axioms

MDPI

Article

The Regional Enlarged Observability for Hilfer Fractional Differential Equations

Abu Bakr Elbukhari [1,2,*,†], Zhenbin Fan [1,†] and Gang Li [1,†]

[1] School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China; zbfan@yzu.edu.cn (Z.F.); gli@yzu.edu.cn (G.L.)
[2] Department of Mathematics, Faculty of Education, University of Khartoum, Khartoum, Omdurman 406, Sudan
* Correspondence: mrbakri123@hotmail.com; Tel.: +86-13270-5513-72
† These authors contributed equally to this work.

Abstract: In this paper, we investigate the concept of regional enlarged observability (ReEnOb) for fractional differential equations (FDEs) with the Hilfer derivative. To proceed this, we develop an approach based on the Hilbert uniqueness method (HUM). We mainly reconstruct the initial state ν_0^1 on an internal subregion ω from the whole domain Ω with knowledge of the initial information of the system and some given measurements. This approach shows that it is possible to obtain the desired state between two profiles in some selective internal subregions. Our findings develop and generalize some known results. Finally, we give two examples to support our theoretical results.

Keywords: Hilfer fractional derivatives; fractional diffusion systems; regional enlarged observability; Hilbert uniqueness method

MSC: 35R11; 33B07; 93C20; 44A10

check for updates

Citation: Elbukhari, A.B.; Fan, Z.; Li, G. The Regional Enlarged Observability for Hilfer Fractional Differential Equations. *Axioms* **2023**, *12*, 648. https://doi.org/10.3390/axioms12070648

Academic Editors: Hatıra Günerhan, Francisco Martínez González and Mohammed K. A. Kaabar

Received: 30 May 2023
Revised: 20 June 2023
Accepted: 22 June 2023
Published: 29 June 2023

1. Introduction

In recent decades, fractional calculus theory has proven to be a significant tool for the formulation of several problems in science and engineering, where fractional derivatives and integrals can be utilized to describe the characteristics of various real materials in various scientific disciplines; see, e.g., [1–5]. This theory has recently received a large amount of consideration by many academics; we mention Euler, Laplace, Riemann, Liouville, Marchaud, Riesz, and Hilfer; see, e.g., [6–8]. Distributed parameter systems can be analysed in terms of controllability, observability, and stability, which lead to numerous applications. However, one of the most basic concerns in system analysis and control is observability, which is concerned with the reconstruction of a system's initial state that is taken from measurements on a system by means of so-called sensors; see, [9]. Amouroux et al. [10] developed two approaches to investigate regional observability (ReOb) for distributed systems. The first is state-space-based, and the second allows for estimating the state on the considered subregion. El Jai et al. [11] introduced the concept of regional strategic sensors for a class of distributed systems and presented the sensor characterization for various geometrical situations. In [12], Al-Saphory et al. considered and analysed the notion of regional gradient strategic sensors, and the results applied to a two-dimensional linear infinite distributed system in Hilbert space.

In a problem governed by a diffusion system, it is commonly known that the positioning of sensors is restricted by severe practical restrictions. In fact, observation processes are generally restricted to subsets, boundaries, or points [13,14]. This indicates that the operators of the observation can be unbounded in their state spaces.

Recently, the study of ReOb for partial differential equations (PDEs) has received considerable attention in the literature. Zerrik et al. [15] reviewed regional boundary observability for a two-dimensional diffusion system. In [16], Chen investigated infinite time exact observability for the Volterra system in Hilbert spaces. Chen and Yi [17] studied the observability and admissibility of Volterra systems in Hilbert spaces. Zouiten et al. [18] studied the following ReEnOb for a linear parabolic system.

$$\begin{cases} \frac{\partial}{\partial t}\nu(y,t) = A\nu(y,t) & \text{in } \Omega\times[0,T], \\ \nu(\xi,t) = 0 & \text{on } \Sigma_T, \\ \nu(y,0) = \nu_0(y) & \text{in } \Omega, \\ \mathfrak{M}(t) = C\nu(t), & t\in[0,T], \end{cases} \tag{1}$$

where A is an infinitesimal operator and generates a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ on the state space $L^2(\Omega)$, Ω is an open bound of $L^2(\Omega)$, and $\mathfrak{M}$ is the output function (OuPuFu), which represents the measurements. The authors used the HUM approach to reconstruct the initial state between two profiles in an internal subregion.

More recently, many researchers have investigated the ReOb for fractional differential equations (FDEs). In [19], Zguaid and El Alaoui investigated the notion of the regional boundary observability of Caputo fractional systems. Zguaid et al. [20] studied ReOb for a class of linear time-fractional systems using the HUM approach and proved that the considered approach allows to transform the ReOb problem into a solvability one. Regional gradient observability for Caputo fractional diffusion systems is considered in [21]. In [22], Ge et al. presented the notion of the regional gradient observability for Riemann–Liouville (R-L) diffusion systems for the first time. Cai et al. [23] investigated the concept of exact and approximate ReOb of Hadamard–Caputo diffusion systems using the HUM approach. Zguaid and El Alaoui [24] investigated the notion of regional boundary observability of R-L linear diffusion systems by using an extension of HUM.

On the other hand, some works concerning the concept of ReEnOb-FDEs have recently been conducted. In [25], Zouiten et al. studied the ReEnOb for R-L fractional evolution equations with R-L derivatives:

$$\begin{cases} {}^{RL}_{0}\mathrm{D}^{\eta}_{t}\nu(y,t) = A\nu(y,t) & \text{in } \Omega\times[0,T], \\ \nu(\xi,t) = 0 & \text{on } \Sigma_T, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\eta}\nu(y,t) = \nu_0(y) & \text{in } \Omega, \\ \mathfrak{M}(t) = C\nu(t), & t\in[0,T], \end{cases} \tag{2}$$

where Ω is an open bound of $\mathbb{R}^n (n = 1,2,3)$, with the regular boundary $\partial\Omega$ and ${}^{RL}_{0}\mathrm{D}^{\eta}_{t}$ and ${}_0I_t^{1-\eta}$ are R-L fractional derivatives and R-L fractional integrals of orders $0 < \eta \leq 1$ and $1-\eta$, respectively. The authors developed an approach based on HUM allowing them to reconstruct the initial state between two given functions in an internal subregion of the whole domain. In [26], Zouiten and Boutoulout investigated the ReEnOb for the following Caputo fractional diffusion system in a Hilbert space

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\eta}_{t}\nu(y,t) = A\nu(y,t) & \text{in } \Omega\times[0,T], \\ \nu(\xi,t) = 0 & \text{on } \Sigma_T, \\ \nu(y,t) = \nu_0(y) & \text{in } \Omega, \\ \mathfrak{M}(t) = C\nu(t), & t\in[0,T]. \end{cases} \tag{3}$$

The HUM approach for fractional differential systems is used for the process of reconstructing the initial state between two profiles in a considered subregion of the whole domain.

Inspired and motivated by the above discussion, in this manuscript we extend the investigation of the notion of the ReEnOb for sub-diffusion systems with fractional derivatives, augmented and restricted by some measurements given by the so-called OuPuFu.

We note that FDEs have been widely used for modelling in various science and engineering fields due to their well-described systems and high accuracy, as well as yielding better results compared with systems with integer differentiation. Therefore, the results obtained from Systems (2) and (3) are better than those of System (1). Moreover, use the Hilfer fractional derivative as we know it has two parameters and contains Caputo and R-L derivatives in its definition. Thus, our findings can be seen as a generalization of the mentioned results.

This paper is interested in the concept of ReEnOb for the following sub-diffusion system via Hilfer FDs of order η, type κ and augmented with the OuPuFu (5). We first characterize the ReEnOb of a diffusion system augmented with the OuPuFu in an internal subregion ω of Ω. Moreover, we recognize two types of sensors based on the boundness issue of the observation operator C. Then, we reconstruct the initial state ν_0^1 of the addressed system using an approach that relies on the HUM approach introduced by Lions [27]. The investigation of the addressed problem shows that it is possible to obtain the desired state between two profiles in some selective internal subregions. Let Ω be an open bound of $\mathbb{R}^n(n = 1, 2, 3)$ with the regular boundary $\partial\Omega$, and let $\mathfrak{J} = [0, T]$. The space $\mathfrak{S}_T = \Omega \times \mathfrak{J}$ and $\Sigma_T = \partial\Omega \times \mathfrak{J}$. We consider the following diffusion sub-system:

$$\begin{cases} {}^H_0\mathrm{D}_t^{\eta,\kappa}\nu(y,t) = A\nu(y,t) & \text{in } \mathfrak{S}_T, \\ \nu(\xi,t) = 0 & \text{on } \Sigma_T, \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\zeta}\nu(y,t) = \nu_0(y) & \text{in } \Omega, \end{cases} \tag{4}$$

where ${}^H_0\mathrm{D}_t^{\eta,\kappa}$ stands for the Hilfer fractional derivative (left-sided) of order $0 < \eta < 1$, type $0 \leq \kappa \leq 1$ with respect to time t, the integral ${}_0I_t^{1-\zeta}$, $\zeta = \eta + \kappa - \eta\kappa$, $0 < \zeta \leq 1$ is the left-sided R-L fractional integral operator (1), and the operator A is linear and has a dense domain, so the coefficients are independent of time t. Moreover, operator A is infinitesimal and generates a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ on the state space $L^2(\Omega)$, which is a Hilbert space. Here, the initial state $\nu_0 \in L^2(\Omega)$ is assumed to be unknown. The measurements and information of System (4) are obtained by the OuPuFu below:

$$\mathfrak{M}(t) = C\nu(t), \qquad t \in \mathfrak{J}, \tag{5}$$

where C is the observation operator, and it is a linear, not necessary a bounded, operator determined by the number of sensors or their structure, with a dense domain $\mathcal{D}(C) \subseteq L^2(\Omega)$ with range in the observation space $\mathcal{O} = L^2(\mathfrak{J}; \mathbb{R}^q)$ ($q \in \mathbb{N}$ is the number of considered sensors), and $\mathcal{O}$ is a Hilbert space.

This paper is arranged as follows: In Section 2, we review the definitions, basic concepts, and lemmas utilized throughout this paper. In Section 3, we characterize the ReEnOb. Moreover, we present some remarks, then introduce and prove the main theorem of the ReOb of the Hilfer diffusion System (4). In Section 4, the HUM approach is introduced and applied in the reconstruction process of the initial state of System (4). In addition, two theoretical illustrative examples are given to support our results. In Section 5, we give some conclusions.

2. Preliminaries

In this section, we review the essential definitions, notations, and basic facts utilized throughout this paper.

Definition 1. *(See [7]) The R-L fractional integral (left-sided) of order η for a function $f : \mathfrak{J} \to \mathbb{R}$ is defined as*

$${}_0I_t^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(s)\mathrm{d}s, \quad 0 < \eta < 1.$$

Definition 2. *(See [7]) The R-L fractional integral (right-sided) of order η for a function $f : \mathfrak{J} \to \mathbb{R}$ is defined as*

$$ {}_tI_T^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_t^T (s-t)^{\eta-1} f(s) \mathrm{d}s, \qquad 0 < \eta < 1. $$

Definition 3. *(See [1,28]) The R-L fractional derivative (left-sided) and R-L fractional derivative (right-sided) of order $0 < \eta < 1$ with respect to t for a function f are defined as*

$$ \begin{aligned} {}^{RL}_{0}\mathrm{D}_t^{\eta} f(t) =& \left({}_0I_t^{1-\eta} f(t) \right)' \\ =& \frac{1}{\Gamma(1-\eta)} \left(\int_0^t (t-s)^{-\eta} f(s) \mathrm{d}s \right)' \quad \text{for a.e. } t \in \mathfrak{J}, \end{aligned} $$

and

$$ \begin{aligned} {}^{RL}_{t}\mathrm{D}_T^{\eta} f(t) =& - \left({}_tI_T^{1-\eta} f(t) \right)' \\ =& - \frac{1}{\Gamma(1-\eta)} \left(\int_t^T (s-t)^{-\eta} f(s) \mathrm{d}s \right)' \quad \text{for a.e. } t \in \mathfrak{J}, \end{aligned} $$

respectively, where the notation $'$ stands for differentiation.

Definition 4. *(See [1,28]) The Hilfer fractional derivative (left-sided) and the Hilfer fractional derivative (right-sided) of order $0 < \eta < 1$, type $0 \leq \kappa \leq 1$ with respect to t for a function f are respectively defined by*

$$ \begin{aligned} {}^{H}_{0}\mathrm{D}_t^{\eta,\kappa} f(t) =& \left({}_0I_t^{\kappa(1-\eta)} \left({}_0I_t^{1-\zeta} f \right)' \right)(t) \\ =& {}_0I_t^{\zeta-\eta} \, {}^{RL}_{0}\mathrm{D}_t^{\zeta} f(t) \\ =& \frac{1}{\Gamma(\zeta-\eta)\Gamma(1-\zeta)} \int_0^t (t-s)^{(\zeta-\eta)-1} \left(\int_0^s (s-\tau)^{-\zeta} f(\tau) \mathrm{d}\tau \right)' \mathrm{d}s, \end{aligned} $$

for almost everywhere $t \in \mathfrak{J}$, where $\zeta = \eta + \kappa - \eta\kappa$, $0 < \zeta \leq 1$, $\zeta \leq \eta$, and $\zeta > \kappa$.

$$ \begin{aligned} {}^{H}_{t}\mathrm{D}_T^{\eta,\kappa} f(t) =& - \left({}_tI_T^{\kappa(1-\eta)} \left({}_tI_T^{1-\zeta} f \right)' \right)(t) \\ =& - {}_tI_T^{\zeta-\eta} \, {}^{RL}_{t}\mathrm{D}_T^{\zeta} f(t) \\ =& - \frac{1}{\Gamma(\zeta-\eta)\Gamma(1-\zeta)} \int_t^T (s-t)^{(\zeta-\eta)-1} \left(\int_s^T (\tau-s)^{-\zeta} f(\tau) \mathrm{d}\tau \right)' \mathrm{d}s, \end{aligned} $$

for a.e. $t \in \mathfrak{J}$.

Next, we recall a mild solution for the following Hilfer fractional evolution equation; see [29].

Lemma 1. *Let $\mathcal{X} = L^2(\Omega)$ be a Hilbert space, for any $u_0 \in \mathcal{X}$, $0 < \eta < 1$, $0 \leq \kappa \leq 1$ and $f \in \mathfrak{J} \times \mathcal{X} \longrightarrow \mathcal{X}$, the function $u \in L^2(\mathfrak{J}; \mathcal{X})$ is said to be a mild solution of the following system*

$$ \begin{cases} {}^{H}_{0}\mathrm{D}_t^{\eta,\kappa} u(t) = Au(t) + f(t,u), & t \in \mathfrak{J}, \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\zeta} u(t) = u_0, \end{cases} \tag{6} $$

if u fulfils

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\zeta-\eta)} \int_0^t (t-s)^{(\zeta-\eta)-1} s^{\eta-1} \int_0^\infty \eta\theta M_\eta(\theta) Q(s^\eta\theta)) u_0 \mathrm{d}\theta \mathrm{d}s \\ &+ \int_0^t \int_0^\infty \eta\theta M_\eta(\theta) Q((t-s)^\eta\theta)(t-s)^{\eta-1} f(s,u(s)) \mathrm{d}\theta \mathrm{d}s, \end{aligned} \tag{7}$$

where $P_\eta(t) = \int_0^\infty \eta\theta M_\eta(\theta) Q(t^\eta\theta)\mathrm{d}\theta$, *and the function* $M_\eta(\theta) = \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-\rho n)}$, *where* $0 < \rho < 1$, $\theta \in \mathbb{C}$ *is the Wright function, which fulfils the following equality:*

$$\int_0^\infty \theta^\iota M_\eta(\theta) d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\eta\iota)} \quad \textit{for} \quad \iota \geq 0, \quad \theta \geq 0.$$

Remark 1. *(See Remark 2.14 in [29]) Let* $0 < \eta < 1$, $0 \leq \kappa \leq 1$, $0 < \zeta \leq 1$ *and* $t \in \mathfrak{J}$*; thus, we have*

$$ {}^{RL}_{0}\mathrm{D}_t^{\zeta-\eta} \mathcal{S}_{\eta,\kappa}(t) = \mathcal{R}_\eta(t), \quad t \in (0,T], \tag{8}$$

where

$$\mathcal{R}_\eta(t) = t^{\eta-1} P_\eta(t), \tag{9}$$

and

$$\mathcal{S}_{\eta,\kappa}(t) = {}_0I_t^{\zeta-\eta} \mathcal{R}_\eta(t). \tag{10}$$

We can rewrite the equality in (7) as follows:

$$u(t) = \mathcal{S}_{\eta,\kappa}(t) u_0 + \int_0^t \mathcal{R}_\eta(t-s) f(s,u(s)) \mathrm{d}s. \tag{11}$$

Note that if the non-linear term of System (6) is zero, then the mild solution (11) becomes $u(\cdot) = \mathcal{S}_{\eta,\kappa}(\cdot)u_0$. Consequently, the mild solution of (4) may alternatively be expressed as

$$v(t) = \mathcal{S}_{\eta,\kappa}(t) v_0, \quad t \in \mathfrak{J}. \tag{12}$$

We give the following lemma, which will be utilized afterwards to prove our results.

Lemma 2. *(See [30]) Let a function* g *be defined on interval* $[S,T]$, $(S < T)$ *and* $S, T \in \mathbb{R}$*, then the reflection operator* $\mathfrak{Q}$ *acting on* g *is*

$$\mathfrak{Q}[g(t)] = g(S+T-t).$$

Lemma 3. *Let* f *be a function defined on the interval* $\mathfrak{J}$ *and let* f *be differentiable and integrable in the Hilfer derivative sense. We now introduce the reflection operator* $\mathfrak{Q}$ *when acting on* f *as follows:*

$$\mathfrak{Q}[f(t)] = f(T-t), \tag{13}$$

Moreover, the following assertions hold,

(i) ${}_0I_t^\eta \mathfrak{Q}[f(t)] = \mathfrak{Q}\left[{}_tI_T^\eta f(t)\right]$.

(ii) $\mathfrak{Q}\left[{}_0I_t^\eta f(t)\right] = {}_tI_T^\eta \mathfrak{Q}[f(t)]$.

(iii) $-{}^H_0\mathrm{D}_t^{\eta,\kappa} \mathfrak{Q}[f(t)] = \mathfrak{Q}\left[{}^H_t\mathrm{D}_T^{\eta,\kappa} f(t)\right]$.

(iv) $\mathfrak{Q}\left[{}^H_0\mathrm{D}_t^{\eta,\kappa} f(t)\right] = -{}^H_t\mathrm{D}_T^{\eta,\kappa} \mathfrak{Q}[f(t)]$.

Note that, assertions (i) and (ii) are given in [25,26]. Here, we state their proof due to the demonstration of assertions (iii) and (iv).

Proof. Our proof is obtained by virtue of Equation (13) and by utilizing changes in the variables, specifically, changes in the role of time.
(i): We show that ${}_0I_t^\eta \mathfrak{Q}[f(t)] = \mathfrak{Q}\left[{}_tI_T^\eta f(t)\right]$. Since

$$\begin{aligned} {}_0I_t^\eta \mathfrak{Q}[f(t)] &= \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} \mathfrak{Q} f(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(T-s) \mathrm{d}s. \end{aligned} \tag{14}$$

Using the change in the variables, let $\tilde{s} = T - s$, then $-\mathrm{d}\tilde{s} = \mathrm{d}s$. Now, for $s = 0$ and $s = t$, we obtain $\tilde{s} = T$ and $\tilde{s} = T - t$, respectively. Let us fix $\mathcal{M} = \frac{1}{\Gamma(\eta)}$. Substituting these values into (14), we obtain

$${}_0I_t^\eta \mathfrak{Q}[f(t)] = -\mathcal{M} \int_T^{T-t} (t - T + \tilde{s})^{\eta-1} f(\tilde{s}) \mathrm{d}\tilde{s},$$

Let $\tilde{s} := s$, we obtain

$${}_0I_t^\eta \mathfrak{Q}[f(t)] = \mathcal{M} \int_{T-t}^T (s - T + t)^{\eta-1} f(s) \mathrm{d}s. \tag{15}$$

We now consider the right-hand side:

$$\begin{aligned} \mathfrak{Q}\left[{}_tI_T^\eta f(t)\right] &= \mathfrak{Q}\left[\mathcal{M} \int_t^T (s-t)^{\eta-1} f(s) \mathrm{d}s\right] \\ &= \mathcal{M} \int_{T-t}^T (s - T + t)^{\eta-1} f(s) \mathrm{d}s. \end{aligned} \tag{16}$$

Consequently, from (15) and (16), we can see that ${}_0I_t^\eta \mathfrak{Q}[f(t)] = \mathfrak{Q}\left[{}_tI_T^\eta f(t)\right]$.
(ii): The proof follows the same way as (i). Considering the left-hand side:

$$\begin{aligned} \mathfrak{Q}\left[{}_0I_t^\eta f(t)\right] =& \mathfrak{Q}\left[\mathcal{M} \int_0^t (t-s)^{\eta-1} f(s) \mathrm{d}s\right] \\ =& \mathcal{M} \int_0^{T-t} (T - t - s)^{\eta-1} f(s) \mathrm{d}s, \end{aligned}$$

and the right-hand side:

$$\begin{aligned} {}_tI_T^\eta \mathfrak{Q}[f(t)] &= \mathcal{M} \int_t^T (s-t)^{\eta-1} \mathfrak{Q} f(s) \mathrm{d}s \\ &= -\mathcal{M} \int_{T-t}^0 (T - s - t)^{\eta-1} f(s) \mathrm{d}s \\ &= \mathcal{M} \int_0^{T-t} (T - t - s)^{\eta-1} f(s) \mathrm{d}s. \end{aligned}$$

(iii): We demonstrate $-{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\mathfrak{Q}[f(t)] = \mathfrak{Q}\left[{}^{H}_{t}\mathrm{D}^{\eta,\kappa}_{T}f(t)\right]$. Let us fix $\tilde{\mathcal{M}} = \frac{1}{\Gamma(\zeta-\eta)\Gamma(1-\zeta)}$, which will be used in the remainder of the proof of this lemma. We first consider the left-hand side:

$$\begin{aligned}-{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\mathfrak{Q}[f(t)] &= -\frac{1}{\Gamma(\zeta-\eta)\Gamma(1-\zeta)}\int_0^t (t-s)^{(\zeta-\eta)-1}\mathfrak{Q}\left(\int_0^s (s-\tau)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s \\ &= -\tilde{\mathcal{M}}\int_0^t (t-s)^{(\zeta-\eta)-1}\left(\int_0^s (s-\tau)^{-\zeta}f(T-\tau)\mathrm{d}\tau\right)'\mathrm{d}s,\end{aligned} \tag{17}$$

Let $\tilde{\tau} = T-\tau$, then $-\mathrm{d}\tilde{\tau} = \mathrm{d}\tau$. Now, for $\tau = 0$ and $\tau = s$, we obtain $\tilde{\tau} = T$ and $\tilde{\tau} = T-s$, respectively. Substituting these values into (17), we obtain

$$-{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\mathfrak{Q}[f(t)] = -\tilde{\mathcal{M}}\int_0^t (t-s)^{(\zeta-\eta)-1}\left(-\int_T^{T-s}(s-T+\tilde{\tau})^{-\zeta}f(\tilde{\tau})\mathrm{d}\tilde{\tau}\right)'\mathrm{d}s, \tag{18}$$

Let $s = T-\tilde{s}$, then $-\mathrm{d}\tilde{s} = \mathrm{d}s$. Now for $s = 0$ and $\tau = s$, we obtain $\tilde{s} = T$ and $\tilde{s} = T-t$, respectively. Substituting these values into (18), we obtain

$$-{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\mathfrak{Q}[f(t)] = \tilde{\mathcal{M}}\int_T^{T-t}(t-T+\tilde{s})^{(\zeta-\eta)-1}\left(\int_{\tilde{s}}^T(\tilde{\tau}-\tilde{s})^{-\zeta}f(\tilde{\tau})\mathrm{d}\tilde{\tau}\right)'\mathrm{d}\tilde{s},$$

Let $\tau := \tilde{\tau}$ and $s := \tilde{s}$, we obtain

$$-{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\mathfrak{Q}[f(t)] = \tilde{\mathcal{M}}\int_T^{T-t}(t-T+s)^{(\zeta-\eta)-1}\left(\int_s^T(\tau-s)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s.$$

On the other hand, we proceed with the right-hand side as follows:

$$\begin{aligned}\mathfrak{Q}\left[{}_{t}\mathrm{D}^{\eta,\kappa}_{T}f(t)\right] &= \mathfrak{Q}\left[-\tilde{\mathcal{M}}\int_t^T(s-t)^{(\zeta-\eta)-1}\left(\int_s^T(\tau-s)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s\right] \\ &= -\tilde{\mathcal{M}}\int_{T-t}^T(s-T+t)^{(\zeta-\eta)-1}\left(\int_s^T(\tau-s)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s \\ &= \tilde{\mathcal{M}}\int_T^{T-t}(s-T+t)^{(\zeta-\eta)-1}\left(\int_s^T(\tau-s)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s,\end{aligned}$$

Hence, $-{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\mathfrak{Q}[f(t)] = \mathfrak{Q}\left[{}_{t}\mathrm{D}^{\eta,\kappa}_{T}f(t)\right]$.

(iv): The proof follows the same way as (iii). We first consider the left-hand side:

$$\begin{aligned}\mathfrak{Q}\left[{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}f(t)\right] &= \mathfrak{Q}\left[\tilde{\mathcal{M}}\int_0^t(t-s)^{(\zeta-\eta)-1}\left(\int_0^s(s-\tau)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s\right] \\ &= \tilde{\mathcal{M}}\int_0^{T-t}(T-t-s)^{(\zeta-\eta)-1}\left(\int_0^s(s-\tau)^{-\zeta}f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s,\end{aligned}$$

then the right-hand side:

$$\begin{aligned}
-{}^{H}_{t}\mathrm{D}^{\eta,\kappa}_{T}\mathfrak{Q}[f(t)] &= -\left[-\tilde{\mathcal{M}}\int_t^T (s-t)^{(\zeta-\eta)-1}\mathfrak{Q}\left(\int_s^T (\tau-s)^{-\zeta} f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s\right] \\
&= \tilde{\mathcal{M}}\int_t^T (s-t)^{(\zeta-\eta)-1}\left(\int_0^s (\tau-s)^{-\zeta} f(T-\tau)\mathrm{d}\tau\right)'\mathrm{d}s \\
&= \tilde{\mathcal{M}}\int_t^T (s-t)^{(\zeta-\eta)-1}\left(-\int_{T-s}^0 (T-\tilde{\tau}-s)^{-\zeta} f(\tilde{\tau})\mathrm{d}\tilde{\tau}\right)'\mathrm{d}s \\
&= -\tilde{\mathcal{M}}\int_{T-t}^0 (T-\tilde{s}-t)^{(\zeta-\eta)-1}\left(\int_0^{T-s} (\tilde{s}-\tilde{\tau})^{-\zeta} f(\tilde{\tau})\mathrm{d}\tilde{\tau}\right)'\mathrm{d}\tilde{s} \\
&= \tilde{\mathcal{M}}\int_0^{T-t} (T-s-t)^{(\zeta-\eta)-1}\left(\int_0^s (s-\tau)^{-\zeta} f(\tau)\mathrm{d}\tau\right)'\mathrm{d}s.
\end{aligned}$$

Consequently, $\mathfrak{Q}\left[{}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t} f(t)\right] = -{}^{H}_{t}\mathrm{D}^{\eta,\kappa}_{T}\mathfrak{Q}[f(t)]$.
Thus, this completes the proof of the lemma. □

Since C is an admissible operator, as we will see later, then the OuPuFu of System (4) is given by

$$\mathfrak{M}(t) = C\mathcal{S}_{\eta,\kappa}(t)\nu_0 = \mathcal{K}_{\eta,\kappa}(t)\nu_0, \quad t \in \mathfrak{J}, \tag{19}$$

where $\mathcal{K}_{\eta,\kappa} : L^2(\Omega) \longrightarrow \mathcal{O}$ is a fractional linear operator. Let us recall the observation space $\mathcal{O} = L^2(\mathfrak{J};\mathbb{R}^q)(q \in \mathbb{N})$. Two cases arise for obtaining the adjoint operator of $\mathcal{K}_{\eta,\kappa}$.

- Case 1. C is bounded. In this case, we can define zonal sensors. Let operator C be from $L^2(\Omega)$ to $\mathcal{O}$. Then, if C^* is adjoint on the other hand, the adjoint of operator $\mathcal{K}_{\eta,\kappa}$ can be obtained by

$$\begin{aligned}
\mathcal{K}^*_{\eta,\kappa} : \mathcal{O} &\longrightarrow L^2(\Omega) \\
\mathfrak{M}^* &\longmapsto \int_0^T \mathcal{S}^*_{\eta,\kappa}(s)C^*\mathfrak{M}^*(s)\mathrm{d}s.
\end{aligned}$$

- Case 2. C is unbounded. We can define pointwise sensors. However, in this case, the operator C can be introduced from $\mathcal{D}(C) \subseteq L^2(\Omega)$ to the observation space $\mathcal{O}$. Then, C^* is adjoint. However, in order to give this case a sense of (5), we make an assumption on C in the following definition, namely, C is an admissible observation operator, as we will see in Definition 5 below.

Definition 5. *(See [18]) The observation operator C is an admissible of* (4) *and* (5)*, if for any $\nu_0 \in \mathcal{D}(C)$ there is a constant $\mathcal{L} > 0$, such that*

$$\int_0^T \|C\mathcal{S}_{\eta,\kappa}(t)\nu_0\|^2\mathrm{d}s \leq \mathcal{L} \parallel \nu_0 \parallel .$$

Note that operator C being admissible assures that the map

$$\nu_0 \longmapsto C\mathcal{S}_{\eta,\kappa}(t)\nu_0 = \mathcal{K}_{\eta,\kappa}(t)\nu_0$$

can be extended to a bounded linear operator from $L^2(\Omega)$ to the space $\mathcal{O}$. Thus, we can introduce $\mathcal{K}^*_{\eta,\kappa}$ as the adjoint of operator $\mathcal{K}_{\eta,\kappa}$ as follows:

$$\begin{aligned}\mathcal{K}^*_{\eta,\kappa} : \mathcal{D}(\mathcal{K}^*_{\eta,\kappa}) \subseteq \mathcal{O} &\longrightarrow L^2(\Omega)\\ \mathfrak{M}^* &\longmapsto \int_0^T \mathcal{S}^*_{\eta,\kappa}(s) C^* \mathfrak{M}^*(s) \mathrm{d}s\end{aligned}$$

3. Characterization of Enlarged Observability

In this section, we will characterize the ReEnOb of System (4) with the output function (5) in the subregion ω of Ω. Let ω be a positive Lebesgue measure, and let us define the restriction mapping (projection mapping) p_ω, as follows:

$$\begin{aligned}p_\omega : L^2(\Omega) &\longrightarrow L^2(\omega)\\ \nu &\longmapsto p_\omega \nu = \nu_{|_\omega}.\end{aligned}$$

We can now define the adjoint p^*_ω of p_ω as follows: $(p^*_\omega \nu)(y) := \nu(y, \cdot)$ when $y \in \omega$, and $(p^*_\omega \nu)(y) := 0$ when $y \in \Omega \backslash \omega$. In addition, we note that the regional exact observability of System (4) with (5) can be achieved at time t in the subregion ω, if $\mathrm{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) = L^2(\omega)$, see, e.g., [25,26,31–33]. Now, let $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, $\gamma_1(\cdot) \leq \gamma_2(\cdot)$ almost everywhere in the subregion ω, be two functions defined in $L^2(\Omega)$. We thus define the following set

$$\mathfrak{Z} = \{\nu \in L^2(\omega) | \gamma_1(\cdot) \leq \nu(\cdot) \leq \gamma_2(\cdot) \text{ almost everywhere in the subregion } \omega\},$$

where $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are given functions in ω. We assume that the initial state is given by

$$\nu_0 = \begin{cases} \nu_0^1 & \text{in } \mathfrak{Z}, \\ \nu_0^2 & \text{in } L^2(\Omega) \backslash \mathfrak{Z}. \end{cases}$$

The main objective of the investigation proposed in this paper is to demonstrate ReEnOb for Hilfer time fractional-order diffusion systems, that is, we will answer the following question: Given the Hilfer fractional diffusion System (4) with (5) in the subregion ω at time $t \in \mathfrak{J}$, can we reconstruct ν_0^1 between $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$?

The following definition will be used in the following.

Definition 6. *If* $\mathrm{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z} = \{0\}$, *then System* (4) *with* (5) *is exactly* $\mathcal{E}$*-observable in the subregion* ω.

Definition 7. *A sensor is exactly* $\mathfrak{Z}$*-strategic in the subregion* ω *if the observed system is exactly* $\mathfrak{Z}$*-observable in subregion* ω.

The following three remarks show that the results obtained in [18,25,26] are particular cases of our results.

Remark 2. *If* $\kappa = 0$ *and* $\eta = 1$, *then the Hilfer fractional diffusion* (4) *corresponds to the normal diffusion process, which is investigated in [18].*

Remark 3. *If* $\kappa = 0$ *and* $0 < \eta < 1$, *then the Hilfer fractional diffusion System* (4) *corresponds to the R-L fractional diffusion process, which is investigated in [25].*

Remark 4. *If* $\kappa = 1$ *and* $0 < \eta < 1$, *then the Hilfer fractional diffusion* (4) *corresponds to the Caputo fractional diffusion process, which is considered in [26].*

The following result can be obtained directly from Definition 7.

Remark 5. *If System (4) with the OuPuFu (5) is exactly $\mathfrak{Z}$-observable in ω_1, then for any subregion ω_2 of ω_1 it is also exactly $\mathfrak{Z}$-observable in ω_2.*

The following remark will be used in the proof of the theorem presented below.

Remark 6. *Let X be a Hilbert space and F a linear subspace of X, then $F \cap F^{\perp} = \{0\}$, where $F^{\perp}$ is the orthogonal complement of F.*

Theorem 1. *The following assertions are equivalent:*

1. *System (4) with the OuPuFu (5) is exactly $\mathfrak{Z}$-observable in the subregion ω.*
2. $\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} \neq \emptyset$.

Proof. We show that Statement 1 implies Statement 2, and Statement 2 implies Statement 1. The following two facts play a key role in the proof.

$$\operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) = \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa})^{\perp}, \tag{20}$$

it follows from Remark 6 that

$$\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa})^{\perp} = \{0\}. \tag{21}$$

We demonstrate that the left-hand side implies the right-hand side, and vice versa:

$$\operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z} = \{0\} \Longleftrightarrow \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} \neq \emptyset.$$

We first show that

$$\operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z} = \{0\} \Longrightarrow \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} \neq \emptyset.$$

Let $y \in \operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z}$, then $y = 0$. From (20), one can see that $y \in \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa})^{\perp} \cap \mathfrak{Z}$. Therefore, it follows from (21) that, $\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa})$ has at least one element, which is zero. Thus, $\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} \neq \emptyset$.

We now prove that statement 2 implies statement 1, that is,

$$\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} \neq \emptyset \Longrightarrow \operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z} = \{0\}.$$

Suppose

$$\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} \neq \emptyset, \tag{22}$$

and

$$\operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z} \neq \{0\}. \tag{23}$$

Now, let $y \in \operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z}$, then $y \neq 0$, $y \in \mathfrak{Z}$ and $y \in \operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega)$. From (20) and (21), we have $y \in \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa})^{\perp}$ and $y \notin \operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa})$, respectively. Consequently, one can see that

$$\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \Big(\operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z}\Big) = \emptyset;$$

therefore,

$$\operatorname{Im}(p_\omega \mathcal{K}^*_{\eta,\kappa}) \cap \mathfrak{Z} = \emptyset,$$

which contradicts (22). Thus, (23) is not true. Consequently,

$$\operatorname{Ker}(\mathcal{K}_{\eta,\kappa} p^*_\omega) \cap \mathfrak{Z} = \{0\}.$$

Therefore, System (4) with (5) is exactly $\mathfrak{Z}$-observable in the subregion ω. This completes the proof. □

4. The Hilbert Uniqueness Method

In this section, we provide an approach for reconstructing the initial state of the system between $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ in subregion ω. Let $\mathfrak{P}$ be a space defined as

$$\mathfrak{P} = \{g \in L^2(\Omega) | g = 0 \text{ in } L^2(\Omega)\backslash\mathfrak{Z}\}. \tag{24}$$

4.1. Pointwise Sensors

Let System (4) be observed by a pointwise sensor $(l, \delta(l - \cdot))$, where $l \in \overline{\Omega}$ is the location of a sensor and δ is the Dirac mass (delta function), which is concentrated in l. Here, the OuPuFu is introduced as

$$\mathfrak{M}(t) = \psi(b, T - t), \quad t \in \mathfrak{J}. \tag{25}$$

Let ψ_0 be in $\mathfrak{P}$; thus, we examine the following system:

$$\begin{cases} {}^H_0\mathrm{D}^{\eta,\kappa}_t \psi(y,t) = A\psi(y,t) & \text{in } \mathfrak{S}_T, \\ \psi(\xi,t) = 0 & \text{on } \Sigma_T, \\ \lim\limits_{t\to 0^+} {}_0I^{1-\zeta}_t \psi(y,t) = \psi_0(y) & \text{in } \Omega. \end{cases} \tag{26}$$

For simplicity of notation, we denote $\psi(y,t) := \psi(t)$. We note that System (26) admits a unique solution $\psi \in L^2(\mathfrak{J}; \mathcal{D}(A)) \cap C(\Omega \times \mathfrak{J})$ given by $\psi(t) = \mathcal{S}_{\eta,\kappa}(t)\varphi_0$, if $\psi_0(x) \in \mathcal{D}(A)$. Let us denote a semi-norm on $\mathfrak{P}$ by

$$\psi_0 \longmapsto \|\psi_0\|^2_{\mathfrak{P}} = \int_0^T \|C\psi(T-t)\|^2 \mathrm{d}t. \tag{27}$$

In the following lemma, we will see that a norm can be defined.

Lemma 4. *If System (4) with OuPuFu (25) is exactly $\mathfrak{Z}$-observable in the subregion ω; consequently, Equation (27) defines a norm in the space $\mathfrak{P}$.*

Proof. Firstly, in light of Theorem 1 and Definition 6, we suppose that System (4) with the OuPuFu (25) is exactly $\mathfrak{Z}$-observable in the space $\mathfrak{P}$. Now, for $\psi_0 \in \mathfrak{P}$ and a semi-norm in $\mathfrak{P}$, we have

$$\|\psi_0\|_{\mathfrak{P}} = 0 \Longrightarrow C\psi(T-t) = 0 \text{ for all } t \in \mathfrak{J}.$$

Let

$$\psi_0 \in L^2(\Omega) \Longrightarrow p_\omega\psi_0 \in L^2(\omega),$$

then,

$$\mathcal{K}_{\eta,\kappa}p^*_\omega p_\omega\psi_0 = C\mathcal{S}_{\eta,\kappa}(t)p^*_\omega p_\omega\psi_0 = 0.$$

Hence,

$$p_\omega\psi_0 \in \mathrm{Ker}(\mathcal{K}_{\eta,\kappa}p^*_\omega).$$

and for $p_\omega\psi_0 \in \mathfrak{Z}$, one has $p_\omega\psi_0 \in \mathrm{Ker}(\mathcal{K}_{\eta,\kappa}p^*_\omega) \cap \mathfrak{Z}$ and $p_\omega\psi_0 = 0$, since the system is exactly $\mathfrak{Z}$-observable in the subregion ω. Consequently, $\psi_0 = 0$ and (27) is a norm. □

We now consider the following system, which is controlled by the solution to System (26), that is,

$$\begin{cases} \mathfrak{Q}\left[-\,{}^H_t\mathrm{D}^{\eta,\kappa}_T \Upsilon(y,t)\right] = A^*\mathfrak{Q}\left[\Upsilon(y,t)\right] + C^*C\mathfrak{Q}[\psi(y,t)] & \text{in } \mathfrak{S}_T, \\ \Upsilon(\xi,t) = 0 & \text{on } \Sigma_T, \\ \lim\limits_{t\to T^-} \mathfrak{Q}\left[{}_tI^{1-\zeta}_T \Upsilon(y,t)\right] = 0 & \text{in } \Omega. \end{cases} \tag{28}$$

Next, for $\psi_0 \in \mathfrak{P}$, we define the operator $\Lambda : \mathfrak{P} \longrightarrow \mathfrak{P}^*$ by

$$\Lambda\psi_0 = \mathcal{P}\left({}_0I_T^{\zeta-\eta}\Upsilon(0)\right), \tag{29}$$

where $\mathcal{P} = p_\omega^* p_\omega$ and $\Upsilon(0) = \Upsilon(y, 0)$.

Next, let us consider the following system:

$$\begin{cases} \mathfrak{Q}\left[-\,{}_t^H\mathrm{D}_T^{\eta,\kappa}\Phi(y,t)\right] = A^*\mathfrak{Q}\left[\Phi(y,t)\right] + C^*\mathfrak{Q}[\mathfrak{M}(t)] & \text{in } \mathfrak{S}_T, \\ \Phi(\xi, t) = 0 & \text{on } \Sigma_T, \\ \lim\limits_{t\to T^-} \mathfrak{Q}\left[{}_tI_T^{1-\zeta}\Phi(y,t)\right] = 0 & \text{in } \Omega. \end{cases} \tag{30}$$

If we choose the initial state ψ_0 of System (26) such that $\Phi(0) = \Upsilon(0)$ in the subregion ω, then one can see that System (30) stands for the adjoint of System (4). Thus, our problem of ReEnOb can be simplified solved in Equation (29), since following Equation (31) is equivalent to Equation (29).

$$\Lambda\psi_0 = \mathcal{P}\left({}_0I_T^{\zeta-\eta}\Phi(0)\right). \tag{31}$$

Theorem 2. *System (4) augmented by (25) is exactly $\mathfrak{Z}$-observable in ω, if Equation (29) has a unique solution $\psi_0 \in \mathfrak{P}$, that coincides with the state v_0^1 observed between functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ in the subregion ω. In addition, $v_0^1 = p_\omega\varphi_0$.*

Proof. We note that, System (4) with (25) is exactly $\mathfrak{Z}$-observable in ω, then the norm $\|\cdot\|_{\mathfrak{P}}$ can be defined on $\mathfrak{P}$ by Lemma 4. Next, we prove that, if Λ is an isomorphism (see [18]), then (29) admits a unique solution in the set $\mathfrak{P}$. For this, we have

$$\begin{aligned} \langle\Lambda\psi_0, \psi_0\rangle_{L^2(\Omega)} &= \left\langle\mathcal{P}\left({}_0I_T^{\zeta-\eta}\Upsilon(0)\right), \psi_0\right\rangle_{L^2(\Omega)} \\ &= \left\langle p_\omega^* p_\omega\left({}_0I_T^{\zeta-\eta}\Upsilon(0)\right), \psi_0\right\rangle_{L^2(\Omega)} \\ &= \left\langle {}_0I_T^{\zeta-\eta}\Upsilon(0), \psi_0\right\rangle_{L^2(\omega)} \end{aligned}$$

We note that the following propositions are important in the following proof.

Proposition 1. *Let $0 < \eta < 1$, $0 \le \kappa \le 1$, $0 < \zeta \le 1$ and $t \in \mathfrak{J}$. Since System (30) is adjoint of (4), then from (9) and (10), we have*

$$\mathcal{R}_\eta^*(t) = t^{\eta-1}P_\eta^*(t),$$

and

$$\mathcal{S}_{\eta,\kappa}^*(t) = {}_0I_t^{\zeta-\eta}\mathcal{R}_\eta^*(t).$$

Therefore, the solution to System (28) is given by

$$\Upsilon(t) = \int_0^{T-t} \mathcal{R}_\eta^*(T-t-s)C^*C\psi(T-s)\mathrm{d}s. \tag{32}$$

Proposition 2. *Let $0 < \eta < 1$, $0 \le \kappa \le 1$, $0 < \zeta \le 1$ and $t \in \mathfrak{J}$, we have*

$${}_0I_T^{\zeta-\eta}\Upsilon(0) = \int_0^T \mathcal{S}_{\eta,\kappa}^*(T-s)C^*C\psi(T-s)\mathrm{d}s. \tag{33}$$

Proof. In view of Fubini's theorem and Equation (32), and for any $\tau \in \mathfrak{J}$, we have

$$
\begin{aligned}
{}_{\tau}I_T^{\zeta-\eta}\Upsilon(\tau) =& \frac{1}{\Gamma(\zeta-\eta)} \int_{\tau}^{T} (t-\tau)^{\zeta-\eta-1}\Upsilon(t)\mathrm{d}t \\
=& \frac{1}{\Gamma(\zeta-\eta)} \int_{\tau}^{T} (t-\tau)^{\zeta-\eta-1} \int_0^{T-t} \mathcal{R}_{\eta}^{*}(T-t-s)C^{*}C\psi(T-s)\mathrm{d}s\mathrm{d}t \\
=& \frac{1}{\Gamma(\zeta-\eta)} \int_0^{T} \left(\int_{\tau}^{T-s} (t-\tau)^{\zeta-\eta-1}\mathcal{R}_{\eta}^{*}(T-t-s)\mathrm{d}t \right) C^{*}C\psi(T-s)\mathrm{d}s
\end{aligned}
$$

Let $u = T-t-s$, then $\mathrm{d}u = -\mathrm{d}t$. Now, for $t = \tau$ and $t = T-s$, we obtain $u = T-\tau-s$ and $u = 0$, respectively. Thus, we obtain

$$
\begin{aligned}
{}_{\tau}I_T^{\zeta-\eta}\Upsilon(\tau) =& \frac{1}{\Gamma(\zeta-\eta)} \int_0^{T} \left(\int_0^{T-\tau-s} (T-s-\tau-u)^{\zeta-\eta-1}\mathcal{R}_{\eta}^{*}(u)\mathrm{d}u \right) C^{*}C\psi(T-s)\mathrm{d}s \\
=& \int_0^{T} \mathcal{S}_{\eta,\kappa}^{*}(T-s-\tau)C^{*}C\psi(T-s)\mathrm{d}s.
\end{aligned}
$$

We now let $\tau = 0$, we obtain

$$
{}_{0}I_T^{\zeta-\eta}\Upsilon(0) = \int_0^{T} \mathcal{S}_{\eta,\kappa}^{*}(T-s)C^{*}C\psi(T-s)\mathrm{d}s.
$$

□

Now, we continue the proof of our theorem

$$
\begin{aligned}
\langle \Lambda\psi_0, \psi_0 \rangle_{L^2(\Omega)} &= \left\langle {}_{0}I_T^{\zeta-\eta}\Upsilon(0), \psi_0 \right\rangle_{L^2(\omega)} \\
&= \left\langle \int_0^{T} \mathcal{S}_{\eta,\kappa}^{*}(T-s)C^{*}C\psi(T-s)\mathrm{d}s, \psi_0 \right\rangle \\
&= \int_0^{T} \langle C\psi(T-s), C\mathcal{S}_{\eta,\kappa}(T-s)\psi_0 \rangle \mathrm{d}s \\
&= \int_0^{T} \langle C\psi(T-s), C\psi(T-s) \rangle \mathrm{d}s \\
&= \int_0^{T} \|C\psi(T-s)\|^2 \mathrm{d}s \\
&= \|\psi_0\|_{\mathcal{G}}^2.
\end{aligned}
$$

Thus, the operator Λ is an isomorphism. Therefore, we establish that Equation (29) has a unique solution, which corresponds to the desired initial state $\nu_0^1 = p_{\omega}\psi_0$. This completes the proof. □

4.2. Zone Sensors

Here we suppose the measurements of System (4) are given by an internal zone sensor defined by $(\mathcal{A}, h)$ with $\mathcal{A} \subset \Omega$ and $h \in L^2(\mathcal{A})$. The system is augmented with the OuPuFu

$$
\mathfrak{M}(t) = \int_{\mathcal{A}} \nu(y, T-t)h(y)\mathrm{d}y. \tag{34}
$$

In this case, we consider System (26), and we assume $\mathfrak{P}$ is given by Equation (24). Then, a semi-norm can be introduced by

$$
\|\varphi_0\|_{\mathfrak{P}}^2 = \int_0^{T} \langle\, \psi(T-t), h \rangle_{L^2(\mathcal{A})}^2 \mathrm{d}t, \tag{35}
$$

and if System (26) with (25) is exactly $\mathfrak{Z}$-observable in a subregion ω of Ω, then a norm can be defined.

In this case, we can introduce the adjoint System of (26) as follows:

$$\begin{cases} \mathfrak{Q}\Big[-\,{}^{H}_{t}\mathrm{D}^{\eta,\kappa}_{T}\Upsilon(y,t)\Big] = A^{*}\mathfrak{Q}\big[\Upsilon(y,t)\big] + \langle\,\mathfrak{Q}[\psi(t)],h\rangle_{L^2(\mathcal{A})}h(y) & \text{in } \mathfrak{S}_T,\\ \Upsilon(\xi,t)=0 & \text{on } \Sigma_T,\\ \lim\limits_{t\to T^-}\mathfrak{Q}\Big[{}_{t}I^{1-\zeta}_{T}\Upsilon(y,t)\Big]=0 & \text{in } \Omega. \end{cases} \tag{36}$$

Thus, the operator Λ can be defined by

$$\begin{aligned} \Lambda : \mathfrak{P} &\longrightarrow \mathfrak{P}^{*} \\ \psi_0 &\longmapsto \Lambda\psi_0 = \mathcal{P}\Big({}_{0}I^{\zeta-\eta}_{T}\Upsilon(0)\Big), \end{aligned} \tag{37}$$

where $\mathcal{P} = p^{*}_{\omega}p_{\omega}$ is a projection operator. For simplicity, let us write $\Upsilon(0) = \Upsilon(y,0)$.

We introduce the following system

$$\begin{cases} \mathfrak{Q}\Big[-\,{}^{H}_{t}\mathrm{D}^{\eta,\kappa}_{T}\Phi(y,t)\Big] = A^{*}\mathfrak{Q}\big[\Phi(y,t)\big] + \langle\,\mathfrak{Q}[\mathfrak{M}(t)],h\rangle_{L^2(\mathcal{A})}p_{\mathcal{A}}h(y) & \text{in } \mathfrak{S}_T,\\ \Phi(\xi,t)=0 & \text{on } \Sigma_T,\\ \lim\limits_{t\to T^-}\mathfrak{Q}\Big[{}_{t}I^{1-\zeta}_{T}\Phi(y,t)\Big]=0 & \text{in } \Omega. \end{cases} \tag{38}$$

If the initial state ψ_0 of System (26) is chosen such that $\Phi(0) = \Upsilon(0)$ in the subregion ω, then one can see that System (38) is the adjoint of System (4); thus, our ReEnOb problem can be simplified and solved by the following equation

$$\Lambda\psi_0 = \mathcal{P}\Big({}_{0}I^{\zeta-\eta}_{T}\Phi(0)\Big), \tag{39}$$

Theorem 3. *If System (4) with OuPuFu (34) is exactly $\mathfrak{Z}$-observable in the subregion ω, then Equation (39) has a unique solution $\psi_0 \in \mathfrak{P}$ that corresponds with the observed initial state ν^1_0 between functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ in the subregion ω.*

Proof. The procedures of the proof are remarkably similar to those of Theorem 2. □

4.3. Examples

Example 1. *In this subsection, we will consider the case where C is unbounded (pointwise sensors). The following time fractional diffusion system can be use to describe a chemical reaction or a heat conduction.*

Let $\Omega_1 = [0,l]$ and $\tilde{\mathfrak{S}}_T = \Omega_1 \times \mathfrak{J}$, we thus consider

$$\begin{cases} {}^{H}_{0}\mathrm{D}^{\eta,\kappa}_{t}\nu(y,t) = \aleph_{\eta,\kappa}\frac{\partial^2}{\partial y^2}\nu(y,t) + f(y,t) & \text{in } \tilde{\mathfrak{S}}_T,\\ \nu(0,t)=h_1(y), \quad \nu(l,t)=h_2(y) & \text{in } \mathfrak{J},\\ \lim\limits_{t\to 0^+} {}_{0}I^{1-\zeta}_{t}\nu(y,t) = \nu_0(y) & \text{in } \Omega_1, \end{cases} \tag{40}$$

where $f(y,t)$ is the density of the sources that transmits the substance in/out the system, $A = \aleph_{\eta,\kappa}\frac{\partial^2}{\partial x^2}$ and $\aleph_{\eta,\kappa}$ represents a constant of physical dimension $[\aleph_{\eta,\kappa}] = cm^2 s^{\eta}$, which only depends on η and is independent of κ.

For simplicity, we assume $\aleph_{\eta,\kappa} = 1$, $f(y,t) = 0$, $h_1(y) = h_2(y) = 0$, and $l = 1$, obtaining $\Omega_1 = [0,1]$ and $\bar{\bar{\mathfrak{S}}}_T = \Omega_1 \times \mathfrak{J}$. Hence, System (40) can be written as follows

$$\begin{cases} {}^{H}_{0}\mathrm{D}_t^{\eta,\kappa}\nu(y,t) = \frac{\partial^2}{\partial y^2}\nu(y,t) & \text{in } \bar{\bar{\mathfrak{S}}}_T, \\ \nu(\xi,t) = 0 & \text{in } [0,T], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\zeta}\nu(y,t) = \nu_0(y) & \text{in } [0,1], \end{cases} \tag{41}$$

augmented with the OuPuFu

$$\mathfrak{M}(t) = C\nu(y,t) = \nu(b,t), \tag{42}$$

where $\frac{1}{4} = b \in [0,1]$, and System (44) has a mild solution $\nu(y,t)$, $t \in \mathfrak{J}$ given by

$$\begin{aligned} \nu(y,t) =& 2\sum_{n=1}^{\infty} t^{\eta+\kappa(1-\eta)-1}E_{\eta,\kappa(\eta-1)-\eta}(-n^2\pi^2t^{\eta})\sin(n\pi y) \\ &\times \int_0^1 \nu_0(y)\sin(n\pi y)\mathrm{d}x, \end{aligned} \tag{43}$$

where $E_{\eta,\kappa}(\cdot)$ stands for the two-parameter Mittag–Leffler function [4], and one can easily see that the operator $\frac{\partial^2}{\partial y^2}$ has a complete set of eigenfunctions $\phi_n = \sin(n\pi y)$ in the Hilbert space $L^2(\Omega_1)$ associated with the eigenvalues $\lambda_n = -n^2\pi$. Let us assume the initial state that needs to be observed in System (44) is given by $\nu_0(y) = \sin(2\pi y)$, $\eta = 0.2$, and $\kappa = 0.4$. Now, for the subregion $\omega_1 = \left[\frac{1}{2}, \frac{2}{3}\right] \subset [0,1]$, the following results hold.

Proposition 3. *There exists a state for which System (44) with the OuPuFu (42) is not weakly observable in Ω_1, but is $\mathfrak{Z}_1$-observable in the subregion ω_1.*

Proof. To show that System (44) with the OuPuFu (42) is not weakly observable in Ω_1, it sufficient to verify that $\nu_0 \in \mathrm{Ker}(\mathcal{K}_{\eta,\kappa})$. From Equation (43) and the assumptions above we can now calculate

$$\begin{aligned} \mathcal{K}_{0.2,0.4}\nu_0 =& 2\sum_{n=1}^{\infty} t^{-0.48}E_{0.2,-0.52}(-n^2\pi^2t^{0.2})\sin\left(\frac{n\pi}{4}\right) \\ &\times \int_0^1 \sin(2\pi y)\sin(n\pi y)\mathrm{d}y \\ =& 0. \end{aligned}$$

Hence, $\nu_0 \in \mathrm{Ker}(\mathcal{K}_{\eta,\kappa})$. As a result, System (44) and (42) is not weakly observable in Ω_1,

$$
\begin{aligned}
\mathcal{K}_{0.2,0.4}p^*_{\omega_1}p_{\omega_1}\nu_0 =& 2\sum_{n=1}^{\infty} t^{-0.48}E_{0.2,-0.52}(-n^2\pi^2t^{0.2})\sin(0.25n\pi) \\
&\times \int_0^1 p^*_{\omega_1}p_{\omega_1}\sin(2\pi y)\sin(n\pi y)\mathrm{d}y \\
=& 2\sum_{n=1}^{\infty} t^{-0.48}E_{0.2,-0.52}(-n^2\pi^2t^{0.2})\sin\left(\frac{n\pi}{4}\right) \\
&\times \int_{\frac{1}{2}}^{\frac{2}{3}} \sin(2\pi y)\sin(n\pi y)\mathrm{d}y \\
=& 2t^{-0.48}E_{0.2,-0.52}(-\pi^2t^{0.2})\sin\left(\frac{\pi}{4}\right) \\
&\times \int_{\frac{1}{2}}^{\frac{2}{3}} \sin(2\pi y)\sin(\pi y)\mathrm{d}y \\
=& \frac{(3\sqrt{3}-8)t^{-0.48}}{6\sqrt{2}\pi}E_{0.2,-0.52}(-\pi^2t^{0.2}) \\
\neq& 0.
\end{aligned}
$$

While on the other hand, this leads us to observe that the initial state ν_0 is weakly observable in the subregion ω_1. In addition, for all $y \in \omega_1$, we have

$$\tilde{\gamma_1} = |\nu^0_{|\omega_1}(y)| - \frac{2}{3} < \nu^0_{|\omega_1}$$

and

$$\tilde{\gamma_2} = |\nu^0_{|\omega_1}(y)| + \frac{2}{3} > \nu^0_{|\omega_1}.$$

Thus, $p_{\omega_1}\nu_0 \in \mathfrak{Z}_1$ and (44) together with (42) is $\mathfrak{Z}_1$-observable in ω_1. This completes the proof. □

Let the space $\mathfrak{P}_1$ be given by

$$\mathfrak{P}_1 = \{g \in L^2(\Omega_1) | g = 0 \quad \text{in} \quad L^2(\Omega_1)\backslash \mathfrak{Z}_1\}.$$

From Lemma 4, we have

$$\psi_0 \longmapsto \|\psi_0\|^2_{\mathfrak{P}_1} = \int_0^T \|C\psi(T-t)\|^2\mathrm{d}t.$$

which defines a norm on $\mathfrak{P}_1$, and thus we can introduce the adjoint System of (44) as follows:

$$
\begin{cases}
\mathfrak{Q}\Big[-{}^H_tD^{\eta,\kappa}_T\Phi(y,t)\Big] = A^*\mathfrak{Q}\big[\Phi(y,t)\big] + \mathfrak{M}(b,T-t) & \text{in} \quad \mathfrak{S}_T, \\
\Phi(\xi,t) = 0 & \text{on} \quad \partial\Omega_1 \times [0,T], \\
\lim_{t\to T^-} \mathfrak{Q}\Big[{}_tI^{1-\zeta}_T\Phi(y,t)\Big] = 0 & \text{in} \quad \Omega_1,
\end{cases}
$$

then, in view of Theorem 2, we can now conclude that $\Lambda\psi_0 = \mathcal{P}\Big({}_0I^{\zeta-\eta}_T\Phi(0)\Big)$ has a unique solution in $\mathfrak{P}_1$, and the initial state ν_0 is observed between functions $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ in the subregion ω_1.

Example 2. *In this example, we consider C as bounded (zone sensors). Considering the following diffusion system*

$$\begin{cases} {}_0^H\mathrm{D}_t^{\eta,\kappa}\nu(y,t) = \frac{\partial^2}{\partial y^2}\nu(y,t) & \text{in} \quad [0,1]\times[0,T], \\ \nu(\xi,t) = 0 & \text{in} \quad [0,T], \\ \lim_{t\to 0^+} {}_0I_t^{1-\zeta}\nu(y,t) = \nu_0(y) & \text{is unknown in} \quad [0,1], \end{cases} \tag{44}$$

augmented with the OuPuFu

$$\mathfrak{M}(t) = \int_0^1 \nu(y,T-t)h(y)\mathrm{d}y = \nu(b,t), \tag{45}$$

where $A = \frac{\partial^2}{\partial y^2}$ with eigenvalues $\lambda_n = -n^2\pi^2$ and the corresponding eigenfunctions $\phi_n(y) = \sin(n\pi y)$. Let us fix $\frac{1}{3} = b \in [0,1] = \Omega_2$ and take any internal subregion $\omega_2 = [\frac{1}{6},\frac{1}{2}]$ of the whole domain. We note that System (44) has a unique mild solution $\nu(y,t)$ in $L^2([0,T];\mathcal{D}(A)) \cap C([0,1]\times[0,T])$.

Proposition 4. *There exists a state for which System (44) with the OuPuFu (45) is not weakly observable in Ω_2, but is $\mathfrak{Z}_2$-observable in the subregion ω_2.*

Proof. To show that System (44) with the OuPuFu (45) is not weakly observable in Ω_2, it is sufficient to verify that $\nu_0 \in \mathrm{Ker}(\mathcal{K}_{\eta,\kappa})$. Thus, we can now derive

$$C\mathcal{S}_{\eta,\kappa}(t)\nu_0 = \mathcal{K}_{\eta,\kappa}(t)\nu_0 = 2\sum_{n=1}^{\infty} t^{\eta+\kappa(1-\eta)-1}E_{\eta,\kappa(\eta-1)-\eta}(-n^2\pi^2t^\eta)\langle\nu_0,\phi_n\rangle\phi_n\left(\frac{1}{3}\right),$$

where $E_{\eta,\kappa}(\cdot)$ stands for the two-parameter Mittag–Leffler function. Now, for all $y \in [0,1]$, $|\phi_n| \leq \sqrt{2}$, the Mittag–Leffler function $E_{\eta,\kappa(\eta-1)-\eta}(-n^2\pi^2t^\eta)$ is continuous with $|E_{\eta,\kappa(\eta-1)-\eta}(-n^2\pi^2t^\eta)| \leq \frac{K}{1+|-n^2\pi^2|t^\eta}$ for $t \geq 0$ with $K > 0$. Hence,

$$|C\mathcal{S}_{\eta,\kappa}(t)\nu_0| = 2\sum_{n=1}^{\infty} \frac{K\sqrt{2}\|\nu_0\|t^{\eta+\kappa(1-\eta)-1}}{1+|-n^2\pi^2|t^\eta}$$

and

$$\begin{aligned} \mathcal{K}^*_{\eta,\kappa}\mathfrak{M}(t) &= 2\sum_{n=1}^{\infty}\phi_n(y)\int_0^{\frac{1}{3}} \sigma^{\eta+\kappa(1-\eta)-1}E_{\eta,\kappa(\eta-1)-\eta}(-n^2\pi^2\sigma^\eta)\langle C^*\mathfrak{M}(\sigma),\phi_n\rangle\mathrm{d}\sigma \\ &= \mathcal{S}^*_{\eta,\kappa}(t)C^*\mathfrak{M}(t). \end{aligned}$$

Thus, the observation operator C is admissible. From the above, we can see that $Ker\mathcal{K}_{\eta,\kappa}(t) \neq 0$, which means System (44) is not observable in the whole domain $[0,1]$. Next, we investigate the observability of the addressed system in the internal subregion ω_2.

$$\mathcal{K}_{\eta,\kappa}p^*_\omega p_\omega\nu_0 = 2\sum_{n=1}^{\infty}\phi_n(y)t^{\eta+\kappa(1-\eta)-1}E_{\eta,\kappa(\eta-1)-\eta}(-n^2\pi^2t^\eta)\langle p^*_\omega p_\omega\nu_0,\phi_n\rangle_{L^2(\omega_2)} \neq 0.$$

Thus, the initial state ν_0 is weakly observable in the subregion ω_2. In addition, for all $y \in \omega_2$, we have

$$\bar{\bar{\gamma}}_1 = |\nu^0_{|\omega_2}(y)| - \frac{2}{3} < \nu^0_{|\omega_2}$$

and

$$\bar{\bar{\gamma}}_2 = |\nu^0_{|\omega_2}(y)| + \frac{2}{3} > \nu^0_{|\omega_2}.$$

Thus, $p_{\omega_2} v_0 \in \mathfrak{Z}_2$ and (44) together with (42) is $\mathfrak{Z}_2$-observable in ω_2. This completes the proof. □

Let the space $\mathfrak{P}_2$ be given by

$$\mathfrak{P}_2 = \{g \in L^2(\Omega_2) | g = 0 \text{ in } L^2(\Omega_2) \backslash \mathfrak{Z}_2\}.$$

From Lemma 4, we have

$$\psi_0 \longmapsto \|\psi_0\|^2_{\mathfrak{P}_2} = \int_0^T \|C\psi(T-t)\|^2 \mathrm{d}t,$$

which defines a norm on $\mathfrak{P}_2$, and we can introduce the adjoint system of (44) *as follows:*

$$\begin{cases} \mathfrak{Q}\Big[-{}^H_t\mathrm{D}^{\eta,\kappa}_T \Phi(y,t) \Big] = A^*\mathfrak{Q}\big[\Phi(y,t)\big] + \langle C^*\mathfrak{M}(t), h\rangle_{L^2(\omega_2)} h(y) & \text{in } \Omega_2 \times [0,T], \\ \Phi(\xi,t) = 0 & \text{on } \partial\Omega_2 \times [0,T], \\ \lim\limits_{t \to T^-} \mathfrak{Q}\Big[{}_t I^{1-\zeta}_T \Phi(y,t) \Big] = 0 & \text{in } \Omega_2, \end{cases}$$

Then, in view of Theorem 3, we can now conclude that $\Lambda\psi_0 = \mathcal{P}\Big({}_0 I^{\zeta-\eta}_T \Phi(0)\Big)$ has a unique solution in $\mathfrak{P}_2$, and the initial state v_0 can be observed between functions $\bar{\bar{\gamma}}_1$ and $\bar{\bar{\gamma}}_2$ in the subregion ω_2.

5. Conclusions

In this manuscript we studied the concept of regional enlarged observability (ReEnOb) for fractional differential equations (FDEs) with Hilfer derivatives. We developed an approach based on the Hilbert uniqueness method (HUM). Based on this approach and with the knowledge of the initial information of the system and some given measurements, we reconstructed the initial state v_0^1 on an internal subregion ω from the whole domain Ω. Our findings show that it is possible to obtain the desired state between two profiles in some selective internal subregions. Finally, we gave two illustrative examples to support our theoretical results. It is of great interest for future works to investigate the ReOb of sub-diffusion systems with the Hilfer derivative in cases where the reconstructed initial state is in a subregion on the boundary of the whole domain. Furthermore, our paper motivates the study of the ReEnOb of sub-diffusion systems via ψ-Hilfer or (k,ψ)-Hilfer fractional derivatives.

Author Contributions: All authors contributed equally in this paper. All authors read and approved the final manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (grant numbers: 11871064; 11571300).

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors are deeply grateful to the two anonymous referees for their great comments and suggestions that improved the quality of this manuscript. This work was supported by the National Natural Science Foundation of China [grant numbers 11871064, 11571300].

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Petráš, I. *Handbook of Fractional Calculus with Applications*; Walter de Gruyter GmbH & Co. KG: Berlin, Germany, 2019.
2. Yang, X.-J. *General Fractional Derivatives: Theory, Methods and Applications*; CRC Press/Taylor and Francis Group: New York, NY, USA, 2019.
3. Baleanu, D.; Lopes, A.M. *Handbook of Fractional Calculus with Applications in Engineering, Life and Social Sciences, Part B*; Walter de Gruyter GmbH & Co. KG: Berlin/Boston, Germany, 2019.
4. Rosa, S.; Torres, D.F. Fractional Modelling and Optimal Control of COVID-19 Transmission in Portugal. *Axioms* **2022**, *11*, 170. [CrossRef]

5. Elbukhari, A.B.; Fan, Z.; Li, G. Existence of Mild Solutions for Nonlocal Evolution Equations with the Hilfer Derivatives. *J. Funct. Spaces* **2023**, *2023*. [CrossRef]
6. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific Pub. Co.: Singapore, 2000.
7. Podlubny, I. *Fractional Differential Equations, Mathematics in Science and Engineering*; Academic Press: New York, NY, USA, 1999.
8. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. In *North-Holland Mathematics Studies*; van Mill, J., Ed.; Elsevier: Amsterdam, The Netherland, 2006.
9. Curtain, R.F.; Zwart, H. *An Introduction to Infinite-Dimensional Linear Systems Theory*; Springer Science and Business Media: New York, NY, USA, 2012.
10. Amouroux, M.; El Jai, A.; Zerrik, E. Regional observability of distributed systems. *Int. J. Syst. Sci.* **1994**, *25*, 301–313. [CrossRef]
11. El Jai, A.; Simon, M.C.; Zerrik, E. Regional observability and sensor structures. *Sens. Actuators Phys.* **1993**, *39*, 95–102. [CrossRef]
12. Al-Saphory, R.A.; Al-Jawari, N.J.; Al-Janabi, A.N. Regional gradient strategic sensors characterizations. *arXiv* **2005**, arXiv:2005.07497v1.
13. Courant, R.; Hilbert, D. *Methods of Mathematical Physics: Partial Differential Equations*; J. Wily & Sons Inc.: Singapore, 2008.
14. Curtain, R.F. On semigroup formulations of unbounded observations and control action for distributed systems. In *Mathematical Theory of Networks and Systems: Proceedings of the MTNS-83 International Symposium, Beer Sheva, Israel, 20–24 June 1983*; Springer: Berlin/Heidelberg, Germany, 1984.
15. Zerrik, E.H.; Bourray, H.; Boutoulout, A. Regional boundary observability: A numerical approach. *Int. J. Appl. Math. Comput. Sci.* **2002**, *12*, 143–151.
16. Chen, J.H. Infinite-time exact observability of Volterra systems in Hilbert space. *Syst. Control Lett.* **2019**, *126*, 28–32. [CrossRef]
17. Chen, J.H.; Yi, N.Y. Infinite-time Admissibility and Exact Observability of Volterra Systems. *SIAM J. Control Optim.* **2021**, *59*, 1275–1292. [CrossRef]
18. Zouiten, H.; Boutoulout, A.; El Alaoui, F.Z. On the regional enlarged observability for linear parabolic Systems. *J. Math. Syst. Sci.* **2017**, *7*, 79–87.
19. Zguaid, K.; El Alaoui, F.Z. Regional boundary observability for linear time-fractional systems. *Partial Differ. Equations Appl. Math.* **2022**, *6*, 100432. [CrossRef]
20. Zguaid, K.; El Alaoui, F.Z.; Boutoulout, A. Regional observability for linear time fractional systems. *Math. Comput. Simul.* **2021**, *185*, 77–87. [CrossRef]
21. Zguaid, K.; El Alaoui, F.Z.; Torres, D.F.M. Regional gradient observability for fractional differential equations with Caputo time-fractional derivatives. *Int. J. Dynam. Control* **2023**. [CrossRef]
22. Ge, F.; Chen, Y.; Kou, C. On the regional gradient observability of time fractional diffusion processes. *Automatica* **2016**, *74*, 1–9. [CrossRef]
23. Cai, R.; Ge, F.; Chen, Y.; Kou, C. Regional observability for Hadamard-Caputo time fractional distributed parameter systems. *Appl. Math. Comput.* **2019**, *360*, 190–202. [CrossRef]
24. Zguaid, K.; El Alaoui, F.Z. Regional boundary observability for Riemann-Liouville linear fractional evolution systems. *Math Comput. Simul.* **2022**, *199*, 272–286. [CrossRef]
25. Zouiten, H.; Boutoulout, A.; Torres, D.F. Regional Enlarged Observability of Fractional Differential Equations with Riemann-Liouville Time Derivatives. *Axioms* **2018**, *7*, 92. [CrossRef]
26. Zouiten, H.; Boutoulout, A.; Torres, D.F. Regional enlarged observability of Caputo fractional differential equations. *Discret. Contin. Dynam. Syst.-S* **2020**, *13*, 1017–1029. [CrossRef]
27. Lions, J.L. Sur la Controlabilite Exacte Elargie. In *Partial Differential Equations and the Calculus of Variations. Progress in Nonlinear Differential Equations and their Applications*; Colombini, F., Marino, A., Modica, L., Spagnolo, S., Eds.; Birkhauser: Boston, MA, USA, 1989; pp. 703–727.
28. Kamochi, R. A new representation formula for the Hilfer fractional derivative and its application. *J. Comput. Appl. Math.* **2016**, *308*, 39–45. [CrossRef]
29. Gu, H.; Trujillo, J.J. Existence of mild solution for evolution equation with Hilfer fractional derivative. *Appl. Math. Comput.* **2015**, *257*, 344–354. [CrossRef]
30. Abdeljawad, T.; Atangana, A.; Gómez-Aguilar, J.F.; Jarad, F. On a more general fractional integration by parts formulae and applications. *Stat. Mech. Appl.* **2019**, *536*, 122494. [CrossRef]
31. Dolecki, S.; Russell, D.L. A general theory of observation and control. *SIAM J. Control Optim.* **1977**, *15*, 185–220. [CrossRef]
32. Pritchard, A.J.; Wirth, A. Unbound control and observation systems and their duality. *SIAM J. Control Optim.* **1978**, *16*, 535–545. [CrossRef]
33. Ge, F.; Chen, Y.; Kou, C. *Regional Analysis of Time Fractional Diffusion Processes*; Springer International Publishing: Berlin, Germany, 2018.