphere S^1_δ centered at the origin in $\mathbb{R}^2$. When $[f]$ has isolated zeros (i.e., $f^{-1}(0) = \{0\}$, where f is a representative of $[f]$), the link is a stable map $\gamma_f : S^2 \to S^1$ and f is topologically equivalent to the cone of γ_f. As a consequence, two finitely determined map germs $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ with isolated zeros are topologically equivalent if their associated links are topologically equivalent. A complete description of topological classification of this case was recently studied in Batista et al. (2017).

In this paper, we assume $[f]$ has non isolated zeros (i.e., $f^{-1}(0) \neq \{0\}$). This case is more complicated than the previous one, because the link is now a stable map $\gamma_f : N \to S^1$, where N is a compact surface with boundary and genus zero, diffeomorphic to S^2 minus $2L$ disks.

Inspired in the works of Arnold, Prishlyak or Sharko (cf. Arnold 2007; Prishlyak 2002; Sharko 2003) we introduced in Batista et al. (2017) the notion of generalized Reeb graph and showed that it turns out to be a complete topological invariant for stable maps from S^2 to S^1. Here, we extend the results of generalized Reeb graph for a stable map $\gamma : N \to S^1$, where N is a manifold with boundary. In Sect. 4 we show that it is also a complete topological invariant, as it happens in the case of surfaces without boundary (cf. Batista et al. 2017).

In Sect. 5 we take special attention to the case that $[f]$ has corank 1. In this case, there exists a representative f written as $f(x, y, z) = (x, h_x(y, z))$ and gives a stabilization of $[h_0] : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The topology of f is now determined by two stabilizations h_x^+, with $x > 0$ and h_x^-, with $x < 0$. We introduce the notion of partial trees associated to h_x^+ and h_x^- and show that the sum of these partial trees is equivalent to the Reeb graph of the link of $[f]$. In the last part of this paper, we apply our results to obtain the topological description of some map germs with Boardman symbol $\Sigma^{2,1}$.

It is important to cite that recently many papers treat that the problem of topological classification of finitely determined map germs $[f] : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by looking at the topological type of the link (cf. Costa and Nuño-Ballesteros 2013; Moya-Pérez and Nuño-Ballesteros 2010, 2014, 2015). However, as far as we know, this paper is the first one which considers the case of non isolated zeros.

All map germs considered here are smooth (C^∞) except otherwise stated. We adopt the usual notation and basic definitions that are common in Singularity theory (e.g., $\mathcal{A}$-equivalence, finite determinacy, stability, etc.) as the readers can find in Wall's survey paper (Wall 1981).

2 The Link of a Finite Determined Map Germ

Definition 2.1 Two smooth map germs $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ are said to be *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $[\psi] : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ and $[\phi] : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $[f] = [\phi] \circ [g] \circ [\psi]^{-1}$. If $[\phi], [\psi]$ are homeomorphism germs instead of diffeomorphism germs, then we say that $[f]$ and $[g]$ are *topologically equivalent* (or *$C^0 - \mathcal{A}$-equivalent*).

Notice that notions of $\mathcal{A}$-equivalence and topological equivalence given in Definition 2.1 also can be applied for mappings (not germs) from $\mathbb{R}^3$ to $\mathbb{R}^2$ in a analogous way. Just replace the words "diffeomorphism germs" (resp. homeomorphism germs) by "diffeomorphisms" (resp. homeomorphisms).

A crucial notion in Singularity Theory is finite determinacy. In fact, if a map germ $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is finite determined, it may be assumed polynomial.

Definition 2.2 A map germ $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is said to be *k-determined* if for any map germ $[g]$ with the same k-jet, $[g]$ is $\mathcal{A}$-equivalent to $[f]$. The germ $[f]$ is said to be *finitely determined* if it is k-determined for some k.

Let $f : U \to \mathbb{R}^2$ be a smooth map, where $U \subset \mathbb{R}^3$ is an open subset. We denote by $S(f) = \{p \in U \mid Jf(p) \text{ does not have rank } 2\}$ the *singular set* of f, where $Jf(p)$ is the Jacobian matrix of f. We also denote the *discriminant set* of f by $\Delta(f) = f(S(f))$. Another important set is $X(f) = \overline{f^{-1}(\Delta(f)) - S(f)}$.

Definition 2.3 Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ be a smooth map and $p \in S(f)$.

1. We say that p is a *definite fold point* if the map germ of f at p is $\mathcal{A}$-equivalent to $(x, y^2 + z^2)$;
2. We say that p is a *indefinite fold point* if the map germ of f at p is $\mathcal{A}$-equivalent to $(x, y^2 - z^2)$.

The next corollary follows from the Mather-Gaffney finite determinacy criterion (Wall 1981), and the well known classification of stable singularities from $\mathbb{R}^3$ to $\mathbb{R}^2$ (see Batista et al. 2017 for details).

Corollary 2.4 *Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there exists a representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ such that*

(i) $S(f) \cap f^{-1}(0) = \{0\}$,
(ii) *the restriction $f|U \setminus \{0\}$ has only definite and indefinite simple fold singularities.*

If $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is a finitely determined map germ, the discriminant $\Delta(f)$ is a plane curve with an isolated singularity at the origin. The number of half branches of $\Delta(f)$ will play a crucial role in the analysis of the topological behavior of f. In addition, when $f^{-1}(0) \neq \{0\}$, this set $f^{-1}(0)$ is also a (space) curve with an even number of half branches.

Denote by $J^r(n, p)$ the r-jet space from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^p, 0)$. For positive integers r and s with $s \geq r$, let $\pi^s_r : J^s(n, p) \to J^r(n, p)$ be the canonical projection defined by $\pi^s_r(j^s f(0)) = j^r f(0)$. For a positive number $\epsilon > 0$ we set

$$\begin{aligned} D^n_\epsilon &= \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \epsilon\}, \quad B^n_\epsilon \\ &= \{x \in \mathbb{R}^n \mid \|x\|^2 < \epsilon\} \quad \text{and} \quad S^{n-1}_\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 = \epsilon\}. \end{aligned}$$

We denote D^n, B^n and S^{n-1} the standard disk, ball and sphere of radius 1, respectively. Fukuda has proved the following theorem in Fukuda (1985):

Theorem 2.5 *For any semialgebraic subset W of $J^r(n, p)$, with $n > p$, there exist an integer s $(s \geq r)$ depending only on n, p and r, and there exists a closed semialgebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ map $f : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s f(0)$ belonging to $(\pi^s_r)^{-1}(W) \setminus \Sigma_W$ and with $f^{-1}(0) \neq \{0\}$ there exist a positive number ϵ_0 and a strictly increasing C^∞ function $\delta : [0, \epsilon_0] \to [0, \infty)$ with $\delta(0) = 0$ such that for every ϵ and δ with $0 < \epsilon \leq \epsilon_0$ and $0 < \delta \leq \delta(\epsilon)$ we have:*

(i) $f^{-1}(0) \cap S^{n-1}_\epsilon$ is an $(n - p - 1)$-dimensional manifold and it is diffeomorphic to $f^{-1}(0) \cap S^{n-1}_{\epsilon_0}$.

(ii) $N_{\epsilon,\delta} := D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ is a C^∞ manifold, in general with boundary and it is diffeomorphic to $N_{\epsilon_0,\delta(\epsilon_0)}$.

(iii) the restriction $f_{\epsilon,\delta} = f|N_{\epsilon,\delta} : N_{\epsilon,\delta} \to S^{p-1}_\delta$ is a topologically stable map (C^∞ stable if (n, p) is a nice pair in Mather's sense) and its topological class is independent of ϵ and δ.

Assuming that $[f]$ is r-determined for some r and taking $W = \{j^r f(0)\}$, we can apply Theorem 2.5 to obtain a representative f of $[f]$ satisfying (i), (ii) and (iii), up to $\mathcal{A}$-equivalence.

Definition 2.6 Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ with $f^{-1}(0) \neq \{0\}$. We say that the stable map $f|N_{\epsilon,\delta} : N_{\epsilon,\delta} \to S^1_\delta$ is the *link* of $[f]$, where $0 < \delta \ll \epsilon \ll 1$ are given in Theorem 2.5.

By Theorem 2.5 the link is well defined up to $\mathcal{A}$-equivalence. However we do not have a cone structure as in the case $f^{-1}(0) = \{0\}$ given in Batista et al. (2017). The topology of the domain of the link can be described easily.

Proposition 2.7 *The manifold $N_{\epsilon,\delta}$ is homeomorphic to S^2 minus $2L$ disks, where $2L$ be the number of half branches of $f^{-1}(0) \neq \{0\}$.*

Proof Let $M = D^3_\epsilon \cap f^{-1}(D^2_\delta)$. Since M is a contractible 3-manifold with boundary, it is homeomorphic to the standard disk D^3. Hence ∂M is homeomorphic to standard sphere S^2. On the other hand, ∂M is equal to the union of $N_{\epsilon,\delta}$ with $2L$ disks. As a consequence, $N_{\epsilon,\delta}$ is homeomorphic to S^2 minus $2L$ disks.

In a recent paper (Batista et al. 2017) we have obtained a cone structure theorem for map germs with $f^{-1}(0) \neq \{0\}$. In order to state this theorem we need the definitions of link diagram and generalized cone.

Definition 2.8 A *link diagram* is a diagram of the form

$$V \xleftarrow{r} N \xrightarrow{\gamma} S^{p-1}$$

where N is a manifold with boundary, γ is a continuous map, V is a contractible space and r is a continuous surjective mapping such that the attaching space $(N \times I) \cup_r V$ is homeomorphic to the closed disk D^n (here we set $I = [0, 1]$ and we identify $N \equiv N \times \{0\} \subset N \times I$).

Definition 2.9 Given a link diagram $V \xleftarrow{r} N \xrightarrow{\gamma} S^{p-1}$ the *generalized cone* is the induced map

$$C(\gamma, r) : (N \times I) \cup_r V \to c(S^{p-1})$$

defined in the obvious way (that is, $[x, t] \mapsto [\gamma(x), t]$ if $(x, t) \in N \times I$ and $[y] \mapsto 0$ if $y \in V$, where $c(S^{p-1})$ is the usual cone of S^{p-1}).

Note that in the particular case that $V = \{0\}$, the generalized cone coincides with the usual notion of cone.

Definition 2.10 Two link diagrams

$$V_0 \xleftarrow{r_0} N_0 \xrightarrow{\gamma_0} S^{p-1} \quad \textit{and} \quad V_1 \xleftarrow{r_1} N_1 \xrightarrow{\gamma_1} S^{p-1}$$

are *topologically equivalent* if there are homeomorphisms $\alpha : V_0 \to V_1, \phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and $\gamma_1 = \psi \circ \gamma_0 \circ \phi^{-1}$.

Theorem 2.11 (Batista et al. 2017) *Let* $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *be a finitely determined map germ such that* $f^{-1}(0) \neq \{0\}$. *For each* $0 < \delta \ll \epsilon \ll 1$ *small enough there exists a continuous and surjective map* $r_{\epsilon,\delta} : N_{\epsilon,\delta} \to V_\epsilon$, *such that:*

1. *The link diagram*

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^1_\delta$$

   is independent of ϵ, δ *up to topological equivalence.*
2. *The restriction* $f|D^3_\epsilon \cap f^{-1}(D^2_\delta) : D^3_\epsilon \cap f^{-1}(D^2_\delta) \to D^2_\delta$ *is topologically equivalent to the generalized cone:*

$$C(f_{\epsilon,\delta}, r_{\epsilon,\delta}) : (N_{\epsilon,\delta} \times I) \cup_{r_{\epsilon,\delta}} V_\epsilon \to c(S^1_\delta),$$

   where $I = [0, \delta]$.

Definition 2.12 Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ with $f^{-1}(0) \neq \{0\}$. The *link diagram of* $[f]$ is the link diagram

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^1_\delta$$

given in Theorem 2.11 for $0 < \delta \ll \epsilon \ll 1$.

Corollary 2.13 *Let* $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *be two finitely determined map germs, with* $f^{-1}(0) \neq \{0\}$ *and* $g^{-1}(0) \neq \{0\}$. *If their associated link diagrams are topologically equivalent then* $[f]$ *and* $[g]$ *are topologically equivalent.*

The proof of Theorem 2.11 is based on the integration of stratified vector fields with respect to the stratification by stable types. In the case $n = 3$ and $p = 2$ the stratification of a representative $f : U \to W$ by stable types $(\mathcal{N}, \mathcal{M})$ is given by:

$$\mathcal{N} = \left\{U \setminus f^{-1}(\Delta(f)), f^{-1}(\Delta(f)) \setminus (S(f) \cup f^{-1}(0)), S(f) \setminus \{0\}, f^{-1}(0) \setminus \{0\}, \{0\}\right\},$$
$$\mathcal{M} = \{W \setminus \Delta(f), \Delta(f) \setminus \{0\}, \{0\}\}.$$

Theorem 2.14 *Let* $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *be two finitely determined map germs, with* $f^{-1}(0) \neq \{0\}$ *and* $g^{-1}(0) \neq \{0\}$. *If their links are topologically equivalent then their link diagrams are also topologically equivalent.*

Proof Let $\gamma_1 : N_{\epsilon,\delta} \to S^1_\delta$ and $\gamma_2 : M_{\epsilon,\delta} \to S^1_\delta$ be the links of $[f]$ and $[g]$, respectively. For simplicity we will put $N_1 = N_{\epsilon,\delta}$ and $N_2 = M_{\epsilon,\delta}$. We also write $V_1 = f^{-1}(0)$, $V_2 = g^{-1}(0)$, and $r_1 : N_1 \to V_1$, $r_2 : N_2 \to V_2$ for the maps given in Theorem 2.11.

Since γ_1 and γ_2 are topologically equivalent, N_1 is homeomorphic to N_2. Thus, N_1 and N_2 have the same number of boundary components, and consequently V_1 is homeomorphic to V_2.

For each $v \in S^1_\delta$, the stratification by stable types of γ_1 induces a stratification $\mathcal{N}_v$ on $\gamma_1^{-1}(v)$. Since r_1 is a regular map when restricted to $N_1 \setminus r_1^{-1}(0)$, each stratum of $\mathcal{N}_v$ that intersects ∂N_1 is diffeomorphically mapped by r_1 in to the half branch of V_1 corresponding to the boundary component intersected by the stratum.

On the other hand, the stratum of $\mathcal{N}_v$ that does not intersect ∂N_1 is either a closed curve, a critical point of γ_1 or a curve whose union with a singular point of γ_1 gives a closed curve. Notice that $S(f)$ is a simply connected curve, then r_1 maps each critical point of γ_1 to the origin. Moreover, since V_1 is simply connected and r_1 is regular when restricted to $N_1 \setminus r_1^{-1}(0)$, r_1 maps every point of the connected components that do not intersect ∂N_1 to the origin. Therefore, r_1 is completely determined by the stratification of N_1.

Since γ_1 and γ_2 are topologically equivalent there exist homeomorphisms $\phi : N_1 \to N_2$ and $\psi : S^1_\delta \to S^1_\delta$ such that $\gamma_2 = \psi \circ \gamma_1 \circ \phi^{-1}$. But ϕ must preserve the stratification by stable types of each level curve. By using the above comments, we can construct another homeomorphism $\alpha : V_1 \to V_2$ such that $r_2 = \alpha \circ r_1 \circ \phi^{-1}$. Hence the link diagrams are topologically equivalent. □

Corollary 2.15 *Let* $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *be two finitely determined map germs, with* $f^{-1}(0) \neq \{0\}$ *and* $g^{-1}(0) \neq \{0\}$. *If their links are topologically equivalent then* $[f]$ *and* $[g]$ *are topologically equivalent.*

3 The Generalized Reeb Graph

The Reeb graph was introduced by Reeb (1946) and it is well known that it is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$ (see Arnold 2007; Sharko 2003). In Batista et al. (2017) we extended the concept of Reeb graph for stable maps from S^2 to S^1. In this section, again we will extend the concept of Reeb graph, but now for stable maps $\gamma : N \to S^1$, where N is a connected and compact surface with boundary.

Let N be a connected and compact surface with boundary ∂N (at first, also we can consider the case $\partial N = \emptyset$). We recall that a smooth map $\gamma : N \to S^1$ is stable if:

1. γ is a Morse function with distinct critical values;
2. γ has no critical points in ∂N;
3. $\gamma|_{\partial N}$ is regular.

Remark 3.1 1. If N is homeomorphic to a sphere S^2 minus $2L$ disks then we have that $\gamma|C_i : C_i \to S^1$ is a diffeomorphism, where C_i's are the connected components of ∂N, $i = 1, \ldots, 2L$.
2. The level curves of γ intersect ∂N transversely.

Let $\gamma : N \to S^1$ be a stable map. Consider the following equivalence relation on N:

$x \sim y \Leftrightarrow \gamma(x) = \gamma(y)$ and x and y belong in the same connected component of $\gamma^{-1}(\gamma(x))$.

Proposition 3.2 *If N is a connected compact surface with boundary, and $\gamma : N \to S^1$ is a stable map. Then the quotient space $N/\sim$ admits a graph structure as follows:*

1. *The vertices are the connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a critical value;*
2. *Each edge is formed by points that correspond to connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a regular value.*

Proof Since γ is stable its critical points are isolated and N being compact, γ has a finite number of critical points. Moreover, N connected implies $N/\sim$ connected.

Let $v_1, \ldots, v_r$ be the critical values of γ. Then,

$$\gamma|N - \gamma^{-1}(\{v_1, \ldots, v_r\}) : N - \gamma^{-1}(\{v_1, \ldots, v_r\}) \to S^1 - \{v_1, \ldots, v_r\}$$

is regular, and the induced map

$$\tilde{\gamma} : (N - \gamma^{-1}(\{v_1, \ldots, v_r\}))/\sim \to S^1 - \{v_1, \ldots, v_r\}$$

is a local homeomorphism. Each connected component of $S^1 - \{v_1, \ldots, v_r\}$ is homeomorphic to an open interval, so each connected component of $(N - \gamma^{-1}(\{v_1, \ldots, v_r\}))/\sim$ is also homeomorphic to an open interval. □

All possibilities for the topological types of level curves of γ are given in Fig. 1.

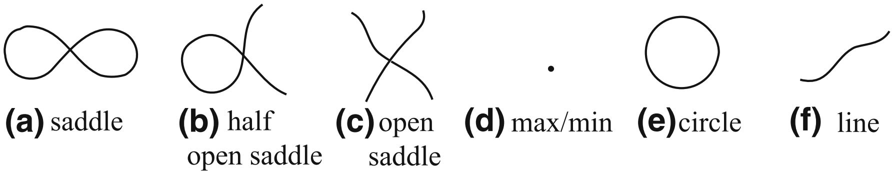


Fig. 1 Level curves

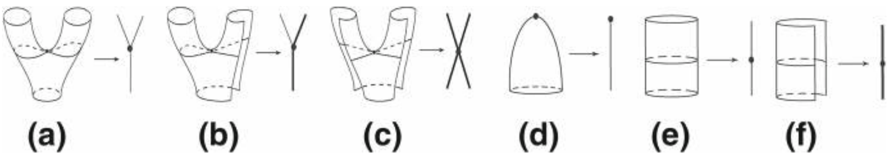


Fig. 2 Incidence rules

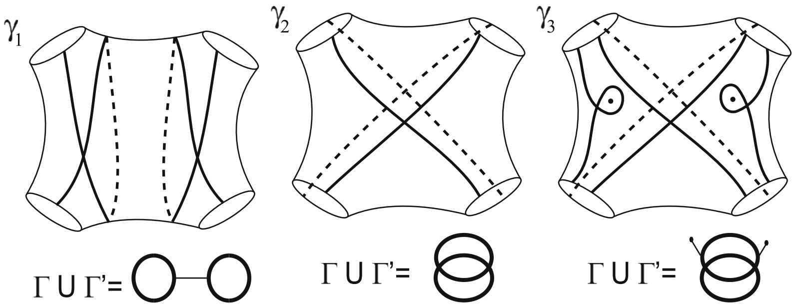


Fig. 3 Graphs $N/\sim$ for stable maps

By Remark 3.1 item (2), the level curves of γ that can intersect ∂N are only (d), (e) and (f) types. Furthermore, by item (1), each level curve of γ can intersect at most once a connected component C_i of ∂N, and these intersections happen in regular points.

By Proposition 3.2 we can associate a graph to $N/\sim$, which will be denoted by Γ_γ. Each edge of Γ_γ can be of two types: one corresponds to connected components of circle type and will be denoted by a slim trace; another corresponds to connected components of interval type and will be denoted by a bold trace. We denote by Γ the subgraph of Γ_γ given by the slim edges with their respective vertices, and by Γ' the subgraph of Γ_γ given by the bold edges with their respective vertices.

Each vertex of the graph can be of six types, depending on if the connected component has a maximum/minimum critical point, a saddle point, a half open saddle point, a open saddle point or a regular point. Then, the possible incidence rules of edges and vertices when $\gamma : N \to S^1$ is stable are given in Fig. 2.

The Fig. 3 represents some possible structures of the graph $N/\sim$ for stable maps. Notice that Γ and Γ' are not necessarily connected graphs.

Let $v_1, \ldots, v_r \in S^1$ be the critical values of $\gamma : N \to S^1$. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 \leq v_1 < \ldots < v_r$ and we label each vertex with the index $i \in \{1, \ldots, r\}$, if it corresponds to the critical value v_i.

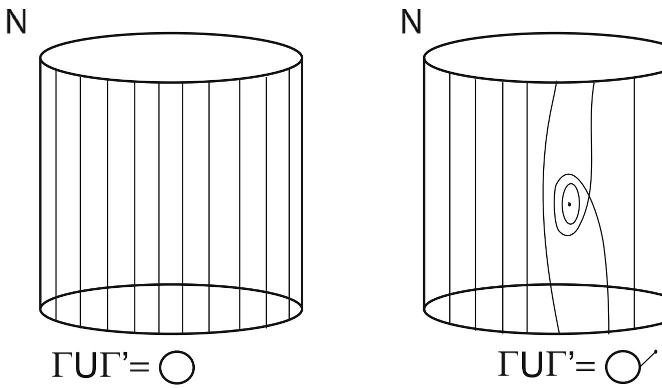


Fig. 4 Reeb graphs with first Betti number greater than zero

Definition 3.3 Let $\gamma : N \to S^1$ be a stable map. The graph given by $N/\sim$ together with the types of edges and the labels of the vertices, as previously defined, is called the *generalized Reeb graph* associated to γ.

It is well-known that Reeb graphs of stable maps $\gamma : S^2 \to \mathbb{R}$ are always trees. In Batista et al. (2017), we show that for stable maps from S^2 to S^1 the generalized Reeb graphs are also trees. When N is a manifold with boundary this is not true anymore as we can see in Fig. 4. We are interested in the case when N is homeomorphic to S^2 minus $2L$ disks. In the next theorem we collect some results whose prove can be found in Batista et al. (2016). For simplicity, from now on we will just call Reeb graph to the generalized Reeb graph, unless otherwise specified.

Theorem 3.4 (Batista et al. 2016) *Let $\gamma : N \to S^1$ be a stable map such that N is homeomorphic to $S^2 - 2L$ disks. Let $\Gamma_\gamma = \Gamma \cup \Gamma'$ be the Reeb graph of γ. We have:*

(1) $\beta_0(\Gamma') \leq L$.
(2) *If $\beta_0(\Gamma') = 1$, then $\beta_1(\Gamma_\gamma) = 2L - 1$.*
(3) *If γ is regular, then $L = 1$ and $\Gamma_\gamma = \Gamma'$, and it is a circle.*

Here we write β_i for the ith-Betti number.

Example 3.5 If $2L = 2$, by Theorem 3.4 $\beta_0(\Gamma') \leq L = 1$ and $\beta_1(\Gamma') = 2L - 1 = 1$. Consequently, $\beta_1(\Gamma) = S'$, where S' is the number of vertices of type (b). If $S' = 0$, $\Gamma_\gamma = \Gamma'$, and it is a circle. If $S' \neq 0$, Γ_γ is a circle of bold trace connected to S' trees of slim trace, as indicated in Fig. 4.

In the remaining of this section we see that the Reeb graph is a complete topological invariant for stable maps $\gamma : N \to S^1$, where N is a surface homeomorphic to S^2 minus $2L$ disks, with $L \geq 0$.

It is obvious that the labeling of vertices of the Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base point on S^1. Different choices will produce either a cyclic permutation or a reversion of the labeling in the Reeb graph. This leads us to the following definition of equivalent Reeb graphs.

Let $\gamma, \delta : N \to S^1$ be two stable maps, with Γ_γ and Γ_δ their respective Reeb graphs. Consider the induced quotient maps $\bar{\gamma} : \Gamma_\gamma \to S^1_\gamma$ and $\bar{\delta} : \Gamma_\delta \to S^1_\delta$, where S^1_γ and S^1_δ are S^1 with the graph structure whose vertices are the critical values of γ, δ, respectively.

Definition 3.6 We say that Γ_γ *is equivalent to* Γ_δ and we denote it by $\Gamma_\gamma \sim \Gamma_\delta$, if there exist graph isomorphisms $j : \Gamma_\gamma \to \Gamma_\delta$ and $l : S^1_\gamma \to S^1_\delta$, such that the following diagram is commutative:

$$\begin{array}{ccc} V_\gamma & \xrightarrow{\bar{\gamma}|V_\gamma} & \Delta_\gamma \\ {\scriptstyle j|_{V_\gamma}}\downarrow & & \downarrow{\scriptstyle l|_{\Delta_\gamma}} \\ V_\delta & \xrightarrow{\bar{\delta}|V_\delta} & \Delta_\delta \end{array}$$

where $V_\gamma = \{\text{vertices of } \Gamma_\gamma\}$, $V_\delta = \{\text{vertices of } \Gamma_\delta\}$ and Δ_γ and Δ_δ are the respective discriminant sets.

Notice that if two Reeb graphs are equivalent then it is possible to pass from one to the other by using two operations: cyclic permutation or reversion.

Theorem 3.7 *Let $\gamma, \delta : N \to S^1$ be two stable maps. If γ and δ are topologically equivalent then their respective Reeb graphs are equivalents.*

Proof Since γ and δ are topologically equivalent there exist homeomorphisms $h : N \to N$ and $k : S^1 \to S^1$ such that $k \circ \gamma \circ h^{-1} = \delta$. Then h maps critical points into critical points and k maps critical values into critical values. Hence h induces a graph isomorphism from Γ_γ to Γ_δ and k induces a graph isomorphism from S^1_γ to S^1_δ which give the equivalence between the Reeb graphs. □

The above theorem allows us to extend the definition of Reeb graph for C^0-stable maps.

Definition 3.8 Let $\gamma : N \to P$ be a continuous map, where N is homeomorphic to S^2 minus $2L$ disks (denoted by $S^2 - 2L$) and P is homeomorphic to S^1. We say that γ is C^0-*stable* if there exist a stable map $\delta : S^2 - 2L \to S^1$ and homeomorphisms $k : N \to S^2 - 2L$, $h : P \to S^1$ such that the following diagram is commutative

$$\begin{array}{ccc} N & \xrightarrow{\gamma} & P \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\ S^2 - 2L & \xrightarrow{\delta} & S^1 \end{array}$$

We say that $y \in P$ is a *critical value* of γ if $h(y)$ is a critical value of δ. Moreover, $N/\sim$ has a graph structure induced by the Reeb graph of δ. We call this graph the *Reeb graph* of γ and also we denote it by Γ_γ. The notion of equivalence of graphs given in Definition 3.6 can be also extended for C^0-stable maps in the obvious way. By Theorem 3.7, the Reeb graph Γ_γ is well defined up to equivalence of graphs.

Theorem 3.9 *Let $\gamma, \delta : N \to S^1$ be stable maps such that $\Gamma_\gamma \sim \Gamma_\delta$, then γ is topologically equivalent to δ.*

The proof of Theorem 3.9 can be found in the PhD thesis of the first named author (Batista 2015), and it is similar to the proof presented in Batista et al. (2017) for the case of stable maps from closed surfaces to S^1. The next result follows from Theorems 3.7 and 3.9:

Corollary 3.10 *The Reeb graph is a complete topological invariant for stable maps from N to S^1, where N is homeomorphic to S^2 minus $2L$ disks.*

4 Topological Classification of Map Germs

Given a finitely determined map germ $[f]$, we define the Reeb graph of $[f]$ as the Reeb graph of the link of $[f]$. It follows from Corollary 2.15 and Theorem 3.9 that if two map germs have equivalent Reeb graphs then they are topologically equivalent. Here we prove the converse of this.

Theorem 4.1 *Let $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be finitely determined map germs, with $f^{-1}(0) \neq \{0\}$ and $g^{-1}(0) \neq \{0\}$. If $[f]$ and $[g]$ are topologically equivalent, then the Reeb graphs of their associated links are equivalent.*

Proof If $S(f) = \{0\}$ and $S(g) = \{0\}$, by Theorem 3.4 the Reeb graphs of $[f]$ and $[g]$ are equivalent.

Consider $S(f) \neq \{0\}$ and $S(g) \neq \{0\}$. By hypothesis, there exist two homeomorphisms germs $[h], [k]$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^3, 0) & \xrightarrow{[f]} & (\mathbb{R}^2, 0) \\ {\scriptstyle [h]}\downarrow & & \downarrow{\scriptstyle [k]} \\ (\mathbb{R}^3, 0) & \xrightarrow{[g]} & (\mathbb{R}^2, 0) \end{array}$$

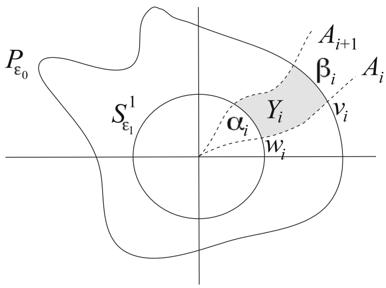
We take representatives f, g, h and k of $[f]$, $[g]$, $[h]$ and $[k]$, respectively and for any small enough $0 < \delta \ll \epsilon \ll 1$, the next diagram is also commutative:

$$\begin{array}{ccc} N_{\epsilon,\delta} & \xrightarrow{\gamma_f} & S^1_\delta \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M_{\epsilon,\delta} & \xrightarrow{g|M_{\epsilon,\delta}} & P_\delta \end{array} \qquad (1)$$

where $M_{\epsilon,\delta} = h(N_{\epsilon,\delta})$ and $P_\delta = k(S^1_\delta)$.

From the commutativity of diagram (1) it follows that $g|M_{\epsilon,\delta}$ is C^0-stable. Choose $\epsilon_0, \epsilon_1 > 0$ and $\delta_0, \delta_1 > 0$ such that $\gamma_f : N_{\epsilon_0,\delta_0} \to S^1_{\delta_0}$ and $\gamma_g : N_{\epsilon_1,\delta_1} \to S^1_{\delta_1}$ are the links of $[f]$ and $[g]$, respectively, and $S^1_{\delta_1} \subset k(D^2_{\delta_0})$. Let $\Gamma_{g|M_{\epsilon_0,\delta_0}}$ be the Reeb graph of $g|M_{\epsilon_0,\delta_0}$. Then, $\Gamma_{g|M_{\epsilon_0,\delta_0}}$ is equivalent to Γ_{γ_f}, where Γ_{γ_f} is the Reeb graph of γ_f.

Fig. 5 Connected component Y_i



Consider $A_1, \ldots, A_n$ the half branches of the discriminant $\Delta(g)$ ordered in the anticlockwise orientation. By the cone structure of f (Theorem 2.11), each half branch A_i intersects P_{δ_0} in a unique point v_i so that $v_1, \ldots, v_n$ are the critical points of $g|M_{\epsilon_0,\delta_0}$. Analogously, each A_i intersects $S^1_{\delta_1}$ in a unique point w_i, where now $w_1, \ldots, w_n$ are the critical points of γ_g. We have a graph isomorphism $l : P_{\delta_0} \to S^1_{\delta_1}$ given by $l(v_i) = w_i, \forall i = 1, \ldots, n$.

Let $C_1, \ldots, C_r$ be the connected components of $g^{-1}(\Delta(g)) - \{0\} = \cup_{i=1}^n g^{-1}(A_i)$. Again by the cone structure of f, each connected component C_j intersects M_{ϵ_0,δ_0} in a unique connected component V_j of some $g^{-1}(v_i)$, so that $V_1, \ldots, V_r$ are the vertices of $\Gamma_{g|M_{\epsilon_0,\delta_0}}$. Finally, each C_j intersects N_{ϵ_1,δ_1} in a unique connected component W_j of $g^{-1}(w_i)$, in such a way that $W_1, \ldots, W_r$ are now the vertices of Γ_{γ_g}. We have a bijection φ defined by $\varphi(V_j) = W_j, \forall j = 1, \ldots, r$. In order to have a graph isomorphism between $\Gamma_{g|M_{\epsilon_0,\delta_0}}$ and Γ_{γ_g} we need to show that φ is edge preserving.

Consider $U = k(D^2_{\delta_0}) - (\Delta(g) \cup B^2_{\delta_1})$, and let Y_i be one of its connected components limited by two consecutive half branches A_i and A_{i+1}. We denote by α_i and β_i the arcs of $S^1_{\delta_1}$ and P_{δ_0} respectively, which bound $Y_i, \forall i = 1, \ldots, n$ (see Fig. 5). Note that the connected components of $g^{-1}(\alpha_i)$ and $g^{-1}(\beta_i)$ give all the edges of the graphs Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0,\delta_0}}$, respectively.

Take X any connected component of $f^{-1}(Y_i)$, for some $1 \le i \le n$. Since $g|X : X \to Y_i$ is regular, the induced map $\tilde{g} : X/\sim \to Y_i$ is a local homeomorphism and hence, a covering space. But Y_i is simply connected, so $\tilde{g}$ is in fact a homeomorphism. We deduce that the boundary of $X/\sim$ has two components: one is an edge of Γ_{γ_g} given by the quotient of $X \cap g^{-1}(\alpha_i)$ and the other is an edge of $\Gamma_{g|M_{\epsilon_0,\delta_0}}$ given by the quotient of $X \cap g^{-1}(\beta_i)$.

Notice that all the edges of Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0,\delta_0}}$ can be obtained in this way, hence we have a bijection between the edges of Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0,\delta_0}}$ which is compatible with the above bijection φ defined between the vertices. □

Corollary 4.2 *Let $[f], [g] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be finitely determined map germs, with $f^{-1}(0) \neq \{0\}$ and $g^{-1}(0) \neq \{0\}$. The following statements are equivalent:*

1. *$[f]$ and $[g]$ are topologically equivalent;*
2. *the links of $[f]$ and $[g]$ are topologically equivalent;*
3. *the Reeb graphs of $[f]$ and $[g]$ are equivalent.*

5 The Corank 1 Case

If $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ has corank 1, after a coordinate change in the source, we can assume that a representative f of $[f]$ is given by

$$f(x, y, z) = (x, h_x(y, z)). \tag{2}$$

We say that $[f]$ is a *stabilization* of $[h_0]$ if there is a representative $f : U = (-\epsilon, \epsilon) \times V \to \mathbb{R}^2$ such that for any x, with $0 < |x| < \epsilon$, $h_x : V \subset \mathbb{R}^2 \to \mathbb{R}$ is stable (i.e., it is a Morse function with distinct critical values). The proofs of the following results can be found in Batista et al. (2017).

Lemma 5.1 *Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ as in (2). Then:*

(1) $[h_0] : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a finitely determined map germ.
(2) $[f]$ is a stabilization of $[h_0]$.
(3) Let f be a representative of $[f]$. Then, $S(f)$, $X(f)$ and $\Delta(f)$ are transverse to the planes $\{x\} \times \mathbb{R}^2$ and to the lines $\{x\} \times \mathbb{R}$, respectively, with $0 < |x| < \epsilon$ and ϵ small enough.

Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ as in (2). We take a representative $f : U = (-\epsilon, \epsilon) \times V \to \mathbb{R}^2$ such that for any x, with $0 < |x| < \epsilon$, h_x is stable and h_0 is regular in $V - \{0\}$.

Because of stability, all the functions h_x are $\mathcal{A}$-equivalent if $-\epsilon < x < 0$ and we will denote by h_x^- one of these functions. Analogously, all functions h_x are $\mathcal{A}$-equivalent if $0 < x < \epsilon$ and we will denote by h_x^+ one of these functions. The next lemma shows the connectedness of the subgraph Γ' of the Reeb graph Γ_{γ_f}:

Lemma 5.2 *Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ with $f^{-1}(0) \neq \{0\}$. Consider $\gamma_f : N_{\epsilon,\delta} \to S^1_\delta$ the link of $[f]$ with $\Gamma_{\gamma_f} = \Gamma \cup \Gamma'$ its Reeb graph. Then Γ' is connected and $S'' = 2L - 2$, where S'' is the number of vertices of type open saddle and $2L$ is the number of boundary components of $N_{\epsilon,\delta}$.*

Proof Consider a representative f of $[f]$ such that $f(x, y, z) = (x, h_x(y, z))$ and $N_{\epsilon,\delta}$ homeomorphic to S^2 minus $2L$ disks. Then by Lemma 5.1 $S(f)$ and $\Delta(f)$ are transverse to the planes $\{x\}\times\mathbb{R}^2$ and to the lines $\{x\}\times\mathbb{R}$, respectively, with $0 < |x| < \epsilon$ and ϵ small enough.

Since $S(f)$ and $\Delta(f)$ are simply connected, $\{0\} \times \mathbb{R}^2$ intersects $S(f)$ only at the origin. Similarly, $\{0\} \times \mathbb{R}$ intersects $\Delta(f)$ only at the origin. Therefore, every point $(0, h_0(y, z))$ is a regular value of f if $h_0(y, z) \neq 0$. Thus $(0, \epsilon)$ and $(0, -\epsilon)$ are regular values of γ_f.

Since the only level curves of γ_f in $\{0\} \times \mathbb{R}^2$ are $\gamma_f^{-1}(0, \epsilon)$ and $\gamma_f^{-1}(0, -\epsilon)$, and the restriction of γ_f to each boundary component of $\partial N_{\epsilon,\delta}$ is a diffeomorphism, it follows that both $\gamma_f^{-1}(0, \epsilon)$ and $\gamma_f^{-1}(0, -\epsilon)$ have at least $2L$ connected components of interval type, which alternate in $N_{\epsilon,\delta}$.

Let $a, b \in S^1_\delta$. Consider α and β two connected components of the level curves $\gamma_f^{-1}(a)$ and $\gamma_f^{-1}(b)$, respectively, which intersect some component of $\partial N_{\epsilon,\delta}$. Then,

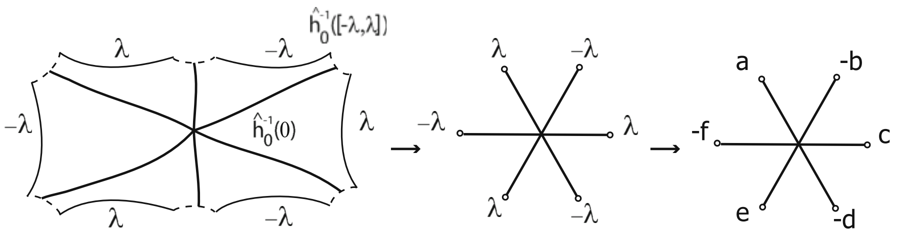


Fig. 6 Labeled vertices following orientation and sign

we can take a path in $\partial N_{\epsilon,\delta} \cup \gamma_f^{-1}(0,\epsilon) \cup \gamma_f^{-1}(0,-\epsilon)$ connecting α and β. Thus, $\beta_0(\Gamma') = 1$ and, by Theorem 3.4, $\beta_1(\Gamma') = 2L - 1$.

On the other hand, since $\chi(\Gamma') = -S''$, it follows that $\beta_1(\Gamma') = S'' + 1$. Therefore $S'' = 2L - 2$. □

Remark 5.3 Note that $\gamma_f^{-1}(0,\epsilon)$ and $\gamma_f^{-1}(0,-\epsilon)$ do not have connected components of circle type. In fact, consider $M = D_\epsilon^2 \cap h_0^{-1}(D_\delta^1)$. We have that M is a contractible 2-manifold with boundary. Hence, M is homeomorphic to the disk D^2. Therefore ∂M is homeomorphic to S^1. Since $[h_0]$ is finitely determined, Theorem 2.5 is valid and as a consequence, $D_\epsilon^2 \cap h_0^{-1}(S_\delta^0)$ is homeomorphic to $2L$ intervals.

Now we define the partial trees of h_x^+ and h_x^-. We follow the same notation as in the paragraph before Lemma 5.2. Take $x \in [-\delta, \delta]$ and choose $\lambda > 0$ such that $D_\rho^2 \cap (h_x)^{-1}([-\lambda, \lambda]) \subset V$ and $\{x\} \times [-\lambda, \lambda]$ intersects all the positive half branches (resp. negative) of $\Delta(f)$ if $x > 0$ (resp. $x < 0$), where $\rho > 0$ is such that D_ρ^2 contains all the critical points of h_x in its interior.

Consider the restriction $h_x : D_\rho^2 \cap (h_x)^{-1}([-\lambda, \lambda]) \to [-\lambda, \lambda]$ and the equivalence relation used to define Reeb graphs. For $x \neq 0$, h_x is stable and $h_x^{-1}([-\lambda, \lambda])/\sim$ admits a graph structure similar to the Reeb graph. If $x = 0$ we can also define a graph for h_0 which consists of one vertex corresponding to the level curve $h_0^{-1}(0)$ and $2L$ vertices corresponding to the level curves $h_0^{-1}(\{-\lambda, \lambda\})$.

Since the boundary of $h_0^{-1}([-\lambda, \lambda])$ is homeomorphic to S^1, we can choose one of the vertices given by the level curves $h_0^{-1}(\{-\lambda, \lambda\})$ as a initial point and choose also an orientation to $h_0^{-1}([-\lambda, \lambda])$.

We can label the initial point with the letter $\pm a$ (according to the sign of λ) and the other boundary vertices with the other letters $\pm b, \pm c, \pm d \ldots$, always following the chosen orientation and the sign of λ, as illustrated in Fig. 6. We will denote the vertices that correspond to the connected components of $h_x^{-1}(\{-\lambda, \lambda\})$ by "$\circ$".

By Lemma 5.1 the map germ $[f]$ is a stabilization of $[h_0]$ and the values $-\lambda$ and λ are regular for both functions h_0 and h_x. Hence, we can define a canonic bijection between the boundary vertices set of the Reeb graph of h_0 and the boundary vertices set of the Reeb graph of h_x. We will denote by h_x^+ and h_x^- the topological equivalence classes of functions h_x, with $0 < |x| \leq \delta$, depending on the sign of x.

Definition 5.4 Given $0 < |x| < \delta$, consider $\lambda > 0$, $\rho > 0$ and $h_x : D_\rho^2 \cap (h_x)^{-1}([-\lambda, \lambda]) \to [-\lambda, \lambda]$ as above. We define the *partial tree of* h_x^+ as the Reeb

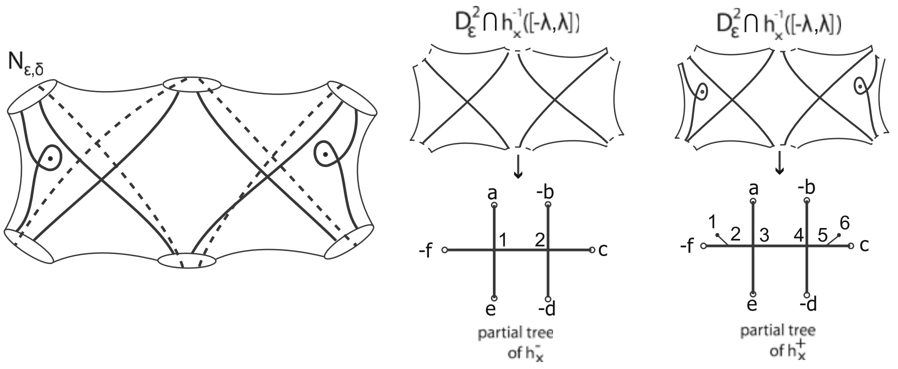
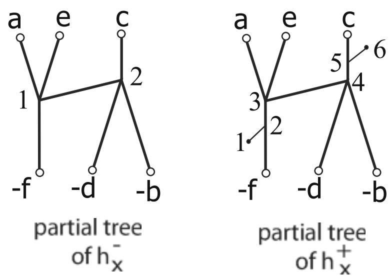


Fig. 7 Partial trees

Fig. 8 Corresponding partial trees of Fig. 7



graph of h_x with $x > 0$ together with the labeling of its boundary vertices, denoted by $\Gamma_{x>0}$. Analogously, we define the *partial tree of* h_x^- as the Reeb graph of $-h_x$ with $x < 0$, denoted by $\Gamma_{x<0}$.

Lemma 5.5 *The graphs* $\Gamma_{x<0}$ *and* $\Gamma_{x>0}$ *are trees.*

The proof of this lemma follows from $\beta_1(\Gamma_{x>0}) \leq \beta_1(M) = 0$ and $\beta_1(\Gamma_{x>0}) \leq \beta_1(M) = 0$, where $M = D^2_\rho \cap (h_x)^{-1}([-\lambda, \lambda])$.

Example 5.6 The Fig. 7 illustrates an example of $\Gamma_{x>0}$ and $\Gamma_{x<0}$. The Fig. 8 illustrates the same example, but the partial trees are drawn more conveniently.

Definition 5.7 Let $u_1 < \ldots < u_r$ and $v_1 < \ldots < v_s$ be the vertices corresponding to the level curves of critical values of $\Gamma_{x>0}$ and $\Gamma_{x<0}$, respectively. Consider Γ_{sum} the graph obtained by connecting the edge that incides in the vertex of type "$\circ$" of $\Gamma_{x>0}$ with the edge that incides in the vertex with same label of $\Gamma_{x<0}$. We relabel each vertex v_i by $u_{r+s+1-i}$, where $i = 1, \ldots, s$. We call Γ_{sum} the *sum of partial trees* of h_x^+ and h_x^- (see Fig. 9).

The main result of this section is the following:

Theorem 5.8 *Let* $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *be a corank 1 finitely determined map germ with* $f^{-1}(0) \neq \{0\}$*, and take a representative* f *such that* $f(x, y, z) = (x, h_x(y, z))$. *Then,* Γ_{sum} *is equivalent to the Reeb graph of* $[f]$.

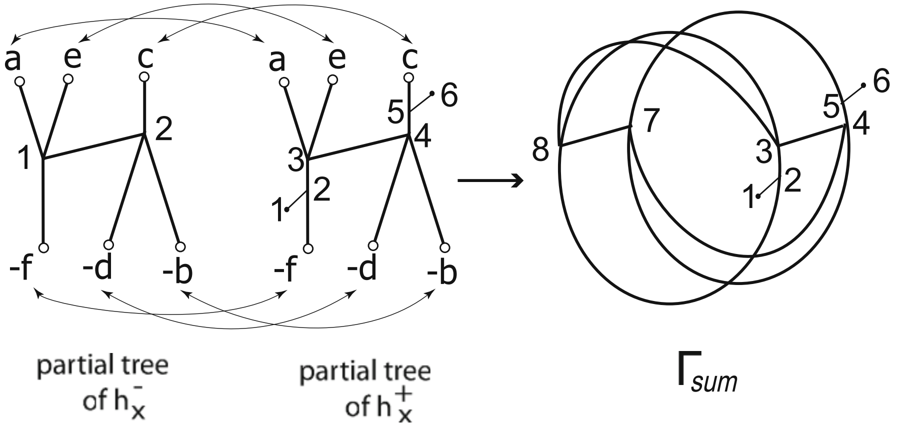


Fig. 9 Sum of partial trees

Proof If $S(f) = \{0\}$ we have $S(h_x) = \emptyset$ for any $x \neq 0$ small enough. Then both partial trees are formed just by one edge of bold trace and two vertices. Consequently, Γ_{sum} is a circle of bold trace and the result holds.

If $S(f) \neq \{0\}$, take $0 < \delta \ll \epsilon \ll 1$ and $\lambda > 0$ small enough. Let $V \subset \mathbb{R}^2$ be a neighborhood of the origin such that the following four conditions are satisfied:

(i) $\gamma_f : N_{\epsilon,\delta} \to S^1_\delta$ is the link of $[f]$;
(ii) $h_x|V : V \to \mathbb{R}$ is stable for all $x \in (-\lambda, \lambda)$, $x \neq 0$;
(iii) $\{x\} \times V$ intercepts all the half branches of $S(f)$ with the same sign of x;
(iv) $N_{\epsilon,\delta} \subset (-\lambda, \lambda) \times V$;
(v) $h_0^{-1}(0)$ is a curve and h_0 is regular on $V - \{0\}$.

From (v), $S(f) \cap (\{0\} \times \mathbb{R}^2) = \{0\}$ and $\Delta(f) \cap (\{0\} \times \mathbb{R}) = \{0\}$. Hence $(0, \delta)$ and $(0, -\delta)$ are regular values of γ_f. Moreover, since γ_f is surjective, both points $(0, -\delta)$, $(0, \delta)$ belong to the image of γ_f.

Let $A_1, \ldots, A_n$ be the half branches of $\Delta(f)$ considered in the anti-clockwise orientation and such that $(0, -\delta)$ is the base point. We also assume that $A_1, \ldots, A_r$ are on the half plane $x > 0$ and that $A_{r+1}, \ldots, A_n$ are on the half plane $x < 0$. By the conic structure of f, each half branch A_i intersects S^1_δ in a unique point v_i, so that $v_1 < \cdots < v_n$ are the critical points of γ_f in the chosen orientation. By transversality of $\Delta(f)$ to the vertical lines $\{x\} \times \mathbb{R}$, given $\delta < x < \lambda$ we have that each half branch A_i also intersects $\{x\} \times \mathbb{R}$ in a unique point w_i. But now $w_1 < \cdots < w_r$ are the critical values of h_x^+ and $w_n < \cdots < w_{r+1}$ are the critical values of h_x^-.

Since each critical value corresponds to a unique vertex, there exists a bijection given by $\varphi(v_i) = w_i$ for $i \in \{1, \ldots, n\}$ between the vertices of Γ_{γ_f} and the vertices of Γ_{sum}. Moreover, this bijection is compatible with the labels of the vertices as in Definition 5.7.

To finish the proof, we show that there is also a bijection between the edges compatible with φ. Consider the following sets (Fig. 10):

- U_i is the set of points limited by A_i, A_{i+1}, S^1_δ and $\{x\} \times \mathbb{R}$;

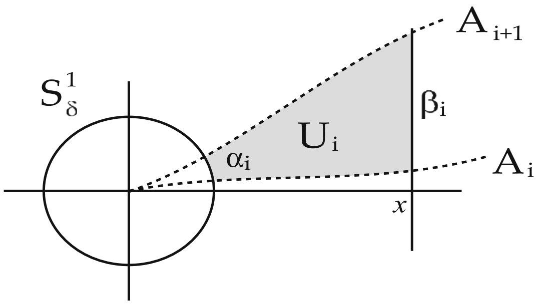


Fig. 10 Connected component U_i

- α_i is the arc of S^1_δ limited by A_i and A_{i+1};
- β_i is the segment of line of $\{x\} \times \mathbb{R}$ limited by A_i and A_{i+1};
- $Y_i = U_i \cup \alpha_i \cup \beta_i$

Where $\delta < x < \lambda$ if $1 \leq i < r$ and $-\lambda < x < -\delta$ if $r + 1 \leq i < n$.

Each one of the connected components of $f^{-1}(\alpha_i)$ and $f^{-1}(\beta_i)$ gives an edge for the graphs Γ_{γ_f} and Γ_{sum}, respectively.

Let X be any connected component of $f^{-1}(Y_i)$. Notice that $f|X : X \to Y_i$ is regular. So, the induced map $\tilde{f} : X/\sim \to Y_i$ is a local homeomorphism and hence, a covering map. Since Y_i is simply connected and X is connected, $\tilde{f}$ is a homeomorphism. Hence, $X/\sim$ contains only one edge of Γ_{γ_f} corresponding to $X \cap f^{-1}(\alpha_i)$, and also only one edge of Γ_{sum} corresponding to $X \cap f^{-1}(\beta_i)$. Moreover, since $\gamma_f^{-1}(0, \delta)$ has L connected components, all of interval type, the arc of S^1_δ delimited by A_s and A_{s+1} corresponds to L edges of Γ_{γ_f}. The same holds for $\gamma_f^{-1}(0, -\delta)$. We associate each one of these edges with the edges of Γ_{sum}. In this way, we can define a bijection ϕ between the edges of Γ_{γ_f} and the edges of Γ_{sum}, which is compatible with φ. Hence the graphs Γ_{γ_f} and Γ_{sum} are equivalent. □

5.1 Germs with Boardman Symbol $\Sigma^{2,1}$

We consider map germs whose representative has the form $f(x, y, z) = (x, \widetilde{h}_x(y) \pm z^2)$, with

$$\widetilde{h}_x(y) = y^{k+1} + a_{k-1}(x)y^{k-1} + a_{k-2}(x)y^{k-2} + \cdots + a_1(x)y.$$

We have two cases:

(i) Boardman symbol $\Sigma^{2,1}$ and k even. The zero-set of f has two half branches. By Proposition 2.7, $N_{\epsilon,\delta}$ is homeomorphic to a cylinder. Hence, Γ_{γ_f} is formed by a circle of bold trace attached to S' trees of slim trace, where S' is the number of half-open saddles of γ_f. If $h_x(y, z) = \widetilde{h}_x(y) + z^2$ every vertex of type maximum/minimum is smaller than the half-open saddle vertex to which it is connected.

Remark 5.9 The Reeb graph of $[f]$ when $S' = 0$ (as in Fig. 4), is realized by $f(x, y, z) = (x, y)$.

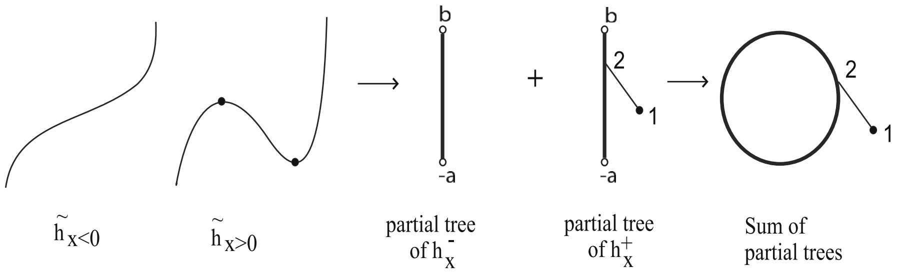


Fig. 11 Sum of partial trees for $f(x, y, z) = (x, y^3 + x^2 y + z^2)$

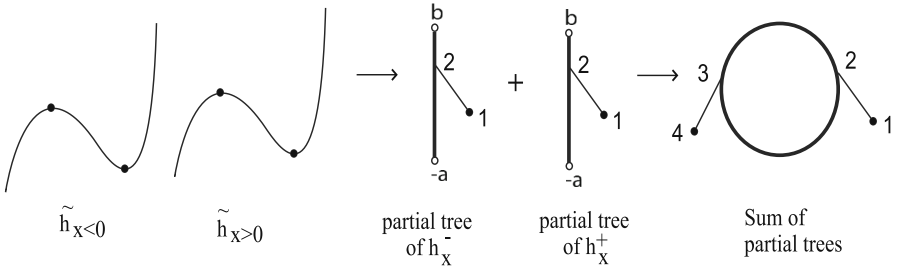


Fig. 12 Sum of partial trees for $f(x, y, z) = (x, y^3 - x^2 y + z^2)$

Example 5.10 Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ such that a representative of $[f]$ is $f(x, y, z) = (x, y^3 + a(x)y + z^2)$. Then there are two possibilities for the Reeb graph of γ_f, according to its number of half-open saddles S':

- If $S' = 1$ then f is topologically equivalent to $(x, y^3 + x^2 y + z^2)$ (see Fig. 11).
- If $S' = 2$ then f is topologically equivalent to $(x, y^3 - x^2 y + z^2)$ (see Fig. 12).

(ii) Boardman symbol $\Sigma^{2,1}$ and k odd. By Lemma 5.2, $\beta_0(\Gamma') = 1$ and $\beta_0(\Gamma) + \beta_0(\Gamma') = 1 + S'$, then $\beta_0(\Gamma) = S'$. Given $x \neq 0$ small enough, denote by $h_x(y, z) = \widetilde{h}_x(y) - z^2$, and consider $\alpha_1, \ldots, \alpha_r$, the connected components of the level curves of h_x of type half-open saddle.

Since $\beta_0(\Gamma) = S'$, each α_i divides $h_x^{-1}([-\lambda, \lambda])$ in three connected components, such that two of them intersect $\partial h_x^{-1}([-\lambda, \lambda])$. We will denote by B_i the one component which does not intersect $\partial h_x^{-1}([-\lambda, \lambda])$.

Lemma 5.11 *Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ with Boardman symbol $\Sigma^{2,1}$ and $f^{-1}(0) \neq \{0\}$. Then, all the critical values of type saddle or maximum/minimum of $h_x|\overline{B_i}$ are bigger than v_i, where $v_i = h_x(\alpha_i)$.*

Proof Let $\Gamma_{h_x|\overline{B_i}}$ be the Reeb graph of $h_x|\overline{B_i}$, for some $1 \leq i \leq r$. Notice that $\Gamma_{h_x|\overline{B_i}}$ is a tree formed by vertices of type: boundary, saddles or maximum/minimum, because $\overline{B_i}$ is homeomorphic to a disk. In particular, the points belonging to $\partial B_i \cap \alpha_i$ correspond to the only vertex of boundary type in $\Gamma_{h_x|\overline{B_i}}$.

Since $v_i = h_x(\alpha_i)$ is a minimum value of $h_x|\overline{B_i}$, the vertex corresponding to $h_x^{-1}(v_i)$ can be connected just to a vertex corresponding to a critical value bigger than v_i. Then,

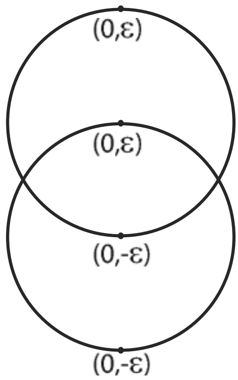


Fig. 13 Structure of Γ'

if $\Gamma_{h_x|\overline{B_i}}$ have just one saddle vertex, it cannot correspond to a critical value smaller than v_i.

Assume there exist $w_1 < \cdots < w_r$ critical values of $h_x|\overline{B_i}$, with $r > 1$, whose level curves are of saddle type and such that $w_1 < v_i$. Then, w_1 has degree 3. There are three possibilities for the vertices which are connected to the vertex corresponding to w_1:

- 3 vertices of saddle type;
- 2 vertices of saddle type and 1 vertex of maximum/minimum type;
- 1 vertex of saddle type and 2 vertices of maximum/minimum type.

However, since $w_1 < w_j$ for all $j = 2, \ldots, r$ and each critical value of type maximum/minimum is a local maximum value of $h_x|\overline{B_i}$, it follows that none of these possibilities holds. Otherwise, the vertex corresponding to w_1 would be connected to three vertices with bigger critical values, but this is not possible by Arnold (2007). □

By Lemma 5.2, γ_f has two level curves with connected components of open saddle type. Then Γ' have the structure illustrated in Fig. 13.

Remark 5.12 (1) The Reeb graph formed only by bold traces (as in Fig. 13), is realized by $f(x, y, z) = (x, y^2 - z^2)$.

(2) Since all critical points of γ_f belong to the plane $z = 0$, the level curves of γ_f are contained in the connected components that intersect $z = 0$. Consequently, the Reeb graph of γ_f admits vertices of half-open saddle type only on the edges containing the points $\gamma_f^{-1}(0, \epsilon)/\sim$.

(3) Each vertex of half-open saddle type has critical value bigger than those of vertices of open saddle type of Γ_{γ_f} and all minimum values of h_x correspond to vertices of saddle, half-open saddle or open saddle types. Therefore, the vertices of Γ_{γ_f} of open saddle type correspond to the global minimum values of h_x.

As a consequence, we deduce the partial trees of γ_f from the graph of $\widetilde{h}_x$.

Example 5.13 Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ with a representative given by $f(x, y, z) = (x, y^4 + a(x)y^2 + b(x)y - z^2)$. Then there are 3 possibilities for the Reeb graph of γ_f, according to the number S' and position of the half-open saddles:

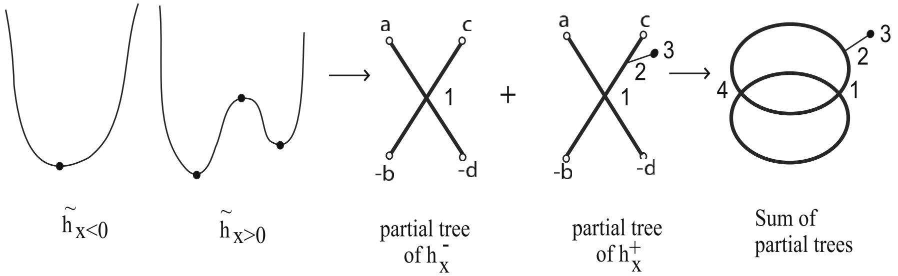


Fig. 14 Sum of partial trees for $f(x, y, z) = (x, y^4 + xy^2 + x^3y - z^2)$

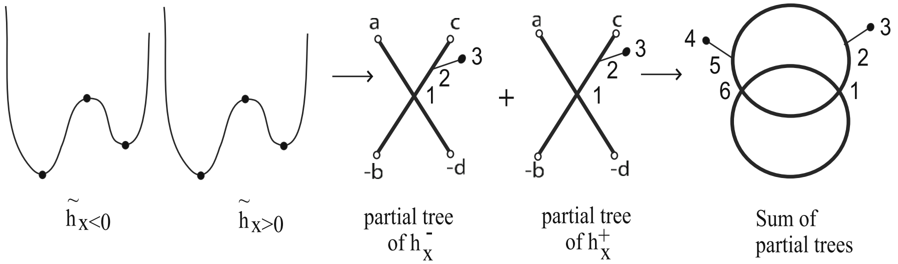


Fig. 15 Sum of partial trees for $f(x, y, z) = (x, y^4 + \frac{1}{3}x^2y^3 - 2x^4y^2 - x^6y - z^2)$

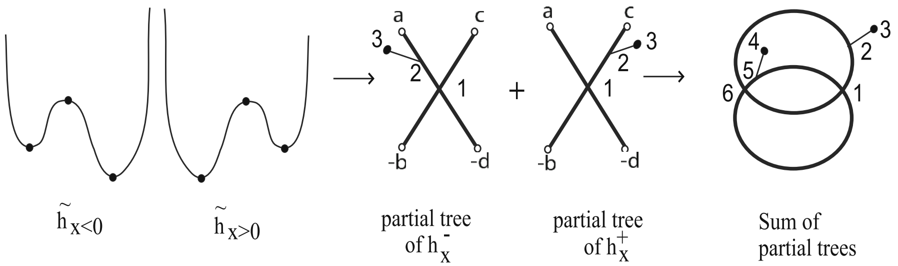


Fig. 16 Sum of partial trees for $f(x, y, z) = (x, y^4 + \frac{1}{3}x^2y^3 - 2x^2y^2 - x^3y - z^2)$

- If $S' = 1$ then f is topologically equivalent to $(x, y^4 + xy^2 + x^3y - z^2)$ (see Fig. 14).
- If $S' = 2$ such that the half-open saddles are on the same side of the global minimum, then f is topologically equivalent to $(x, y^4 + \frac{1}{3}x^2y^3 - 2x^4y^2 - x^6y - z^2)$ (see Fig. 15).
- If $S' = 2$ such that the half-open saddles are on the opposite side of the global minimum, then f is topologically equivalent to $(x, y^4 + \frac{1}{3}x^2y^3 - 2x^2y^2 - x^3y - z^2)$ (see Fig. 16).

Theorem 5.14 *Let* $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ *be a corank 1 finitely determined map germ with* $f^{-1}(0) \neq \{0\}$, *Boardman symbol* $\Sigma^{2,1}$ *and multiplicity* ≤ 5. *Then all the possibilities for the Reeb graph of the link of* $[f]$ *are realized and are presented in Table 1.*

Table 1 Corank 1 finitely determined map germ with $f^{-1}(0) \neq \{0\}$, Boardman symbol $\Sigma^{2,1}$ and multiplicity ≤ 5

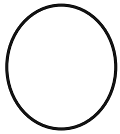
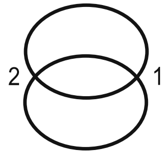
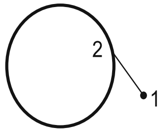
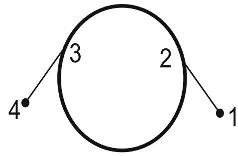
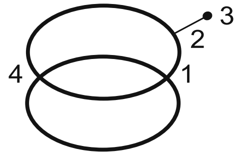
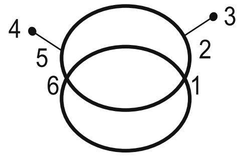
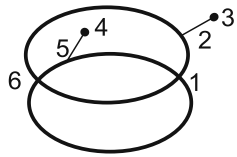
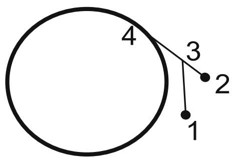
Germ	Graph
(x, y)	
$(x, y^2 - z^2)$	
$(x, y^3 + x^2y + z^2)$	
$(x, y^3 - x^2y + z^2)$	
$(x, y^4 + xy^2 + x^3y - z^2)$	
$\left(x, y^4 + \frac{1}{3}x^2y^3 - 2x^4y^2 - x^6y - z^2\right)$	
$\left(x, y^4 + \frac{1}{3}x^2y^3 - 2x^2y^2 - x^3y - z^2\right)$	
$\left(x, y^5 - \frac{1}{3}x^3y^3 - \frac{5}{3}xy^3 + x^4y + z^2\right)$	

Table 1 continued

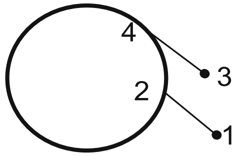
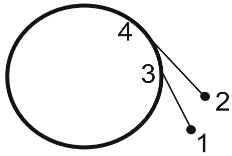
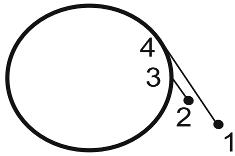
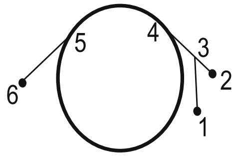
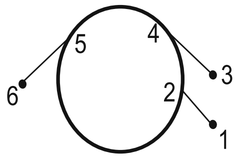
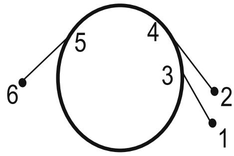
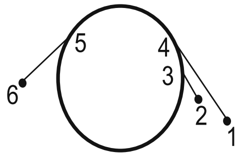
Germ	Graph
$\left(x,\, y^5+\frac{1}{3}x^3y^3+2xy^3+x^4y+x^2y+z^2\right)$	
$\left(x,\, y^5+15x^3y^3+56x^6y+z^2\right)$	
$\left(x,\, y^5+3xy^3-2x^2y^3-x^2y^2+\frac{9}{5}x^2y+z^2\right)$	
$\left(x,\, y^5-\frac{5}{3}x^4y^3+\frac{1}{3}x^3y^3-x^7y+z^2\right)$	
$\left(x,\, y^5-\frac{35}{8}xy^4+x^3y^3+5x^2y^3-\frac{7}{4}x^4y^2+3x^5y+z^2\right)$	
$\left(x,\, y^5+\frac{5}{4}xy^4-\frac{55}{12}x^3y^3+\frac{589}{1500}x^2y^3-\frac{55}{8}x^4y^2-\frac{81}{25}x^5y+z^2\right)$	
$\left(x,\, y^5+\frac{5}{4}xy^4+2x^3y^3+\frac{1}{2}x^4y^2+x^6y+z^2\right)$	

Table 1 continued

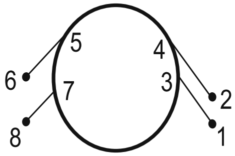
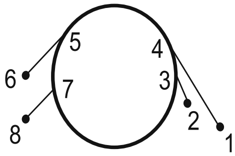
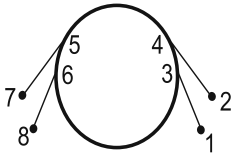
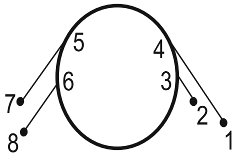
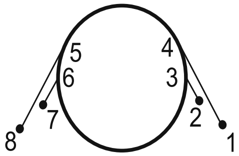
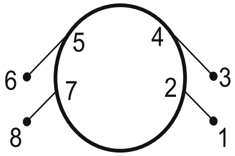
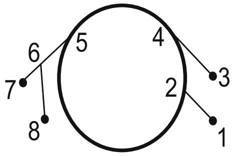
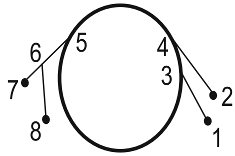
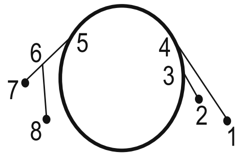
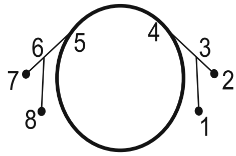
Germ	Graph
$\left(x, y^5 + \frac{1}{4}x^4y^4 + \frac{31}{8}x^2y^4 + \frac{31}{30}x^5y^3 - \frac{11}{4}x^4y^3 - \frac{33}{40}x^7y^2 - \frac{315}{8}x^6y^2 - \frac{63}{8}x^9y + z^2\right)$	
$\left(x, y^5 - \frac{3}{4}x^2y^4 - 10xy^4 + 8x^3y^3 + 25x^2y^3 - \frac{45}{2} + x^4y^2 + z^2\right)$	
$\left(x, y^5 + \frac{11}{8}xy^4 - \frac{31}{6}x^2y^3 - 5x^3y^2 + 6x^4y + z^2\right)$	
$\left(x, y^5 + \frac{7}{4}x^2y^4 - \frac{7}{32}xy^4 - \frac{51}{125}x^3y^3 - \frac{1}{160}x^2y^3 - \frac{1}{125}x^4y^2 - \frac{1}{640}x^3y^2 + \frac{1}{125}x^5y + z^2\right)$	
$\left(x, y^5 + \frac{15}{8}x^2y^4 - \frac{5}{3}x^4y^3 - \frac{15}{4}x^6y^2 + z^2\right)$	
$\left(x, y^5 + \frac{9}{4}xy^4 + \frac{7}{3}x^3y^3 + \frac{4}{3}x^2y^3 - \frac{5}{2}x^4y^2 - 6x^6y + z^2\right)$	
$\left(x, y^5 + 2xy^4 - \frac{4}{3}x^3y^3 - x^2y^3 - x^4y^2 - x^6y + z^2\right)$	

Table 1 continued

Germ	Graph
$\left(x, y^5 - \frac{109}{20}xy^4 - 2x^3y^4 - \frac{227}{50}x^2y^3 - \frac{28}{5}x^4y^2 - 46x^3y^2 + \frac{4423}{100}x^5y + 3x^5y + z^2\right)$	
$\left(x, y^5 + \frac{5}{4}x^2y^4 + \frac{5}{3}x^4y^3 - \frac{2}{3}x^3y^3 - x^3y - 2x^5y + z^2\right)$	
$\left(x, y^5 + \frac{5}{4}xy^4 + \frac{5}{3}x^3y^3 - \frac{1}{3}x^2y^3 - \frac{1}{2}x^3y^2 - x^5y + z^2\right)$	

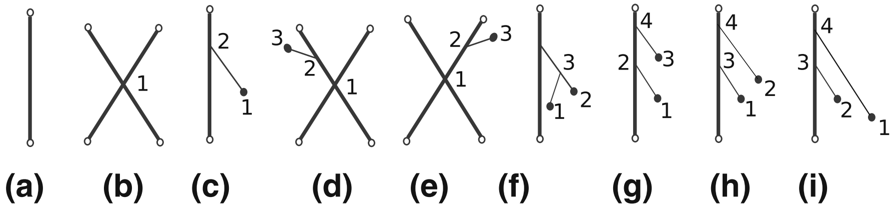


Fig. 17 Possible partial trees

Proof Let $[f] : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a map germ with $f^{-1}(0) \neq \{0\}$, Boardman symbol $\Sigma^{2,1}$ and multiplicity ≤ 5. By Lemma 5.11 and Remark 5.12 all the possibilities for the partial trees of the link of $[f]$ are given in Fig. 17. If $[f]$ has odd multiplicity, the possible Reeb graphs of $[f]$ can be obtained by taking the combinations among (a), (c), (f), (g), (h) and (i) models of partial trees. On the other hand, if $[f]$ has even multiplicity, the possible Reeb graphs of $[f]$ can be obtained by taking the combinations among (b), (d) and (e) models of partial trees. In this way, all the possibilities for the Reeb graph of $[f]$ are given in Table 1.

Also, by Examples 5.10 and 5.13, and Remarks 5.9 and 5.12, the first seven graphs in the table are realized by the respective germs. For the remaining graphs in the Table 1, it is sufficient to verify that they are equivalent to the sum of the partial trees of their respective germs as illustrated in Fig. 18 for $f(x, y, z) = (x, y^5 - \frac{1}{3}x^3y^3 - \frac{5}{3}xy^3 + x^4y + z^2)$. $\square$

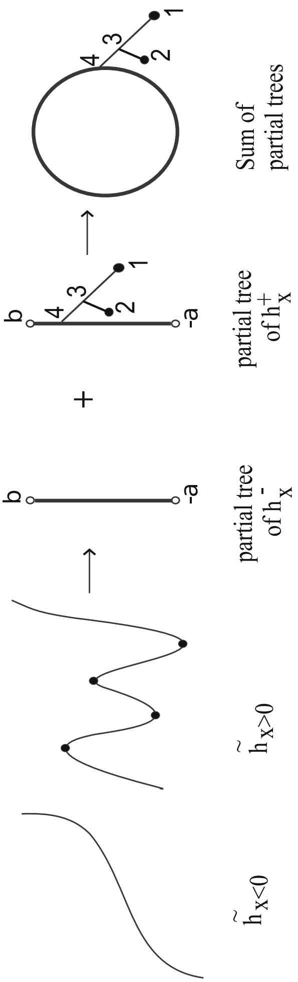


Fig. 18 Sum of partial trees for $f(x, y, z) = (x, y^5 - \frac{1}{3}x^3y^3 - \frac{5}{3}xy^3 + x^4y + z^2)$

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