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THE REFLECTED IMPEDANCE OF A CIRCULAR COIL IN THE PROXIMITY OF A SEMI-INFINITE MEDIUM

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The Reflected Impedance of a Circular Coil in the Proximity of a Semi-Infinite Medium

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Abstract—Most analyses on a circular coil when used in the eddy current method for nondestructive testing are empirical. Theories based on simple models are often inadequate to account for some experimental observations when the spacing between the coil and the material became small. In the present paper this problem is formulated as a boundary value problem. Wave equations of the magnetic vector potential are solved.

The change in the coil impedance, when placed above a semi-infinite medium, is obtained by means of the induced voltage method, which is shown to depend only on the ϕ component of the magnetic vector potential. This change in impedance is found to be dependent on a number of factors: the shape and size of the coil; the spacing between the coil and the metal; the thickness, conductivity, and composition of the material, etc.

Numerical computations are discussed for a few selected materials in connection with experimental results obtained elsewhere. The comparison made lent support to the present analysis. Extension of this method to the case of a stratified media is included.

I. Introduction

When a coil carrying a time-varying current is placed in the proximity of a material, an eddy current is induced in the material. Subsequently, a pronounced change in the coil impedance is observed. The magnitude of the change depends on a number of factors: the shape and size of the coil; the spacing between the probe and the metal; the thickness, conductivity, and composition of the material; etc. This change is also a function of frequency, if a sinusoidal current is employed. Because of this variation in the coil impedance, the eddy current method is used to determine the homogeneity and other characteristics of the material as a nondestructive testing technique [1].

The impedance analysis in eddy current testing came into use in the early 1940's. Förster [2] made a theoretical analysis and experimental tests of the effect of a metal rod or tube as the core of a circular coil. Hanstock [3] gave a plausible account about the state of the eddy current method and initiated the concept of an equivalent electric circuit.

Waidelich and Renken [4] made an analysis for the impedance of a coil near a conducting surface by using the image approach. Their theoretical treatment yielded results that closely approximated their experimental data when the spacing between the probe and metal is relatively large and the frequency is relatively high.

Waidelich [5] also made a study of a pulsed eddy current used in coating thickness measurements with magnetic field equations as a starting point. Further work was carried out by Waidelich and others [6] in determining the impedance of a coil on a conducting plane.

It was felt that a basic theoretical study was needed to give a clear understanding of the many factors causing the impedance change of a coil when placed near a conducting medium and the degree to which these factors influenced the change. The purpose of this study is to fulfill this need and to develop analytical relationships by which a quantitative evaluation of the coil impedance change may be made. Although the scope of the work is limited to a theoretical treatment of the circular coil, the method of analysis is sufficiently comprehensive to allow its application to the situation where the coil is in juxtaposition to a multilayered type media.

In addition to the analytical results, numerical computations of the reflected impedance for a few selected materials are discussed in connection with experimental results obtained elsewhere. The comparison made in a few instances lends support to the present analysis.

A method of evaluating the reflected impedance for a stratified media, consisting of an upper stratum of finite thickness and a lower stratum of infinite thickness is developed. Extension of this method to many layered media seems quite natural.

II. A Circular Coil in the Proximity of a Semi-Infinite Medium

Consider a vanishingly thin, circular coil of radius a in air, oriented with its axis in the z direction and situated at $z=h$ of the cylindrical coordinate system (ρ, ϕ, z). Let it be assumed that the current density in the coil is of sinusoidal variation, has a ϕ component, namely, $i_\phi = I\delta(\rho-a)\delta(z-h)$, and has the same phase throughout the entire coil.

The input impedance of this coil, formulated by the method of induced voltage, is readily shown to depend only on the ϕ component of the magnetic vector potential A_ϕ.

When such a coil is brought in the proximity of a semi-infinite medium which is characterized by its di-

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electric constants ϵ_1, σ, and permeability μ_1, and which fills out the region $z<0$, the vector potential must satisfy the following wave equations in the air and in the infinite medium respectively:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right)-\frac{A_\phi}{\rho^2}+\frac{\partial^2 A_\phi}{\partial z^2}+\mathrm{k}_0{}^2A_\phi = -\mu I\delta(\rho-a)\delta(z-h) \quad (1)$$

$$\frac{1}{\rho}\frac{\partial}{\sigma\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right)-\frac{A_\phi}{\rho^2}+\frac{\partial^2 A_\phi}{\partial z^2}+\mathrm{k}_1{}^2A_\phi = 0. \quad (2)$$

At the surface of the plane medium, the continuity of the tangential components of the electric and magnetic fields across the interface requires that

$$\frac{1}{\mu_0}\frac{\partial A_\phi}{\partial z}\bigg|_{r=0^+}=\frac{1}{\mu_1}\frac{\partial A_\phi}{\partial z}\bigg|_{r=0^-} \quad (3)$$

$$A_\phi\big|_{r=0^+}=A_\phi\big|_{r=0^-}. \quad (4)$$

For $z>0$, the solution as presented in Appendix A becomes

$$A_\phi=\frac{\mu_0Ia}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{-\lambda_0|z-h|}\frac{\xi d\xi}{\lambda_0} + \frac{\mu_0Ia}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{-\lambda_0(z+h)}\left\{\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1}\right\}\frac{\xi d\xi}{\lambda_0} \quad (5)$$

and for $z<0$,

$$A_\phi=\frac{\mu_0Ia}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{\lambda_1z-\lambda_0h}\frac{2\mu_1\xi d\xi}{\mu_1\lambda_0+\mu_0\lambda_1} \quad (6)$$

where

$$\lambda_0=\sqrt{\xi^2-k_0{}^2} \quad\text{and}\quad \lambda_1=\sqrt{\xi^2-k_1{}^2}.$$

It is noted that the factor in the braces of the second term in (5) is analogous to a reflection coefficient. It can be shown that this is indeed a reflection coefficient.

The solution of the magnetic potential in the air can be written

$$A_\phi=\frac{\mu_0Ia}{4\pi}\int_{-\pi}^{\pi}d\phi_0\cos(\phi_0-\phi)\frac{e^{-jk_0R}}{R}-\frac{\mu_0Ia}{4\pi}\cdot\int_{-\pi}^{\pi}d\phi\ \cos(\phi_0-\phi)\frac{e^{-jk_0R'}}{R'}-\frac{\mu_0Ia}{4\pi}\cdot\int_{-\pi}^{\pi}d\phi_0\cos(\phi_0-\phi)\cdot\int_0^\infty\frac{2\mu_1\xi d\xi}{\mu_1\lambda_0+\mu_0\lambda_1}J_0(\xi v)e^{-\lambda_0(z+h)} \quad (7)$$

where $R'=\sqrt{(z+h)^2+\rho^2+a^2-2a\rho\ \cos\ (\phi^0-\phi)}$. By means of a transformation due to Van der Pol [7] when applied to the second term of (5) which represents the reflected field, it can be shown that

$$A_\phi=\frac{\mu_0Ia}{4\pi}\int_{-\pi}^{\pi}d\phi\cos(\phi_0-\phi)\left\{\frac{e^{-jk_0R}}{R}-\frac{e^{-jk_0R'}}{R'}-\frac{1}{\pi\mu_0}\int_\tau\frac{e^{-jk_0R''}}{R''}\frac{\partial^2}{\partial\zeta^2}\left(\frac{e^{-jk_1r'}}{r'}\right)d\tau\right\}. \quad (8)$$

The cylindrical volume element $d\tau=sdsd\alpha d\zeta$ and the domain of the volume integration is $0\leq s\leq\infty$, $0\leq\alpha\leq 2\pi$, $0\leq\zeta\leq\infty$. The distance $r'=\sqrt{s^2+(\mu_0\zeta)^2}$, and the quantity R'' is a distance defined by

$$R''=\sqrt{r^2+s^2-2sr\cos\alpha+(\mu_1\zeta+z+h)^2}.$$

The domain of integration can be interpreted as the half space below the image coil. This is the shaded part of Fig. 1. ζ is measured positive downwards.

In (8) the first term is the contribution due to the coil, and the second is that due to the image coil with the current flowing in the same direction as in the actual coil. The third term can be regarded as the effect due to a secondary source of the type $\partial^2/\partial\zeta^2(e^{-jkr'}/r')$ with its amplitude factor depending on $e^{-jk_0R''}/R''$. This effect is obtained from the space inside the second medium measured from the plane which contains the image coil over all the second medium.

The voltage induced on this coil is

$$V_i=j\omega a\int_{-\pi}^{\pi}A_\phi d\phi. \quad (9)$$

It follows, then, that the impedance of the coil in the proximity of a semi-infinite medium becomes

$$Z=\frac{i\omega a}{I}\int_{-\pi}^{\pi}A_\phi d\phi. \quad (10)$$

Upon substituting the value of A_ϕ from (5) and letting $z=h$ and $\rho=a$, it can be written as follows:

$$Z=\frac{j\omega\mu_0a^2}{2}\int_{-\pi}^{\pi}d\phi\int_0^\infty|J_1(\xi a)|^2\frac{\xi d\xi}{\lambda_0} + \int_0^\infty|J_1(\xi a)|^2e^{-2h\lambda_0}\left(\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1}\right)\frac{\xi d\xi}{\lambda_0}\bigg\}. \quad (11)$$

It is noted that the first term in the bracket is the coil impedance when the conducting medium below the coil is absent and is the same impedance function obtained for a coil in air. The second term is the reflected impedance due to the semi-infinite medium. The first term is singular at the coil, whereas the second is regular. We are interested only in the change in impedance due to the presence of another medium. Hence the second term is of immediate concern to us. Upon integrating with respect to ϕ, the reflected impedance becomes

$$Z_r=j\omega\mu_0\pi a^2\int_0^\infty|J_1(\xi a)|^2e^{-2h\lambda_0}\left[\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1}\right]\frac{\xi d\xi}{\lambda_0}. \quad (12)$$

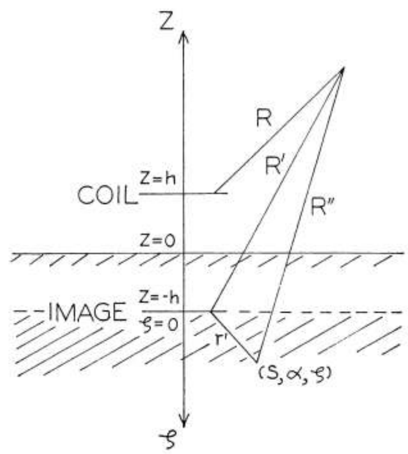


Fig. 1. Geometry of the coil and its image with respect to a conducting surface. The domain of the volume integration in (8) is the half space below the image coil.

III. Extension to Layered Media

Equation (7) clearly indicates that the wave has a spherical symmetry. This suggests that an alternate procedure is possible. Because of the fact that the boundary involved is a plane, it seems plausible that one could solve the problem by expanding the spherical wave into plane waves [8], especially since the theory of the reflection and refraction of plane waves is well known.

For a plane wave with horizontal polarization impinging upon the plane medium in an oblique incidence, where ϕ_0 is the incident angle and θ_1 the refracted angle, the transmission line model gives as characteristic impedances in air $\eta_0/\cos\theta_0$ and the plane medium $\eta_1/\cos\theta_1$ where $\eta_0=\sqrt{\mu_0/\epsilon_0}$ and $\eta_1=\sqrt{\mu_1/(\epsilon_1-j(\sigma/\omega)}$. After putting $\epsilon^*=\epsilon_1-j(\sigma/\omega)$, the coefficient of reflection in air is given by

$$\Gamma=\frac{\sqrt{\mu_1/\epsilon_1^*}\cos\theta-\sqrt{\mu_0/\epsilon_0}\cos\theta_2}{\sqrt{\mu_1/\epsilon_1^*}\cos\theta+\sqrt{\mu_0/\epsilon_0}\cos\theta_2}. \qquad (13)$$

As is shown in Appendix B, this can be reduced to

$$\Gamma=\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1} \qquad (14)$$

which is exactly the factor in the braces in the integrand of the second term in (5). This indicates that such a transmission line model based on the horizontal rather than the vertical polarization of a plane wave is the relevant one in this case.

When the semi-infinite plane medium is replaced by a layered media consisting of an upper stratum of thickness t characterized by material constants ϵ_1, μ_1, and σ_1 and a lower stratum of infinite thickness characterized by ϵ_2, μ_2, and σ_2, the coefficient of reflection is given by

$$\Gamma=\frac{\Gamma_{01}+\Gamma_{12}e^{-2\lambda_1 t}}{1+\Gamma_{01}\Gamma_{12}e^{-2\lambda_1 t}} \qquad (15)$$

where

$$\Gamma_{01}=\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1}$$

and

$$\Gamma_{12}=\frac{\mu_2\lambda_1-\mu_1\lambda_2}{\mu_2\lambda_1+\mu_1\lambda_2}.$$

This formula can be arrived at directly by considering the analogous transmission line problem. (See details in Appendix B.) Replacing the expression in the braces in (5) by the above expression, one obtains the solution of the magnetic potential for the layered problem.

This method of using the transmission line analog can be extended to multilayered problems.

IV. Discussion

The reflected impedance given in (12) is resolved into real and imaginary parts. They are evaluated by means of quadrature methods [9], [10]. A number of parameters were used in the actual computation. These are:

1) Radius of the coil—five different sizes are used. They are: 0.01, 0.05, 0.10, 0.50, and 1.00 meter.

2) Coil to material spacing—four different values are used. They are 0.001, 0.01, 0.10, and 1.00 meter.

3) Material constants—permittivities, permeabilities, and conductivities.

4) Frequencies—ranging from 10 cycles to 10 KMC per second.

When a coil is brought to the vicinity of a nonferromagnetic material, its input resistance tends to increase whereas its input inductance tends to decrease in comparison with the input values of the coil with the material absent. The computed values as obtained from (12) seem to agree with this statement. As an example the reflected resistance R_r of the coil in the proximity of a material is always positive and becomes larger with increasing frequency (Fig. 2); on the other hand, the reflected reactance X_r is negative and becomes even more so as the frequency increases (Fig. 3).

The change in the reflected impedance depends also on the size of the coil. At a given frequency and a fixed coil-to-material spacing, the greater the coil radius the larger will be the reflected resistance (Fig. 2) and the more capacitive will be the reflected reactance (Fig. 3). In terms of the input impedance, the coil will become more resistive and less inductive.

On the other hand, for a given coil at a given frequency, as the coil is brought closer and closer to the material, the reflected resistance becomes more and more pronounced (Fig. 4) and the reflected reactance becomes highly capacitive (Fig. 5). Again, in terms of the input coil impedance, it becomes more resistive and less inductive, as the coil-to-material spacing is reduced.

Figures 6 and 7 show the reflected resistance and reactance vs. frequency with conductivity as parameter.

The $-X_r$ curves (Fig. 7), as a group, increase uniformly with increasing frequency. At a given frequency, $-X_r$ is directly proportional to the conductivity. However, it appears that the difference in the reflected reactance becomes practically negligible as frequency becomes higher and higher.

The reflected resistance R_r (Fig. 6) curves show a rather peculiar behavior. At frequencies below one kilocycle per second, the reflected resistance is directly proportional to the conductivity. As frequency increases, the R_r curve for lower conductivity crosses the one for higher conductivity. For instance, at a frequency about one kilocycle, for conductivities between 5.7×10^7 and 4.8×10^6 mhos per meter, it appears that the reflected resistance is inversely proportional to the conductivity. However, for the conductivity smaller than 4.8×10^6 mhos per meter, the R_r becomes again smaller proportionally. At a still higher frequency, some of the latter group begin to cross those of higher conductivities. This phenomenon seems rather persistent throughout, until they all approach practically the same value at very high frequency.

It seems only natural to conjecture that for a given conductivity a particular value of frequency can be found at which the change in the coil resistance with the presence of the material of this conductivity is a maximum or a minimum.

Results are also obtained for a material with conductivity equal to 1.0×10^6 mhos per meter with various permeabilities greater than that of air. Figure 8 shows that the reflected resistance of the coil varies inversely with the permeability at low-frequency range and becomes just the opposite at higher frequency range and approaches the same value at extremely high frequencies. From Fig. 9, where the ratio of the reflected reactance and the reflected resistance is plotted against the frequency, one may see that at low-frequency range the reflected reactance is positive and becomes negative at the higher frequency range. This ratio varies directly with the permeability. This indicates that when a coil is brought to the vicinity of such material, its inductance increases at low frequencies, whereas for frequencies above a critical value its inductance will decrease.

The effect of changing dielectric constants of a second medium was also determined. However, computations show that a change in the dielectric constant has little effect on the reflected reactance or reflected resistance.

Comparisons with experimental results obtained by the University of Missouri group [4] are presented in Figs. 10 and 11. The conditions under which such comparisons are made are not identical.[1] Nevertheless, similar trends are observed in both curves.

[1] The measured results were taken from the report by Waidelich and Renken which was published in part in the *Proc. Nat'l Elec. Conf.* (See [4], Figs. 4 and 5.) These measurements were obtained by using a circular probe consisting of 40 turns of No. 34 wire wound 1/64 inch deep and 3/64 inch wide on a 1/2 inch piece of polystyrene rod. The computed results, using the same parameter, however, were based on a circular coil of infinitely small thickness.

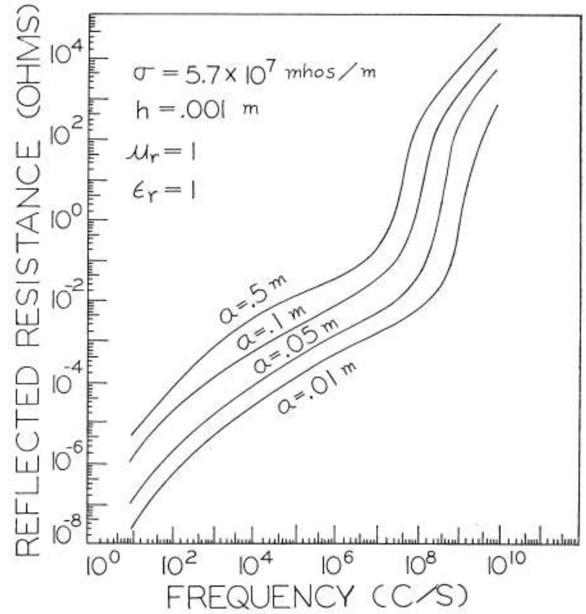


Fig. 2. Variation of the reflected resistance of a circular coils with various coil sizes a at a constant coil-to-metal spacing h above copper.

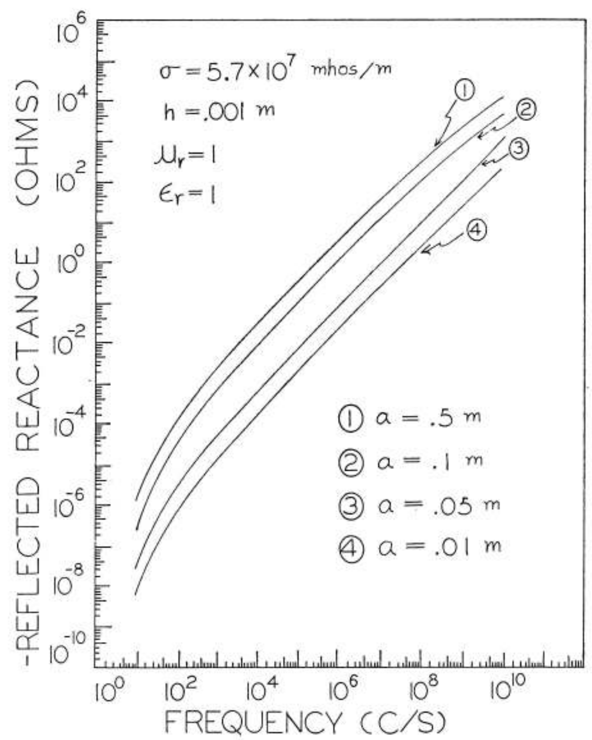


Fig. 3. Variation of the negative reflected reactance of circular coils with various coil sizes a at a constant coil-to-metal spacing h above copper.

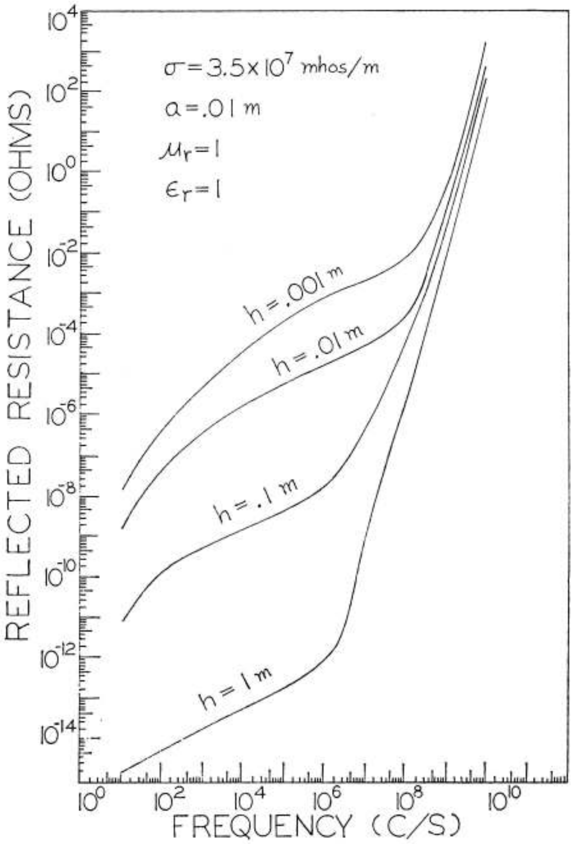


Fig. 4. Variation of the reflected resistance of a circular coil of radius $a=0.01$ meter at various coil-to-metal spacing h above aluminum.

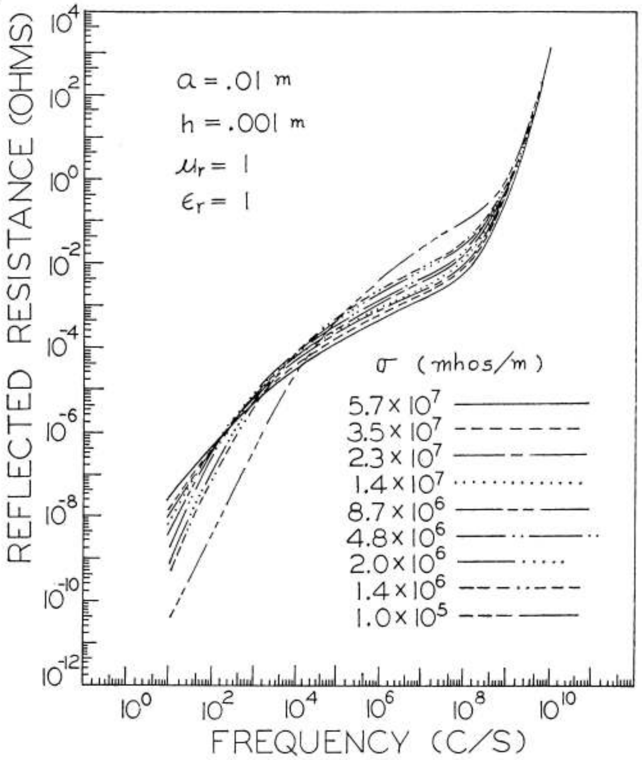


Fig. 6. Variation of the reflected resistance of a circular coil of radius a and spacing h above metals at various conductivities.

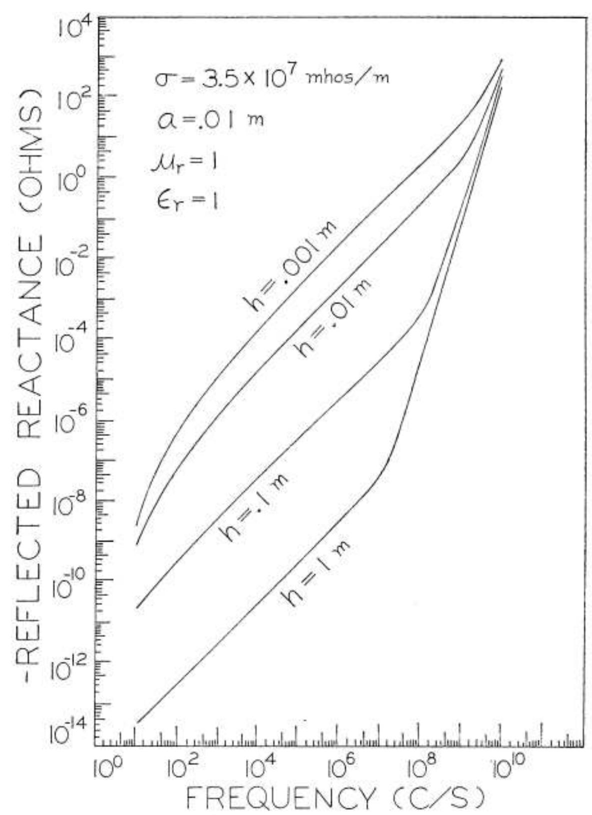


Fig. 5. Variation of the negative reflected reactance of a circular coil of radius $a=0.01$ meter at various coil-to-metal spacing h above aluminum.

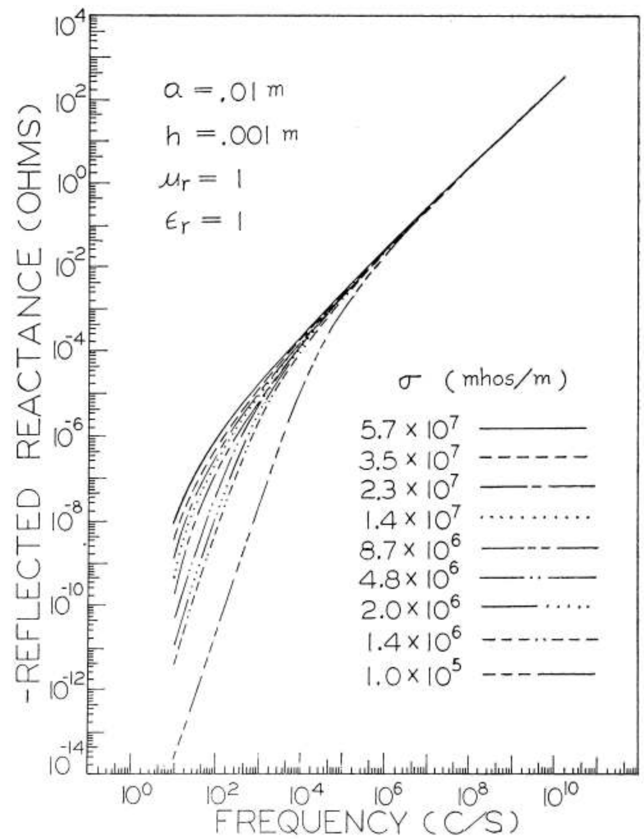


Fig. 7. Variation of the negative reflected reactance of a circular coil of radius a and spacing h above metals at various conductivities.

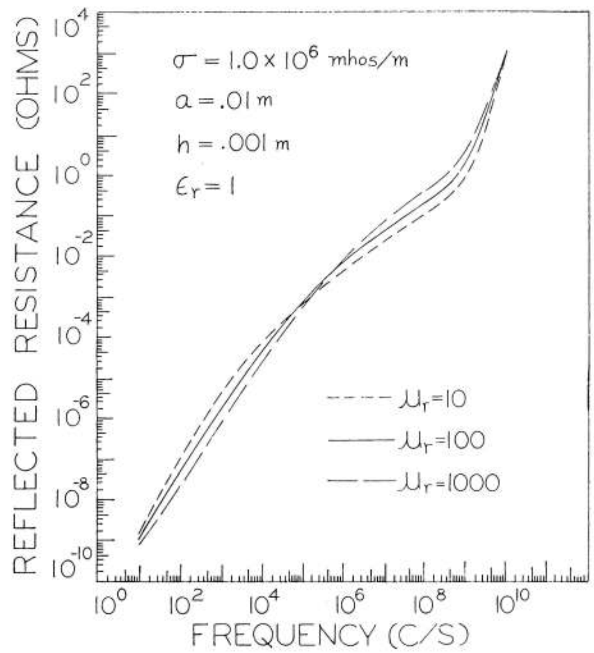


Fig. 8. Variation of the reflected resistance of a circular coil of radius a and spacing h above a conductor with changing permeabilities.

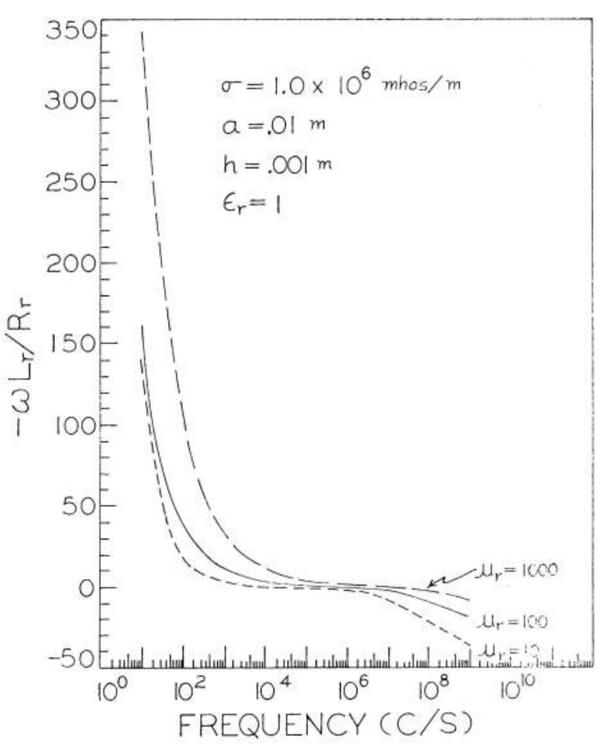


Fig. 9. Ratio of the negative reflected reactance to the reflected resistance above materials with various permeabilities.

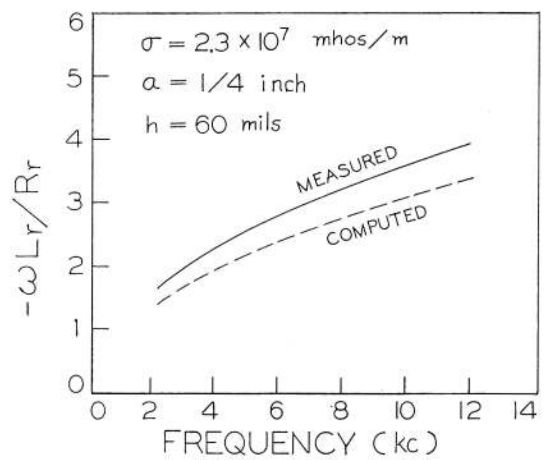


Fig. 10. Comparison of the measured result by Waidelich and Renken and the computed result for a coil of radius $a=\frac{1}{4}$ inch coil-to-metal spacing $h=60$ mils above aluminum.

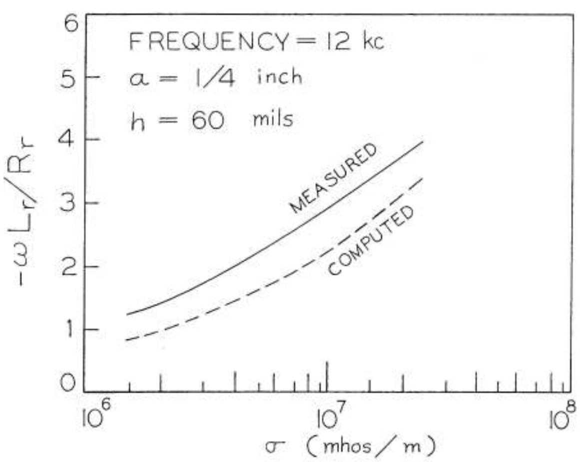


Fig. 11. Comparison of the measured result by Waidelich and Renken and the computed result for a coil of radius $a=\frac{1}{4}$ inch coil-to-metal spacing $h=60$ mils above various conductors at frequency equal to 12 kc.

V. Conclusions

1) The current density assumed in this analysis is a Dirac function. The solution A_ϕ obtained is thus the space impulse response of the system consisting of the coil, its surrounding medium, and the nearby medium of semi-infinite dimension. In this study, interest is directed to the reflected impedance in the coil due to the presence of a semi-infinite material. The self-impedance of the coil, though singular, does not come into the picture. When an actual loop of wire with a finite cross section is used and the current distribution is known, one could resort to the familiar circuit concept which involves a convolution integral so that the actual response may be obtained. Furthermore, one could think of the solution in the present analysis as the Green's function solution. To obtain solutions for an actual wire carrying current of some other wave shape, a superposition integral with the Green's function as kernel is called for.

2) Both the permittivity and permeability of the semi-infinite medium are assumed real in the analysis. In order to account for loss other than conductive, complex permittivity and permeability may be introduced in the computation of the reflected impedance.

3) An alternate method of solution is suggested in Section III. Such a method leads naturally to the solution of the reflected impedance of a circular coil above stratified medium. By means of the transmission line analog, the multilayered problem can be solved.

4) Another possible extension in connection with the present analysis is to investigate the effect of a plane medium in uniform motion upon the impedance of a coil above the medium.

5) Extension of this analysis to a pulsed current source other than the sinusoidal as assumed in this paper can also be fruitful.

Appendix A

Determination of the Magnetic Vector Potential for the Coil in the Proximity of a Semi-Infinite Medium

The wave equations (1) and (2) which the vector potential satisfies in the air and in the infinite medium are

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right) - \frac{A_\phi}{\rho^2} + \frac{\partial^2 A_\phi}{\partial z^2} + k_0^2 A_\phi = -\mu_0 I\delta(\rho - a)\delta(z - h) \quad (16)$$

and

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right) - \frac{A_\phi}{\rho^2} + \frac{\partial^2 A_\phi}{\partial z^2} + k_1^2 A_\phi = 0 \quad (17)$$

where

$$k_0^2 = \omega^2\mu_0\epsilon_0, \quad \text{and} \quad k_1^2 = \omega^2\mu_1\epsilon_1 - j\omega\mu_1\sigma.$$

The solution for the inhomogeneous equation (16) by means of successive Hankel's and Fourier transforms is as follows:

$$A_\phi = \frac{\mu_0 I a}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{-\sqrt{\xi - k_0}\,(z-h)}\frac{\xi d\xi}{\sqrt{\xi^2 - k_0^2}}. \quad (18)$$

The solution of the homogeneous equation (17) is of the following form:

$$A_\phi = CJ_1(\xi\rho)e^{\sqrt{\xi^2-k^2}z}$$

and, in general

$$A_\phi = \int_0^\infty g(\xi)J_1(\xi\rho)^{-\sqrt{\xi^2-k^2}z}d\xi \quad (19)$$

where $g(\xi)$ is an amplitude factor to be determined.

Now, for the problem at hand, one must distinguish three regions.

1) For $z > h$ in air, there are both a primary wave due to the actual excitation in the coil and a secondary effect due to the currents induced in the medium. One may write according to (18)

$$A_\phi\big|_{\text{Prim.}} = \frac{\mu_0 I a}{2}\int_0^\infty J_1(\rho\xi)J_1(\xi a) \cdot e^{-\sqrt{\xi^2-k_0^2}(z-h)}\frac{\xi d\xi}{\sqrt{\xi^2 - k_0^2}}. \quad (20)$$

For the secondary wave, one may look for a representation as follows:

$$A_\phi\big|_{\text{Sec.}} = \frac{\mu_0 I a}{2}\int_0^\infty F(\xi)J_1(\xi\rho)J_1(\xi a)e^{-\sqrt{\xi^2-k_0^2}(z+h)}d\xi \quad (21)$$

where $F(\xi)$ is an amplitude factor indicating the "spectral" distribution of the eigenfunctions of ξ space. It is yet to be determined. On the other hand, it could also be looked upon as the coefficient of reflection in the language of wave propagation.

2) For the air layer $0<z<h$, nothing is essentially different from those in the first region, except that in the expression for the primary wave one factor of the exponent is written as $z-h$, for z is smaller in this region than h. Thus, the negative sense is still preserved in the exponent. Thus,

$$A_\phi\big|_{\text{Prim.}} = \frac{\mu_0 I a}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a) \cdot e^{\sqrt{\xi^2-k_0^2}(z-h)}\frac{\xi d\xi}{\sqrt{\xi^2 - k_0^2}} \quad (22)$$

$$A_\phi\big|_{\text{Sec.}} = \frac{\mu_0 I a}{2}\int_0^\infty F(\xi)J_1(\xi\rho)J_1(\xi a)e^{-\sqrt{\epsilon^2-k_0^2}(z+h)}d\xi. \quad (23)$$

3) Infinite medium $0>z>-\infty$. There is no primary excitation in this region. To satisfy the differential equation (17) for the infinite medium with $k_1^2=\omega^2\mu_1\epsilon_1+j\omega\mu_1\sigma$ we write

$$A_\phi = \frac{\mu_0 I a}{2}\int_0^\infty G(\xi)J_1(\xi\rho)J_1(\xi a)e^{\sqrt{\xi^2-k_1^2}z-\sqrt{\xi^2-k_0^2}h}d\xi. \quad (24)$$

The factor $e^{-\sqrt{\xi^2-k_{j0}^2}h}$ is included for convenience of computation later. It could be included in $G(\xi)$ which is arbitrary.

The continuity of the tangential components of $\overline{E}$ and $\overline{H}$ across the interface requires that

$$\frac{1}{\mu_0}\frac{\partial A_\phi}{\partial z}\bigg|_{z=0^+} = \frac{1}{\mu_1}\frac{\partial A_\phi}{\partial z}\bigg|_{z=0^-}, \qquad A_\phi\big|_{z=0^+}A_\phi\big|_{z=0^-}. \quad (25)$$

From the boundary conditions of (25) and (22), (23), and (24), one obtains

$$\frac{\xi}{\sqrt{\xi^2 - k_0^2}} + F(\xi) = G(\xi) \quad (26)$$

$$\mu_0(\xi - \sqrt{\xi^2 - k_0^2}\,F(\xi)) = \mu_1\sqrt{\xi^2 - k_1^2}\,G(\xi). \quad (27)$$

Thus, from (26) and (27)

$$F(\xi) = \left\{\frac{\mu_1\sqrt{\xi^2 - k_0^2} - \mu_0\sqrt{\xi^2 - k_1^2}}{\mu_1\sqrt{\xi^2 - k_0^2} + \mu_0\sqrt{\xi^2 - k_1^2}}\right\} \frac{\xi}{\sqrt{\xi^2 - k_0^2}} \tag{28}$$

$$G(\xi) = \frac{2\mu_1\xi}{\mu_1\sqrt{\xi^2 - k_0^2} + \mu_0\sqrt{\xi^2 - k_1^2}} \cdot \tag{29}$$

Therefore, for $z>0$,

$$\begin{aligned} A_\phi = {} & \frac{\mu_0 Ia}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{-\sqrt{\xi^2-k_0^2}(z-h)}\frac{\xi d\xi}{\sqrt{\xi^2 - k_0^2}} \\ & + \frac{\mu_0 Ia}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{\xi^2-k_0^2(z+h)} \\ & \cdot \frac{\mu_1\sqrt{\xi^2 - k_0^2} - \mu_0\sqrt{\xi^2 - k_1^2}}{\mu_1\sqrt{\xi^2 - k_0^2} + \mu_0\sqrt{\xi^2 - k_1^2}}\frac{\xi d\xi}{\sqrt{\xi^2 - k_0^2}} \end{aligned} \tag{30}$$

and for $z<0$,

$$\begin{aligned} A_\phi = {} & \frac{\mu_0 I_a}{2}\int_0^\infty J_1(\xi\rho)J_1(\xi a)e^{\sqrt{\xi^2-k_1^2}z-\sqrt{\xi^2-k_0^2}h} \\ & \cdot \frac{2\mu_1\xi d\xi}{\mu_1\sqrt{\xi^2 - k_0^2} + \mu_0\sqrt{\xi^2 - k_1^2}} \cdot \end{aligned} \tag{31}$$

By means of the Bessel function identity

$$\begin{aligned} J_1(\xi\rho)J_1(\xi a) = {} & \frac{1}{2\pi} \\ & \cdot \int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)J_0(\xi\sqrt{\rho^2 - a^2 - 2a\rho\cos(\phi_0 - \phi)}) \end{aligned} \tag{32}$$

one can rewrite the solutions of A_ϕ as follows: for $z>0$:

$$\begin{aligned} A_\phi = {} & \frac{\mu_0 I_a}{2}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\int_0^{\pi} J_0(\xi r)e^{-\lambda_0|z-h|}\frac{\xi d\xi}{\lambda_0} \\ & + \frac{\mu_0 I_a}{2}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\int_0^\infty J_0(\xi r)e^{-\lambda_0(z+h)} \\ & \cdot \left\{\frac{\mu_1\lambda_0 - \mu_0\lambda_1}{\mu_1\lambda_0 + \mu_0\lambda_1}\right\}\frac{\xi d\xi}{\lambda_0} \end{aligned} \tag{33}$$

where

$$r = \sqrt{\rho^2 + a^2 - 2a\rho\cos(\phi_0 - \phi)}$$

$$\lambda_0 = \sqrt{\xi^2 - k_0^2}$$

$$\lambda_1 = \xi^2 - k_1^2.$$

And for $z<0$,

$$\begin{aligned} A_\phi = {} & \frac{\mu_0 I_a}{4\pi}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\int_0^\infty J_0(\xi r)e^{\lambda_1\xi z-\lambda_0 h} \\ & \cdot \frac{2\mu_1\xi d\xi}{\mu_1\lambda_0 + \mu_0\lambda_1} \cdot \end{aligned} \tag{34}$$

Note that

$$\int_0^\infty J_0(\xi r)e^{-\lambda_0|z-h|}\frac{\xi d\xi}{\lambda_0} = \frac{e^{-jk_0R}}{R} \tag{35}$$

where

$$R = \sqrt{(z-h)^2 + \rho^2 + a^2 - 2a\rho\cos(\phi_0 - \phi)}.$$

The second term in (33) can be decomposed as follows:

$$\begin{aligned} & \frac{\mu_0 I_a}{4\pi}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\int_0^\infty J_0(\xi r)e^{-\lambda(z+h)} \\ & \cdot \left\{\frac{2\mu_1\lambda_0}{\mu_1\lambda_0 + \mu_0\lambda_1} - 1\right\}\frac{\xi d\xi}{\lambda_0} \cdot \end{aligned} \tag{36}$$

It follows that (33) can be written as follows:

$$\begin{aligned} A_\phi = {} & \frac{\mu_0 I_a}{4\pi}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\frac{e^{-jk_0k}}{R} \\ & - \frac{\mu_0 I_a}{4\pi}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\frac{e^{-jk_0R'}}{R'} \\ & + \frac{\mu_0 I_a}{4\pi}\int_{-\pi}^{\pi} d\phi_0 \cos(\phi_0 - \phi)\int_0^\infty \\ & \cdot \frac{2\mu_1\xi d\xi}{\mu_1\lambda_0 + \mu_0\lambda_1}J_0(\xi r)e^{-\lambda_0(\xi+h)} \end{aligned} \tag{37}$$

where

$$R' = \sqrt{(z+h)^2 + \rho^2 + a^2 - 2a\rho\cos(\phi_0 - \phi)}.$$

This is (7).

Appendix B

Determination of the Magnetic Vector Potential for a Multilayered Problem

A spherical wave of the form e^{-jkR}/R can be expanded into the plane waves as follows:

$$\frac{e^{-jkR}}{R} = \frac{1}{2\pi j}\iint \exp\left[i(k_x x + k_y y + k^\xi z)\right]\frac{dk_x dk_y}{k_z} \tag{38}$$

where the k_i's, for $i=x$, y, z, are the Cartesian components of the wave vector $\bar{k}$, and the direction of propagation is in the z direction. By a change of coordinates, both the (k_x, k_y, k_z) and (x, y, z) can be transformed into cylindrical systems (ξ, ψ, k_z) and (r, ϕ, z). In other words, $k_x = \xi\cos\psi$, $k_y = \xi\sin\psi$, and $dk_x dk_y = \xi d\xi d\psi$; $x = r\cos\phi$, $y = r\sin\phi$. Thus $k_x x + k_y y = \xi r\cos(\psi - \phi)$. Equation (38) can be written as follows:

$$\frac{e^{-jkR}}{R} = \frac{1}{2\pi j}\int_0^\infty \frac{\xi d\xi e^{j_z z}}{kz}\int_0^{2\pi} e^{j\xi r\cos(\psi-\phi)}d\psi. \tag{39}$$

Upon integration with respect to ψ by means of a table of integrals, and due to the fact that $k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{k^2 - \xi^2}$, (39) becomes

$$\frac{e^{-jkR}}{R} = \int_0^\infty \frac{\xi d\xi e^{-\sqrt{\xi^2-k^2}z}}{\sqrt{\xi^2 - k^2}}J_0(\xi r) \tag{40}$$

which was used in Sections II and III.

For a plane wave with horizontal polarization impinging upon the plane medium in an oblique incidence, where θ_0 is the incident angle and θ_1 the refracted angle, the transmission line model gives as characteristic impedances in air $\eta_0/\cos\theta_0$ and the plane medium $\eta_1/\cos\theta_1$, where $\eta_0=\sqrt{\mu_0/\epsilon_0}$ and $\eta_1=\sqrt{\mu_1/(\epsilon_1-j(\sigma/\omega)}$. After putting $\epsilon_1^*=\epsilon_1-j(\sigma/\omega)$, the coefficient of reflection in air is given by

$$\Gamma=\frac{\dfrac{\eta_1}{\cos\theta_1}-\dfrac{\eta_0}{\cos\theta_0}}{\dfrac{\eta_1}{\cos\theta_1}+\dfrac{\eta_0}{\cos\theta_0}}$$

$$=\frac{\eta_1\cos\theta_0-\eta_0\cos\theta_1}{\eta_1\cos\theta_0+\eta_0\cos\theta_1}$$

$$=\frac{\sqrt{\mu_1/\epsilon_1^*}\cos\theta_0-\sqrt{\mu_0/\epsilon_0}\cos\theta_1}{\sqrt{\mu_1/\epsilon_1^*}\cos\theta_0+\sqrt{\mu_0/\epsilon_0}\cos\theta_1}\cdot \tag{41}$$

It is shown in the first paragraph of this section how (k_x, k_y, k_z) for a plane wave is transformed into (ξ, ψ, k_z) and k_z into $\sqrt{k^2-\xi^2}$. In addition, knowing $k_z=k\cos\theta$ where θ is the angle that the wave vector $\bar{k}$ makes with the polar axis, it therefore follows that

$$k_{z_0}=k_0\cos\theta_0$$
$$k_{z_1}=k_1\cos\theta_1. \tag{42}$$

In air $k_0=\omega\sqrt{\mu_0\epsilon_0}$, in the medium $k_1=\omega\sqrt{\mu_1\epsilon_1^*}$, hence

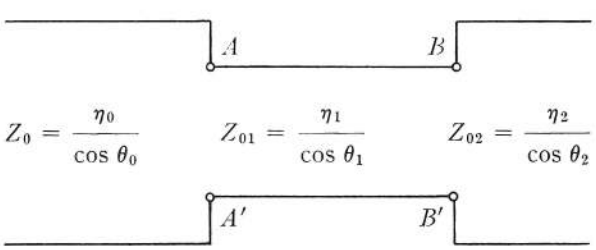
$$k_{z_0}=\sqrt{k_0^2-\xi^2}$$
$$k_{z_1}=\sqrt{k_1^2-\xi^2} \tag{43}$$

where ξ's are same in both media such that the boundary conditions can be matched. Upon substituting various quantities in (41), the coefficient of reflection can be reduced to

$$\Gamma=\frac{\mu_1\sqrt{\xi^2-k_0^2}-\mu_0\sqrt{\xi^2-k_1^2}}{\mu_1\sqrt{\xi^2-k_0^2}+\mu_0\sqrt{\xi^2-k_1^2}} \tag{44}$$

which is the same as the coefficient of reflection in (5). In view of the agreement with the result obtained in the analysis, the choice of the transmission line model based on the horizontal rather than the vertical polarization of a plane wave is the relevant one in this case.

Such a choice also holds true in the case where the semi-infinite plane medium is replaced by layered media, consisting of an upper stratum of thickness t characterized by material constants ϵ_1, μ_1, and σ_1, and a lower stratum of infinite thickness characterized by ϵ_2, μ_2, and σ_2, the transmission line analog would be as follows:



$$Z_0=\frac{\eta_0}{\cos\theta_0}\qquad Z_{01}=\frac{\eta_1}{\cos\theta_1}\qquad Z_{02}=\frac{\eta_2}{\cos\theta_2}$$

where Z_0, Z_{01}, and Z_{02} are the characteristic impedances of the three lines, respectively.

The input impedance at AA' is

$$Z_{AA'}=Z_{01}\frac{1+\Gamma_{12}e^{-2\lambda_1 t}}{1-\Gamma_{12}e^{-2\lambda_1 t}} \tag{45}$$

where

$$\Gamma_{12}=\frac{\mu_2\lambda_1-\mu_1\lambda_2}{\mu_2\lambda_1+\mu_1\lambda_2}$$

$$\lambda_1=\sqrt{\xi^2-k_1^2}\quad\text{and}\quad\lambda_2=\sqrt{\xi^2-k_2^2}.$$

The coefficient of reflection at AA' becomes

$$\Gamma=\frac{Z_{AA'}-Z_0}{Z_{AA'}+Z_0}=\frac{\Gamma_{01}+\Gamma_{12}e^{-2\lambda_1 t}}{1+\Gamma_{01}\Gamma_{12}e^{-2\lambda_1 t}} \tag{46}$$

where

$$\Gamma_{01}=\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1}\quad\text{and}\quad\lambda_0=\sqrt{\xi^2-k_0^2}.$$

On the other hand, from the present analysis point of view, the magnetic potential A_0 satisfies the following wave equation in the three regions as previously indicated:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right)-\frac{A_\phi}{\rho^2}+\frac{\partial^2 A_\phi}{\partial\ ^2}+k_0^2A_\phi$$
$$=-\mu_1 I\delta(\rho-a)\delta(z-h) \tag{47}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right)-\frac{A_\phi}{\rho^2}+\frac{\partial^2 A_\phi}{\partial z^2}-k_1^2A_\phi=0 \tag{48}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial\rho}\right)-\frac{A_\phi}{\rho^2}+\frac{\partial^2 A_\phi}{\partial\ ^2}-k_2^2A_\phi=0. \tag{49}$$

The boundary conditions are

$$\frac{1}{\mu_0}\frac{\partial A_\phi}{\partial\ }\bigg|_{z=0^+}=\frac{1}{\mu_1}\frac{\partial A_\phi}{\partial z}\bigg|_{z=0^-}$$
$$A_\phi\big|_{z=0^+}=A_\phi\big|_{z=0^-}$$
$$\frac{1}{\mu_1}\frac{\partial A_\phi}{\partial\ }\bigg|_{z=-t^+}=\frac{1}{\mu_2}\frac{\partial A_\phi}{\partial z}\bigg|_{z=-t^-}$$
$$A_\phi\big|_{z=-t^+}=A_\phi\big|_{z=-t^-}. \tag{50}$$

By extending the procedure of Appendix A, it follows that for $z>0$,

$$A_\phi=\frac{\mu_0 I_a}{2}\int_0^\infty d\xi J_1(\xi\rho)J_1(\xi a)\cdot\left\{\frac{\xi}{\lambda_0}e^{\lambda_0(z-z_0)}+F(\xi)e^{-\lambda_0 z}\right\}. \tag{51}$$

For $0<Z<-t$,

$$A_\phi=\frac{\mu_0 I_a}{2}\int_0^\infty d\xi J_1(\xi\rho)J_1(\xi a)\left\{G(\xi)e^{\lambda_1 z}+H(\xi)e^{-\lambda_1 z}\right\}. \tag{52}$$

And for $z<-t$

$$A_\phi=\frac{\mu_0 I_a}{2}\int_0^\infty d\xi J_1(\xi\rho)J_1(\xi a)I(\xi)e^{\lambda_2(z+t)}. \tag{53}$$

By virtue of the boundary conditions, one obtains the following set of equations:

$$F(\xi) - \{G(\xi) + H(\xi)\} = \frac{\xi}{\lambda_0} e^{-\lambda_0 z_0}$$

$$G(\xi)e^{-\lambda_1 t} + H(\xi)e^{\lambda_1 t} = I(\xi)$$

$$\frac{\lambda_0}{\mu_0} F(\xi) + \frac{\lambda_1}{\mu_1} G(\xi) - \frac{\lambda_1}{\mu_1} H(\xi) = \frac{\xi}{\mu_0} e^{-\lambda_0 z_0}$$

$$\frac{\mu_2\lambda_1}{\mu_1\lambda_2} \{G(\xi)e^{-\lambda_1 t} - H(\xi)e^{\lambda_1 t}\} = I(\xi). \tag{54}$$

From this set of equations, one may solve for $F(\xi)$, $G(\xi)$, $H(\xi)$, and $I(\xi)$. Among these, $F(\xi)$ is of particular interest to us. From (54), $F(\xi)$ is found as follows:

$$F(\xi) = \frac{\xi}{\lambda} e^{-\lambda_0 z_0} \left\{\frac{\Gamma_{01} + \Gamma_{12}e^{-2\lambda_1 t}}{1 + \Gamma_{01}\Gamma_{12}e^{-2\lambda_1 t}}\right\}. \tag{55}$$

The expression in the braces is the same as (46).

List of Symbols

a = radius of a circular coil.
A_ϕ = ϕ-component of the magnetic vector potential.
B_ρ, B_ϕ, B_z = cylindrical components of the magnetic flux density.
E_ρ, E_ϕ, E_z = cylindrical components of the electric field intensity.
f = frequency.
h = coil to material spacing.
I = current in a coil.
$J_1(\xi a)$ = Bessel function of first kind and first order.
$\bar{k}$ = wave vector.
k_x, k_y, k_z = rectangular components of the wave vector.
k_ρ, k_ϕ, k_z = cylindrical components of the wave vector.
$k_0^2 = \omega^2\mu_0\epsilon_0$
$k_i^2 = \omega^2\mu_i\epsilon_i - j\omega\mu_i\sigma_i$
Q = charge of a coil.
$r = \sqrt{\rho^2 + a^2 - 2a\rho \cos(\phi_0 - \phi)}$.
$r' = \sqrt{s^2 + (\mu_0\zeta)^2}$
$R = \sqrt{(z-h)^2 + \rho^2 + a^2 - 2a\rho \cos(\phi_0 - \phi)}$.
$R' = \sqrt{(z+h)^2 + \rho^2 + a^2 - 2a\rho \cos(\psi_0 - \phi)}$.
$R'' = \sqrt{r^2 + s^2 - 2sr \cos \alpha + (\mu_1\zeta + z + h)^2}$.
R_r = reflected resistance of a coil.
s, α, ζ = cylindrical coordinates in the region below the image coil.
V = scalar potential.
V_i = induced voltage of a coil.
X_r = reflected reactance of a coil.
Z = input impedance of a coil.
Z_r = reflected impedance of a coil.
Γ = coefficient of reflection.
Γ_{01} = coefficient of reflection in the air due to another medium.
Γ_{12} = coefficient of reflection in medium one due to medium two.
$\delta(z-h)$, $\delta(\rho - a)$ = Dirac functions.
ϵ_0 = permittivity of the air.
ϵ_i = permittivity of medium i.
ϵ_r = relative permittivity of medium i.
η = wave impedance of a medium, $\sqrt{\mu/\epsilon}$.
$\lambda_0 = \sqrt{\xi^2 - k_0^2}$
$\lambda_i = \sqrt{\xi^2 - k_i^2}$
μ_0 = permeability of the air.
μ_i = permeability of medium i.
μ_r = relative permeability of medium i.
ρ, ϕ, z = cylindrical coordinates.
σ_i = conductivity of medium i.
ω = angular frequency in radians per second.

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