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THE REGULAR RING AND THE MAXIMAL RING OF QUOTIENTS OF A FINITE BAER *-RING

BY

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ABSTRACT. Necessary and sufficient conditions are obtained for extending the involution of a Baer *-ring to its maximal ring of quotients. Berberian's construction of the regular ring of a Baer *-ring is generalized and this ring is identified (under suitable hypotheses) with the maximal ring of quotients.

1. **Introduction.** J.-E. Roos has noted [9, pp. A122–A123] that if A is a finite Baer *-ring satisfying the (EP)-axiom and the (SR)-axiom (this and other terminology is explained in §2 below), then the involution of A can be extended to its maximal ring of quotients, and if A is an AW *-algebra, its maximal ring of quotients can then be identified with its regular ring. We are thus led to pose the following problem: *Determine conditions on a Baer *-ring which make its involution extendible to its maximal ring of quotients in such a way that the maximal ring of quotients can be identified with the regular ring.*

Our approach to this problem is as follows. We first obtain a necessary and sufficient condition for the involution of a Baer *-ring to be extendible to its maximal (right) ring of quotients, viz. that it satisfy *Utumi's condition*: *Every nonzero right ideal whose left annihilator is zero is large.* We then obtain sufficient conditions for a Baer *-ring to satisfy this condition–one formulation is that it be finite, satisfy LP $\sim$ RP (note that this much is required just to define the regular ring) and the (WEP)-axiom, and have a 2-proper involution (this and a great deal more was assumed by Berberian to establish regularity of the regular ring). Finally, we generalize the construction of the regular ring, obtaining regularity through an identification with the maximal ring of quotients (all of which requires only the above-mentioned hypotheses).

2. **Preliminaries.** Throughout this paper, A will denote a *-ring with unity. An extension B of A is a *right ring of quotients* of A, written $B \geqslant A$, if for

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each pair $x, y \in B$ with $x \neq 0$, there exists $a \in A$ such that $xa \neq 0$ and $ya \in B$. (More precisely, $B \geqslant A$ if there exists an embedding $\sigma: A \rightarrow B$ such that, for each pair $x, y \in B$ with $x \neq 0$, there exists $a \in A$ such that $x\sigma(a) \neq 0$ and $y\sigma(a) \in A$. We follow the usual practice of suppressing σ and identifying A with $\sigma(A)$. There should be no confusion, for the embedding intended will always be clear from the context.) If $B_2 \geqslant A$ and $B_1 \geqslant A$, we say that B_1 can be *embedded* in B_2 *over* A if there exists a monomorphism $\sigma: B_1 \rightarrow B_2$ such that $\sigma(a) = a$ for all $a \in A$. If σ is surjective, we say that B_1 is *isomorphic* to B_2 *over* A and write $B_1 \cong_A B_2$; if, moreover, B_1 and B_2 are $*$-rings and σ is a $*$-isomorphism, we write $B_1 \overset{*}{\cong}_A B_2$.

Let $D(A)$ be the set of all right ideals I of A such that $A \geqslant I$ (via the identity embedding) and let $F(A)$ be the set of all right A-module homomorphisms $\theta: I \rightarrow A$, where I varies over $D(A)$. (*Notation*. If $\theta \in F(A)$, we write M_θ for its domain.) Since $D(A)$ is closed under finite intersections and $\theta^{-1}(I) \in D(A)$ for any $\theta \in F(A), I \in D(A)$ [10, p. 3], we may define operations on $F(A)$ as follows:

$$(\theta + \sigma)(x) = \theta(x) + \sigma(x), \quad x \in M_\theta \cap M_\sigma; \qquad (\theta\sigma)(x) = \theta(\sigma(x)), \quad x \in \sigma^{-1}(M_\theta).$$

An equivalence relation is defined on $F(A)$ by putting $\theta \equiv \sigma$ whenever there exists $I \in D(A)$ such that $\theta = \sigma$ on I; we denote the equivalence class of θ by $\hat{\theta}$ and write Q for the set of all such equivalence classes. The operations on $F(A)$ are extended to Q in the obvious way: $\hat{\theta} + \hat{\sigma} = (\theta + \sigma)^\wedge$, $(\theta\sigma)^\wedge = \hat{\theta}\hat{\sigma}$. Finally, we embed A in Q by identifying each $a \in A$ with the equivalence class of left multiplication by a.

(2.1) LEMMA [10, pp. 2, 4]. (i) *If* $\theta \in F(A)$ *and* $x = \hat{\theta} \in Q$, *then* $xa = \theta(a)$. (ii) *Let* $B \geqslant A$. *If* σ *is a ring endomorphism of* B *leaving* A *elementwise fixed, then* σ *is the identity on* B.

It follows [10, p. 4]:

(2.2) THEOREM [UTUMI]. Q *is a maximal right ring of quotients of* A *in the following sense*: $Q \geqslant A$, *and if* $B \geqslant A$, *then* B *can be embedded in* Q *over* A.

Since Q is clearly unique up to isomorphism over A, we shall refer to it as *the* maximal right ring of quotients of A. Left rings of quotients and Q_λ, *the maximal left ring of quotients of* A, are defined similarly. A *two-sided ring of quotients of* A is a ring B which is both a left and a right ring of quotients of A *with respect to the same embedding.*

We write $R(S)$ for the *right annihilator* of a subset S of A, i.e.,

$$R(S) = \{a \in A: sa = 0 \text{ for all } s \in S\},$$

and $L(S)$ for the *left annihilator* of S.

$$Z(A) = \{a \in A: R(\{a\}) \text{ is large}\}$$

is called the (*right*) *singular ideal* of A (a right ideal I is *large* if $I \cap J \neq 0$ for every nonzero right ideal J; if $Z(A) = 0$, then a right ideal I is large if and only if $I \in D(A)$; cf. [4, p. 58]). If $Z(A) = 0$, then Q is *regular* (i.e., for every $x \in Q$, there exists $y \in Q$ such that $x = xyx$) [5, p. 893].

We write $\widetilde{A}$ for the set of projections in A, i.e.,

$$\widetilde{A} = \{e \in A: e^2 = e = e^*\}.$$

If, for some $x \in A$, there exists $e \in \widetilde{A}$ such that $R(\{x\}) = (1 - e)A$, then e is unique and is called the *right projection* of x; we write $e = \mathrm{RP}(x)$. $\mathrm{RP}(x)$ is the minimal projection in A such that $x\mathrm{RP}(x) = x$ (here $e \leqslant f$ means $ef = e = fe$). The *left projection* of x is defined similarly and is denoted (when it exists) by $\mathrm{LP}(x)$. A *partial isometry* in A is an element w such that $w = ww^*w$. $e, f \in \widetilde{A}$ are *equivalent*, $e \sim f$, if there exists a partial isometry w such that $w^*w = e$ and $ww^* = f$; e is then called the *initial* and f the *final projection* of w. $e, f \in \widetilde{A}$ are *orthogonal* if $ef = 0$. Partial isometries in A are said to be *addable* in A if, whenever (w_ι) is a family of partial isometries in A with orthogonal initial projections (e_ι) and orthogonal final projections (f_ι), there exists a partial isometry w in A whose initial projection is $\sup e_\iota$ and whose final projection is $\sup f_\iota$, such that $we_\iota = w_\iota = fw_\iota$. A is *finite* if $e \sim 1$ implies $e = 1$; A is *strongly finite* if $xy = 1$ implies $yx = 1$. We say that A has an *n-proper involution* if $x_1x_1^* + \cdots + x_nx_n^* = 0$ implies $x_1 = \cdots = x_n = 0$.

We will consider the following axioms on *-rings (recall that the *commutant* of a subset S of A is the set $S' = \{a \in A: sa = as \text{ for all } s \in S\}$; the commutant of S' is denoted simply S''):

LP $\sim$ RP. For every $x \in A$, $\mathrm{LP}(x) \sim \mathrm{RP}(x)$.

(WEP)-Axiom. For every nonzero $x \in A$, there exists $y \in \{x^*x\}''$ such that $0 \neq x^*xy^*y \in \widetilde{A}$.

(EP)-Axiom. For every nonzero $x \in A$, there exists $y \in \{x^*x\}''$ such that $y = y^*$ and $0 \neq x^*xy^2 \in \widetilde{A}$.

(SR)-Axiom. For every $x \in A$, there exists $r \in \{x^*x\}''$ such that $r = r^*$ and $x^*x = r^2$.

A *Rickart *-ring* is a *-ring A in which every element has both a left and a right projection. In such a ring, $\widetilde{A}$ is a lattice; if it is complete, A is called a *Baer *-ring*. A Rickart *-ring has a unity element and a 1-proper involution [2,

p. 13]; furthermore, it is easy to see that its singular ideal is zero.

3. **Extending the involution.** We will utilize the following theorem, proved by Utumi [11, pp. 144–145]:

(3.1) THEOREM. *Suppose* $Z(A) = 0$ *and* $B \geqslant A$. *Then* (i) Q *satisfies Utumi's condition*; (ii) $Q \cong_A Q_\lambda$ *if and only if* A *satisfies Utumi's condition*; (iii) *if* A *satisfies Utumi's condition, so does* B; (iv) *if* B *is a two-sided ring of quotients of* A *and* B *satisfies Utumi's condition, then* A *satisfies Utumi's condition.*

(3.2) THEOREM. *The involution of* A *can be extended to* Q *if and only if* Q *is a two-sided ring of quotients of* A. *If* $Z(A) = 0$, *this is equivalent to each of the following*: (i) A *satisfies Utumi's condition*; (ii) $Q \cong_A Q_\lambda$. *The extension* (*when it exists*) *is unique.*

PROOF. If $*$ and $\#$ are involutions on Q extending that of A, then the mapping $x \mapsto x^{*\#}$ is a ring endomorphism of Q leaving A elementwise fixed. Thus, $x = x^{*\#}$ by (2.1), and hence $x^\# = x^*$. Suppose now that Q is a two-sided ring of quotients of A, and for each $x \in Q$, set $I_x = \{a \in A\colon a^*x \in A\}$ and $\theta_x(a) = (a^*x)^*$, $a \in I_x$. Then $I_x \in D(A)$ and $\theta_x \in F(A)$; hence, we may define a mapping $*$ on Q which extends the involution of A by putting $x^* = \hat{\theta}_x$. To show that this mapping is an involution for Q, fix $x, y \in Q$.

(i) If $a \in I_x \cap I_y = M_{\theta_x} \cap M_{\theta_y}$, then

$$\theta_{x+y}(a) = [a^*(x+y)]^* = (a^*x)^* + (a^*y)^* = \theta_x(a) + \theta_y(a),$$

which proves $(x + y)^* = x^* + y^*$.

(ii) If $a \in I_{xy} \cap I_x$, then

$$\theta_{xy}(a) = [a^*(xy)]^* = [(a^*x)y]^* = \theta_y((a^*x)^*) = \theta_y(\theta_x(a)),$$

hence $(xy)^* = y^*x^*$.

(iii) If we write $x = \hat{\theta}$, $\theta \in F(A)$, the assertion $x = x^{**}$ is equivalent to $\theta_{x^*} \equiv \theta$. But for $a \in I_{x^*} \cap M_\theta$, we have $\theta_{x^*}(a)^* = a^*x^* = (xa)^* = \theta(a)^*$.

This proves one implication; the converse follows from the following more general (and obvious) fact: a ring of quotients of A having an involution extending that of A is a two-sided ring of quotients of A. The statements for rings with zero singular ideal are evident from (3.1).

The next result will enable us to apply (3.2) to Baer $*$-rings.

(3.3) THEOREM. *If* B *is a two-sided ring of quotients of a Baer* $*$*-ring* A, *then for each* $x \in B$, *there exist* $e, f \in \widetilde{A}$ *such that* $L(\{x\}) = B(1 - e)$ *and* $R(\{x\}) = (1 - f)B$.

PROOF. It is easy to see that the unity element for A is also a unity element for B, and for each $x \in B$, $I = \{a \in A: xa \in A\} \in D(A)$. Set $e = \sup\{\mathrm{LP}(xa): a \in I\}$; we claim: (1) $ex = x$ and (2) $yx = 0$ if and only if $ye = 0$. This will obviously prove the assertion for e; the corresponding assertion for f follows by symmetry. For each $a \in I$, $exa = xa$ since $e \geqslant \mathrm{LP}(xa)$; thus, $(ex - x)a = 0$, so $ex - x = 0$ results from $B \geqslant I$. If $ye = 0$, then $yx = y(ex) = 0$. Conversely, suppose $yx = 0$ and put $J = \{a \in A: ay \in A\}$. Since B is a left ring of quotients of A and A is a left ring of quotients of J, B is clearly a left ring of quotients of J. But for $a \in I$, $b \in J$, we have $(by)(xa) = b(yx)a = 0$; since $by, xa \in A$, this implies $(by)\mathrm{LP}(xa) = 0$. Varying a over I, it follows that $bye = 0$, which implies $ye = 0$.

(3.4) COROLLARY. *If $B \geqslant A$, where A is a Baer *-ring whose involution is extendible to B, then B is a Baer *-ring with no new projections (i.e., $\widetilde{B} = \widetilde{A}$).*

PROOF. B is a Rickart *-ring by (3.3). Moreover, if $f \in \widetilde{B}$, there exists $e \in \widetilde{A}$ such that $B(1 - f) = L(\{f\}) = B(1 - e)$; thus, $1 - f = 1 - e$, or $f = e \in \widetilde{A}$. Therefore, $\widetilde{A} = \widetilde{B}$; since $\widetilde{B}$ is complete, B is a Baer *-ring.

(3.5) COROLLARY. *If A is a Baer *-ring whose involution is extendible to Q (i.e., if A satisfies Utumi's condition), then Q is a regular Baer *-ring with no new projections; in particular, Q and A are strongly finite.*

PROOF. A regular Baer *-ring is strongly finite [6, p. 532].

We now determine a large class of Baer *-rings which satisfy Utumi's condition:

(3.6) THEOREM. *A Baer *-ring A satisfies Utumi's condition if*

(i) *$\widetilde{A}$ is an upper continuous lattice (i.e., $e_\alpha \uparrow e$ implies $e_\alpha \cap f \uparrow e \cap f$),*

(ii) *the involution of A is 2-proper, and*

(iii) *for each $x \in A$, the principal right ideal xA contains an orthogonal family (e_α) of projections with* $\sup e_\alpha = \mathrm{LP}(x)$.

PROOF. Let I be a right ideal with $L(I) = 0$ and let J be any nonzero right ideal; we must show that $I \cap J \neq 0$. Since J contains a nonzero projection f, it suffices to find a projection $e \in I$ such that $e \cap f \neq 0$. Suppose, to the contrary, that $e \cap f = 0$ for every projection $e \in I$. Then it suffices to show

$$(*) \qquad \mathrm{LP}(x) \cap f = 0, \qquad x \in I.$$

For, hypothesis (ii) and the fact that $L(I) = 0$ imply $(\mathrm{LP}(x))_{x \in I} \uparrow 1$ [3, pp. 21, 225]; hence, $\mathrm{LP}(x) \cap f \uparrow f$ by upper continuity, implying $f = 0$ by $(*)$, a contradiction. To prove $(*)$, we first obtain by (iii) an orthogonal family of projections in $xA \subseteq I$ with sup $\mathrm{LP}(x)$; passing to the net (e_α) of finite sums, we have $e_\alpha \uparrow \mathrm{LP}(x)$, with $e_\alpha \in I$. Then by upper continuity, $0 = e_\alpha \cap f \uparrow \mathrm{LP}(x) \cap f$.

(3.7) COROLLARY. *If A is a finite Baer *-ring with a 2-proper involution, either of the following hypotheses implies that A satisfies Utumi's condition (and hence that Q is a regular Baer *-ring with no new projections)*: (i) *A satisfies the* (WEP)*-axiom and* $\mathrm{LP} \sim \mathrm{RP}$; (ii) *$A$ satisfies the* (EP)*-axiom and the* (SR)*-axiom.*

PROOF. (i) [3, p. 44, Exercise 7; p. 83, Exercise 13; p. 185]; (ii) [7, p. 99].

4. The ring of closed right operators. For the remainder of the paper, we assume that A *is a finite Baer *-ring satisfying* $\mathrm{LP} \sim \mathrm{RP}$. In this section, we extend A to a ring which may, under very mild hypotheses, be identified with the maximal ring of quotients of A. In the next section, we will show that if A also satisfies Utumi's condition, then this ring is the regular ring of A.

A family $(e_\alpha)_{\alpha \in I}$ in $\widetilde{A}$, indexed by an increasingly directed set I, is called a *strongly dense domain* in A (briefly, an SDD). If $(e_\alpha)_{\alpha \in I}$ and $(f_\beta)_{\beta \in J}$ are SDD's, then so is $(e_\alpha \cap f_\beta)_{(\alpha,\beta) \in I \times J}$ with the product ordering of indices: $(\alpha', \beta') \geqslant (\alpha, \beta)$ if $\alpha' \geqslant \alpha$, $\beta' \geqslant \beta$ [3, p. 185, Exercise 3]. (For simplicity, we omit the index set in the future.) A *right operator* (RO) for A is a family of pairs (x_α, e_α), where (e_α) is an SDD and $\alpha' \geqslant \alpha$ implies $x_{\alpha'} e_\alpha = x_\alpha e_\alpha$. It follows that $x_\alpha(e_\alpha \cap e_\beta) = x_\beta(e_\alpha \cap e_\beta)$ for all indices α, β. {*Proof.* Fix α, β and choose $\gamma \geqslant \alpha, \beta$. Since $e_\gamma \geqslant e_\alpha, e_\beta$,

$$x_\alpha(e_\alpha \cap e_\beta) = x_\alpha e_\alpha(e_\alpha \cap e_\beta) = x_\gamma e_\alpha(e_\alpha \cap e_\beta)$$
$$= x_\gamma(e_\alpha \cap e_\beta) = x_\gamma e_\beta(e_\alpha \cap e_\beta) = x_\beta e_\beta(e_\alpha \cap e_\beta) = x_\beta(e_\alpha \cap e_\beta).$$

This argument illustrates the principle technique used in handling RO's.} Two RO's (x_α, e_α), (y_β, f_β) are *equivalent*, $(x_\alpha, e_\alpha) \equiv (y_\beta, f_\beta)$, if $x_\alpha(e_\alpha \cap f_\beta) = y_\beta(e_\alpha \cap f_\beta)$ for all α, β. It is not hard to see that this is equivalent to the existence of an auxiliary SDD (g_γ) such that $x_\alpha(e_\alpha \cap f_\beta \cap g_\gamma) = y_\beta(e_\alpha \cap f_\beta \cap g_\gamma)$ for all α, β, γ. Equivalence is particularly simple when the index sets involved are the same (and, as we shall see, they may always be chosen this way): $(x_\alpha, e_\alpha) \equiv (y_\beta, f_\beta)$ if and only if there exists an SDD (g_α) such that $x_\alpha g_\alpha = y_\alpha g_\alpha$ for all α. (Note that this implies the following: If (x_α, e_α) is an RO and (f_α) an SDD such that $x_\alpha f_\alpha = 0$ for all α, then $(x_\alpha, e_\alpha) \equiv (0, e_\alpha)$, or $[x_\alpha, e_\alpha] = 0$ in the notation which follows.) The relation $\equiv$ is an equivalence relation on the set of all RO's; we denote the equivalence class of (x_α, e_α) by

$[x_\alpha, e_\alpha]$ and call it a *closed right operator* (CRO) for A. The set of all CRO's for A will be denoted by C_ρ.

Ring operations for C_ρ are defined essentially componentwise:

$$[x_\alpha, e_\alpha] + [y_\beta, f_\beta] = [x_\alpha + y_\beta, e_\alpha \cap f_\beta], \quad [x_\alpha, e_\alpha][y_\beta, f_\beta] = [x_\alpha y_\beta, f_\beta \cap y_\beta^{-1}(e_\alpha)],$$

where, following Berberian, we write $x^{-1}(e) = 1 - \mathrm{RP}[(1-e)x]$ (thus $x^{-1}(e)$ is the largest projection g such that $exg = xg$). The peculiar definition of multiplication is necessitated by the fact that $(x_\alpha y_\beta, e_\alpha \cap f_\beta)$ is not in general an RO. It will require a considerable amount of work to legitimatize this definition; in contrast, things are quite simple for addition, and we omit further details.

(4.1) LEMMA. *Let* (x_α, e_α) *be an RO,* (f_β) *an SDD, and put* $x_{\alpha\beta} = x_\alpha$ *for all* α, β. *Then* $(x_{\alpha\beta}, e_\alpha \cap f_\beta)$ *is an RO equivalent to* (x_α, e_α).

Note how (4.1) simplifies things: since $[x_\alpha, e_\alpha] = [x_{\alpha\beta}, g_{\alpha\beta}]$ and $[y_\beta, f_\beta] = [y_{\alpha\beta}, g_{\alpha\beta}]$, where $x_{\alpha\beta} = x_\alpha$, $y_{\alpha\beta} = y_\beta$, and $g_{\alpha\beta} = e_\alpha \cap f_\beta$, *we may assume, when dealing with a finite number of CRO's, that the index sets and SDD's are the same.*

(4.2) LEMMA. *If* (x_α, e_α) *is an RO,* (f_α) *an SDD, and* $g_\alpha = e_\alpha \cap x_\alpha^{-1}(f_\alpha)$, *then* (g_α) *is an SDD; in particular, if* (f_α) *is an SDD, then* $x_\alpha^{-1}(f_\alpha)$ *is an SDD for any* $x \in A$.

PROOF. [3, p. 214, Lemma 5; p. 185, Exercise 4].

(4.3) LEMMA. *Let* (x_α, e_α) *be an RO and let* $(g_{\alpha\beta})$ *be an SDD whose index set is the direct product of the index set for* (x_α, e_α) *with some increasingly directed set (indexed by* β*), having the property that* $g_{\alpha\beta} \leqslant e_\alpha$ *for all* α, β. *Then* $(x_{\alpha\beta}, g_{\alpha\beta})$ *is an RO equivalent to* (x_α, e_α), *where* $x_{\alpha\beta} = x_\alpha$ *for all* α.

PROOF. Straightforward calculation.

(4.4) LEMMA. *If* (e_α) *is an SDD and* (y_β, f_β) *an RO, then* $g_{\alpha\beta} = f_\beta \cap y_\beta^{-1}(e_\alpha)$ *defines an SDD* $(g_{\alpha\beta})$ *(with the product index set). Hence, if* (x_α, e_α) *is an RO, so is* $(x_\alpha y_\beta, f_\beta \cap y_\beta^{-1}(e_\alpha))$.

PROOF. It is not hard to see that $g_{\alpha\beta} \uparrow$. {Note that if $\beta' \geqslant \beta$, then $f_{\beta'} \cap y_{\beta'}^{-1}(e_\alpha) \geqslant f_\beta \cap y_\beta^{-1}(e_\alpha)$ follows from the fact that $e_\alpha y_{\beta'}[f_\beta \cap y_\beta^{-1}(e_\alpha)] = y_{\beta'}[f_\beta \cap y_\beta^{-1}(e_\alpha)]$ and the maximality property of $y_{\beta'}^{-1}(e_\alpha)$.} By (4.3) and upper continuity [3, pp. 80, 185],

$$\sup_{\alpha,\beta} g_{\alpha\beta} = \sup_{\beta}\left\{\sup_{\alpha}[f_\beta \cap y_\beta^{-1}(e_\alpha)]\right\}$$

$$= \sup_{\beta}\left\{f_\beta \cap \sup_{\alpha} y_\beta^{-1}(e_\alpha)\right\} = \sup_{\beta}(f_\beta \cap 1) = \sup_{\beta} f_\beta = 1.$$

Finally, if $(\alpha', \beta') \geqslant (\alpha, \beta)$,

$$x_{\alpha'}y_{\beta'}g_{\alpha\beta} = x_{\alpha'}y_{\beta'}f_\beta g_{\alpha\beta} = x_{\alpha'}y_\beta f_\beta g_{\alpha\beta} = x_{\alpha'}y_\beta y_\beta^{-1}(e_\alpha)g_{\alpha\beta}$$

$$= x_{\alpha'}e_\alpha y_\beta y_\beta^{-1}(e_\alpha)g_{\alpha\beta} = x_\alpha e_\alpha y_\beta y_\beta^{-1}(e_\alpha)g_{\alpha\beta} = x_\alpha y_\beta g_{\alpha\beta}.$$

(4.5) LEMMA. *If* $(x_\alpha, e_\alpha) \equiv (r_\gamma, g_\gamma)$ *and* $(y_\beta, f_\beta) \equiv (s_\delta, h_\delta)$, *then* $(x_\alpha, e_\alpha)(y_\beta, f_\beta) \equiv (r_\gamma, g_\gamma)(s_\delta, h_\delta)$.

PROOF. The formula

$$k_{\alpha\beta\gamma\delta} = [f_\beta \cap y_\beta^{-1}(e_\alpha)] \cap [h_\delta \cap s_\delta^{-1}(g_\gamma)] \cap s_\delta^{-1}(e_\alpha \cap g_\gamma)$$

defines an SDD with the product ordering of indices. Indeed, for fixed β, δ, we have $e_\alpha \cap g_\gamma \uparrow 1$, $y_\beta^{-1}(e_\alpha) \uparrow 1$, $s_\delta^{-1}(g_\gamma) \uparrow 1$, and $s_\delta^{-1}(e_\alpha \cap g_\gamma) \uparrow 1$ as $(\alpha, \gamma) \uparrow$; therefore,

$$\sup_{\alpha,\beta,\gamma,\delta} k_{\alpha\beta\gamma\delta} = \sup_{\beta,\delta}\{(f_\beta \cap 1) \cap (h_\delta \cap 1) \cap 1\} = 1.$$

Now put $u_{\alpha\beta\gamma\delta} = x_\alpha y_\beta$ and $v_{\alpha\beta\gamma\delta} = r_\gamma s_\delta$. Then by (4.3), $(u_{\alpha\beta\gamma\delta}, k_{\alpha\beta\gamma\delta})$ is an RO equivalent to $(x_\alpha y_\beta, f_\beta \cap y_\beta^{-1}(e_\alpha))$ and $(v_{\alpha\beta\gamma\delta}, k_{\alpha\beta\gamma\delta})$ is an RO equivalent to $(r_\gamma s_\delta, h_\delta \cap s_\delta^{-1}(g_\gamma))$; thus, it suffices to show that $(u_{\alpha\beta\gamma\delta}, k_{\alpha\beta\gamma\delta}) \equiv (v_{\alpha\beta\gamma\delta}, k_{\alpha\beta\gamma\delta})$. But

$$u_{\alpha\beta\gamma\delta}k_{\alpha\beta\gamma\delta} = x_\alpha y_\beta(f_\beta \cap h_\delta)k_{\alpha\beta\gamma\delta} = x_\alpha s_\delta(f_\beta \cap h_\delta)k_{\alpha\beta\gamma\delta}$$

$$= x_\alpha s_\delta s_\delta^{-1}(e_\alpha \cap g_\gamma)k_{\alpha\beta\gamma\delta} = x_\alpha(e_\alpha \cap g_\gamma)s_\delta s_\delta^{-1}(e_\alpha \cap g_\gamma)k_{\alpha\beta\gamma\delta}$$

$$= r_\gamma(e_\alpha \cap g_\gamma)s_\delta s_\delta^{-1}(e_\alpha \cap g_\delta)k_{\alpha\beta\gamma\delta} = r_\gamma s_\delta k_{\alpha\beta\gamma\delta} = v_{\alpha\beta\gamma\delta}k_{\alpha\beta\gamma\delta}.$$

As noted earlier, things are much simpler when the index sets are the same:

(4.6) LEMMA. $(x_\alpha, e_\alpha) + (y_\alpha, f_\alpha) \equiv (x_\alpha + y_\alpha, e_\alpha \cap f_\alpha)$, *and* $(x_\alpha, e_\alpha)(y_\alpha, f_\alpha) \equiv (x_\alpha y_\alpha, f_\alpha \cap y_\alpha^{-1}(e_\alpha))$.

PROOF. The formula

$$k_{\alpha\beta\gamma} = [f_\beta \cap y_\beta^{-1}(e_\alpha)] \cap [f_\gamma \cap y_\gamma^{-1}(e_\gamma)] \cap [(f_\alpha \cap f_\gamma) \cap y_\gamma^{-1}(e_\alpha \cap e_\gamma)]$$

defines an SDD such that $x_\alpha y_\beta k_{\alpha\beta\gamma} = x_\gamma y_\gamma k_{\alpha\beta\gamma}$.

We embed A in C_ρ by identifying each $a \in A$ with the RO $(a, 1)$ obtained by taking $I = \{1\}$ as the index set and defining $x_1 = a$, $e_1 = 1$.

(4.7) THEOREM. *If* A *is a finite Baer *-ring satisfying* LP $\sim$ RP, *then* C_ρ *is a ring with unity and* $C_\rho \geqslant A$.

PROOF. The first statement follows routinely from (4.6), while the second is an immediate consequence of the following lemma.

(4.8) LEMMA. *If* $x = [x_\alpha, e_\alpha]$, *then* $xe_\beta = x_\beta e_\beta$ *for any fixed index* β.

PROOF. Clearly, $e_\beta = [y_\alpha, e_\alpha]$ and $x_\beta e_\beta = [z_\alpha, e_\alpha]$, where $y_\alpha = e_\beta$ and $z_\alpha = x_\beta e_\beta$ for all α. By (4.6), $xe_\beta = [x_\alpha, e_\alpha][y_\alpha, e_\alpha] = [x_\alpha y_\alpha, g_\alpha]$ for a suitable SDD (g_α), and it suffices to find an SDD (f_α) such that $x_\alpha y_\alpha f_\alpha = z_\alpha f_\alpha$, i.e., $x_\alpha e_\beta f_\alpha = x_\beta e_\beta f_\alpha$. This may be accomplished by setting $f_\alpha = 1$ if $\alpha \geqslant \beta$, and $f_\alpha = 0$ otherwise.

(4.9) THEOREM. *Let* A *be a finite Baer *-ring satisfying* LP $\sim$ RP. *If every* $I \in D(A)$ *contains an SDD, then* $C_\rho \cong_A Q$; *in particular,* C_ρ *is regular.*

PROOF. Let $x \in Q$, say $x = \hat{\theta}$, and let (e_α) be an SDD in M_θ. Setting $x_\alpha = \theta(e_\alpha)$, it follows that (x_α, e_α) is an RO. We want to define $\Psi\colon Q \to C_\rho$ by $\Psi(x) = [x_\alpha, e_\alpha]$. To see that this is possible, suppose that also $x = \hat{\phi}$ and (f_β) is an SDD in M_ϕ, and put $y_\beta = \phi(f_\beta)$. Choose $I \in D(A)$ such that $\theta = \phi$ on I and let (g_γ) be an SDD in I. Then $k_{\alpha\beta\gamma} = e_\alpha \cap f_\beta \cap g_\gamma$ defines an SDD in I such that

$$\begin{aligned} x_\alpha k_{\alpha\beta\gamma} &= \theta(e_\alpha)k_{\alpha\beta\gamma} = \theta(e_\alpha k_{\alpha\beta\gamma}) = \theta(k_{\alpha\beta\gamma}) \\ &= \phi(k_{\alpha\beta\gamma}) = \phi(f_\beta k_{\alpha\beta\gamma}) = \phi(f_\beta)k_{\alpha\beta\gamma} = y_\beta k_{\alpha\beta\gamma}; \end{aligned}$$

hence $[x_\alpha, e_\alpha] = [y_\beta, f_\beta]$. It is easy to see that $\Psi(x + y) = \Psi(x) + \Psi(y)$ and $\Psi(a) = a$ for all $x, y \in Q$, $a \in A$. To show that $\Psi(xy) = \Psi(x)\Psi(y)$, write $x = \hat{\theta}$, $y = \hat{\phi}$, and $\Psi(x) = [x_\alpha, e_\alpha]$, $\Psi(y) = [y_\alpha, e_\alpha]$, where (e_α) is an SDD in $M_\theta \cap M_\phi$. Let (f_β) be any SDD in $M_{\theta\phi}$ and put $g_{\alpha\beta} = f_\beta \cap e_\alpha \cap y_\alpha^{-1}(e_\alpha)$, $z_{\alpha\beta} = (\theta\phi)(g_{\alpha\beta})$; thus, $(g_{\alpha\beta})$ is an SDD in $M_{\theta\phi}$ and $\Psi(xy) = [z_{\alpha\beta}, g_{\alpha\beta}]$ by definition. Furthermore, by (4.3),

$$\Psi(x)\Psi(y) = [x_\alpha, e_\alpha][y_\alpha, e_\alpha] = [x_\alpha y_\alpha, e_\alpha \cap y_\alpha^{-1}(e_\alpha)] = [x_\alpha y_\alpha, g_{\alpha\beta}],$$

and it suffices to note that $z_{\alpha\beta} g_{\alpha\beta} = x_\alpha y_\alpha g_{\alpha\beta}$ for all α, β. It remains to show that Ψ is a bijection. By (2.2) and (4.7), there exists a monomorphism $\Phi\colon C \to Q$ such that $\Phi(a) = a$ for all $a \in A$. Since $\Phi\Psi$ is a ring endomorphism of Q whose restriction to A is the identity, $\Phi\Psi$ is the identity map on Q by (2.1). Similarly, $\Psi\Phi$ is the identity map on C; in particular, Ψ is a bijection.

A *-ring is said to *contain sufficiently many projections* if each of its nonzero one-sided ideals contains a nonzero projection.

(4.10) COROLLARY. *If A is a finite Baer *-ring satisfying* LP $\sim$ RP *and containing sufficiently many projections, then $C_\rho \cong_A Q$.*

PROOF. Let $M \in D(A)$; it suffices to show that M contains an SDD. Let $(f_\rho)_{\rho \in J}$ be a maximal family of nonzero orthogonal projections in M. If $\sup f_\rho \neq 1$, then $f = 1 - \sup f_\rho \neq 0$. Since $fA \neq 0$ and M is large, it follows that $fA \cap M \neq 0$; hence, there exists a nonzero projection $g \in fA \cap M$. Since $gf_\rho = 0$ for all ρ, this contradicts maximality and proves $\sup f_\rho = 1$. The required SDD is obtained by taking finite sums of the f_ρ.

It is clear how the preceding arguments may be modified to define a *ring* C_λ *of closed left operators* and an identification of C_λ with Q_λ. Thus, a *left operator* (LO) is a family of pairs (e_α, x_α) such that (e_α) is an SDD and $\alpha' \geqslant \alpha$ implies $e_\alpha x_{\alpha'} = e_\alpha x_\alpha$. Two LO's (e_α, x_α), (f_β, y_β) are *equivalent* if $(e_\alpha \cap f_\beta)x_\alpha = (e_\alpha \cap f_\beta)y_\beta$ for all α, β. The equivalence class $[e_\alpha, x_\alpha]$ is a *closed left operator* (CLO), and the set C_λ of all CLO's is made into a ring by defining

$$[e_\alpha, x_\alpha] + [f_\beta, y_\beta] = [e_\alpha \cap f_\beta, x_\alpha + y_\beta], \quad [e_\alpha, x_\alpha][f_\beta, y_\beta] = [e_\alpha \cap x_\alpha^{-1}(f_\beta), \ \ x_\alpha y_\beta].$$

Finally, A is embedded in C_λ by identifying a with $[1, a]$.

The final result of this section will be used in §5.

(4.11) PROPOSITION. *If (e_α) is an SDD, then $M = \bigcup e_\alpha A \in D(A)$.*

PROOF. Since $M \subseteq A \subseteq C_\rho$, it suffices to show that $C_\rho \geqslant M$. Let $x, y \in C_\rho$ with $x \neq 0$, say $x = [x_\beta, f_\beta]$, $y = [y_\beta, f_\beta]$. Now, $g_{\alpha\beta} = e_\alpha \cap f_\beta \cap y_\beta^{-1}(e_\alpha)$ defines an SDD such that $x = [x_{\alpha\beta}, g_{\alpha\beta}]$ and $y = [y_{\alpha\beta}, g_{\alpha\beta}]$, where $x_{\alpha\beta} = x_\beta$ and $y_{\alpha\beta} = y_\beta$ for all α, β (cf. (4.33)); moreover, since $x \neq 0$, there exist indices γ, δ such that $x_{\gamma\delta} g_{\gamma\delta} \neq 0$. It follows by (4.8) that $xg_{\gamma\delta} = x_{\gamma\delta} g_{\gamma\delta} \neq 0$, while $g_{\gamma\delta} \in e_\gamma A \subseteq M$ and

$$yg_{\gamma\delta} = y_{\gamma\delta} g_{\gamma\delta} = y_\delta g_{\gamma\delta} = y_\delta y_\delta^{-1}(e_\gamma) g_{\gamma\delta} = e_\gamma(y_\delta y_\delta^{-1}(e_\gamma) g_{\gamma\delta}) \in e_\gamma A \subseteq M.$$

5. **The regular ring.** The ring of closed right operators does not in general have an involution, a defect which may be remedied by slight modifications in the definitions of §4. We shall see that the ring obtained in this manner is equivalent to C_ρ (and Q) when A satisfies Utumi's condition and contains sufficiently many projections. The details may be filled in by utilizing the results of §4. (Note that this construction is a direct generalization of Berberian's construction of the regular ring in [2].)

An *operator* for A is a family of pairs $\{x_\alpha, e_\alpha\}$ such that (e_α) is an SDD and (x_α, e_α), (x_α^*, e_α) are RO's. Two operators $\{x_\alpha, e_\alpha\}$, $\{y_\beta, f_\beta\}$ are *equivalent*, $\{x_\alpha, e_\alpha\} \equiv \{y_\beta, f_\beta\}$, if $(x_\alpha, e_\alpha) \equiv (y_\beta, f_\beta)$ and $(x_\alpha^*, e_\alpha) \equiv (y_\beta^*, f_\beta)$;

equivalently (cf. [3, p. 219]), if $(x_\alpha, e_\alpha) \equiv (y_\beta, f_\beta)$ (thus, two operators which are equivalent as RO's are also equivalent as operators; we express this by saying that, in testing for equivalence, adjoints take care of themselves). The relation $\equiv$ is, of course, an equivalence relation; we write $\langle x_\alpha, e_\alpha \rangle$ for the equivalence class of $\{x_\alpha, e_\alpha\}$ and call it a *closed operator* (CO) for A. The set of all CO's will be denoted by C, and we embed A in C by identifying a with the CO $\langle a, 1 \rangle$, indexed by a singleton. We define ring operations and an involution on C (extending that of A) as follows:

$$\langle x_\alpha, e_\alpha \rangle + \langle y_\beta, f_\beta \rangle = \langle x_\alpha + y_\beta, e_\alpha \cap f_\beta \rangle;$$

$$\langle x_\alpha, e_\alpha \rangle \langle y_\beta, f_\beta \rangle = \langle x_\alpha y_\beta, [f_\beta \cap y_\beta^{-1}(e_\alpha)] \cap [e_\alpha \cap (x_\alpha^*)^{-1}(f_\beta)] \rangle;$$

$$\langle x_\alpha, e_\alpha \rangle^* = \langle x_\alpha^*, e_\alpha \rangle.$$

It follows (see (4.8) and [3, Proposition 1, p. 219]) that if $x = \langle x_\alpha, e_\alpha \rangle$, then $e_\beta x = e_\beta x_\beta$ and $x e_\beta = x_\beta e_\beta$ for any fixed index β. Thus:

(5.1) THEOREM. *If A is a finite Baer *-ring satisfying* LP $\sim$ RP, *then C is a *-ring with unity containing A as a *-subring*; *moreover, C is a two-sided ring of quotients of A and a Baer *-ring with no new projections.*

PROOF. The last assertion follows from (3.4).

(5.2) THEOREM. *Let A be a finite Baer *-ring containing sufficiently many projections and satisfying* LP $\sim$ RP *and Utumi's condition*; *equip C with the unique involution extending that of A. Then the mapping Ψ: $\langle x_\alpha, e_\alpha \rangle \rightarrow [x_\alpha, e_\alpha]$ is a *-isomorphism of C onto C_ρ*; *in particular, C is a regular Baer *-ring with no new projections, and $C \overset{*}{\cong}_A C \overset{*}{\cong}_A Q \overset{*}{\cong}_A Q_\lambda$.*

PROOF. Since adjoints take care of themselves, Ψ is injective. The only other nonobvious point is surjectivity. But, $C_\rho \cong_A Q \cong_A Q_\lambda \cong_A C_\lambda$ by (3.2) and (4.10); let Φ be an isomorphism of C_ρ onto C_λ over A. Suppose $x = [x_\alpha, e_\alpha] \in C_\rho$; we will define a CO z such that $\Psi(z) = x$. Writing $[f_\beta, y_\beta]$ for the LO $\Phi(x)$, we have by (4.8) and its dual,

$$(f_\beta y_\beta) e_\alpha = [f_\beta \Phi(x)] e_\alpha = f_\beta \Phi(x) \Phi(e_\alpha) = f_\beta \Phi(x e_\alpha) = f_\beta (x_\alpha e_\alpha);$$

we define $z_{\alpha\beta} = f_\beta y_\beta e_\alpha = f_\beta x_\alpha e_\alpha$. Now, $(e_\alpha \cap x_\alpha^{-1}(f_\beta))$ and $(f_\beta \cap (y_\beta^*)^{-1}(e_\alpha))$ are SDD's by (4.4), so $g_{\alpha\beta} = [e_\alpha \cap x_\alpha^{-1}(f_\beta)] \cap [f_\beta \cap (y_\beta^*)^{-1}(e_\alpha)]$ defines an SDD; we will show that $\{z_{\alpha\beta}, g_{\alpha\beta}\}$ is an operator and $z = \langle z_{\alpha\beta}, g_{\alpha\beta} \rangle$ the required CO. If $(\alpha', \beta') \geqslant (\alpha, \beta)$, then

$$z_{\alpha'\beta'} g_{\alpha\beta} = f_{\beta'} x_{\alpha'} e_{\alpha'} g_{\alpha\beta} = f_{\beta'} x_{\alpha'} e_\alpha g_{\alpha\beta} = f_{\beta'} x_\alpha g_{\alpha\beta}$$

$$= f_{\beta'} x_\alpha x_\alpha^{-1}(f_\beta) g_{\alpha\beta} = f_\beta x_\alpha x_\alpha^{-1}(f_\beta) g_{\alpha\beta} = f_\beta x_\alpha g_{\alpha\beta} = f_\beta x_\alpha e_\alpha g_{\alpha\beta} = z_{\alpha\beta} g_{\alpha\beta}.$$

Thus, $(z_{\alpha\beta}, g_{\alpha\beta})$ is an RO, and a similar calculation shows that $(z^*_{\alpha\beta}, g_{\alpha\beta})$ is an RO. Finally,

$$\begin{aligned} z_{\alpha\beta}(g_{\alpha\beta} \cap e_\gamma) &= f_\beta x_\alpha e_\alpha(g_{\alpha\beta} \cap e_\gamma) = f_\beta x_\alpha(g_{\alpha\beta} \cap e_\gamma) = f_\beta x_\alpha x_\alpha^{-1}(f_\beta)(g_{\alpha\beta} \cap e_\gamma) \\ &= x_\alpha x_\alpha^{-1}(f_\beta)(g_{\alpha\beta} \cap e_\gamma) = x_\alpha(g_{\alpha\beta} \cap e_\gamma) = x_\alpha(e_\alpha \cap e_\gamma)(g_{\alpha\beta} \cap e_\gamma) \\ &= x_\gamma(e_\alpha \cap e_\gamma)(g_{\alpha\beta} \cap e_\gamma) = x_\gamma(g_{\alpha\beta} \cap e_\gamma). \end{aligned}$$

We denote by A the class of all finite Baer $*$-rings A such that (i) A satisfies the (EP)-axiom and the (SR)-axiom, (ii) partial isometries are addable in A, (iii) $1 + a^*a$ is invertible for all $a \in A$, (iv) A contains a central element ι such that $\iota^2 = -1$ and $\iota^* = -\iota$, and (v) if $u \in A$ is unitary (i.e., $u^*u = 1 = uu^*$) and $\mathrm{RP}(1 - u) = 1$, then there exists a sequence of projections $e_\kappa \in \{u\}''$ such that (e_κ) is an SDD and $(1 - u)e_\kappa$ is invertible in $e_\kappa A e_\kappa$ for all κ. Berberian showed [3, p. 235] that if $A \in \mathsf{A}$, then there exists a unique ring R, called the *regular ring* of A, such that (1) R is a regular Baer $*$-ring containing A and having no new unitary elements, (2) R has a 2-proper involution, and (3) the element ι of A is also central in R.

(5.3) THEOREM. *If $A \in \mathsf{A}$, then C is the regular ring of A; more precisely, $C \overset{*}{\cong}_A R$.*

PROOF. By (5.2) and (3.7), C is a regular Baer $*$-ring with no new projections (cf. [3, p. 224] and [7, p. 99]). Since (2) and (3) above are straightforward, it will suffice to show that A contains no new unitaries. Suppose, then, that $u = \langle x_\alpha, e_\alpha\rangle$ is unitary. Since $u^*u = 1$, there exists an SDD (f_α) such that $\{x^*_\alpha x_\alpha, f_\alpha\} \equiv \{1, 1\}$; thus, $(x^*_\alpha x_\alpha)(f_\alpha \cap 1) = 1(f_\alpha \cap 1)$ for all α; changing notation (and noting that $u = \langle x_\alpha, e_\alpha\rangle = \langle x_\alpha, e_\alpha \cap f_\alpha\rangle$), we may assume that $x^*_\alpha x_\alpha e_\alpha = e_\alpha$ for all α.

By (4.11), $M = \bigcup e_\alpha A$ is a large right ideal in A; therefore (see the proof of (4.10)), M contains an orthogonal family (f_ρ) of projections with $\sup f_\rho = 1$. We define an SDD (h_α) by setting $h_\alpha = \sup\{f_\rho : f_\rho \leqslant e_\alpha\}$. Now, for each ρ, put $v_\rho = x_\alpha f_\rho$, where α is *any* index such that $e_\alpha \geqslant f_\rho$. (Note that if also $e_\beta \geqslant f_\rho$, then

$$x_\alpha f_\rho = x_\alpha e_\alpha f_\rho = x_\gamma e_\alpha f_\rho = x_\gamma f_\rho = x_\gamma e_\beta f_\rho = x_\beta e_\beta f_\rho = x_\beta f_\rho,$$

where $\gamma \geqslant \alpha, \beta$.) Since

$$v^*_\rho v_\rho = (f_\rho x^*_\alpha)(x_\alpha f_\rho) = f_\rho(x^*_\alpha x_\alpha e_\alpha f_\rho) = f_\rho e_\alpha f_\rho = f_\rho,$$

(v_ρ) is a family of partial isometries [3, p. 10] with initial projections (f_ρ). If $\rho \neq \rho'$, then, choosing γ such that $f_\rho, f_{\rho'} \leqslant e_\gamma$, we have

$$(v_\rho v_\rho^*)(v_{\rho'} v_{\rho'}^*) = (x_\gamma f_\rho x_\gamma^*)(x_\gamma f_{\rho'} x_\gamma^*) = x_\gamma f_\rho (x_\gamma^* x_\gamma e_\gamma) f_{\rho'} x_\gamma^*$$

$$= x_\gamma f_\rho e_\gamma f_{\rho'} x_\gamma^* = x_\gamma f_\rho f_{\rho'} x_\gamma^* = 0;$$

hence, the final projections of the (v_ρ) are also orthogonal. If D denotes the dimension function for A (cf. [3, Chapter 6], then

$$D(\sup v_\rho v_\rho^*) = \Sigma\, D(v_\rho v_\rho^*) = \Sigma\, D(f_\rho) = D(\sup f_\rho) = 1;$$

thus, the partial isometries (v_ρ) may be added to obtain $v \in A$ such that $v^*v = 1$, $vv^* = 1$, and $vf_\rho = v_\rho$ for all ρ (note, in particular, that v is unitary). Furthermore, if $f_\rho \leqslant e_\alpha$, then $(v - x_\alpha)f_\rho = vf_\rho - x_\alpha f_\rho = 0$; holding α fixed and taking sup over ρ, it follows that $(v - x_\alpha)h_\alpha = 0$. Therefore, $u = \langle x_\alpha, e_\alpha\rangle = \langle v, 1\rangle \in A$.

For further details, the reader is referred to [8], which also contains background source material in two appendices.

Added in proof. In March, 1974, the author received a preprint of an article by Izidor Hafner, containing some of the same results as this paper.

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