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Physica A 241 (1997) 637–648

PHYSICA A

The reference state for finite coherent states[1]

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Received 1 July 1996

Abstract

We propose a reference state for finite-dimensional coherent states, which is easy to deal with in comparison to former suggestions which we briefly review. We also advance explicit calculations which shows that the phase of the overlap of finite coherent state has a structure analogous to the usual infinite-dimensional continuous coherent states.

1. Introduction

The theory of coherent states has been of great importance in various branches of physics since its earliest formulations back in the 1920s by Schrödinger [1] and later in the 1960s by Glauber [2], Sudarshan [3] and Klauder [4]. These are the so-called field coherent states, that we shall call continuous coherent states (CCS), for reasons that will soon be obvious. Many generalization have been proposed since then [5–7]. They are all dynamic in nature, in the sense that the coherent states for a given physical quantum system depend strongly on the Hamiltonian operator. More recently, there has been some proposals of a "kinematic" generalization of coherent states for finite dimensional spaces based on Schwinger's quantum kinematics [8–11]. We shall call them "finite coherent states". (FCS). Both the constructions have, in common, though, the definition of a "reference state" which acted upon by a suitable operator, generates the whole set of coherent states.

The choice of a reference state for the finite case is quite ambiguous. In the present work we discuss this problem and propose an alternative definition which is sure to

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[1]Partially supported by CNPq, Brazil.

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PII S0378-4371(96)00170-2

solve several problems. As we shall show, the present definition is both natural and easy to deal with. In Section 2, we review very briefly the continuous formulation only to establish notations and to compare it to the finite theory. In Section 3, we also review the finite-space formalism based on Schwinger's quantum kinematics. We also discuss the problem of definition of finite coherent states as a problem of choosing a reference state. We review some earlier choices in the literature as well, and present our own. In Section 4, we make some explicit calculations, showing that the overlap phase of the FCS and the CCS have a similar structure. We conclude this paper in Section 5, where we make further suggestions. Proofs of mathematical identities used in the text are given in the Appendices.

2. The continuous coherent states

Let $|q(x)\rangle$ be the continuous indexed set of position eigenkets of a single quantum mechanical particle in one dimension and $|p(x)\rangle$ the set of momentum states together with the position and momentum observables following the usual relations (eigenvalue equations):

$$\hat{Q}|q(x)\rangle = x|q(x)\rangle \tag{2.1a}$$

and

$$\hat{P}|p(x)\rangle = x|p(x)\rangle\,, \tag{2.1b}$$

$$[\hat{Q}, \hat{P}] = i\hat{I} \quad \text{(Heisenberg relation)}, \tag{2.2}$$

(we use $\hbar = 1$ units from now on)

$$\langle q(x)|q(x')\rangle = \delta(x - x') \quad \text{(normalization)}, \tag{2.3}$$

$$\hat{I} = \int_{-\infty}^{+\infty} \mathrm{d}x|q(x)\rangle\langle q(x)| = \int_{-\infty}^{+\infty} \mathrm{d}x|p(x)\rangle\langle p(x)| \quad \text{(completeness)}, \tag{2.4}$$

$$\langle q(x)|p(x')\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{ixx'} \quad \text{(overlap equation)}. \tag{2.5}$$

We use here a slightly different notation from the usual one, as can be seen above, where the $|q(x)\rangle$ ket stands for a position eigenket with eigenvalue x. The same happens for the momentum eigenket $|p(x)\rangle$. It pays off, at least for two reasons: the first is to set the CCS in a form as analogous as possible to the FCS formalism. Another reason is that some equations can be written in a more compact and elegant way. For example, the Fourier transform operator may be written as

$$\hat{F} = \int_{-\infty}^{+\infty} dx |p(x)\rangle\langle q(x)| . \tag{2.6}$$

We can write the transition operators in position and momentum spaces as

$$\hat{V}_\xi = e^{i\xi\hat{Q}} \qquad \text{and} \qquad \hat{U}_\eta = e^{i\eta\hat{P}}, \tag{2.7}$$

so that

$$\hat{V}_\xi |q(x)\rangle = |q(x-\xi)\rangle \qquad \text{and} \qquad \hat{U}_\eta |p(x)\rangle = |p(x-\eta)\rangle . \tag{2.8}$$

Eqs. (2.1) and (2.8) give us the well known relation [8].

$$\hat{V}_\xi \hat{U}_\eta = e^{i\xi\eta} \hat{U}_\eta \hat{V}_\xi . \tag{2.9}$$

We also define the usual non-hermitian "lowering" operator

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + \hat{P}) \tag{2.10}$$

together with its conjugate and the number operator

$$\hat{N} = \hat{a}^+ \cdot \hat{a} , \tag{2.11}$$

followed by its eigenstates $\{|n\rangle\}$, such that

$$\hat{N}|n\rangle = n|n\rangle \quad n = 0, 1, 2, \ldots \tag{2.12}$$

and

$$\hat{a}|0\rangle = 0 . \tag{2.13}$$

The "vacuum" state $|0\rangle$ is an eigenstate of the Fourier transform operator:

$$\hat{F}|0\rangle = |0\rangle , \tag{2.14}$$

as a matter of fact it can be shown that (Appendix A)

$$\hat{F}|n\rangle = (i)^n |n\rangle . \tag{2.15}$$

The CCS may be defined as

$$|\xi, \eta\rangle = \hat{D}[\xi, \eta]|0\rangle , \tag{2.16}$$

where the displacement operator is

$$\hat{D}[\xi, \eta] = e^{-i\xi\eta/2} \hat{V}_\xi \hat{U}_{-\eta} \tag{2.17}$$

and the vaccum state is the reference state: $|0, 0\rangle = |0\rangle$ for the CCS.

So, there is an infinite continuous set of coherent states in a one-to-one relation to the points of the "classical" phase space plane. These states have many important properties such as over-completeness, minimum uncertainty for position and momentum measurements and the dynamic fact that "once a coherent state, always a coherent state" for the evolution under a harmonic oscillator Hamiltonian [7].

One particularly important relation is the overlap of two coherent states:

$$\langle \xi, \eta | \xi', \eta' \rangle = \mathrm{e}^{(-1/4)[(\xi-\xi')^2+(\eta-\eta')^2]}\,\mathrm{e}^{(\mathrm{i}/2)(\xi'\eta-\xi\eta')}. \tag{2.18}$$

The complex number above has a phase which is half the symplectic area defined on a one degree-of-freedom classical phase space by the vectors

$$V = \xi\frac{\partial}{\partial q} + \eta\frac{\partial}{\partial p} \quad \text{and} \quad V' = \xi'\frac{\partial}{\partial q} + \eta'\frac{\partial}{\partial p} \tag{2.19}$$

and the symplectic two-form [12]

$$\Omega = \mathrm{d}q \wedge \mathrm{d}p. \tag{2.20}$$

This is one of the facts, besides the minimum uncertainty relation mentioned above, that shows a close connection between coherent states and classical mechanics.

We will be able to show in the next sections an analogous structure for the FCS.

3. Finite phase spaces and coherent states

In this section we make an option for self-containedness and review some known results, the main references being [8,9].

Let $W^{(N)}$ be a finite N-dimensional quantum space spanned by a basis of "position vectors" $\{|\mu_k^{(N)}\rangle\}$ $k = 0, 1, 2, \dots, N-1$ or "momentum states" $\{|\nu_k^{(N)}\rangle\}$ such that (We use from now on the sum over repeated upper and lower indices convention.)

$$|\mu_k^{(N)}\rangle\langle\mu^{k(N)}| = |\nu_k^{(N)}\rangle\langle\nu^{k(N)}| = \hat{I}^{(N)} \quad \text{(completeness relation)}, \tag{3.1}$$

$$\langle\mu^{k(N)}|\mu_j^{(N)}\rangle = \langle\nu^{k(N)}|\nu_j^{(N)}\rangle = \delta_j^k \quad \text{(basis orthonormality)} \tag{3.2}$$

and

$$\langle\mu^{k(N)}|\nu_j^{(N)}\rangle = F_j^{k(N)} = \nu_j^{k(N)}/\sqrt{N} = \mathrm{e}^{2\pi i jk/N}/\sqrt{N} \quad \text{(overlap equation)}. \tag{3.3}$$

The $F_j^{k(N)}$ are matrix elements of the finite N-dimensional Fourier transform operator:

$$\hat{F}^{(N)} = |\nu_k^{(N)}\rangle\langle\mu^{k(N)}|. \tag{3.4}$$

The set of indices takes values in the ring Z_N [9] of MOD N integers: $k = [0]_N, [1]_N, \dots, [N-1]_N$ so that the finite phase space is just $Z_N \times Z_N$.

Translation operators U and V defined by

$$\hat{U}|\nu_k^{(N)}\rangle = |\nu_{k+1}^{(N)}\rangle \quad \text{and} \quad \hat{V}|\mu_k^{(N)}\rangle = |\mu_{k-1}^{(N)}\rangle, \tag{3.5a}$$

$$\hat{U}|\mu_k^{(N)}\rangle = \nu_k|\mu_k^{(N)}\rangle \quad \text{and} \quad \hat{V}|\nu_k^{(N)}\rangle = \nu_k|\nu_k^{(N)}\rangle, \tag{3.5b}$$

So that

$$\hat{V}^m \cdot \hat{U}^n = \nu^{mn}\,\hat{U}^m \cdot \hat{V}^n \tag{3.6}$$

which is the discrete analogue of Eq. (2.9). We can define then, the finite coherent states as

$$|m, n^{(N)}\rangle = \hat{D}^{(N)}_{mn} |0^{(N)}\rangle , \tag{3.7}$$

where $|0^{(N)}\rangle$ is a suitable reference state, and

$$\hat{D}^{(N)}_{mn} = v^{-mn/2\,(N)} \hat{U}^m \cdot \hat{V}^{-n} . \tag{3.8}$$

A natural choice for this state would be an eigenstate of $\hat{F}^{(N)}$ with eigenvalue one.

But, as with continuous states, this choice is not unique. It can be shown that the $\hat{F}^{(N)}$ operator has a $[(N+4)/4]$ degeneracy [13] for eigenvalue one, where $[x]$ means "the largest integer not greater than x".

There are at least two former suggestions we would like to discuss before introducing our own. In [14,15] the authors introduce the following operator:

$$\hat{A}^{(N)} = (\hat{V} - \hat{V}^{\dagger}) - \mathrm{i}(\hat{U} - \hat{U}^{\dagger}) \tag{3.9}$$

as a finite-dimensional analogue of the lowering operator $\hat{a}$ and define $|0^{N)}\rangle$ by

$$\hat{A}^{(N)} |0^{(N)}\rangle = 0 . \tag{3.10}$$

Though the $\hat{A}^{(N)}$ operator has some very interesting properties, that we hope to discuss deeply in a future paper), Eq, (3.10) is still ambiguous as can be seen by considering some examples:

For $N = 2$, $\hat{A}^{(N)}$ is the null operator, because

$$\hat{U}^{\dagger(2)} = \hat{U}^{(2)} \quad \text{and} \quad \hat{V}^{\dagger(2)} = \hat{V}^{(2)} .$$

For $N = 4$ we have a double degeneracy. The states

$$|X\rangle = \frac{1}{\sqrt{2}} (|\mu^{(4)}_0\rangle + |\mu^{(4)}_2\rangle)$$

and

$$|Y\rangle = \tfrac{1}{2}(|\mu^{(4)}_0\rangle + |\mu^{(4)}_1\rangle - |\mu^{(4)}_2\rangle + |\mu^{(4)}_3\rangle)$$

both satisfy

$$\hat{A}^{(N)}|X\rangle = \hat{A}^{(N)}|Y\rangle = 0$$

together with

$$\hat{F}^{(4)}|X\rangle = |X\rangle \qquad \hat{F}^{(4)}|Y\rangle = |Y\rangle .$$

In [14], the authors also suggest the eigenstates of the following hermitian operator:

$$\hat{h}^{(N)} = 2\hat{I}^{(N)} - \tfrac{1}{2}(\hat{U}^{\dagger(N)} + \hat{U}^{(N)} + \hat{V}^{\dagger(N)} + \hat{V}^{(N)}) \tag{3.11}$$

as possible candidates for a reference state. The $\hat{h}^{(N)}$ operator would be a kind of finite-dimensional version of the oscillator Hamiltonian

$$\hat{H} = \tfrac{1}{2}(\hat{Q}^2 + \hat{P}^2) .$$

Again, let us consider some examples: For $N = 2$ we can compute that the states

$$|X^-\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}[(1+\sqrt{2})|\mu_0^{(2)}\rangle + |\mu_1^{(2)}\rangle]$$

and

$$|X_+\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}[-|\mu_0^{(2)}\rangle + (1+\sqrt{2})|\mu_1^{(2)}\rangle]$$

satisfy

$$\hat{h}^{(2)}|X_+^-\rangle = (2 \pm \sqrt{2})|X_+^-\rangle .$$

Thus, the $\hat{h}^{(2)}$ operator "lifts up" the degeneracy we had for the $\hat{A}^{(2)}$ operator, so we may choose

$$|0^{(2)}\rangle = |X_-\rangle .$$

In the same way for $N = 4$, we have

$$\hat{h}^{(4)}|X\rangle = 2|X\rangle - \sqrt{2}|y\rangle$$

and

$$\hat{h}^{(4)}|y\rangle = -\sqrt{2}|X\rangle + 2|y\rangle$$

Thus, the reduced matrix of $\hat{h}^{(4)}$ in the two-dimensional subspace spanned by $|X\rangle$ and is $|Y\rangle$:

$$\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}.$$

By diagonalization, we find easily that the states

$$|X'\rangle = \tfrac{1}{\sqrt{2}}(|X\rangle - |y\rangle)$$

and

$$|Y'\rangle = \tfrac{1}{\sqrt{2}}(|X\rangle + |y\rangle)$$

satisfy

$$\hat{h}^{(4)}|X'\rangle = (2+\sqrt{2})|X'\rangle$$

and

$$\hat{h}^{(4)}|Y'\rangle = (2-\sqrt{2})|Y'\rangle$$

In this way, we can observe again, that the $\hat{h}^{(4)}$ operator lifts up the degeneracy we had for $\hat{A}^{(4)}$. These examples are encouraging, but we lack a proof for all N. We can prove, though, that indeed $\hat{h}^{(N)}$ commutes with $\hat{F}^{(N)}$ and so they share the same eigenvectors. (see Appendix B) It still remains to be proved, though, that $\hat{h}^{(N)}$ is non-degenerate for the eigenvalue-one subspace of $\hat{F}^{(N)}$.

Anyway, it is a very cumbersome procedure to follow if we are to analytically diagonalize $\hat{h}^{(N)}$. Another suggestion was given in [11], based on a work of Mehta [13]: Define the following N states in the usual continuous infinite-dimensional space:

$$|Q_j^{(N)}\rangle = (\pi)^{1/4} \sum_{s=-\infty}^{+\infty} \left|q\left[\left(\frac{2\pi}{N}\right)^{1/2}\right](sN+j)\right\rangle . \quad (3.12)$$

Mehta, in his work, proved essentially the following result:

$$F_k^{j(N)} \langle Q^{k(N)}|S\rangle = (\mathrm{i})^S \langle Q^{j(N)}|S\rangle \quad \text{(no sum over } s). \quad (3.13)$$

We can define then a state $|0^{(N)}\rangle$ such that

$$C_N \langle \mu^{j(N)}|0^{(N)}\rangle = \langle Q^{j(N)}|0\rangle \quad \text{(for all } j). \quad (3.14)$$

The constant C_N is determined by normalization of $|0^{(N)}\rangle$.

The reference state defined this way certainly satisfies the correct eigenvalue equation for $\hat{F}^{(N)}$, but it is very difficult to treat analytically. In fact, its components are given by an infinite sum of Gaussians:

$$\langle \mu^{j(N)}|0^{(N)}\rangle = (C_N)^{-1} \sum_{s=-\infty}^{+\infty} \mathrm{e}^{(-\pi^{(sN+j)^2/2)}}, \quad (3.15)$$

where we used here the well-known "Gaussian-like" position (or momentum) representation for the ground state of the harmonic oscillator:

$$\langle q(x)|0\rangle = \langle p(x)|0\rangle = (\pi)^{-1/4} \mathrm{e}^{-x^2/2} . \quad (3.16)$$

We suggest a much simpler algorithm for the choice of a reference state that belongs to the "eigenvalue-one" subspace of $\hat{F}^{(N)}$. For this, we follow an algebraic formalism also introduced by Mehta. First, observe that

$$v_k^{j(N)} v_m^{-k(N)} = \sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} k(j-m)/N} = N\delta_m^j , \quad (3.17)$$

which means that

$$\hat{F}^{(N)} \cdot \hat{F}^{\dagger(N)} = \hat{I}^{(N)} \quad (3.18)$$

and that

$$\langle \mu^{j(N)}|\hat{F}^{2(N)}|\mu_k^{(N)}\rangle = \delta_{-k}^j . \quad (3.19)$$

Remember that the indices are Z_N-valued, so these equations are all MOD N relations. (Thus, $-k$ stands for $N-k$, for instance and the sum in (3.17) may run from any $t \in Z$ to $N+t$).

It follows that

$$\hat{F}^{3(N)} = \hat{F}^{\dagger(N)}$$

and so

$$\hat{F}^{4(N)} = \hat{I}^{(N)}. \tag{3.20}$$

The above equation implies that the eigenvalues of $\hat{F}^{(N)}$ are the fourth-roots of unity:

$$v_k^{(4)} = e^{2\pi ik/N}$$

just as in the continuous theory. Let us define the following four operators:

$$\hat{B}_j^{(N)} = \tfrac{1}{4} v_{-jk}^{(4)} \hat{F}^{k(N)} \quad \text{where } j,k = 0,1,2,3. \tag{3.21}$$

Notice that the k-index above is a Z_4, which stands both as a dummy index in the sum and as the kth power of $\hat{F}^{(N)}$.
Using (3.17) for $N = 4$ makes it possible to invert Eq. (3.21):

$$\hat{F}^{j(N)} = \hat{B}_k^{(N)} v^{Kj(4)} \tag{3.22}$$

and calculate the product:

$$\begin{aligned}\hat{B}_k^{(N)} \cdot \hat{B}_n^{(N)} &= \tfrac{1}{16} v_{-jk}^{(4)} v_{-nm}^{(4)} \hat{F}^{j(N)} \cdot \hat{F}^{m(N)} \quad (\text{making } j + m = s)\\ &= \tfrac{1}{16} v_{-sk}^{(4)} v_{km}^{(4)} v_{-n}^{m(4)} \hat{F}^{s(N)} = \tfrac{1}{4} v_{-sk}^{(4)} \hat{F}^{s(N)} \delta_{kn} = \hat{B}_k^{(N)} \delta_{kn}\end{aligned}$$

that shows that the $\hat{B}_k^{(N)}$ are idempotent.
The above result together with Eq. (3.22) for $j = 1$ gives us

$$\hat{F}^{(N)} \hat{B}_j^{(N)} = \hat{B}_k^{(N)} \cdot \hat{B}_j^{(N)} v^{k(4)} = \hat{B}_k^{(N)} \delta_{kj} v^{k(4)} = v_j^{(4)} \hat{B}_j^{(N)}.$$

In particular,

$$\hat{F}^{(N)} \hat{B}_0^{(N)} = \hat{B}_0^{(N)}.$$

So, if we define

$$|0^{(N)}\rangle = \| \hat{B}_0^{(N)} |\mu_0^{(N)}\rangle \|^{-1} \cdot \hat{B}_0^{(N)} |\mu_0^{(N)}\rangle \tag{3.23}$$

as our normalized reference state it will certainly be an eigenvalue-one state of $\hat{F}^{(N)}$. Its components in the $\{|\mu_k^{(N)}\rangle\}$ basis can be easily computed. Eq. (3.21) for $j = 0$ gives us

$$\hat{B}_0^{(N)} = \tfrac{1}{4}(\hat{I}^{(N)} + \hat{F}^{(N)} + \hat{F}^{2(N)} + \hat{F}^{t(N)}, \tag{3.24}$$

since

$$\hat{B}_0^{2(N)} = \hat{B}_0^{(N)},$$

then

$$\| \hat{B}_0^{(N)} |\mu_0^{(N)}\rangle \|^2 = \langle \mu^{0(N)} | \hat{B}_0^{2(N)} | \mu_0^{(N)}\rangle = \mu^{0(N)} | \hat{B}_0^{(N)} | \mu_0^{(N)}\rangle.$$

While

$$\begin{aligned}\langle \mu^{j(N)} | \hat{B}_0^{(N)} | \mu_0^{(N)}\rangle &= \tfrac{1}{4}(\langle \mu^{j(N)} | \mu_0^{(N)}\rangle + \langle \mu^{j(N)} | \hat{F}^{(N)} | \mu_0^{(N)}\rangle + \langle \mu^{j(N)} | \hat{F}^{2(N)} | \mu_0^{(N)}\rangle\\ &\quad + \langle \mu^{j(N)} | \hat{F}^{t(N)} | \mu_0^{(N)}\rangle) = \tfrac{1}{2}\left(\tfrac{1}{\sqrt{N}} + \delta_0^j\right).\end{aligned}$$

So,

$$\langle \mu^{j(N)} | 0^{(N)} \rangle = [2(N + \sqrt{N})]^{-1/2} (1 + \sqrt{N}\delta_0^j). \tag{3.25}$$

4. The overlap of finite coherent states

Let us define N^2 unitary operators as

$$\hat{X}_{rs}^{(N)} = \hat{U}^{r(N)} \cdot \hat{V}^{s(N)}. \tag{4.1}$$

Because of (3.6), these operators obey the relations below:

$$\hat{X}_{rs}^{t(N)} = v_{rs}^{(N)} \hat{X}_{-r-s}^{(N)} \tag{4.2}$$

and

$$\hat{X}_{rs}^{(N)} \cdot \hat{X}_{pq}^{(N)} = v_{sp}^{(N)} \hat{X}_{r+ps+q}^{(N)}. \tag{4.3}$$

Eq. (3.8) can be redefined as

$$\hat{D}_{pq}^{(N)} = \hat{X}_{p-q}^{(N)} v_{-pq/2}^{(N)}. \tag{4.4}$$

Using (3.5) we have

$$\hat{X}_{rs}^{(N)} | \mu_k^{(N)} \rangle = v_{r(k-s)}^{(N)} | \mu_{k-s}^{(N)} \rangle. \tag{4.5}$$

The overlap of two coherent states is then

$$\langle p, q^{(N)} | r, s^{(N)} \rangle = \langle 0^{(N)} | \hat{X}_{p-q}^{t(N)} \cdot \hat{X}_{r-s}^{(N)} | 0^{(N)} \rangle \cdot v_{pq/2}^{(N)} \cdot v_{-rs/2}^{(N)}.$$

Because of (3.23) the reference state is

$$|0^{(N)}\rangle = [2(N + \sqrt{N})]^{-1/2} | \mu_j^{(N)} \rangle \lambda^{j(N)} \tag{4.6}$$

with

$$\lambda^{j(N)} = 1 + \sqrt{N}\delta_0^j. \tag{4.7}$$

So,

$$\langle p, q^{(N)} | r, s^{(N)} \rangle = [2(N + \sqrt{N})]^{-1} \lambda_{k-q+s}^{(N)} \lambda^{k(N)} v_{(pq-rs)}^{(N)} / v_{(r-p)(k+s)}^{(N)},$$

where we used Eqs. (4.2)–(4.5). Taking $k = m - s$ we get

$$\langle p, q^{(N)} | r, s^{(N)} \rangle = [2(N + \sqrt{N})]^{-1} \lambda_{m-q}^{(N)} \lambda^{m-s(N)} v_{(pq-rs)/2}^{(N)} v_{(r-p)m}^{(N)}.$$

But

$$\begin{aligned}
\lambda_{m-q}^{(N)} \lambda^{m-s(N)} v_{(r-p)m}^{(N)} &= \sum_{m=0}^{N-1} (1 + \sqrt{N}\delta_{m-q}^0)(1 + \sqrt{N}\delta_0^{m-s}) v_{(r-p)m}^{(N)} \\
&= v_m^{r(N)} v_{-p}^{m(N)} + \sqrt{N} v_m^{r-p(N)} \delta_s^m + \sqrt{N} v_m^{r-p(N)} \delta_q^m + N v_m^{r-p(N)} \delta_m^s \delta_q^m \\
&= N\delta_p^r + \sqrt{N}(v_s^{r-p(N)} + v_q^{r-p(N)}) + N v_q^{r-p(N)} \delta_q^s.
\end{aligned}$$

This result can be summarized as

$$\lambda_{m-q}^{(N)}\lambda^{m-s(N)}v_{(r-p)m}^{(N)}$$
$$= 2(N+\sqrt{N}) \quad \text{if } r=p \text{ and } q=s$$
$$= (N+2\sqrt{N})v_q^{r-p(N)} \quad \text{if } r\neq p \text{ and } q=s\,,$$
$$= N+2\sqrt{N} \quad \text{if } r=p \text{ and } q\neq s\,,$$
$$= 2\sqrt{N}\cos[\pi(r-p)(s-q)/N]\,v_{(r-p)(s+q)/2}^{(N)} \quad \text{if } r\neq p \text{ and } q\neq s\,,$$

where we made use of the identity

$$e^{i\alpha}+e^{i\beta} = 2\cos[(\alpha-\beta)/2]e^{i(\alpha+\beta)/2}\,.$$

So finally,

$$\langle p, q^{(N)}[r, s^{(N)}\rangle$$
$$= 1 \quad \text{if } r=p \text{ and } q=s$$
$$= v_{q(r-p)/2}^{(N)}(N+2\sqrt{N})/2(N+\sqrt{N}) \quad \text{if } r\neq p \text{ and } q=s$$
$$= v_{p(q-s)/2}^{(N)}(N+2\sqrt{N})/2(N+\sqrt{N}) \quad \text{if } r=p \text{ and } q\neq s$$
$$= \cos[\pi(r-p)(s-q)/N]\,v_{(rq-ps)/2}^{(N)} \quad \text{if } r\neq p \text{ and } q\neq s\,. \tag{4.8}$$

By inspection of all possibilities above, we can see that if the overlap is non-null, then its phase is given by

$$\text{PHASE}\langle p, q^{(N)}|r, s^{(N)}\rangle = (2\pi/N)(rq-ps)/2 \tag{4.9}$$

in a close analogy to Eq. (2.18) for the CCS

5. Conclusions

In our view, the FCS formalism is an alternative definition of coherent states for finite-dimensional quantum systems (like spin systems, for instance). Its kinematic nature makes the theory more flexible than usual group-theoretical approach [7], because it does not depend on dynamic details specifically, such as the, Hamiltonian of the physical system.

We expect this work to help clear up some aspects of theory. In particular, to reveal some links with classical mechanics in analogy to the continuous theory (CCS). Eq. (3.23), (3.25) and (4.9) are the central results of this work. The two first equations state an umabiguous and clear definition for the reference state of finite coherent states. It has several advantages over former definitions. It can be expressed analytically very easily on the $\{|\mu_k^{(N)}\rangle\}$ basis, making it possible to compute exact results like that of Eq. (4.9). The other way around, Eq. (4.9) make us more confident that our choice for the reference is appropriate.

The idea that the finite phase space has a kind of "pre-symplectic" structure is not a new one [16], so, our results of Section 4 may be important on this respect. The $A^{(N)}$ operator discussed in Section 3 has very interesting properties that are similar to its "continuous" counterpart and may be closely related (as we suspect) to the eigenvalue structure of the finite Fourier transform [17]. We also hope that research in this route may lead to a better understanding of the $\hat{h}$ operator.

Appendix. A

Let us prove first that

$$\langle q(x)|n\rangle = (\mathrm{i})^n \langle p(x)|n\rangle . \tag{A.1}$$

In fact, using that

$$|n\rangle = (n!)^{-1/2} (\hat{a}^{\dagger})^n |0\rangle$$

and the well-known relations

$$\langle q(x)|\hat{Q}|\Psi\rangle = x\langle q(x)|\Psi\rangle \tag{A.2}$$

$$\langle q(x)|\hat{P}|\Psi\rangle = -\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}x}\langle q(x)|\Psi\rangle \tag{A.3}$$

$$\langle p(x)|\hat{P}|\Psi\rangle = x\langle p(x)|\Psi\rangle \tag{A.4}$$

$$\langle p(x)|\hat{Q}|\Psi\rangle = \mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}x}\langle p(x)|\Psi\rangle \tag{A.5}$$

together with (2.6), (2.10) and (3.16) gives us

$$\begin{aligned}\langle q(x)|n\rangle &= (n!)^{-1/2}\langle q(x)|(\hat{a}^{\dagger})^n|0\rangle \\ &= [2(n!)]^{-1/2}(x - \tfrac{\mathrm{d}}{\mathrm{d}x})^n\langle q(x)|0\rangle = [2(n!)]^{-1/2}(x - \tfrac{\mathrm{d}}{\mathrm{d}x})^n\langle p(x)|0\rangle \\ &= [2(n!)]^{-1/2}\langle p(x)|(\hat{P} + \mathrm{i}\hat{Q})^n|0\rangle = (n!)^{-1/2}(\mathrm{i})^n\langle p(x)|(\hat{a}^{\dagger})^n|0\rangle \\ &= (\mathrm{i})^n\langle p(x)|n\rangle .\end{aligned}$$

so,

$$\begin{aligned}\hat{F}|n\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}x\langle p(x)|q(x)\rangle\langle q(x)|n\rangle \\ &= (\mathrm{i})^n \int_{-\infty}^{+\infty} \mathrm{d}x|p(x)\rangle\langle q(x)|n\rangle = (\mathrm{i})^n|n\rangle .\end{aligned}$$

Appendix. B

Let us prove first, that

$$\hat{F}^{t(N)} \cdot \hat{U}^{(N)} \cdot \hat{F}^{(N)} = \hat{V}^{t(N)}. \tag{B.1}$$

This follows from Eqs. (3.1)–(3.5). In fact,

$$|\mu_j^{(N)}\rangle\langle v^{j(N)}|\hat{U}^{(N)}|v_k^{(N)}\rangle\langle\mu^{k(N)}| = |\mu_j^{(N)}\rangle\delta_{k+1}^j\langle\mu^{k(N)}|$$
$$= |\mu_{k+1}^{(N)}\rangle\langle\mu^{k(N)}| = \hat{V}^{t(N)}.$$

In a similar way

$$\hat{F}^{t(N)} \cdot \hat{V}^{(N)} \cdot \hat{F}^{(N)} = \hat{U}^{(N)}. \tag{B.2}$$

Eq. (B.1) and (B.2) (and their adjoints) give us

$$\hat{F}^{t(N)} \cdot \hat{h}^{(N)} \cdot \hat{F}^{(N)} = \hat{h}^{(N)}, \tag{B.3}$$

which is the result we wanted to prove.

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