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THE REGULARISATION OF THE N-WELL PROBLEM BY FINITE ELEMENTS AND BY SINGULAR PERTURBATION ARE SCALING EQUIVALENT IN TWO DIMENSIONS

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Abstract. Let $K := SO(2)\,A_1 \cup SO(2)\,A_2 \ldots SO(2)\,A_N$ where $A_1, A_2, \ldots, A_N$ are matrices of non-zero determinant. We establish a sharp relation between the following two minimisation problems in two dimensions. Firstly the N-well problem with surface energy. Let $p \in [1,2]$, $\Omega \subset \mathbb{R}^2$ be a convex polytopal region. Define

$$I_\epsilon^p(u) = \int_\Omega d^p(Du(z), K) + \epsilon \left|D^2 u(z)\right|^2 \mathrm{d}L^2 z$$

and let A_F denote the subspace of functions in $W^{2,2}(\Omega)$ that satisfy the affine boundary condition $Du = F$ on $\partial\Omega$ (in the sense of trace), where $F \notin K$. We consider the scaling (with respect to ϵ) of

$$m_\epsilon^p := \inf_{u \in A_F} I_\epsilon^p(u).$$

Secondly the finite element approximation to the N-well problem without surface energy. We will show there exists a space of functions $\mathcal{D}_F^h$ where each function $v \in \mathcal{D}_F^h$ is piecewise affine on a regular (non-degenerate) h-triangulation and satisfies the affine boundary condition $v = l_F$ on $\partial\Omega$ (where l_F is affine with $Dl_F = F$) such that for

$$\alpha_p(h) := \inf_{v \in \mathcal{D}_F^h} \int_\Omega d^p(Dv(z), K)\,\mathrm{d}L^2 z$$

there exists positive constants $\mathcal{C}_1 < 1 < \mathcal{C}_2$ (depending on $A_1, \ldots, A_N$, p) for which the following holds true

$$\mathcal{C}_1 \alpha_p\left(\sqrt{\epsilon}\right) \le m_\epsilon^p \le \mathcal{C}_2 \alpha_p\left(\sqrt{\epsilon}\right) \text{ for all } \epsilon > 0.$$

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1. Introduction

The main goal of this paper is to show the equivalence in two dimensions (in the sense of scaling) of two different regularisations of a non-convex variational problem that forms a model of crystalline microstructure,

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specifically regularisation by second order gradients (otherwise known as *singular perturbation*) and regularisation by discretation *via* finite elements.

We focus on the simplest problem with non-trivial symmetries, the N-well problem in two dimensions. To set the scene let us take the Ball-James [3,4], Chipot-Kinderlehrer [6] approach to crystal microstructure. We have an energy function $\mathcal{I}$ on the space of deformations $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ which has the form

$$\mathcal{I}(u) = \int_\Omega W(Du(x))\, \mathrm{d}L^2 x, \tag{1.1}$$

where W is the stored energy density function that describes the various properties of the material. The function W has its minimum on a set of matrices known as the *wells*

$$K = SO(3)\, A_1 \cup SO(3)\, A_2 \ldots SO(3)\, A_N. \tag{1.2}$$

Roughly speaking the $A_1, A_2, \ldots, A_N$ are symmetry related and represent the lattice states of the material.

Since w must be invariant with respect to rotation of the ambient space the wells K must have form (1.2). Functional $\mathcal{I}$ is minimised over the space of functions that have affine boundary condition $F \notin K$.

A key point is that functional $\mathcal{I}$ is not weakly lower semi-continuous. Minimising sequences form finer and finer oscillations, as is to be expected in any model designed to capture properties of microstructure.

Surprisingly for certain choices of K of the form (1.2) in two or three dimensions, the *quasiconvex hull* (see [27] for precise definitions and more information) of K (which we denote K^{qc}) is sufficiently rich to allow for the existence of $F \in K^{qc} \backslash K$ for which there exists an exact minimiser of $\mathcal{I}$ over a space of function with boundary conditions F. Specifically if $K = SO(2) \cup SO(2)\, H$ where $H = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and $\mu\lambda \geq 1$ [35], or $K = SO(2)\, A_1 \cup SO(2)\, A_2 \ldots SO(2)\, A_k$ where $A_1, A_2, \ldots, A_k$ satisfy a certain condition [14], or if K are the so call cubic to tetragonal wells $K = SO(3)\, U_1 \cup SO(3)\, U_2 \cup SO(3)\, U_3$ where for $\lambda > 1$

$$U_1 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}, U_2 = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } U_3 = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$$

[15], then in these cases there is an exact minimiser to $\mathcal{I}$ for some $F \in K^{qc} \backslash K$. This follows from work of Müller-Šverák [29,30], Sychev [33,34], Kirchheim [19,20] and Conti-Dolzmann-Kirchheim [11], see also Dacorogna-Marcellini [12] for a different approach to some related problems. The approach of Müller-Šverák uses the theory of "convex integration" (denoted by CI from this point) developed by Gromov, it is one of the simplest results of the theory.

Functional $\mathcal{I}$ does not constrain oscillations of the gradient, it does not give a length scale or any restriction on the fine geometry of the microstructure. For many materials, the observed length scale of the microstructure is many orders larger than the atomic scale and for these materials functional $\mathcal{I}$ is only a first approximation. To overcome this the following adaption of the functional $\mathcal{I}$ is commonly made, see [27], Section 6,

$$\mathcal{I}_\epsilon(u) = \int_\Omega W(Du(z)) + \epsilon \left|D^2 u(z)\right|^2 \mathrm{d}L^2 z.$$

Roughly speaking this is a regularisation of $\mathcal{I}$ that constrains the minimiser u of $\mathcal{I}$ to have less than M interfaces when typically M will be a negative power of ϵ that depends on K and W. For example if we take $K = SO(2) \cup SO(2)\, H$ (with $\det(H) = 1$) and $W(\cdot) \sim d(\cdot, K)$ then using the characterisation of Šverák [35] (as will be explained later) we have the upper bound of $\inf \mathcal{I}_\epsilon \leq c\epsilon^{\frac{1}{6}}$. Let $v(z) := u(\sqrt{\epsilon} z)\, \epsilon^{-\frac{1}{2}}$, then $\int_{\epsilon^{-\frac{1}{2}}\Omega} \left|D^2 v\right|^2 \leq m_\epsilon \epsilon^{-1} \leq c\epsilon^{-\frac{5}{6}}$. Now v satisfies the elliptic Euler Lagrange equation $\mathrm{div} DW(Dv) + \Delta^2 v = 0$ which by standard elliptic regularity means Dv is Holder with Holder exponent $O(1)$, thus each interface running through $\epsilon^{-\frac{1}{2}}\Omega$ contributes $O\left(\epsilon^{-\frac{1}{2}}\right)$ to $\int_{\epsilon^{-\frac{1}{2}}\Omega} \left|D^2 v\right|^2$ so we have at most $M \leq c\epsilon^{\frac{1}{2}}\epsilon^{-\frac{5}{6}} = c\epsilon^{-\frac{1}{3}}$ such interfaces.

There have been a number of studies of simplified versions of functional $\mathcal{I}_\epsilon$ [8,22,26]. However these works focus on the case where $\mathcal{I}_\epsilon$ acts on scalar functions and the wells of $\mathcal{I}_\epsilon$ are given by two matrices. In this case (scaling) sharp upper and lower bounds have been proved. For functional with wells that have rotational invariance, *i.e.* of the form (1.2), nothing is known about the energy of minimisers.

Another way to constrain oscillation in the gradient is to minimise $\mathcal{I}$ directly over the space of functions that are piecewise affine on a h sized triangular grid. This is known as the finite element approximation of $\mathcal{I}$. There have been many studies of finite element approximations to functionals of the form $\mathcal{I}$, again for the simplified case where the wells are given by sets of two or three matrices [5,7,23,24].

Our main achievement in this paper is to show that for the specific stored energy function $W(\cdot) \sim d^p(\cdot, K)$ (for some $p \in [1,2]$) these two regularisations are scaling equivalent.

For the case where the wells of $\mathcal{I}$ are given by sets of two or three matrices (and $W(\cdot) \sim d(\cdot, K)$) it is possible to calculate the scaling of the energy of $\mathcal{I}_\epsilon$ and the scaling of the energy of the finite element approximation to $\mathcal{I}$ [7,23]. To be more specific given matrices A, B with $\mathrm{rank}(A-B)=1$ using methods from [7] it can be shown that for wells $K_1 = \{A, B\}$ the functional $\mathcal{I}^1$ minimised over the space of functions that are piecewise affine on a h-sized triangular grid[1] and have affine boundary condition $F_0 = \mu_0 A + (1-\mu_0) B$ (for some $\mu_0 \in (0,1)$) scales like $\sqrt{h}$. Strictly speaking the functional studied in [7] acts on scalar functions but the method works for the case stated above with minor modifications. In [23] three rank-1 connected matrices were considered, expanding on the methods of [7] it was shown in [23] that if functional $\mathcal{I}^2$ has wells $K_2 = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ (and $W(\cdot) \sim d(\cdot, K_2)$) then over the space of piecewise affine functions with boundary condition $F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ the energy of functional $\mathcal{I}^2$ scales like $h^{\frac{1}{3}}$. Using very similar methods to [7] and [23] it is possible to show that functionals $\mathcal{I}^a_\epsilon := \int_\Omega \mathrm{d}(Du, K_a) + \epsilon \left|D^2 u\right|^2$ for $a = 1, 2$ are such that their energy scales like $\inf \mathcal{I}^1_\epsilon \sim \epsilon^{\frac{1}{4}}$ and $\inf \mathcal{I}^2_\epsilon \sim \epsilon^{\frac{1}{6}}$.

Thus for functionals whose wells are given by sets of two or three matrices our main theorem is of no interest, for in these cases we can calculate the scaling and it can be seen instantly that the energy of functional $\mathcal{I}^a$ taken over a space of function that are piecewise affine on a grid of size $\sqrt{\epsilon}$ scales in the same way as the energy of functional $\mathcal{I}^a_\epsilon$. The point of this paper is that we study functional $\mathcal{I}_\epsilon$ with wells

$$K = SO(2) A_1 \cup SO(2) A_2 \ldots SO(2) A_N$$

and for these wells the scaling of the energy of $\mathcal{I}_\epsilon$ and the scaling of the energy of $\mathcal{I}$ over the space of piecewise affine functions are *completely unknown.* In this case our main theorem tells us that these two problem, one discrete and one continuous, are scaling equivalent.

1.1. Background and statement of main result

To state our theorem we need to give some background. Given a polytopal region Ω and some small constant $\varsigma \in (0,1)$ we say a collection of disjoint triangles $\{\tau_i\}$ is an (h,ς)-triangulation of Ω if $\bigcup_i \overline{\tau_i} = \Omega$ and every triangle τ_i contains a ball of radius ςh and has diameter less than $\varsigma^{-1} h$. Given $w \in S^1$ we denote by $\triangle^\varsigma_h(w)$ the set of regular triangulations with respect to axis $\langle w \rangle$, $w^\perp$ axis, by this we mean every triangle τ_i of distance $\varsigma^{-1} h$ from $\partial\Omega$ is a right angle triangle with sides parallel to $\langle w \rangle$, $w^\perp$. Finally we let $\mathcal{F}^{\varsigma,h}_F(w)$ denote the space of functions that are piecewise affine on some triangulation in $\triangle^\varsigma_h(w)$ and satisfy the affine boundary condition $u = l_F$ on $\partial\Omega$, where l_F is a fixed affine function with $Dl_F = F$.

Given two connected subsets of matrices $M, N \subset M^{2\times 2}$ we say M and N are *rank*-1 *connected* if and only if there exists $A \in M$ and $B \in N$ and $v \in S^1$ such that $Av = Bv$. The set of *rank*-1 *directions* connecting M, N are the set of vectors $v \in S^1$ satisfying $Av = Bv$ for some $A \in M$, $B \in N$.

[1]Whose edges are not parallel to the rank-1 connections between A and B.

For given triangulation $\{\tau_i\}$ and function $u \in \mathcal{F}_F^{\varsigma,h}(w)$ and triangle τ_i we define the *neighbouring gradients* by

$$N_i(u) = \begin{cases} \left\{Du_{\lfloor\tau_j} : \overline{\tau_j} \cap \overline{\tau_i} \neq \emptyset\right\} & \text{for } i \text{ such that } \overline{\tau_i} \cap \partial\Omega = \emptyset \\ \left\{Du_{\lfloor\tau_j} : \overline{\tau_j} \cap \overline{\tau_i} \neq \emptyset\right\} \cup \{F\} & \text{for } i \text{ such that } \overline{\tau_i} \cap \partial\Omega \neq \emptyset. \end{cases} \tag{1.3}$$

And for $u \in \mathcal{F}_F^{\varsigma,h}$ we define the *jump triangles* by

$$J(u) := \left\{i : \exists\, A, B \in N_i(u) \ \text{ such that } \ |A - B| > \varsigma^{-1}\right\}. \tag{1.4}$$

Let σ be the minimum of the absolute values of the eigenvalues of $A_1, \ldots, A_N$. Let $w_1 \in S^1$ be such that for some $w_2 \in w_1^{\perp}$ we have that $w_1, w_2, \frac{w_1-w_2}{|w_1-w_2|}$ are not in the set of rank-1 directions connecting $SO(2)A_i$ to $SO(2)A_j$ for any $i \neq j$, let $\varsigma \in \left(0, 10^{-1}\sigma\right)$ we define function space

$$\mathcal{D}_F^{\varsigma,h}(w_1) := \left\{ v \in \mathcal{F}_F^{\varsigma,h}(w_1) : \sum_{i\in J(v)} \sum_{M\in N_i(v)} \left|Dv_{\lfloor\tau_i} - M\right|^2 \leq \varsigma^{-1} h^{-2} \int_\Omega d^p(Dv, K) \right\}. \tag{1.5}$$

When there is no ambiguity we will denote these function spaces just as $\mathcal{F}_F^{\varsigma,h}$ or $\mathcal{D}_F^{\varsigma,h}$. Clearly $\inf_{v\in\mathcal{F}_F^{\varsigma,h}} I_0^p(v) \leq \inf_{v\in\mathcal{D}_F^{\varsigma,h}} I_0^p(v)$, the main reason for introducing function space $\mathcal{D}_F^{\varsigma,h}$ is that with our methods we *can not* show the sharpness of the lower bound

$$\inf_{v\in\mathcal{F}_F^{\varsigma,\sqrt{\epsilon}}} I_0^p(v) \leq c \inf_{v\in A_F} I_\epsilon(v) \tag{1.6}$$

(where A_F is the subspace of functions in $u \in W^{2,2}(\Omega)$ with $Du = F$ in the sense of trace). So instead we will prove the stronger lower bound $\inf_{v\in\mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}} I_0^p(v) \leq c \inf_{u\in A_F} I_\epsilon(u)$ and it turns out that function space $\mathcal{D}_F^{\varsigma,h}$ has enough structure to allow us to show the upper bound[2]

$$\inf_{u\in A_F} I_\epsilon^p(u) \leq c \inf_{v\in\mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}} I_0^p(v). \tag{1.7}$$

Our main theorem is the following.

Theorem 1.1. *Let $K := SO(2)A_1 \cup SO(2)A_2 \ldots SO(2)A_N$ where $A_1, A_2, \ldots, A_N \in M^{2\times 2}$ are matrices of non-zero determinant. Let σ be the minimum of the absolute values of the eigenvalues of $A_1, \ldots, A_N$.*

Let $w_1 \in S^1$ be such that for some $w_2 \in w_1^{\perp}$, the vectors $w_1, w_2, \frac{w_1-w_2}{|w_1-w_2|}$ are not in the set of rank-1 directions connecting $SO(2)A_i$ to $SO(2)A_j$ for any $i \neq j$. Let $\Omega \subset \mathbb{R}^2$ be a polytopal convex domain. For $p \in [1,2]$ define

$$I_\epsilon^p(u) := \int_\Omega d^p(Du(z), K) + \epsilon \left|D^2u(z)\right|^2 \mathrm{d}L^2 z.$$

Let $F \notin K$ and let A_F denote the subspace of functions in $W^{2,2}(\Omega)$ that have boundary condition $Du = F$ on $\partial\Omega$ in the sense of trace. For $\varsigma \in \left(0, 10^{-1}\sigma\right)$ let function space $\mathcal{D}_F^{\varsigma,h}(w_1)$ be defined by (1.5). If we define

$$\alpha_p(h) := \inf_{v\in\mathcal{D}_F^{\varsigma,h}(w_1)} I_0^p(v) \ \text{ and } m_\epsilon^p := \inf_{u\in A_F} I_\epsilon^p(u)$$

then there are positive constants $\mathcal{C}_1 < 1 < \mathcal{C}_2$ (depending only on σ, ς, p) for which

$$\mathcal{C}_1 \alpha_p\left(\sqrt{\epsilon}\right) \leq m_\epsilon^p \leq \mathcal{C}_2 \alpha_p\left(\sqrt{\epsilon}\right) \ \text{ for all } \epsilon > 0. \tag{1.8}$$

[2]For further explanation as to why function space $\mathcal{D}_F^{\varsigma,h}$ allows us to prove (1.7) where as with our methods we can not show the same inequality for the larger function space $\mathcal{F}_F^{\varsigma,h}$ see Section 2.2.

The point of introducing parameter ς into the definition of $\mathcal{D}_F^{\varsigma,h}$ is that we would like to use the greater flexibility it allows for a *potential* future improvement of our main result. To explain this further note that the definition of $\mathcal{D}_F^{\varsigma,h}$ gives us the inclusion

$$\mathcal{L}_F^{\varsigma,h} := \left\{ v \in \mathcal{F}_F^{\varsigma,h} : \|Dv\|_{L^\infty(\Omega)} \leq 4^{-1}\varsigma^{-1} \right\} \subset \mathcal{D}_F^{\varsigma,h},$$

so clearly we have the upper bound (1.7) for this function space. Given the results of [28] it seems reasonable to hope that minimisers of a functional equivalent[3] to I_ϵ^p for $p > 1$ are Q-Lipschitz (for some possibly large[4] Q independent of ϵ) inside the whole domain Ω. In [28] this has only been proved for $p = 2$ in an interior domain. If such a result could be proved the methods of this paper would allow us to show the lower bound $\inf_{v\in\mathcal{L}_F^{\varsigma,h}} I_\epsilon^p(v) \leq c \inf_{u\in A_F} I_\epsilon^p(u)$ which together with the upper bound would imply for $p > 1$ the equivalence of the scaling of m_ϵ^p to the scaling of I_ϵ^p over the space of Lipschitz finite elements. This is the principle reason for introducing parameter ς.

In truth our main motivation for establishing Theorem 1.1 was that we hoped to use it as a tool to understanding the minimiser of I_ϵ^p. To explain this further we will simplify and take $K = SO(2) \cup SO(2)H$ where H is a diagonal matrix of determinant 1 and we take $p = 1$.

As mentioned, nothing is known about the *minimiser* of the functional I_ϵ^1. In particular it is completely unknown if for very small ϵ the minimiser is something like the absolute minimiser of I_0 provided by CI[5]. In some sense this might seem reasonable, we refer to the $\int |D^2u|^2$ term as the "surface energy" and the $\int d(Du, K)$ term as the "bulk energy", as $\epsilon \to 0$ the surface energy becomes less and less important, the main thing to be minimised is the bulk energy and of course CI solutions have zero bulk energy.

This question is best expressed by considering the scaling of m_ϵ^1. An upper bound of $m_\epsilon^1 \leq c\epsilon^{\frac{1}{6}}$ is provided by the standard double laminate which follows from the characterisation of the quasiconvex hull of $SO(2) \cup SO(2)H$ provided by [35], see Figure 1.

If $m_\epsilon \sim \epsilon^{\frac{1}{6}+\alpha}$ for $\alpha > 0$ then the minimiser will have to take a very different form than the double laminate. On the other hand if $\alpha = 0$ then energetically the minimiser does no better than the double laminate.

This question is important because CI solutions are important, many counter examples to natural conjectures in PDE have been achieved *via* CI [13,19,31,32]. Minimising functional I_ϵ is the simplest problem that constrains oscillation in some slight way where we can hope to see the effect of the existence of exact minimisers of (1.1).

In the proof of Theorem 1.1 we have to work quite hard to establish the result for $p = 1$, we do so because functional I_ϵ^1 is particularly clean in the sense that it is not necessary to consider laminates with "domain branching" to construct upper bounds (contrast this with the case $p = 2$ [8,22]) as such the upper bound is given by $c\epsilon^{\frac{1}{6}}$ and is domain independent.

Let $w_1 \in S^1$ be such that for $w_2 \in w_1^\perp$ we have $w_1, w_2, \frac{w_1 - w_2}{|w_1 - w_2|}$ do not belong to the rank-1 connections between $SO(2)$ and $SO(2)H$. If $\tilde{u} \in \mathcal{F}_F^{\varsigma,h}(w_1)$ and $\tau_1, \tau_2 \in \triangle_h^\varsigma(w_1)$ are such that $d\left(D\tilde{u}_{\lfloor\tau_1}, SO(2)\right) \approx 0$ and $d\left(D\tilde{u}_{\lfloor\tau_2}, SO(2)H\right) \approx 0$, it is not too hard to see τ_1 can not touch τ_2, *i.e.* there must be a triangle τ_3 between τ_1 and τ_2 for which $d\left(Du_{\lfloor\tau_3}, K\right) \geq o(1)$.

For example if we have an interpolant of a laminate, and triangle τ_i cuts through an interface of the laminate the affine map we get from interpolating the laminate on the corners of τ_i will have its linear part some distance from the wells. See Figure 2.

[3]In order to apply the result of [28] we need a functional that is quadratic at infinity in a strong sense, but given $W(\cdot) \sim d^p(\cdot, K)$ it is easy to construct a function $\widetilde{W}$ such that $W = \widetilde{W}$ in a large ball B_R and $\|\widetilde{W} - W\|_{L^\infty} \leq c$ while $\widetilde{W}(M) = |M|^p$ for any $M \notin B_{2R}$. Defining $\widetilde{I}_\epsilon^p(v) = \int_\Omega \widetilde{W}(Dv) + \epsilon |D^2v|^2$ we have $\left|\widetilde{I}_\epsilon^p(v) - I_\epsilon^p(v)\right| \leq c$ for any $v \in W^{2,2}(\Omega)$ so obviously the energy of $\widetilde{I}_\epsilon^p$ and I_ϵ^p scale the same way with respect to ϵ and for the case $p = 2$ it is possible to apply the results of [28] to the minimiser of $\widetilde{I}_\epsilon^p$.

[4]Found *via* a compactness argument.

[5]We know it can not *be* a function u with $I_0(u) = 0$ because the result of Dolzmann-Müller [16], that any u with this property and with the property that Du is a BV has to be laminate.

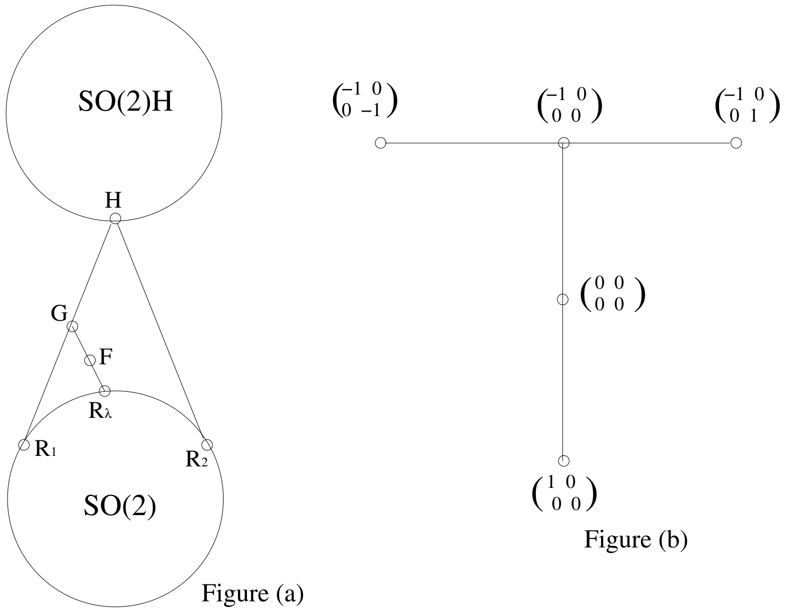

FIGURE 1. Rank-1 connections between sets of matrices.

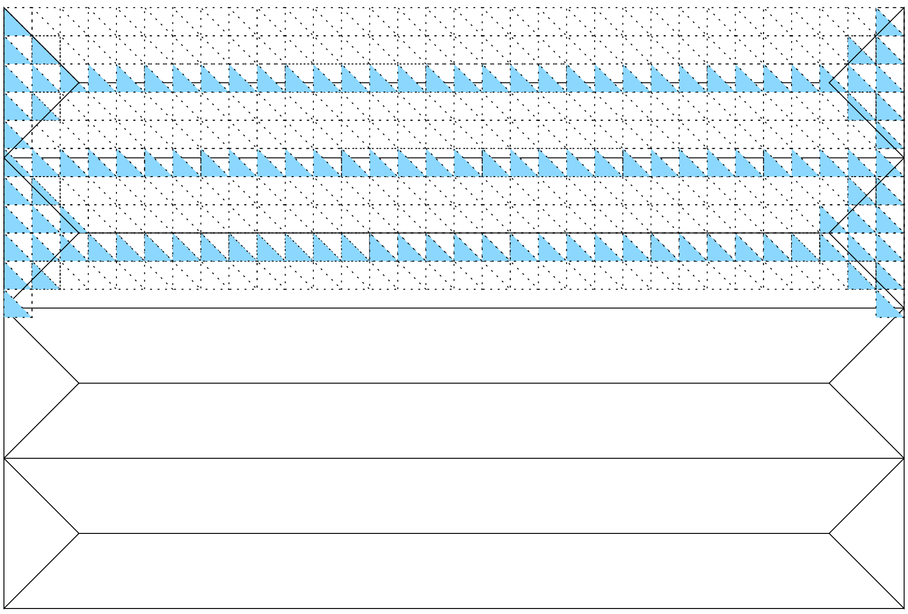

FIGURE 2. A finite element approximation to a laminate.

So we can not lower the energy of I_0 over $\mathcal{F}_F^{\varsigma,h}(w_1)$ by simply making a laminate type function with finer layers, there is a competition between the surface energy as given by the error contributed from the interfaces and the bulk energy which in the case of the laminate is the width of the interpolation layer.

As mentioned for functional $\mathcal{I}^2$ with wells $\left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}$ in [23] it was shown the energy of $\mathcal{I}^2$ over the space $\mathcal{F}^{\varsigma,h}_{F_1}$ (where $F_1 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$) scales like $h^{\frac{1}{3}}$. From Šverák's characterisation [35] we know the exact arrangement of rank-1 connections between the matrices in the set $SO(2) \cup SO(2)H$ and a matrix in the interior of the quasiconvex hull of $SO(2) \cup SO(2)H$, see Figure 1a. As we can see from Figures 1a and 1b, the finite well functional $\mathcal{I}^2$ precisely mimics these rank-1 connections.

Conjecture 1.1. *Let $K = SO(2) \cup SO(2)H$ where H is a diagonal matrix with eigenvalues σ, σ^{-1}. Let $w_1 \in S^1$ and $w_2 \in w_1^{\perp}$ be such that w_1, w_2, $\frac{w_1-w_2}{|w_1-w_2|}$ are not in the set of rank-1 connections between $SO(2)$ and $SO(2)H$. Let Ω be a polytopal convex region, $\varsigma \in (0, 10^{-1}\sigma)$. Given $F \in \mathrm{int}(K^{qc})$, let function space $\mathcal{F}^{\varsigma,h}_F(w_1)$ denote the space of functions that are piecewise affine on some regular triangulation $\{\tau_i\} \in \triangle^{\varsigma}_h(w_1)$. There exists $c_0 = c_0(\sigma, \varsigma) > 0$ such that*

$$\inf_{u\in\mathcal{F}^{\varsigma,h}_F} I^1_0(u) \geq c_0 h^{\frac{1}{3}} \text{ for all } h > 0.$$

So from Theorem 1.1, if Conjecture 1.1 could be proved it would imply the scaling $m^1_\epsilon \sim \epsilon^{\frac{1}{6}}$. Unfortunately even though the minimisation of I^1_0 over $\mathcal{F}^{\varsigma,h}_F$ is discrete problem, it appears to be quite hard to prove lower bounds.

2. Sketch of the Proof

Written out in detail, the proof of Theorem 1.1 is not short, however the basic ideas are quite simple. We give a sketch of the proof based on two lemmas that are only *approximate principles,* by this we mean that either we can not prove them, or only a weaker form hold true. This may be a bit unconventional, but it seems to us to be the best way to get to the heart of the matter without being flooded with details.

2.1. Lower bound

We focus on the case $p = 1$ and take $\Omega = Q_1(0)$. Let $M = \left[\epsilon^{-\frac{1}{2}}\right]$. We cut the square Ω into M^2 sub-squares of side length $\frac{1}{M}$, let $c_1, c_2, \ldots, c_{M^2}$ be the centres of these squares. So $Q_1(0) = \bigcup_{i=1}^{M^2} \overline{Q_{\frac{1}{M}}(c_i)}$. Let $\mathcal{C}_1 = \mathcal{C}_1(\sigma)$ be some small constant we decide on later. Now we define the "bad" squares to be $B := \left\{i : \int_{Q_{\frac{1}{M}}(c_i)} \left|D^2u\right|^2 \geq \mathcal{C}_1\right\}$.

Approximate principle 1. For any $i \in \{1, 2, \ldots, M^2\} \backslash B$ define $v_i(z) = u\left(c_i + \frac{z}{M}\right) M$ we have that there exists affine function L_i with $DL_i \in K$ such that

$$\|v_i - L_i\|_{L^\infty(Q_1(0))} \leq c \int_{Q_1(0)} d(Dv_i, K) + \left|D^2v_i\right|^2. \tag{2.1}$$

Approximate principle 2. The minimiser u of I_ϵ is a Lipschitz.

Let us make it once again clear we can not prove either *approximate principle,* they are simply a device to show the strategy of the proof. Now we split every sub-square $Q_{\frac{1}{M}}(c_i)$ into two right angle triangles, denote them τ_i, τ_{i+M^2} so the set $\{\tau_1, \tau_2, \ldots, \tau_{2M^2}\}$ is a triangulation of Ω. Let $\tilde{u}$ be the piecewise affine function we obtain from u by defining $\tilde{u}_{\lfloor\tau_i}$ to be the affine map we get from interpolating u on the corners of τ_i.

Now for any $i \notin B$ let $\omega_1^i, \omega_2^i, \omega_3^i$ denotes the corners of τ_i, so $l, q \in \{1,2,3\}$

$$\left| D\tilde{u}_{\lfloor \tau_i} \left(\frac{\omega_l^i - \omega_q^i}{\left|\omega_l^i - \omega_q^i\right|} \right) - DL_i \left(\frac{\omega_l^i - \omega_q^i}{\left|\omega_l^i - \omega_q^i\right|} \right) \right| \leq M \left| \left(u\left(\omega_l^i\right) - L_i\left(\omega_l^i\right)\right) - \left(u\left(\omega_q^i\right) - L_i\left(\omega_q^i\right)\right) \right|$$
$$\overset{(2.1)}{\leq} c \int_{Q_{\frac{1}{M}}(c_i)} M^2 d\left(Du, K\right) + \left|D^2 u\right|^2. \tag{2.2}$$

Since (2.2) holds true for every $l, q \in \{1,2\}$ we have $\left|D\tilde{u}_{\lfloor \tau_i} - DL_i\right| \leq c \int_{Q_{\frac{1}{M}}(c_i)} M^2 d\left(Du, K\right) + \left|D^2 u\right|^2$. In exactly the same way $\left|D\tilde{u}_{\lfloor \tau_{i+M^2}} - DL_{i+M^2}\right| \leq c \int_{Q_{\frac{1}{M}}(c_i)} M^2 d\left(Du, K\right) + \left|D^2 u\right|^2$. So

$$\sum_{i \in \{1,2,\ldots,M^2\} \setminus B} \left|D\tilde{u}_{\lfloor \tau_i} - DL_i\right| L^2\left(\tau_i\right) + \left|D\tilde{u}_{\lfloor \tau_{i+M^2}} - DL_{i+M^2}\right| L^2\left(\tau_{i+M^2}\right) \leq c m_\epsilon^1. \tag{2.3}$$

Now for any $i \in B$, since u is Lipschitz, for $l, q \in \{1,2,3\}$ we have

$$\left| D\tilde{u}_{\lfloor \tau_i} \left(\frac{\omega_l^i - \omega_q^i}{\left|\omega_l^i - \omega_q^i\right|} \right) \right| = \left| \frac{u\left(\omega_l^i\right) - u\left(\omega_q^i\right)}{\left|\omega_l^i - \omega_q^i\right|} \right| \leq c$$

thus $d\left(D\tilde{u}_{\lfloor \tau_i}, K\right) \leq c$ and in the same way $d\left(D\tilde{u}_{\lfloor \tau_{i+M^2}}, K\right) \leq c$ so

$$\sum_{i \in B} \left|D\tilde{u}_{\lfloor \tau_i} - DL_i\right| L^2\left(\tau_i\right) + \left|D\tilde{u}_{\lfloor \tau_{i+M^2}} - DL_{i+M^2}\right| L^2\left(\tau_{i+M^2}\right) \leq \frac{c}{M^2} \sum_{i \in B} \int_{Q_{\frac{1}{M}}(c_i)} \left|D^2 u\right|^2 \leq c m_\epsilon^1. \tag{2.4}$$

So as $\{\tau_i\}$ is a $\left(\sqrt{\epsilon}, 10^{-1}\sigma\right)$-triangulation and from (2.3), (2.4) we have $\alpha\left(\sqrt{\epsilon}\right) \leq c m_\epsilon^1$ which establishes the lower bound.

It is easy to construct a counter example to the "morally true" Lemma 1, however as a substitute we have Proposition 5.1, see Section 5. Since $i \in B$ it should seem reasonable that there exists k_0 such that

$$\int_{Q_1(0)} d\left(Dv_i, SO(2) A_{k_0}\right) \leq c \int_{Q_1(0)} d\left(Dv_i, K\right). \tag{2.5}$$

This follows from a kind of capacity type argument that is Step 1 of Proposition 5.1. Alternatively imagine we had slightly more integrability of $D^2 v_i$ and hence that $\left(\int_{Q_1(0)} \left|D^2 v_i\right|^{2+\delta}\right)^{\frac{1}{2+\delta}}$ is "small" (in fact v_i satisfies a fourth order elliptic PDE coming from the Euler Lagrange equation of u so we could indeed establish such higher integrability *via* reverse Holder inequalities), then by Sobolev embedding we would have that Dv_i stays in a neighbourhood of some well $SO(2) A_{k_0}$ and so (2.5) trivially follows.

Now if we were considering the $d^p(\cdot, K)$ distance from the wells then we could apply Theorem 3.1 to obtain sharp L^p control of the distance of Dv_i from a matrix in K. For the $p = 1$ case Theorem 3.1 is false [10] and so we need to use the fact that the "tangent space" to the set $SO(2)$ around the identity is the set of skew symmetric matrices. This allows us to apply the Korn type Poincaré inequality given by Lemma 3.1 to gain sharp control of the L^1 distance of v_i from the affine function.

Note that Proposition 5.1 is not enough since in the argument given in (2.2) we need to control the function exactly at the corners of the triangles. The trick to overcome this is the following. Let $v : Q_M(0) \to \mathbb{R}^2$ be defined by $v(z) = u\left(\frac{z}{M}\right) M$. By the Co-area formula we can find a grid of squares of side length 1, labelled

$S_1, S_2, \ldots, S_{M^2-4M}$ such that for each i there exists affine function L_i with $DL_i \in K$ such that

$$c\int_{\partial S_i} |v - L_i| + \left|D^2 v\right|^2 + d\left(Dv, SO\left(2\right)\mathrm{sym}\left(DL_i\right)\right) \leq \int_{N_1(S_i)} d\left(Dv, K\right) + \left|D^2 v\right|^2 =: \alpha_i \tag{2.6}$$

(where $\mathrm{sym}\,(A)$ denotes the symmetric part of matrix A we obtain by polar decomposition). We can split S_i into disjoint triangles τ_i, τ_{i+M^2}. Let a_i, b_i, c_i be the corners of τ_i where $[a_i, b_i] \cup [b_i, c_i] = \partial\tau_i \cap \partial S_i$. The important point is that Dv along $[a_i, b_i]$ varies by at most $\sqrt{\alpha_i}$ and so its not hard to show $Dv\,(z) \in B_{c\sqrt{\alpha_i}}\,(DL_i)$ for all $z \in [a_i, b_i]$. For simplicity let us assume $\mathrm{sym}\,(DL_i) = Id$.

Given $\tilde{b}_i \in [a_i, b_i]$, by trigonometry this allows to conclude

$$\left|v\left(a_i\right) - v\left(\tilde{b}_i\right)\right| \geq \left(1 - c\alpha_i\right)\left|a_i - \tilde{b}_i\right|.$$

And very easily from (2.6) (since we have assumed $\mathrm{sym}\,(DL_i) = Id$) we have

$$\left|v\left(a_i\right) - v\left(\tilde{b}_i\right)\right| \leq \left(1 + c\alpha_i\right)\left|a_i - \tilde{b}_i\right|.$$

The point $\tilde{b}_i$ can be easily chosen so that $\left|v\left(\tilde{b}_i\right) - L_i\left(\tilde{b}_i\right)\right| \leq c\alpha_i$. In exactly the same way we can find $\tilde{c}_i \in [a_i, c_i]$ such that $|v\,(\tilde{c}_i) - L_i\,(\tilde{c}_i)| \leq c\alpha_i$ and $||v\,(a_i) - v\,(\tilde{c}_i)| - |a_i - \tilde{c}_i|| \leq c\alpha_i$. Let $\gamma_1 = \left|a_i - \tilde{b}_i\right|$ and $\gamma_2 = |a_i - \tilde{c}_i|$ so (defining $N_\delta\,(A) := \{x : d\,(x, A) < \delta\}$) we have

$$v\left(a_i\right) \in N_{c\alpha_i}\left(\partial B_{\gamma_1}\left(\tilde{b}_i\right)\right) \cap N_{c\alpha_i}\left(\partial B_{\gamma_2}\left(\tilde{c}_i\right)\right). \tag{2.7}$$

See Figure 4. From (2.7) it is not hard to show $v\,(a_i) \in B_{c\alpha_i}\,(L_i\,(a_i))$. We can control the corners b_i, c_i in the same way. Therefor if we define l_i to be the affine map we get from interpolating v on $\{a_i, b_i, c_i\}$ we have $d\,(Dl_i, DL_i) \leq c\alpha_i$. Since $\sum_i \alpha_i \leq c\epsilon^{-1} m_\epsilon^p$ this gives the lower bound.

2.2. Upper bound

To obtain the upper bound we will have to convert a function v that is piecewise affine on a $(\sqrt{\epsilon}, \varsigma)$-triangulation into a function $u \in W^{2,2}\,(\Omega)$ with affine boundary condition $Du = F$ on $\partial\Omega$ (in the sense of trace), recall we denote the space of such functions by A_F. The most natural way to do this is to convolve v with a function $\psi_{\sqrt{\epsilon}}$ where $\psi_{\sqrt{\epsilon}}\,(z) := \epsilon^{-1}\psi\left(\frac{z}{\sqrt{\epsilon}}\right)$ and $\psi \in C_0^\infty\,(B_1\,(0) : \mathbb{R}_+)$ with $\psi = 1$ on $B_{\frac{1}{2}}\,(0)$.

Let $G_0 := \left\{i : d\left(Dv_{\lfloor\tau_i}, K\right) \leq \frac{d(SO(2), SO(2)H)}{8}\right\}$ and define $E\,(x) := \left\{i : \overline{\tau_i} \cap B_{\sqrt{\epsilon}}\,(x) \neq \emptyset\right\}$. Suppose $x \in \Omega$ is such that $E\,(x) \subset G_0$, for simplicity we will assume $d\left(Dv_{\lfloor\tau_i}, SO\left(2\right)\right) = d\left(Dv_{\lfloor\tau_i}, K\right)$ for every $i \in E\,(x)$. Since for any $k, l \in E\,(x)$ with $H^1\,(\overline{\tau}_k \cap \overline{\tau}_l) > 0$ we have that there exists $w \in S^1$ such that $Dv_{\lfloor\tau_k} w = Dv_{\lfloor\tau_l} w$ and thus $\left|Dv_{\lfloor\tau_k} - Dv_{\lfloor\tau_l}\right| \leq c\left(d\left(Dv_{\lfloor\tau_k}, SO\left(2\right)\right) + d\left(Dv_{\lfloor\tau_l}, SO\left(2\right)\right)\right)$ because if $Dv_{\lfloor\tau_k} \in SO\,(2)$ and $Dv_{\lfloor\tau_l} \in SO\,(2)$ the fact that $Dv_{\lfloor\tau_k} w = Dv_{\lfloor\tau_l} w$ would imply $Dv_{\lfloor\tau_k} = Dv_{\lfloor\tau_l}$, so the difference between $Dv_{\lfloor\tau_k}$ and $Dv_{\lfloor\tau_l}$ is controlled by the distance of these matrices from $SO\,(2)$.

A relatively easy generalisation of this is that for any x where $E\,(x) \subset G_0$

$$\left|Dv_{\lfloor\tau_k} - Dv_{\lfloor\tau_l}\right| \leq c\max\left\{d\left(Dv_{\lfloor\tau_i}, K\right) : i \in E\left(x\right)\right\} \text{ for any } k, l \in E\left(x\right). \tag{2.8}$$

Now $Du(x) = \sum_{i\in E(x)} Dv_{\lfloor\tau_i} \int_{\tau_i} \psi_{\sqrt{\epsilon}}(z-x)\,\mathrm{d}L^2 z$. Let's pick $i_0 \in E(x)$ we then have

$$\begin{aligned} \left|Du(x) - Dv_{\lfloor\tau_{i_0}}\right| &= \left|\sum_{i\in E(x)} \left(Dv_{\lfloor\tau_i} - Dv_{\lfloor\tau_{i_0}}\right)\int_{\tau_i}\psi_{\sqrt{\epsilon}}(z-x)\,\mathrm{d}L^2 z\right| \\ &\overset{(2.8)}{\leq} c\max\left\{d\left(Dv_{\lfloor\tau_i}, K\right) : i\in E(x)\right\}. \end{aligned} \tag{2.9}$$

So for any $x\in\Omega$ such that $E(x)\subset G_0$, $d(Du(x),K)$ is comparable to $d\left(Dv_{\lfloor\tau_{i_0}}, K\right)$ with error given by $\max\left\{d\left(Dv_{\lfloor\tau_i},K\right) : i\in E(x)\right\}$ and thus $\int_{\{x:E(x)\subset G_0\}} d^p(Du(z),K)\,\mathrm{d}L^2 z \leq c\sum_i d^p\left(Dv_{\lfloor\tau_i},K\right)$.

Since $|Du(x)| \leq c\sum_{i\in E(x)}\left|Dv_{\lfloor\tau_i}\right|$ thus $d^p(Du(x),K) \leq c\left(\sum_{i\in E(x)} d^p\left(Dv_{\lfloor\tau_i},K\right)+1\right)$ so as $L^2(\{x\in\Omega : E(x)\not\subset G_0\}) \leq cL^2\left(\bigcup_{i\notin G_0}\tau_i\right) \leq cm_\epsilon^p$ we have $\int_{\{x:E(x)\not\subset G_0\}} d^p(Du(x),K) \leq cm_\epsilon^p$.

So all that remains is to control the $\int_\Omega \left|D^2u\right|^2$ term. For $x\in\Omega$ such that $E(x)\subset G_0$ this is relatively easy since

$$D^2u(x) = -\int Dv(z)\otimes D\psi_{\sqrt{\epsilon}}(z-x)\,\mathrm{d}L^2 z \tag{2.10}$$

and as $\int D\psi_{\sqrt{\epsilon}}(z-x)\,\mathrm{d}L^2 z = 0$ we have

$$\begin{aligned} D^2u(x) &= -\int\left(Dv(z) - Dv_{\lfloor\tau_{i_0}}\right)\otimes D\psi_{\sqrt{\epsilon}}(z-x)\,\mathrm{d}L^2 z \\ &\leq c\epsilon^{-\frac{1}{2}}\max\left\{\left|Dv_{\lfloor\tau_j} - Dv_{\lfloor\tau_{i_0}}\right| : j\in E(x)\right\}. \end{aligned}$$

So

$$\begin{aligned} \left|D^2u(x)\right|^2 &\leq c\epsilon^{-1}\left(\max\left\{\left|Dv_{\lfloor\tau_j} - Dv_{\lfloor\tau_{i_0}}\right| : j\in E(x)\right\}\right)^p \\ &\overset{(2.8)}{\leq} c\epsilon^{-1}\max\left\{d^p\left(Dv_{\lfloor\tau_i},K\right) : i\in E(x)\right\}. \end{aligned}$$

Thus

$$\int_{\{x:E(x)\subset G_0\}} \left|D^2u(x)\right|^2\mathrm{d}L^2 x \leq c\epsilon^{-1}\sum_i d^p\left(Dv_{\lfloor\tau_i},K\right)L^2(\tau_i) \leq c\epsilon^{-1}m_\epsilon^p.$$

So far everything goes well simply by using (2.8), however for $x\in\Omega$ such that $E(x)\not\subset G_0$ we have a problem because the quantity we are interested in is $\left|D^2u(x)\right|^2$ and from equation (2.10), if the jump from $Dv_{\lfloor\tau_i}$ to $Dv_{\lfloor\tau_l}$ is much greater than 1 we can not estimate $\left|D^2u\right|^2$ by any L^1 control of the distance of Dv from K. Quite simply if we have an arbitrary function $v\in\mathcal{F}_F^{(\varsigma,\sqrt{\epsilon})}$ and we form function u by convolving it with $\psi_{\sqrt{\epsilon}}$ it could be the case that $\int_\Omega d^p(Du,K) + \left|D^2u\right|^2 \gg m_\epsilon^p$. In order for the estimate we want to hold true we need some condition that bounds the square of all the jumps of order >1 by the quantity $\epsilon^{-1}m_\epsilon^p$. The way we deal with this problem is by circumventing it: in establishing the lower bound we showed that from a function $u\in A_F$ we can create a function $\tilde{u}$ that is piecewise affine on a $(\sqrt{\epsilon},\varsigma)$ triangulation and $\int_\Omega d(D\tilde{u},K) \leq cm_\epsilon^p$, if we were smarter we could show the function $\tilde{u}$ that we created had even stronger properties. For example if u was Lipschitz then $\tilde{u}$ would also be Lipschitz and our problems would be over. Unfortunately we can not prove u is Lipschitz, however what we have for free is that $\int_\Omega\left|D^2u\right|^2 \leq \epsilon^{-1}m_\epsilon^p$. It turns out that for sufficiently careful choice of triangulation this is strong enough for us to be able to construct a function $\tilde{u}$ such that if we define

$N_i(\tilde{u})$, $J(\tilde{u})$ by (1.3), (1.4) we have that

$$\sum_{i\in J(\tilde{u})}\sum_{M\in N_i(\tilde{u})}\left|D\tilde{u}_{\lfloor \tau_i}-M\right|^2\leq c\epsilon^{-1}m_\epsilon^p. \tag{2.11}$$

So we define a function space we call $\mathcal{D}_F^{(\varsigma,h)}$ to be the set of piecewise affine functions in $\mathcal{F}_F^{(\varsigma,h)}$ that satisfies (2.11) and we will show in the "lower bound" part of Theorem 1.1 that given $u\in A_F$ with $I_\epsilon^p(u)\leq cm_\epsilon^p$ we can construct function $\tilde{u}\in\mathcal{D}_F^{(\varsigma,\sqrt{\epsilon})}$ from it such that $\int_\Omega d^p(D\tilde{u},K)\leq cm_\epsilon^p$.

To prove the "upper bound" we will need to show that if $v\in\mathcal{D}_F^{(\varsigma,\sqrt{\epsilon})}$ then we can construct function $u\in A_F$ and $I_\epsilon^p(u)\leq c\int_\Omega d^p(Dv,K)$. It turns out that proceeding in the "naive" way and simply defining $u=v*\psi_{\sqrt{\epsilon}}$ inequality (2.11) is strong enough to conclude $\int_\Omega\left|D^2u\right|^2\leq\epsilon^{-1}m_\epsilon^p$, in some sense from equation (2.10) this should come as no great surprise. Since we have already shown $\int_\Omega d^p(Du,K)\leq m_\epsilon^p$ the upper bound is completed.

3. Background

We will need a couple of not so well known Poincaré inequalities. Firstly a Korn type Poincaré inequality from [21], for a form more convenient for our purposes we refer to Theorem 6.5 [1]. The lemma we state is highly simplified version of Theorem 6.5.

Lemma 3.1. *Let $u\in W^{1,1}(\Omega:\mathbb{R}^m)$ we have a constant $c_0=c_0(n)$ such that for any $B_r(x)\subset\Omega$ there exists vectors $a_{x,r}\in\mathbb{R}^m$ and matrix $b_{x,r}\in M^{m\times n}$*

$$\int_{B_r(x)}\left|u(z)-b_{x,r}\cdot(z-x)-a_{x,r}\right|\mathrm{d}L^nz\leq c_0r\int_{B_r(x)}\left|\frac{Du(z)+Du^T(z)}{2}\right|\mathrm{d}L^nz.$$

Secondly a version of the more standard Poincaré inequality.

Lemma 3.2. *Let $a_0>0$ be a fixed small constant. Let $p\geq 1$. Suppose $u\in W^{1,p}(B_1(0))$ is such that $L^n(\{x:u(x)=0\})>a_0$. There exists constant $c_1=c_1(a_0,n)$*

$$\int_{B_1(0)}|u(z)|^p\,\mathrm{d}L^nz\leq c_1\int_{B_1(0)}|Du(z)|^p\,\mathrm{d}L^nz. \tag{3.1}$$

Proof of Lemma 3.2. Since this lemma is essentially standard we only sketch its proof. Suppose (3.1) is false, then we have a sequence $u_n\in W^{1,p}(B_1(0))$ such that

$$\left(\int_{B_1(0)}|u_n(z)|^p\,\mathrm{d}L^nz\right)\left(\int_{B_1(0)}|Du_n(z)|^p\,\mathrm{d}L^nz\right)^{-1}\to\infty. \tag{3.2}$$

Let $w_n(x):=u_n(x)\left(\int_{B_1(0)}|u_n(z)|^p\,\mathrm{d}L^nz\right)^{-1}$. So $\|w_n\|_{L^p(B_1(0))}=1$ and $\|Dw_n\|_{L^p(B_1(0))}\overset{(21)}{\to}0$ as $n\to\infty$. By BV compactness theorem (see Thm. 3.22 [2]) there exists a subsequence of w_n that has a limit $w\in BV(B_1(0))$ where $|Dw|(B_1(0))=0$ and $\int_{B_1(0)}w=1$ with $L^2(\{x:w(x)=0\})\geq a_0$, which is a contradiction. $\square$

A theorem that we will use many times is the following [18].

Theorem 3.1 (Friesecke, James, Müller)**.** *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n\geq 2$. Let $q>1$. There exists a constant $C(U,q)$ with the following property. For each $v\in W^{1,q}(U,\mathbb{R}^n)$ there exists an associated rotation $R\in SO(n)$ such that*

$$\|Dv-R\|_{L^q(U)}\leq C(U,q)\,\|\mathrm{dist}(Dv,SO(n))\|_{L^q(U)}. \tag{3.3}$$

4. Rough lower bounds on m_ϵ^p

Lemma 4.1. *Let $p \geq 1$, define*

$$m_\epsilon^p := \inf_{u \in A_F} \int_\Omega d^p\left(Du\left(z\right), K\right) + \epsilon \left|D^2 u\left(z\right)\right|^2 \mathrm{d}L^2 z. \tag{4.1}$$

We have positive constant c_1 (depending only on σ, p) such that

$$m_\epsilon^p \geq c_1 \epsilon^{\frac{1}{2}} \text{ for all } \epsilon > 0. \tag{4.2}$$

Proof. Let

$$d_0 := \frac{1}{4} \inf\left\{\left|A - B\right| : A \in SO\left(2\right) A_i, B \in SO\left(2\right) A_j, i \neq j\right\}. \tag{4.3}$$

By density of smooth functions in $W^{2,2}\left(\Omega\right)$ we can find a smooth function u satisfying $u\left(x\right) = l_F\left(x\right)$ for $x \in \partial\Omega$ with

$$\int_\Omega d^p\left(Du\left(x\right), K\right) + \epsilon \left|D^2 u\left(x\right)\right|^2 \mathrm{d}L^2 x \leq \max\left\{2m_\epsilon^p, c_1 \epsilon^{\frac{1}{2}}\right\}. \tag{4.4}$$

Now suppose (4.2) is false, so for some small positive constant $c_1 < d_0$ we have $m_\epsilon^p \leq c_1 \epsilon^{\frac{1}{2}}$. By Cauchy Schwartz inequality we have

$$\int_\Omega d^{\frac{p}{2}}\left(Du\left(x\right), K\right)\left|D^2 u\left(x\right)\right| \mathrm{d}L^2 x \leq 2c_1. \tag{4.5}$$

Let $U_i := \left\{x \in \Omega : d\left(Du\left(x\right), SO\left(2\right) A_i\right) < c_1\right\}$. There must exists $i_0 \in \{1, 2, \ldots, N\}$ such that $L^2\left(U_{i_0}\right) \geq \frac{L^2(\Omega) - c\epsilon^{\frac{1}{2p}}}{N}$. Let $E\left(x\right) = d^{\frac{p}{2}}\left(Du\left(x\right), K\right)\left|D^2 u\left(x\right)\right|$ and $\psi_z : \mathbb{R}^2 \to [0, 2\pi)$ be defined by $\left|x - z\right| \mathrm{e}^{i\psi_z(x)} = x - z$. Note ψ_z is smooth in $\mathbb{R}^2 \backslash \left\{(z_1, z_2 + \lambda) : \lambda \in \mathbb{R}_+\right\} =: \mathbb{U}_z$ and $\left|D\psi_z\left(x\right)\right| \leq \frac{1}{|x-z|}$ for any $x \in \mathbb{U}_z$. Let $c_0 := \sup\left\{\int_\Omega \frac{1}{|z-x|} \mathrm{d}L^2 z : x \in \Omega\right\}$. We know *via* Fubini theorem

$$\begin{aligned}
\int_\Omega \int_\Omega E\left(x\right)\left|D\psi_z\left(x\right)\right| \mathrm{d}L^2 x \, \mathrm{d}L^2 z &\leq \int_\Omega E\left(x\right)\left(\int_\Omega \frac{1}{\left|z - x\right|} \mathrm{d}L^2 z\right) \mathrm{d}L^2 x \\
&\leq c_0 \int_\Omega E\left(x\right) \mathrm{d}L^2 x \\
&\overset{(4.5)}{\leq} 2c_0 c_1.
\end{aligned}$$

So we can find a subset $G \subset \Omega$ such that $L^2\left(\Omega \backslash G\right) \leq 2c_0 c_1^{\frac{1}{3}}$ and for every $z \in G$ we have

$$\int_\Omega E\left(x\right)\left|D\psi_z\left(x\right)\right| \mathrm{d}L^2 x \leq c_1^{\frac{2}{3}}.$$

Now by the Co-area formula, for each $z \in G$ we can find $\Psi_z \subset [0, 2\pi)$ with $L^1\left([0, 2\pi) \backslash \Psi_z\right) \leq c_1^{\frac{1}{3}}$ for every $\theta \in \Psi_z$ we have $\int_{(z + \langle \mathrm{e}^{\mathrm{i}\theta}\rangle) \cap \Omega} E\left(x\right) \mathrm{d}H^1 x \leq c_1^{\frac{1}{3}}$. We can assume c_1 is sufficiently small so $G \cap U_{i_0} \neq \emptyset$. Now we claim for each $z \in G \cap U_{i_0}$ we have that

$$\sup\left\{d\left(Du\left(x\right), SO\left(2\right) A_{i_0}\right) : x \in \left(\bigcup_{\theta \in \Psi_z}\left(z + \langle \mathrm{e}^{\mathrm{i}\theta}\rangle\right)\right) \cap \Omega\right\} \leq 4c_1^{\frac{2}{3(2+p)}}. \tag{4.6}$$

Suppose (4.6) is false. So there exists $z_0 \in G \cap U_{i_0}$ and $\theta_0 \in \Psi_{z_0}$ with $z_1 \in \left(z_0 + \langle e^{i\theta}\rangle\right) \cap \Omega$ such that $d\left(Du\left(z_1\right), SO\left(2\right) A_{i_0}\right) > 4c_1^{\frac{2}{3(2+p)}}$. So as $d\left(Du\left(z_0\right), SO\left(2\right) A_{i_0}\right) < c_1$ we can find $z_2, z_3 \in [z_0, z_1]$ with the properties

$$d\left(Du\left(z_2\right), SO\left(2\right) A_{i_0}\right) = c_1^{\frac{2}{3(2+p)}} \text{ and } d\left(Du\left(z_3\right), SO\left(2\right) A_{i_0}\right) = 4c_1^{\frac{2}{3(2+p)}}.$$

In addition we have

$$d\left(Du\left(z\right), SO\left(2\right) A_{i_0}\right) \in \left[c_1^{\frac{2}{3(2+p)}}, 4c_1^{\frac{2}{3(2+p)}}\right] \text{ for any } z \in [z_2, z_3]. \tag{4.7}$$

So $c_1^{\frac{1}{3}} \geq \int_{z_2}^{z_3} E\left(z\right) \mathrm{d}H^1 z \geq c_1^{\frac{p}{3(2+p)}} \int_{z_2}^{z_3} \left|D^2 u\left(z\right)\right| \mathrm{d}H^1 z \geq 3c_1^{\frac{1}{3}}$ which is a contradiction. So pick $z_0 \in G \cap U_{i_0}$ and let $\Lambda = \left(\bigcup_{\theta \in \Psi_{z_0}} \left(z_0 + \langle e^{i\theta}\rangle\right)\right) \cap \Omega$. Note that

$$\begin{aligned} L^2\left(\Omega \backslash \Lambda\right) &\leq L^2\left(\left(\bigcup_{\theta \in [0,2\pi) \backslash \Psi_{z_0}} \left(z_0 + \langle e^{i\theta}\rangle\right)\right) \cap B_{\mathrm{diam}(\Omega)}\left(0\right)\right) \\ &\leq 2\pi \mathrm{diam}\left(\Omega\right) c_1^{\frac{1}{3}}. \end{aligned} \tag{4.8}$$

So as for any $x \in \Omega \backslash \Lambda$ we have $d\left(Du\left(x\right), SO\left(2\right) A_{i_0}\right) \leq d\left(Du\left(x\right), K\right) + c$ thus

$$\begin{aligned} \int_\Omega d\left(Du\left(x\right), SO\left(2\right) A_{i_0}\right) \mathrm{d}L^2 x &\leq \int_\Omega d\left(Du\left(x\right), K\right) \mathrm{d}L^2 x + cL^2\left(\Omega \backslash \Lambda\right) \\ &\overset{(4.8)}{\leq} 2\pi \mathrm{diam}\left(\Omega\right) c_1^{\frac{1}{3}} + c\epsilon^{\frac{1}{2p}}. \end{aligned}$$

So applying Proposition 2.6 [9] we have that there exists $R_0 \in SO\left(2\right)$ such that

$$\int_\Omega \left|Du\left(x\right) - R_0 A_{i_0}\right| \mathrm{d}L^2 x \leq cc_1^{\frac{1}{6}}.$$

Since $R_0 A_{i_0} \neq F$ there must exist $w \in S^1$ such that $R_0 A_{i_0} w \neq Fw$. We must be able to find $m \in w^\perp \cap B_{\frac{\mathrm{diam}(\Omega)}{10}}\left(0\right)$ such that $\int_{\Omega \cap (m + \langle w\rangle)} \left|Du\left(z\right) - R_0 A_{i_0}\right| \mathrm{d}L^1 z \leq cc_1^{\frac{1}{12}}$. Let a, b denote the endpoints of $\Omega \cap \left(c + \langle m\rangle\right)$. We have

$$\left|F\left(a-b\right) - R_0 A_{i_0}\left(a-b\right)\right| \leq \left|\int_a^b \left(Du\left(z\right) - R_0 A_{i_0}\right) w \, \mathrm{d}L^1 z\right| \leq cc_1^{\frac{1}{12}}$$

which is a contradiction assuming c_1 is chosen small enough. □

5. Proof of Theorem 1.1

Proposition 5.1. *Let $K = SO\left(2\right) A_1 \cup \ldots SO\left(2\right) A_N$, let σ be the minimum of the absolute values of the eigenvalues of A_i. Let $p \geq 1$, suppose $u \in W^{2,2}\left(B_1\left(0\right) : \mathbb{R}^2\right)$ satisfies the following properties*

$$\int_{B_1(0)} d^p\left(Du\left(z\right), K\right) \mathrm{d}L^2 z \leq \beta \tag{5.1}$$

$$\int_{B_1(0)} \left|D^2 u\left(z\right)\right|^2 \mathrm{d}L^2 z \leq \beta \tag{5.2}$$

then in the case $p > 1$ there exists matrix $M \in K$ such that

$$\int_{B_1(0)} |Du(z) - M|^p \, \mathrm{d}L^2 z \leq c\beta. \tag{5.3}$$

And for the case $p = 1$ there exists $i_0 \in \{1, 2, \ldots, N\}$ and affine function $L : B_1(0) \to \mathbb{R}^2$ with $DL \in SO(2) A_{i_0}$ such that

$$\int_{B_{\sigma^2}(0)} |u(z) - L(z)| \, \mathrm{d}L^2 z \leq c\beta \tag{5.4}$$

and

$$\int_{B_1(0)} d(Du(z), SO(2) A_{i_0}) \, \mathrm{d}L^2 z \leq c\beta. \tag{5.5}$$

Proof.
Step 1. Recall definition (4.3) of d_0, let $d_1 = \frac{\sigma}{10} d_0$ and let

$$U_i := \{x \in B_1(0) : d(Du(x), SO(2) A_i) < d_1\} \text{ for } i = 1, 2, \ldots, N. \tag{5.6}$$

We will show there exists $i_0 \in \{1, 2, \ldots, N\}$ such that

$$L^2(B_1(0) \backslash U_{i_0}) \leq c\beta. \tag{5.7}$$

As a consequence we will establish (5.5).
Proof of Step 1. Since for any $x \in B_1(0) \backslash \left(\bigcup_{i=1}^N U_i\right)$ we have $d(Du(x), K) > d_1$. So

$$L^2\left(B_1(0) \backslash \left(\bigcup_{i=1}^N U_i\right)\right) \leq \frac{1}{d_1^p} \int_{B_1(0)} d^p(Du(z), K) \, \mathrm{d}L^2 z \overset{(5.1)}{\leq} c\beta \tag{5.8}$$

which implies there must exists $i_0 \in \{1, 2, \ldots, N\}$ such that $L^2(U_{i_0}) \geq \frac{c}{N}$.

Let $\gamma \in \left(0, \frac{d_1}{4}\right)$ be some very small number. We define

$$\mathcal{S}(z) := \begin{cases} z - 3\gamma & \text{for } z > 3\gamma \\ 0 & \text{for } z \leq 3\gamma \end{cases}$$

and $\mathcal{T}(z) = \int \mathcal{S}(x) \psi_\gamma(z - x) \, \mathrm{d}L^1 z$ where $\psi_\gamma(z) := \psi\left(\frac{z}{\gamma}\right) \gamma^{-1}$ and ψ is the standard one dimensional convolution kernel with $\int \psi = 1$ and $\psi \equiv 1$ on $\left[-\frac{1}{4}, \frac{1}{4}\right]$. Let $\mathcal{P}_0 : M^{2\times 2} \to \mathbb{R}$ be defined by $\mathcal{P}_0(M) = \mathcal{T}(d(M, SO(2) A_{i_0}) - d_1)$ note that $\mathcal{P}_0$ is smooth and Lipschitz. We define $f(z) := \mathcal{P}_0(Du(z))$ it is easy to see that $f \in W^{1,2}(B_1(0))$ and we have $|Df(z)| \leq c \left|D^2 u(z)\right|$, hence $\int_{B_1(0)} |Df(z)|^2 \, \mathrm{d}L^2 z \leq c\beta$. We also know we have $f(z) = 0$ for any $z \in U_{i_0}$ and so by Lemma 3.2 we have that $\int_{B_1(0)} |f(z)|^2 \, \mathrm{d}L^2 z \leq c\beta$. As $f(z) \geq d_1$ for any $z \in \bigcup_{i \in \{1,2,\ldots,N\} \backslash \{i_0\}} U_i$ together with (5.8) this implies (5.7).

Note $(d(Du(z), K) + c)^p \leq d^p(Du(z), K) + c$

$$\begin{aligned} &\int_{B_1(0)} d^p(Du(z), SO(2) A_{i_0}) \, \mathrm{d}L^2 z \\ &\qquad \leq \int_{B_1(0)} d^p(Du(z), K) \, \mathrm{d}L^2 z + cL^2(B_1(0) \backslash U_{i_0}) \\ &\qquad \overset{(5.1),(5.7)}{\leq} c\beta. \end{aligned} \tag{5.9}$$

Now for $p>1$ by Theorem 3.1 there exists $R_0\in SO(2)$ such that $\int_{B_1(0)}|Du(z)-R_0A_{i_0}|^p\,\mathrm{d}L^2z\le c\beta$ which establishes (5.3). Obviously inequality (5.9) also gives (5.5) for $p=1$.

Step 2. Let P_0 be the affine function with $P_0(0)=0$, $DP_0=A_{i_0}^{-1}$. Define $v:B_\sigma(0)\to\mathbb{R}^2$ by $v(z)=u(P_0(z))$. We will show there exists and affine function L_1 such that

$$\int_{B_\sigma(0)}|v(z)-L_1(z)|\,\mathrm{d}L^2z\le c\beta. \tag{5.10}$$

Proof of Step 2. Firstly we apply the truncation theorem Proposition A.1. [18]. So there exists a Lipschitz function $\tilde{v}$ with $\|D\tilde{v}\|_{L^\infty(B_\sigma(0))}\le C$ and

$$L^2(\{x\in B_\sigma(0):\tilde{v}(x)\neq v(x)\})\le c\int_{\{x\in B_\sigma(0):|Dv(x)|>C\}}|Dv(z)|\,\mathrm{d}L^2z\le c\beta \tag{5.11}$$

and

$$\|Dv-D\tilde{v}\|_{L^1(B_\sigma(0))}\le c\int_{\{x\in B_\sigma(0):|Dv(x)|>C\}}|Dv(z)|\,\mathrm{d}L^2z\le c\beta. \tag{5.12}$$

Note

$$\begin{aligned}\int_{B_\sigma(0)}d(D\tilde{v}(z),SO(2))\,\mathrm{d}L^2z &\overset{(5.12)}{\le} \int_{B_\sigma(0)}d(Dv(z),SO(2))\,\mathrm{d}L^2z+c\beta\\ &\overset{(5.9)}{\le} c\beta.\end{aligned} \tag{5.13}$$

Thus by Theorem 3.1 we have that there exists R_0 such that

$$\begin{aligned}\int_{B_\sigma(0)}|D\tilde{v}(x)-R_0|^2\,\mathrm{d}L^2x &\le c\int_{B_\sigma(0)}d^2(D\tilde{v}(x),SO(2))\,\mathrm{d}L^2x\\ &\overset{(5.13)}{\le} c\beta.\end{aligned} \tag{5.14}$$

Let l_{R_0} be an affine function with $Dl_{R_0}=R_0$ and $l_{R_0}(0)=0$, we define $w(x)=\tilde{v}(l_{R_0}(x))$. So from (5.14) we have

$$\int_{B_\sigma(0)}|Dw(x)-Id|^2\,\mathrm{d}L^2x\le c\beta. \tag{5.15}$$

Now linearising $d(\cdot,SO(2))$ near the identity we have

$$\begin{aligned}d(G,SO(2)) &= \left|\frac{1}{2}\left(G+G^T\right)-Id\right|+O\left(|G-Id|^2\right)\\ &= |\mathrm{sym}(G-Id)|+O\left(|G-Id|^2\right).\end{aligned}$$

So we have

$$\begin{aligned}\int_{B_\sigma(0)}|\mathrm{sym}(Dw(x)-Id)|\,\mathrm{d}L^2x &\le c\int_{B_\sigma(0)}|Dw(x)-Id|^2\,\mathrm{d}L^2x\\ &\quad+c\int_{B_\sigma(0)}d(Dw(x),SO(2))\,\mathrm{d}L^2x\\ &\overset{(5.15),(5.13)}{\le} c\beta.\end{aligned}$$

Now by Lemma 3.1 we have that there exists an affine function $L_0 : B_\sigma(0) \to \mathbb{R}^2$ such that

$$\int_{B_\sigma(0)} |w(x) - x - L_0(x)| \, \mathrm{d}L^2 x \le c\beta \tag{5.16}$$

which gives us an affine function $L_1 : B_\sigma(0) \to \mathbb{R}^2$ with the property that

$$\int_{B_\sigma(0)} |\tilde{v}(x) - L_1(x)| \, \mathrm{d}L^2 x \le c\beta. \tag{5.17}$$

Now note by Lemma 3.2 we know that

$$\int_{B_\sigma(0)} |\tilde{v}(x) - v(x)| \, \mathrm{d}L^2 x \le \int_{B_\sigma(0)} |D\tilde{v}(x) - Dv(x)| \, \mathrm{d}L^2 x \overset{(5.12)}{\le} c\beta. \tag{5.18}$$

Thus

$$\int_{B_\sigma(0)} |v(x) - L_1(x)| \, \mathrm{d}L^2 x \overset{(5.17),(5.18)}{\le} c\beta.$$

Step 3. We will show there exists $R_0 \in SO(2)$ such that

$$|DL_1 - R_0| \le c\beta. \tag{5.19}$$

Proof of Step 3. It is immediate from (5.2) that $\int_{B_\sigma(0)} |D^2 v(x)|^2 \, \mathrm{d}L^2 x \le c\beta$. And so by Holder $\int_{B_\sigma(0)} |D^2 v(x)| \, \mathrm{d}L^2 x \le c\sqrt{\beta}$. We also know that

$$\int_{B_\sigma(0)} d(Dv(x), SO(2)) \, \mathrm{d}L^2 x \overset{(5.9)}{\le} c\beta. \tag{5.20}$$

Let $\mathcal{C}_3$ be some large positive number we decide on later

$$H_0 := \{x \in B_\sigma(0) : |L_1(z) - v(z)| \le \mathcal{C}_3 \beta\}. \tag{5.21}$$

Assuming constant $\mathcal{C}_3$ is large enough we have from (5.10) that

$$L^2(B_\sigma(0) \backslash H_0) \le \frac{\sigma^2}{1000}. \tag{5.22}$$

Let $w \in S^1$. We define

$$G_w^1 := \left\{ y \in P_{w^\perp}\left(B_{\frac{\sigma}{2}}(0)\right) : \int_{P_{w^\perp}^{-1}(y) \cap B_{\frac{\sigma}{2}}(0)} d(Dv(z), SO(2)) \, \mathrm{d}H^1 z \le \mathcal{C}_3 \beta \right\}$$

and

$$G_w^2 := \left\{ y \in P_{w^\perp}\left(B_{\frac{\sigma}{2}}(0)\right) : \int_{P_{w^\perp}^{-1}(y) \cap B_{\frac{\sigma}{2}}(0)} \left|D^2 v(z)\right|^2 \mathrm{d}H^1 z \le \mathcal{C}_3 \beta \right\}.$$

Assuming $\mathcal{C}_3$ was chosen large enough we have that

$$L^1\left(P_{w^\perp}\left(B_{\frac{\sigma}{2}}(0)\right) \backslash G_w^1\right) \le \frac{\sigma^2}{1000} \text{ and } L^1\left(P_{w^\perp}\left(B_{\frac{\sigma}{2}}(0)\right) \backslash G_w^2\right) \le \frac{\sigma^2}{1000}.$$

Now by (5.22) we can pick $y\in G_w^1\cap G_w^2$ such that

$$L^1\left(P_{w^\perp}^{-1}(y)\cap B_{\frac{\sigma}{2}}(0)\cap H_0\right)>\frac{\sigma}{100}\cdot$$

So we can pick $a,b\in P_{w^\perp}^{-1}(y)\cap B_{\frac{\sigma}{2}}(0)\cap H_0$ such that $|a-b|>\frac{\sigma}{100}$. We have that

$$\int_{[a,b]} d\left(Dv\left(z\right),SO\left(2\right)\right)\mathrm{d}H^1z\le c\beta \tag{5.23}$$

and

$$\int_{[a,b]}\left|D^2v\left(z\right)\right|\mathrm{d}H^1z\le c\sqrt{\beta}. \tag{5.24}$$

For each $z\in[a,b]$ let $R(z)\in SO(2)$ be such that $d\left(Dv\left(z\right),SO\left(2\right)\right)=\left|Dv\left(z\right)-R\left(z\right)\right|$. From (5.23) and (5.24) we have that there exists $R_0\in SO(2)$ such that

$$\sup\left\{\left|Dv\left(z\right)-R_0\right|:z\in[a,b]\right\}\le c\sqrt{\beta}. \tag{5.25}$$

Now note

$$\begin{aligned}(v(a)-v(b))\cdot R_0v_1 &= \left(\int_{[a,b]}Dv\left(z\right)v_1\mathrm{d}H^1z\right)\cdot R_0v_1\\ &\overset{(5.23)}{\geq} \int_{[a,b]}R\left(z\right)e_1\cdot R_0e_1\mathrm{d}H^1z-c\beta.\end{aligned} \tag{5.26}$$

By definition of $R(z)$, we have that $\left|Dv\left(z\right)-R\left(z\right)\right|\le\left|Dv\left(z\right)-R_0\right|\overset{(5.25)}{\leq}c\sqrt{\beta}$. So

$$\left|R\left(z\right)-R_0\right|\le\left|Dv\left(z\right)-R_0\right|+\left|Dv\left(z\right)-R\left(z\right)\right|\overset{(5.25)}{\leq}c\sqrt{\beta}.$$

Let $\psi\in[0,2\pi)$ be such that $R_0=\begin{pmatrix}\sin\psi & \cos\psi\\ -\cos\psi & \sin\psi\end{pmatrix}$ and $\psi\left(z\right)\in[0,2\pi)$ be such that $R\left(z\right)=\begin{pmatrix}\sin\psi\left(z\right) & \cos\psi\left(z\right)\\ -\cos\psi\left(z\right) & \sin\psi\left(z\right)\end{pmatrix}$. We know $\sup\left\{\left|\psi-\psi\left(z\right)\right|:z\in[a,b]\right\}\le c\sqrt{\beta}$ so

$$\begin{aligned}\int_{[a,b]}R\left(z\right)e_1\cdot R_0e_1\mathrm{d}H^1z &= \int_{[a,b]}\cos\left(\psi\left(z\right)-\psi\right)\mathrm{d}H^1z\\ &\geq |a-b|-c\beta.\end{aligned}$$

Putting this together with (5.26) we have $(v(a)-v(b))\cdot R_0v_1\ge|a-b|-c\beta$ which of course implies

$$\left|v\left(a\right)-v\left(b\right)\right|\ge|a-b|-c\beta. \tag{5.27}$$

Now

$$\begin{aligned}\left|v\left(a\right)-v\left(b\right)\right| &\le \int_{[a,b]}\left|Dv\left(z\right)\frac{a-b}{|a-b|}\right|\mathrm{d}H^1z\\ &\le |a-b|+c\beta.\end{aligned} \tag{5.28}$$

Since $a, b \in H_0$ we have

$$||L_1(a-b)| - |a-b|| \overset{(5.21)}{\leq} ||v(a) - v(b)| - |a-b|| + c\beta \overset{(5.27),(5.28)}{\leq} c\beta$$

which gives

$$||L_1(w)| - 1| \leq c\beta \text{ for all } w \in S^1. \tag{5.29}$$

Let us take three points x_1, x_2, x_3 that form the corners of an equilateral triangle, *i.e.* $|x_i - x_j| = 1$ for $i, j \in \{1,2,3\}$. So $L_1(x_1)$, $L_1(x_2)$, $L_1(x_3)$ form the corners of a triangle which we denote by T_1.

Let θ_i denote the angle of the triangle T_1 at the corner $L_1(x_i)$. Let $A_1 = |L_1(x_2) - L_1(x_3)|$, $A_2 = |L_1(x_1) - L_1(x_3)|$, $A_3 = |L_1(x_1) - L_1(x_2)|$. Now by the law of sins $\frac{\sin\theta_1}{A_1} = \frac{\sin\theta_2}{A_2} = \frac{\sin\theta_3}{A_3}$. Let $i, j \in \{1,2,3\}$, $\frac{\sin\theta_i}{A_i} = \frac{\sin\theta_j}{A_j} = \frac{\sin\theta_j}{A_i} + \sin\theta_j\left(\frac{1}{A_j} - \frac{1}{A_i}\right)$. So $\frac{\sin\theta_i - \sin\theta_j}{A_i} = \sin\theta_j\left(\frac{A_i - A_j}{A_j A_i}\right)$. Note $A_1 = |L_1(x_1 - x_3)| \overset{(5.29)}{\in} (1 - c\beta, 1 + c\beta)$. In the same way $1 - c\beta \leq A_i \leq 1 + c\beta$ for $i = 2, 3$ so

$$|\sin\theta_i - \sin\theta_j| \leq c|A_i - A_j| < c\beta. \tag{5.30}$$

Now assuming β is small enough we must have $\theta_i \in \left(0, \frac{999\pi}{2000}\right)$ for $i = 1,2,3$ since otherwise $\max\{|L_1(x_i) - L_1(x_j)| : i, j \in \{1,2,3\}, i \neq j\} > \sqrt{2} - \frac{1}{50}$ which contradicts (5.29). So

$$|\theta_i - \theta_j| \leq c|\sin\theta_i - \sin\theta_j| \overset{(5.29)}{\leq} c\beta.$$

Since $\theta_1 + \theta_2 + \theta_3 = \pi$ this gives $\left|\theta_i - \frac{\pi}{3}\right| \leq c\beta$ for $i = 1,2,3$ which implies there exists rotation $R_0 \in SO(2)$ such that $|DL_1 - R_0| \leq c\beta$ which completes the proof of Step 3.

Proof of Proposition 5.1 *completed.* Let L_0 be the affine function with $L_0(0) = L_1(0)$ and $DL_0 = R_0$ where $R_0 \in SO(2)$ satisfies (5.19) of Step 3. So from (5.10) we know

$$\int_{B_\sigma(0)} |v(x) - L_0(x)| \,\mathrm{d}L^2 x \leq c\beta. \tag{5.31}$$

As $u(z) = v\left(P_0^{-1}(z)\right)$ we have that

$$\int_{B_{\sigma^2}(0)} \left|u(z) - L_0\left(P_0^{-1}(z)\right)\right| \mathrm{d}L^2 z = \int_{B_{\sigma^2}(0)} \left|v\left(P_0^{-1}(z)\right) - L_0\left(P_0^{-1}(z)\right)\right| \mathrm{d}L^2 z \overset{(5.31)}{\leq} c\beta.$$

Define $L := L_0 \cdot P_0^{-1}$, so $DL = DL_0 \cdot DP_0^{-1} = R_0 A_0 \in K$ so L satisfies (5.4) which completes the proof of Proposition 5.1. □

Proposition 5.2. *Let $w_1 \in S^1$ be such that for some $w_2 \in w_1^\perp$ we have that $w_1, w_2, \frac{w_1 - w_2}{|w_1 - w_2|}$ are not in the set of rank-1 directions connecting $SO(2)A_i$ to $SO(2)A_j$ for any $i \neq j$. Let $p \geq 1$, we will show that for some enough $\varsigma = \varsigma(\sigma)$ we can find $\tilde{u} \in \mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}(w_1)$ such that*

$$\int_\Omega d^p\left(D\tilde{u}(z), K\right) \mathrm{d}L^2 z \leq c m_\epsilon^p. \tag{5.32}$$

Proof. The main idea for the proof is to take a function $u \in A_F$ with $I_\epsilon^p(u) \leq 2m_\epsilon^p$ and to *find* a regular triangulation $\{\tau_i\} \in \triangle_{\sqrt{\epsilon}}^{\varsigma}(w_1)$ (recall notation from Sect. 1.1) such that when we define $\tilde{u}$ to be the piecewise affine interpolation of u on $\{\tau_i\}$ then we have $\int_\Omega d^p(D\tilde{u}, K)\,\mathrm{d}L^2 z \leq c m_\epsilon^p$ and $\tilde{u} \in \mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}$. In order for $\tilde{u}$ to satisfy these properties we will need Du to have controlled surface and bulk energies on the set $\bigcup_i \partial\tau_i$.

However as $(\Omega\backslash N_{\varsigma^{-1}\sqrt{\epsilon}}(\partial\Omega))\cap\bigcup_i\partial\tau_i$ is the intersection of three sets of evenly spaced parallel lines that are of order $O(\sqrt{\epsilon})$ apart, by applying the Co-area formula to all possible shifted copies of these sets of lines we can find a triangulation with the properties we want. The rest of the proof is just a matter of harvesting the inequalities we need.

Let $\mathcal{C}_0=\mathcal{C}_0(\sigma,\varsigma)$ be some small number we decide on later. We claim we can assume

$$m_\epsilon^p\leq\mathcal{C}_0. \tag{5.33}$$

Suppose (5.33) is false, then we define $\tilde{u}=l_F$, clearly $l_F\in\mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}$ and $\int_\Omega d^p(Dl_F,K)\,\mathrm{d}L^2z\leq c$, so inequality (5.32) is satisfied. So we can assume (5.33) or there is nothing to show.

Let $u\in A_F$ be such that $I_\epsilon^p(u)\leq cm_\epsilon^p$. So we $\int_\Omega\left|D^2u(z)\right|^2\mathrm{d}L^2z\leq c\epsilon^{-1}m_\epsilon^p$. Define $v(z):=\frac{u(\sqrt{\epsilon}z)}{\sqrt{\epsilon}}$. Recall $\Omega_{\epsilon^{-\frac{1}{2}}}=\epsilon^{-\frac{1}{2}}\Omega$. Note

$$\int_{\Omega_{\epsilon^{-\frac{1}{2}}}}d^p(Dv(z),K)\,\mathrm{d}L^2z\leq c\epsilon^{-1}m_\epsilon^p \tag{5.34}$$

and

$$\int_{\Omega_{\epsilon^{-\frac{1}{2}}}}\left|D^2v(z)\right|^2\mathrm{d}L^2z\leq c\epsilon^{-1}m_\epsilon^p. \tag{5.35}$$

Let $T_t^1:=\{kw_2+\langle w_1\rangle:k\in\mathbb{Z}\}+tw_2$ and $T_t^2:=\{kw_1+\langle w_2\rangle:k\in\mathbb{Z}\}+tw_1$. Define $\mathbb{L}_1:\Omega_{\epsilon^{-\frac{1}{2}}}\to[0,1]$ to be such that $\mathbb{L}_1^{-1}(s)=T_s^1\cap\Omega_{\epsilon^{-\frac{1}{2}}}$ and $\mathbb{L}_2:\Omega_{\epsilon^{-\frac{1}{2}}}\to[0,1]$ to be such that $\mathbb{L}_1^{-1}(s)=T_s^1\cap\Omega_{\epsilon^{-\frac{1}{2}}}$. It is easy to see $|D\mathbb{L}_1|\leq1$, $|D\mathbb{L}_2|\leq1$.

Now $Dv=F$ in the sense of trace on $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$. By Theorem 2, Section 5.3 [17], this implies

$$\lim_{r\to0}\fint_{B_r(x)\cap\Omega_{\epsilon^{-\frac{1}{2}}}}|Dv(z)-F(z)|\,\mathrm{d}L^2z=0\text{ for }H^1a.e.\ x\in\partial\Omega_{\epsilon^{-\frac{1}{2}}}. \tag{5.36}$$

Let $\mathsf{S}_1,\ldots,\mathsf{S}_{p_0}$ denote the sides of $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$. For simplicity we make the assumption that none of the sides $\mathsf{S}_1,\mathsf{S}_2,\ldots,\mathsf{S}_{p_0}$ are parallel to w_1, w_2. Let $i\in\{1,\ldots,p_0\}$, there exists $\widetilde{\mathsf{S}}_i\subset\mathsf{S}_i$ with $L^1\left(\mathsf{S}_i\backslash\widetilde{\mathsf{S}}_i\right)=0$ such that for any $x\in\widetilde{\mathsf{S}}_i$ we can find $r_x\in(0,\epsilon)$ with the property that for any $r\in(0,r_x]$ we have $\int_{B_r(x)\cap\Omega_{\epsilon^{-\frac{1}{2}}}}|Dv(z)-F(z)|\,\mathrm{d}L^2z\leq\epsilon r^2$.

So there exists $\delta\in(0,1)$ such that for each i we can find subset $\mathbb{S}_i\subset\widetilde{\mathsf{S}}_i$ with $L^1\left(\widetilde{\mathsf{S}}_i\backslash\mathbb{S}_i\right)\leq\epsilon$ and for each $x\in\mathbb{S}_i$, $r_x\geq\delta$.

Let $q\in\{1,2\}$, $i\in\{1,\ldots,p_0\}$. The set of intervals $\left\{P_{w_q^\perp}(B_\delta(x)):x\in\mathbb{S}_i\right\}$ forms a cover of $P_{w_q^\perp}(\mathbb{S}_i)$ and so by the $5r$ Covering Theorem, Theorem 2.1 [25] we can extract a subset $\{x_1,x_2,\ldots,x_{J_0}\}\subset\mathbb{S}_i$ such that

$$\left\{P_{w_q^\perp}\left(B_{\frac{\delta}{5}}(x_n)\right):n\in\{1,2,\ldots,J_0\}\right\}\text{ are disjoint} \tag{5.37}$$

and

$$P_{w_q^\perp}(\mathbb{S}_i)\subset\bigcup_{n=1}^{J_0}P_{w_q^\perp}(B_\delta(x_n)). \tag{5.38}$$

Let $\mathsf{C}_n^i:=\left\{z\in B_\delta(x_n)\cap\Omega_{\epsilon^{-\frac{1}{2}}}:|Dv(z)-F|\leq1\right\}$ so $L^2\left(B_\delta(x_n)\backslash\mathsf{C}_n^i\right)\leq\epsilon\delta^2$. This implies

$$L^1\left(P_{w_q^\perp}(B_\delta(x_n))\backslash P_{w_q^\perp}\left(\mathsf{C}_n^i\right)\right)\leq c\epsilon\delta. \tag{5.39}$$

Let $\Sigma_i = \bigcup_{n=1}^{J_0} \mathsf{C}_n^i$. We have

$$\begin{aligned} L^1\left(P_{w_q^\perp}\left(\mathsf{S}_i\cap H\left(0,w_q\right)\right)\setminus P_{w_q^\perp}\left(\Sigma_i\cap H\left(0,w_q\right)\right)\right) &= L^1\left(P_{w_q^\perp}\left(\mathsf{S}_i\cap H\left(0,w_q\right)\right)\setminus\left(\bigcup_{n=1}^{J_0} P_{w_q^\perp}\left(\mathsf{C}_n^i\cap H\left(0,w_q\right)\right)\right)\right) \\ &\overset{(5.38),(5.39)}{\leq} cJ_0\epsilon\delta \\ &\overset{(5.37)}{\leq} c\epsilon. \end{aligned} \tag{5.40}$$

By exactly the same argument

$$L^1\left(P_{w_q^\perp}\left(\mathsf{S}_i\cap H\left(0,-w_q\right)\right)\setminus P_{w_q^\perp}\left(\Sigma_i\cap H\left(0,-w_q\right)\right)\right)\leq c\epsilon. \tag{5.41}$$

Define

$$\mathsf{A}_0 := \bigcup_{i=1}^{p_0}\Sigma_i \text{ and note that } \mathsf{A}_0\subset N_1\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right). \tag{5.42}$$

Let $q\in\{1,2\}$ and let l be such that $\{l\}=\{1,2\}\setminus\{q\}$. As shown on Figure 3, let

$$Q_1^q = \inf\left\{k\in\mathbb{Z} : \left(kw_l+\langle w_q\rangle\right)\cap\Omega_{\epsilon^{-\frac{1}{2}}}\neq\emptyset\right\}$$

and let

$$Q_2^q = \sup\left\{k\in\mathbb{Z} : \left(kw_l+\langle w_q\rangle\right)\cap\Omega_{\epsilon^{-\frac{1}{2}}}\neq\emptyset\right\}.$$

Step 1. For $q\in\{1,2\}$ and l be such that $\{l\}=\{1,2\}\setminus\{q\}$ define

$$\mathsf{P}_q^+ := \left\{t\in[0,1] : \left(w_q\mathbb{R}_+ + (t+k)\,w_l\right)\cap\mathsf{A}_0\neq\emptyset \text{ for every } k\in\left\{Q_1^q,Q_1^q+1,\ldots,Q_2^q-1\right\}\right\} \tag{5.43}$$

and

$$\mathsf{P}_q^- := \left\{t\in[0,1] : \left(w_q\mathbb{R}_- + (t+k)\,w_l\right)\cap\mathsf{A}_0\neq\emptyset \text{ for every } k\in\left\{Q_1^q,Q_1^q+1,\ldots,Q_2^q-1\right\}\right\} \tag{5.44}$$

we will show $L^1\left([0,1]\setminus\mathsf{P}_q^+\right)\leq c\sqrt{\epsilon}$ and $L^1\left([0,1]\setminus\mathsf{P}_q^-\right)\leq c\sqrt{\epsilon}$.

Proof of Step 1. We argue only for the set P_1^+. For each $t\in[0,1]\setminus\mathsf{P}_1^+$ let

$$N_t := \left\{k : \left(w_1\mathbb{R}_+ + (t+k)\,w_2\right)\cap\mathsf{A}_0=\emptyset,\ k\in\left\{Q_1^1,Q_1^1+1,\ldots,Q_2^1-1\right\}\right\} \tag{5.45}$$

and let[6] $n(t) := \min N_t$.

So $[0,1]\setminus\mathsf{P}_1^+ = \bigcup_{k\in\left\{Q_1^1,Q_1^1+1,\ldots,Q_2^1-1\right\}} n^{-1}(k)$ and thus there must exist k_0 such that

$$\begin{aligned} L^1\left(n^{-1}(k_0)\right) &\geq \frac{L^1\left([0,1]\setminus\mathsf{P}_1^+\right)}{|Q_1^1|+|Q_2^1|} \\ &\geq \frac{\sqrt{\epsilon}}{5}L^1\left([0,1]\setminus\mathsf{P}_1^+\right). \end{aligned} \tag{5.46}$$

However by definition since for every $t\in n^{-1}(k_0)$, $k_0=n(t)\in N_t$ and by (5.45) we have

$$\left(w_1\mathbb{R}_+ + (t+k_0)\,w_2\right)\cap\mathsf{A}_0=\emptyset \text{ for any } t\in n^{-1}(k_0) \tag{5.47}$$

[6] We define $n(t)$ to be the minimum only to produce a well defined function, we could just as well take the maximum.

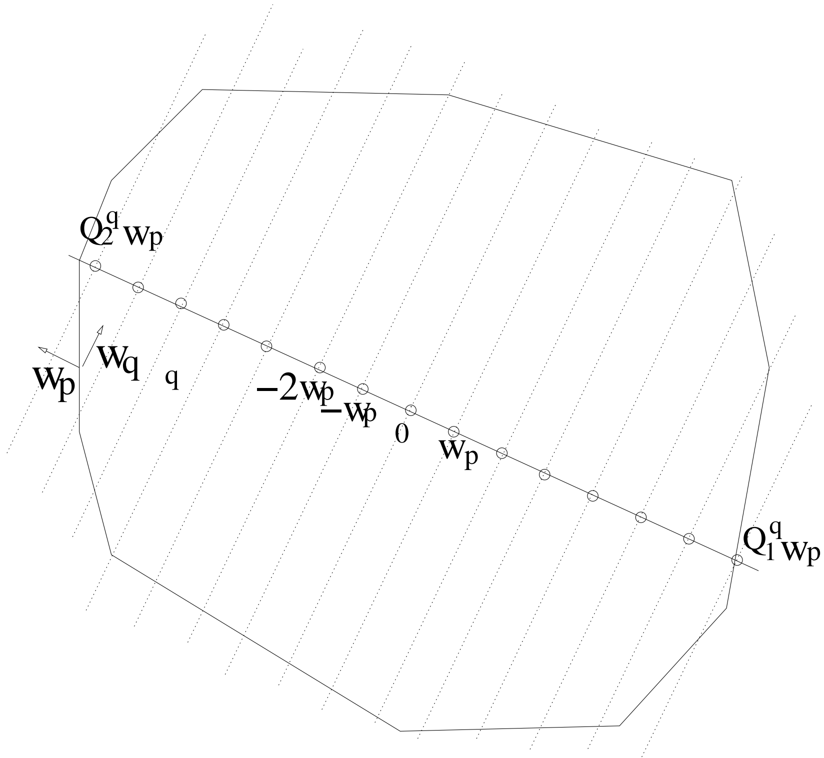

FIGURE 3. Constructing the triangulation for a typical polygonal region.

hence $((t+k_0)\,w_2)\cap P_{w_1^{\perp}}(\mathsf{A}_0\cap H(0,w_1))=\emptyset$ for any $t\in n^{-1}(k_0)$, *i.e.*

$$\left(\left(n^{-1}(k_0)+k_0\right)w_2\right)\cap P_{w_1^{\perp}}(\mathsf{A}_0\cap H(0,w_1))=\emptyset. \tag{5.48}$$

Since $k_0\in\left\{Q_1^1,Q_1^1+1,\ldots,Q_2^1-1\right\}$ we have $\left(n^{-1}(k_0)+k_0\right)w_2\subset P_{w_1^{\perp}}\left(\Omega_{\epsilon^{-\frac{1}{2}}}\right)=P_{w_1^{\perp}}\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)$ and by convexity of Ω this implies $\left(n^{-1}(k_0)+k_0\right)w_2\subset P_{w_1^{\perp}}\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\cap H(0,w_1)\right)$ so for some $a\in\{1,2,\ldots,p_0\}$ we must have

$$L^1\left(P_{w_1^{\perp}}(\mathsf{S}_a\cap H(0,w_1))\cap\left(\left(n^{-1}(k_0)+k_0\right)w_2\right)\right)\geq\frac{L^1\left(n^{-1}(k_0)\right)}{p_0} \tag{5.49}$$

and by (5.48) (and recalling definition (5.42)) we have

$$P_{w_1^{\perp}}(\mathsf{S}_a\cap H(0,w_1))\cap\left(\left(n^{-1}(k_0)+k_0\right)w_2\right)\subset P_{w_1^{\perp}}(\mathsf{S}_a\cap H(0,w_1))\backslash P_{w_1^{\perp}}(\varSigma_a\cap H(0,w_1))$$

and thus from (5.40), (5.49) we have $c\epsilon\geq L^1\left(n^{-1}(k_0)\right)$ by (5.46) $c\sqrt{\epsilon}\geq L^1\left([0,1]\backslash\mathsf{P}_1^{+}\right)$, this completes the proof of Step 1.

Step 2. Let $\{c_i:i=1,2,\ldots,N_0\}$ be an ordering of the set of points

$$\left\{k_1w_1+k_2w_2:k_1,k_2\in\mathbb{Z},k_1w_1+k_2w_2\in\Omega_{\epsilon^{-\frac{1}{2}}}\backslash N_{32\sigma^{-2}}\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)\right\}.$$

Let $\mathcal{C}_1$ be some small positive number we decide on later. Let

$$B_1 := \left\{ i \in \{1,2,\dots,N_0\} : \int_{B_{32\sigma^{-2}}(c_i)} \left|D^2 v(z)\right|^2 \mathrm{d}L^2 z > \mathcal{C}_1 \right\} \tag{5.50}$$

and

$$B_2 := \left\{ i \in \{1,2,\dots,N_0\} : \int_{B_{32\sigma^{-2}}(c_i)} d^p(Dv(z), K) \,\mathrm{d}L^2 z > \mathcal{C}_1 \right\}. \tag{5.51}$$

Note

$$\mathrm{Card}(B_1) + \mathrm{Card}(B_2) \overset{(5.34),(5.35)}{\leq} c\epsilon^{-1} m_\epsilon^p. \tag{5.52}$$

Define $G_0 = \{1,2,\dots,N_0\} \setminus (B_1 \cup B_2)$. For the case $p = 1$, for each $i \in G_0$ by Proposition 5.1 we have the existence of $q(i) \in \{1,2,\dots,N\}$ and an affine function $L_i : B_{32}(c_i) \to \mathbb{R}^2$ with $DL_i \in SO(2) A_{q(i)}$ and

$$\int_{B_{32}(c_i)} |v(z) - L_i(z)| \,\mathrm{d}L^2 z \leq \int_{B_{32\sigma^{-2}}(c_i)} d(Dv(z), K) + \left|D^2 v(z)\right|^2 \mathrm{d}L^2 z \tag{5.53}$$

and

$$\int_{B_{32}(c_i)} d\left(Dv(z), SO(2) A_{q(i)}\right) \mathrm{d}L^2 z \leq \int_{B_{32\sigma^{-2}}(c_i)} d(Dv(z), K) + \left|D^2 v(z)\right|^2 \mathrm{d}L^2 z. \tag{5.54}$$

For $p > 1$ for each $i \in G_0$ by Proposition 5.1 we have a matrix $M_i \in K$ such that

$$\int_{B_{32\sigma^{-2}}(c_i)} |Dv(z) - M_i|^p \,\mathrm{d}L^2 z \leq \int_{B_{32\sigma^{-2}}(c_i)} d^p(Dv(z), K) + \left|D^2 v(z)\right|^2 \mathrm{d}L^2 z. \tag{5.55}$$

Define

$$P(z) = \begin{cases} \sum_{i \in G_0} \chi_{B_{32}(c_i)} \left(|v(z) - L_i(z)| + d\left(Dv(z), SO(2) A_{q(i)}\right)\right), & \text{if } p = 1 \\ 0, & \text{if } p \in (1,2] \end{cases} \tag{5.56}$$

and define

$$Q(z) = \begin{cases} \sum_{i \in G_0} \chi_{B_{32}(c_i)} |Dv(z) - M_i|^p, & \text{if } p \in (1,2] \\ 0, & \text{if } p = 1. \end{cases} \tag{5.57}$$

Note

$$\int_{\Omega_{\epsilon^{-\frac{1}{2}}}} Q(z) + P(z) \,\mathrm{d}L^2 z \leq c\epsilon^{-1} m_\epsilon^p. \tag{5.58}$$

By the Co-area formula we can find $\sigma_1 \in \mathsf{P}_1^+ \cap \mathsf{P}_1^-$ and $\sigma_2 \in \mathsf{P}_2^+ \cap \mathsf{P}_2^-$ such that

$$\int_{\mathbb{L}_i^{-1}(\sigma_i)} d^p(Dv(z), K) + \left|D^2 v(z)\right|^2 \mathrm{d}H^1 z \leq c\epsilon^{-1} m_\epsilon^p \text{ for } i = 1,2 \tag{5.59}$$

and

$$\int_{\mathbb{L}_i^{-1}(\sigma_i)} P(z) + Q(z) \,\mathrm{d}H^1 z \leq c\epsilon^{-1} m_\epsilon^p \text{ for } i = 1,2. \tag{5.60}$$

Now set

$$\mathfrak{A} := \Omega_{\epsilon^{-\frac{1}{2}}} \setminus \left(\mathbb{L}_1^{-1}(\sigma_1) \cup \mathbb{L}_2^{-1}(\sigma_2)\right). \tag{5.61}$$

Let $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_{N_1}$ denote those among them that form complete squares. Let $\{\tau_1, \tau_2, \dots, \tau_{2N_1}\}$ be a collection of right angle triangles with $\overline{\tau_i} \cup \overline{\tau_{i+N_1}} = \overline{\mathcal{R}_i}$ for each $i = 1,2,\dots,N_1$.

Let

$$G_1 := \left\{ i \in \{1, 2, \dots, N_1\} : \overline{\mathcal{R}_i} \cap \{c_i : i \in G_0\} \neq \emptyset \right\}. \tag{5.62}$$

Note that from (5.52) we have

$$\mathrm{Card}\,(G_1) \geq N_1 - c\epsilon^{-1} m_\epsilon^p. \tag{5.63}$$

For each $i \in \{1, 2, \dots, N_1\}$ let l_i denote the affine function we obtain from interpolation of v on the corners of τ_i. We will show

$$\sum_{i \in G_1} d^p\left(Dl_i, K\right) + d^p\left(Dl_{i+N_1}, K\right) \leq c\epsilon^{-1} m_\epsilon^p. \tag{5.64}$$

Proof of Step 2. Case $p > 1$. Firstly we will deal with the simpler case.

For any $i \in G_1$, τ_i has two sides parallel to w_1, w_2. Let $\{a, b, e\}$ denote the corners of τ_i where we have order them so that $\frac{a-b}{|a-b|} = w_1$ and $\frac{e-b}{|e-b|} = w_2$.

$$\begin{aligned} |Dl_i w_1 - M_i w_1| &= |a-b|^{-1} \left| \int_{[a,b]} \left(Dv(z) - M_i\right) w_1 \mathrm{d}H^1 z \right| \\ &\overset{(5.57)}{\leq} c \left(\int_{[a,b]} Q(z)\, \mathrm{d}H^1 z \right)^{\frac{1}{p}}. \end{aligned}$$

So $|Dl_i w_1 - M_i w_1|^p \leq c \int_{[a,b]} Q(z)\, \mathrm{d}H^1 z$, in the same way $|Dl_i w_2 - M_i w_2|^p \leq c \int_{[b,e]} Q(z)\, \mathrm{d}H^1 z$.

Assume without loss of generality $|Dl_i w_1 - M_i w_1| \leq |Dl_i w_2 - M_i w_2|$ so

$$|Dl_i - M_i|^p \leq c \int_{\partial \mathcal{R}_i} Q(z)\, \mathrm{d}H^1 z.$$

So $d^p\left(Dl_i, K\right) \leq c \int_{\partial \mathcal{R}_i} Q(z)\, \mathrm{d}H^1 z$ in exactly the same we have $d^p\left(Dl_{i+N_1}, K\right) \leq c \int_{\partial \mathcal{R}_i} Q(z)\, \mathrm{d}H^1 z$. Thus

$$\sum_{i \in G_1} d^p\left(Dl_i, K\right) + d^p\left(Dl_{i+N_1}, K\right) \leq c \int_{\mathbb{L}^{-1}(\sigma_1) \cup \mathbb{L}^{-1}(\sigma_2)} Q(z)\, \mathrm{d}H^1 z \leq c\epsilon^{-1} m_\epsilon^p.$$

Case $p = 1$**.** Now we tackle the more difficult case. Let $i \in G_1$. So there exists $p(i) \in G_0$ such that $c_{p(i)} \cap \overline{\mathcal{R}_i} \neq \emptyset$. Let

$$\alpha_i = \int_{\partial \mathcal{R}_i} P(z) + |Dv(z)|^2 \mathrm{d}H^1 z + \int_{B_{32\sigma^{-2}}\left(c_{p(i)}\right)} d\left(Dv(z), K\right) + P(z) + \left|D^2 v(z)\right|^2 \mathrm{d}L^2 z. \tag{5.65}$$

So there exists $R_{p(i)} \in SO(2)$ such that $DL_{p(i)} = R_{p(i)} A_{s(i)}$ for some $s(i) \in \{1, 2, \dots, N\}$ (note that $s(i) = q(p(i))$, see (5.54)). Let $\{a, b, d, e\}$ denote that corners of $\mathcal{R}_i$ where $\frac{a-b}{|a-b|} = w_1$, $\frac{e-b}{|e-b|} = w_2$.

By definition of α_i there exists $x_1, x_2 \in [a, b]$, $|x_1 - x_2| > c$, $P(x_1) \leq c\alpha_i$ and $P(x_2) \leq c\alpha_i$. So

$$\left|v(x_1) - L_{p(i)}(x_1)\right| \leq c\alpha_i, \ \left|v(x_2) - L_{p(i)}(x_2)\right| \leq c\alpha_i$$

thus

$$\left|v(x_1) - v(x_2) - R_{p(i)} A_{s(i)}(x_1 - x_2)\right| \leq c\alpha_i. \tag{5.66}$$

Since $\int_{[a,b]} \left|D^2 v(z)\right| \mathrm{d}H^1 z \leq c\sqrt{\alpha_i}$ there exists R_0 such that

$$\sup\left\{\left|Dv(z) - R_0 A_{s(i)}\right| : z \in [a, b]\right\} \leq c\sqrt{\alpha_i}. \tag{5.67}$$

So

$$\left|v(x_1)-v(x_2)-R_0A_{s(i)}(x_1-x_2)\right| = \left|\int_{[x_1,x_2]}\left(Dv(z)-R_0A_{s(i)}\right)\frac{x_1-x_2}{|x_1-x_2|}\mathrm{d}H^1z\right| \overset{(5.67)}{\leq} c\sqrt{\alpha_i}.$$

Putting this together with (5.66) gives

$$\left|R_0-R_{p(i)}\right|\leq c\sqrt{\alpha_i}. \tag{5.68}$$

For $z\in[a,b]$ define $R(z)\in SO(2)$ be such that $d\left(Dv(z),SO(2)A_{s(i)}\right)=\left|Dv(z)-R(z)A_{s(i)}\right|$. So note that $\int_{[a,b]}d(R(z),SO(2))\,\mathrm{d}H^1z\leq c\alpha_i$. Note also that from (5.67) and (5.68) we have

$$\sup\left\{\left|R(z)-R_{p(i)}\right|:z\in[a,b]\right\}\leq c\sqrt{\alpha_i}. \tag{5.69}$$

Arguing as in Step 3, Proposition 5.1. Let $\theta,\theta(z)\in[0,2\pi)$ so that $R(z)=\begin{pmatrix}\sin\theta(z) & -\cos\theta(z)\\ \cos\theta(z) & \sin\theta(z)\end{pmatrix}$ and $R=\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}$. We have

$$\begin{aligned} R(z)e_1\cdot Re_1 &= \cos(\theta(z)-\theta)\\ &\overset{(5.69)}{\geq} 1-c\alpha_i \text{ for any } z\in[a,b]. \end{aligned} \tag{5.70}$$

We can pick point $\tilde{a}\in[a,b]$ with $|b-\tilde{a}|>c$ and $\tilde{e}\in[b,e]$ with $|\tilde{e}-b|>c$ where

$$\left|v(\tilde{e})-L_{p(i)}(\tilde{e})\right|\leq c\alpha_i \text{ and } \left|v(\tilde{a})-L_{p(i)}(\tilde{a})\right|\leq c\alpha_i. \tag{5.71}$$

Let $\gamma_1=|\tilde{a}-b|\left|A_{s(i)}w_1\right|$ and $\gamma_2=|\tilde{e}-b|\left|A_{s(i)}w_2\right|$. We claim

$$v(b)\in N_{c\alpha_i}\left(\partial B_{\gamma_1}(v(\tilde{a}))\right) \tag{5.72}$$

and

$$v(b)\in N_{c\alpha_i}\left(\partial B_{\gamma_2}(v(\tilde{e}))\right). \tag{5.73}$$

To see this note that

$$\begin{aligned} \left|(v(\tilde{a})-v(b))\cdot R_{p(i)}A_{s(i)}(-w_1)\right| &\geq \left|A_{s(i)}w_1\right|^2\left|\int_{[\tilde{a},b]}R(z)e_1\cdot R_{p(i)}e_1\mathrm{d}H^1z\right|-c\alpha_i\\ &\overset{(5.70)}{\geq} \left|A_{s(i)}w_1\right|^2|\tilde{a}-b|(1-c\alpha_i) \end{aligned}$$

which implies $|v(\tilde{a})-v(b)|\geq\left|A_{s(i)}w_1\right||\tilde{a}-b|(1-c\alpha_i)=\gamma_1-c\alpha_i$. Now

$$\begin{aligned} |v(\tilde{a})-v(b)| &\leq \left|\int_{[\tilde{a},b]}-R(z)A_{i_0}w_1\mathrm{d}H^1z\right|+c\alpha_i\\ &\leq \gamma_1+c\alpha_i \end{aligned}$$

which establishes (5.72). Inclusion (5.73) can be shown in exactly the same way. So putting (5.71) together with (5.72), (5.73) we have established that

$$v(b)\in N_{c\alpha_i}\left(\partial B_{\gamma_1}\left(L_{p(i)}(\tilde{a})\right)\right)\cap N_{c\alpha_i}\left(\partial B_{\gamma_2}\left(L_{p(i)}(\tilde{e})\right)\right).$$

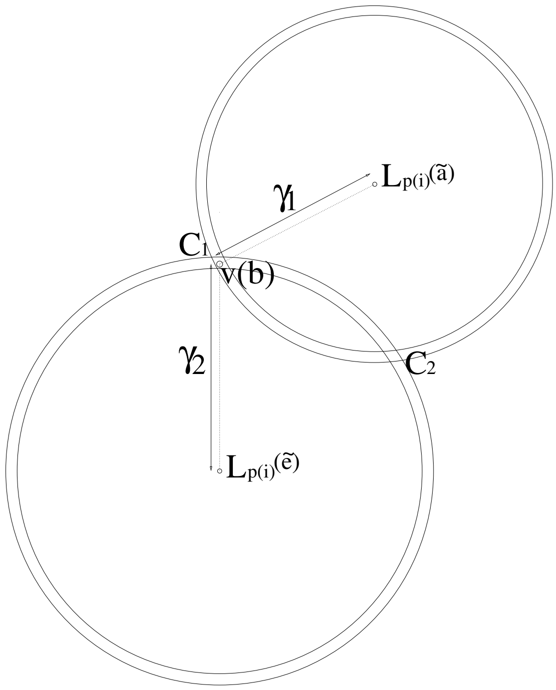

FIGURE 4. Controlling the function of the corners of a triangle.

Now the set $N_{c\alpha_i}\left(\partial B_{\gamma_1}\left(L_{p(i)}\left(\tilde{a}\right)\right)\right)\cap N_{c\alpha_i}\left(\partial B_{\gamma_2}\left(L_{p(i)}\left(\tilde{e}\right)\right)\right)$ consists of two disjoint connected components which we denote C_1 and C_2, see Figure 4. It is quite straightforward to see that $\operatorname{diam}(C_i)\leq c\alpha_i$ for $i=1,2$.

Let C_1 be the component that contains $L_{p(i)}(b)$. We will show $v(b)\in C_1$. We argue by contradiction, suppose $v(b)\in C_2$. By Proposition 5.1, inequality (5.5) (recall $s(i)=q(p(i))$) we know

$$\int_{B_{32}\left(c_{p(i)}\right)} d\left(Dv\left(z\right), SO\left(2\right)A_{s(i)}\right)\mathrm{d}L^2 z \overset{(5.34),(5.35)}{\leq} c\alpha_i.$$

So by Proposition 2.6 [9] we have that there exists $R_0\in SO(2)$ such that

$$\int_{B_{32}\left(c_{p(i)}\right)}\left|Dv\left(z\right)-R_0A_{s(i)}\right|\mathrm{d}L^2 z\leq c\log\left(\alpha_i^{-1}\right)\alpha_i. \tag{5.74}$$

Now by Sobolev embedding theorem there exists matrix M_i such that

$$\left(\int_{B_{32}\left(c_{p(i)}\right)}\left|Dv\left(z\right)-M_i\right|^3\mathrm{d}L^2 z\right)^{\frac{1}{3}}\leq c\left(\int_{B_{32}\left(c_{p(i)}\right)}\left|D^2v\left(z\right)\right|^2\mathrm{d}L^2 z\right)^{\frac{1}{2}}\leq c\sqrt{\alpha_i}. \tag{5.75}$$

So

$$\left|M_i-R_0A_{s(i)}\right| \overset{(5.74),(5.75)}{\leq} c\sqrt{\alpha_i}. \tag{5.76}$$

Let $\Lambda_i : B_{32}\left(c_{p(i)}\right) \to \mathbb{R}^2$ be such that $D\Lambda_i = R_0 A_{s(i)}$ and $\Lambda_i(0) = 0$. Define $w_i(z) = \Lambda_i(z) + \fint_{B_{32}\left(c_{p(i)}\right)} v(x) - \Lambda_i(x)\, \mathrm{d}L^2 x$ so

$$\fint_{B_{32}\left(c_{p(i)}\right)} v(z) - w_i(z)\, \mathrm{d}L^2 z = 0 \tag{5.77}$$

and

$$\begin{aligned}\left(\int_{B_{32}\left(c_{p(i)}\right)} |Dv(z) - Dw_i|^3\, \mathrm{d}L^2 z\right)^{\frac{1}{3}} &\leq \left(\int_{B_{32}\left(c_{p(i)}\right)} |Dv(z) - M_i|^3\, \mathrm{d}L^2 z\right)^{\frac{1}{3}} \\ &\quad + c\left|M_i - R_0 A_{s(i)}\right| \\ &\overset{(5.76),(5.75)}{\leq} c\sqrt{\alpha_i}.\end{aligned}$$

So by Morrey's inequality Theorem 3, Section 4.5.3 [17], together with (5.77) this implies

$$\|v - w_i\|_{L^\infty\left(B_{32}\left(c_{p(i)}\right)\right)} \leq c\sqrt{\alpha_i}. \tag{5.78}$$

Since (5.53), (5.65) $\|v - L_{p(i)}\|_{L^1\left(B_{32}\left(c_{p(i)}\right)\right)} \leq c\alpha_i$ we have $\|w_i - L_{p(i)}\|_{L^1\left(B_{32}\left(c_{p(i)}\right)\right)} \leq c\sqrt{\alpha_i}$. And since w_i and $L_{p(i)}$ are both affine this implies $\left|Dw_i - DL_{p(i)}\right| \leq c\sqrt{\alpha_i}$ and thus $\|w_i - L_{p(i)}\|_{L^\infty\left(B_{32}\left(c_{p(i)}\right)\right)} \leq c\sqrt{\alpha_i}$. Putting this together with (5.78) we have that

$$\|v - L_{p(i)}\|_{L^\infty\left(B_{32}\left(c_{p(i)}\right)\right)} \leq c\sqrt{\alpha_i}. \tag{5.79}$$

Recall we are arguing by contradiction, as we supposed $v(b) \in C_2$, from (5.79) this implies that $L_{p(i)}(b) \in N_{c\sqrt{\alpha}}(C_2)$ however as we also know $L_{p(i)}(b) \in C_1$ and $d(C_1, C_2) > c$ this is a contradiction.

Thus we have that

$$v(b) \in C_1 \subset B_{c\alpha_i}\left(L_{p(i)}(b)\right). \tag{5.80}$$

Arguing in exactly the same way we can establish the same thing for the other corners of $\mathcal{R}_i$, *i.e.* we can show

$$v(a) \in B_{c\alpha_i}\left(L_{p(i)}(a)\right), v(d) \in B_{c\alpha_i}\left(L_{p(i)}(d)\right), v(e) \in B_{c\alpha_i}\left(L_{p(i)}(e)\right). \tag{5.81}$$

Recall l_i and l_{i+N_1} are the affine maps we obtained from interpolating v on the corners of triangle τ_i and τ_{i+N_1} where $\overline{\tau_i} \cup \overline{\tau_{i+N_1}} = \overline{\mathcal{R}_i}$. Recall also that $DL_{p(i)} = R_{p(i)} A_{s(i)}$ where $R_{p(i)} \in SO(2)$, $s(i) \in \{1, 2, \ldots, N\}$. From (5.80) and (5.81) we have

$$\begin{aligned}\left|Dl_i w_1 - R_{p(i)} A_{s(i)} w_1\right| &= \left|\left(\frac{v(a) - v(b)}{|a-b|}\right) - \left(\frac{L_{p(i)}(a-b)}{|a-b|}\right)\right| \\ &\leq c\alpha_i.\end{aligned}$$

In the same way we can show $\left|Dl_i w_2 - R_{p(i)} A_{s(i)} w_2\right| \leq c\alpha_i$ which gives $\left|Dl_i - R_{p(i)} A_{s(i)}\right| \leq c\alpha_i$ and hence $d(Dl_i, K) \leq c\alpha_i$. In exactly the same way we can show $d(Dl_{i+N}, K) \leq c\alpha_i$.

Thus using (5.34), (5.35), (5.58), (5.59) and (5.60) for the last inequality

$$\sum_{i\in G_1} d\left(Dl_i,K\right)+d\left(Dl_{i+N_1},K\right) \overset{(5.65)}{\leq} c\sum_{i\in G_1}\int_{\partial\mathcal{R}_i} P\left(z\right)+\left|D^2v\left(z\right)\right|^2 \mathrm{d}H^1z$$
$$+c\int_{B_{32\sigma^{-2}}\left(c_{p(i)}\right)} d\left(Dv\left(z\right),K\right)+P\left(z\right)+\left|D^2v\left(z\right)\right|^2\mathrm{d}L^2z$$
$$\leq c\int_{\mathbb{L}_1^{-1}(\sigma_1)\cup\mathbb{L}_2^{-1}(\sigma_2)} P\left(z\right)+\left|D^2v\left(z\right)\right|^2\mathrm{d}H^1z$$
$$+c\sum_{i\in G_0}\int_{B_{32\sigma^{-2}}(c_i)} d\left(Dv\left(z\right),K\right)+P\left(z\right)+\left|D^2v\left(z\right)\right|^2\mathrm{d}L^2z$$
$$\leq c\epsilon^{-1}m_\epsilon^1.$$

Thus we have shown (5.64) in the case $p=1$. This completes the proof of Step 2.

Step 3. We will show

$$\sum_{i\in\{1,2,\ldots,N_1\}} d^p\left(Dl_i,K\right)+d^p\left(Dl_{i+N},K\right)\leq c\epsilon^{-1}m_\epsilon^p. \tag{5.82}$$

Proof of Step 3. Let $i\in\{1,2,\ldots,N_1\}\setminus G_1$ and let $\{a_i,b_i,c_i\}$ denote the corners of τ_i where we have ordered them so that $\frac{a_i-b_i}{|a_i-b_i|}=w_1$ and $\frac{c_i-b_i}{|c_i-b_i|}=w_2$. Let Dl_i denote the affine map we obtain from interpolation of v on the corners of τ_i. Note

$$\left|Dl_iw_1\right|^p=\left|\frac{v\left(a_i\right)-v\left(b_i\right)}{\left|a_i-b_i\right|}\right|^p\leq c\int_{a_i}^{b_i}\left|Dv\left(z\right)\right|^p\mathrm{d}H^1z\leq c\int_{\partial\mathcal{R}_i} d^p\left(Dv\left(z\right),K\right)\mathrm{d}H^1z+c.$$

In exactly the same way we have $\left|Dl_iw_2\right|^p\leq c\int_{\partial\mathcal{R}_i} d^p\left(Dv\left(z\right),K\right)\mathrm{d}H^1z+c$ which gives

$$\left|Dl_i\right|^p\leq c\int_{\partial\mathcal{R}_i} d^p\left(Dv\left(z\right),K\right)\mathrm{d}H^1z+c$$

in exactly the same way $\left|Dl_{i+N}\right|^p\leq c\int_{\partial\mathcal{R}_i} d^p\left(Dv\left(z\right),K\right)\mathrm{d}H^1z+c$. As $d^p\left(Dl_i,K\right)\leq c\left|Dl_i\right|^p+c$ and $d^p\left(Dl_{i+N},K\right)\leq c\left|Dl_{i+N}\right|^p+c$ thus

$$\begin{aligned}\sum_{i\in\{1,2,\ldots,N_1\}\setminus G_1} d^p\left(Dl_i,K\right)+d^p\left(Dl_{i+N},K\right) &\leq \sum_{i\in\{1,2,\ldots,N_1\}\setminus G_1} c\left|Dl_i\right|^p+c\left|Dl_{i+N}\right|^p\\ &\quad+c\,\mathrm{Card}\left(\{1,2,\ldots,N_1\}\setminus G_1\right)\\ &\overset{(5.59),(5.63)}{\leq} c\epsilon^{-1}m_\epsilon^p.\end{aligned} \tag{5.83}$$

Putting (5.83) together with (5.64) gives (5.82).

Step 4. Recall $\{\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_{N_1}\}$ denote the connected components of $\mathfrak{A}$ (see (5.61)) that form complete squares, and $\{\tau_1,\tau_2,\ldots,\tau_{2N_1}\}$ are triangles where $\overline{\tau_i}\cup\overline{\tau_{i+N_1}}=\overline{\mathcal{R}_i}$. Let

$$V_0\left(i\right):=\left\{j\in\{1,2,\ldots,2N_1\}:H^1\left(\overline{\tau_i}\cap\overline{\tau_j}\right)>\varsigma\right\}. \tag{5.84}$$

For any $j\in\{1,2,\ldots,2N_1\}$ let l_j denote the affine map we get by interpolating v on the corners of τ_j. Define

$$\Upsilon_0:=\left\{i\in\{1,2,\ldots,2N_1\}:\text{ There exists } j\in V_0\left(i\right)\text{ such that }\left|Dl_i-Dl_j\right|>\varsigma^{-1}\right\}. \tag{5.85}$$

We will show

$$\sum_{i\in\Upsilon_0}\sum_{j\in V_0(i)}|Dl_i - Dl_j|^2 \le c\epsilon^{-1}m_\epsilon^p. \tag{5.86}$$

Proof of Step 4. For any $i \in \{1,2,\dots,2N_1\}$ define

$$\rho(i) := \begin{cases} i & \text{if } i\in\{1,2,\dots,N_1\} \\ i-N_1 & \text{if } i\in\{N_1+1,\dots,2N_1\}. \end{cases}$$

To start we will show that if $i \in \{1,2,\dots,2N_1\}$ and $j \in V_0(i)$ then

$$|Dl_i - Dl_j| \le c\left(\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v(z)\right|^2 \mathrm{d}H^1 z\right)^{\frac{1}{2}}. \tag{5.87}$$

So see this we will argue as follows. Note $\overline{\mathcal{R}_{\rho(i)}}\cup\overline{\mathcal{R}_{\rho(j)}}$ forms a rectangle, thus $\overline{\tau_i}\cup\overline{\tau_j}$ must form a regular parallelogram with two opposite sides that intersect $\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)$, see Figure 5.

Let U_i denote the side of $\partial\tau_i$ that intersects $\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)$ and U_j denote the side of $\partial\tau_j$ that intersects $\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)$. Let $q\in\{1,2\}$ be such that U_i and U_j are parallel to w_q. Now by the fundamental theorem of Calculus (and Holder's inequality) there must exist $M \in M^{2\times 2}$ such that

$$\sup\left\{|Dv(z)-M| : z\in\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)\right\} \le c\left(\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v(z)\right|^2 \mathrm{d}H^1 z\right)^{\frac{1}{2}}. \tag{5.88}$$

Let $\left\{\omega_1^i,\omega_2^i,\omega_3^i\right\}$ denote the corners of τ_i and $\left\{\omega_1^j,\omega_2^j,\omega_3^j\right\}$ the corners of τ_j where we have chosen to label these points such that $\omega_3^i-\omega_2^i=\omega_2^j-\omega_1^j$ and $\omega_1^i=\omega_2^j$, $\omega_2^i=\omega_3^j$, see Figure 5, note $\left\{\omega_3^i,\omega_2^i\right\}=\partial U_i$ and $\left\{\omega_2^j,\omega_1^j\right\}=\partial U_j$, again see Figure 5. Recall we know triangles τ_i, τ_j are conjugate to each other and hence $\left|\omega_3^i-\omega_2^i\right|=\left|\omega_2^j-\omega_1^j\right|$. By definition

$$Dl_i\left(\frac{\omega_3^i-\omega_2^i}{\left|\omega_3^i-\omega_2^i\right|}\right)=\frac{l_i\left(\omega_3^i\right)-l_i\left(\omega_2^i\right)}{\left|\omega_3^i-\omega_2^i\right|}=\frac{v\left(\omega_3^i\right)-v\left(\omega_2^i\right)}{\left|\omega_3^i-\omega_2^i\right|} \tag{5.89}$$

and in the same way

$$Dl_j\left(\frac{\omega_2^j-\omega_1^j}{\left|\omega_2^j-\omega_1^j\right|}\right)=\frac{v\left(\omega_2^j\right)-v\left(\omega_1^j\right)}{\left|\omega_2^j-\omega_1^j\right|}. \tag{5.90}$$

Let l_M denote an affine function with $Dl_M = M$

$$\begin{aligned}\left|v\left(\omega_3^i\right)-v\left(\omega_2^i\right)-l_M\left(\omega_3^i-\omega_2^i\right)\right| &\le \int_{\left[\omega_3^i,\omega_2^i\right]}|Dv(z)-M|\,\mathrm{d}H^1 z \\ &\overset{(5.88)}{\le} c\left(\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v(z)\right|^2 \mathrm{d}H^1 z\right)^{\frac{1}{2}}. \end{aligned} \tag{5.91}$$

In the same way

$$\left|v\left(\omega_2^j\right)-v\left(\omega_1^j\right)-l_M\left(\omega_2^j-\omega_1^j\right)\right| \le c\left(\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v(z)\right|^2 \mathrm{d}H^1 z\right)^{\frac{1}{2}}. \tag{5.92}$$

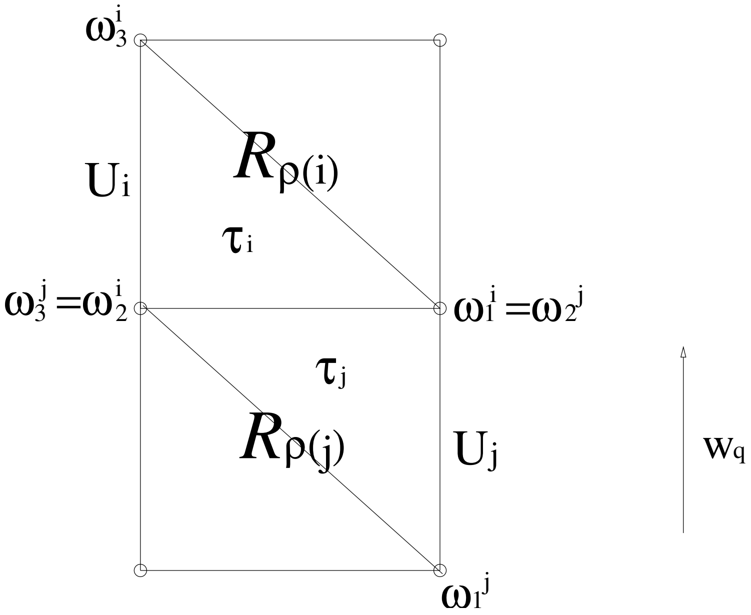

FIGURE 5. Two touching triangles.

Thus as $\omega_2^j-\omega_1^j=\omega_3^i-\omega_2^i$ (see Fig. 5) we have from (5.91), (5.92)

$$\left|\frac{v\left(\omega_3^i\right)-v\left(\omega_2^i\right)}{\left|\omega_3^i-\omega_2^i\right|}-\frac{v\left(\omega_2^j\right)-v\left(\omega_1^j\right)}{\left|\omega_2^j-\omega_1^j\right|}\right|\leq c\left(\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v\left(z\right)\right|^2 \mathrm{d}H^1z\right)^{\frac{1}{2}}.$$

Which from (5.89) and (5.90) implies

$$\left|Dl_i\left(\frac{\omega_3^i-\omega_2^i}{\left|\omega_3^i-\omega_2^i\right|}\right)-Dl_j\left(\frac{\omega_3^i-\omega_2^i}{\left|\omega_3^i-\omega_2^i\right|}\right)\right|\leq c\left(\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v\left(z\right)\right|^2 \mathrm{d}H^1z\right)^{\frac{1}{2}}. \tag{5.93}$$

Recall again (see Fig. 5) the endpoints of $\overline{\tau_i}\cap\overline{\tau_j}$ are given by ω_1^i, ω_2^i. So

$$Dl_i\left(\omega_1^i-\omega_2^i\right)=Dl_j\left(\omega_1^i-\omega_2^i\right) \tag{5.94}$$

and as $\frac{\omega_1^i-\omega_2^i}{\left|\omega_1^i-\omega_2^i\right|}\cdot\frac{\omega_3^i-\omega_2^i}{\left|\omega_3^i-\omega_2^i\right|}=0$ so (5.87) follows from (5.93) and (5.94). Thus

$$\begin{aligned}\sum_{i=1}^{2N_1}\sum_{j\in V_0(i)}\left|Dl_i-Dl_j\right|^2 &\overset{(5.87)}{\leq} \sum_{i=1}^{2N_1}\sum_{j\in V_0(i)}\int_{\partial\left(\mathcal{R}_{\rho(i)}\cup\mathcal{R}_{\rho(j)}\right)}\left|D^2v\left(z\right)\right|^2 \mathrm{d}H^1z\\ &\leq c\int_{\mathbb{L}_1^{-1}(\sigma_1)\cup\mathbb{L}_2^{-1}(\sigma_2)}\left|D^2v\left(z\right)\right|^2 \mathrm{d}H^1z\\ &\overset{(5.59)}{\leq} c\epsilon^{-1}m_\epsilon^p.\end{aligned}$$

Step 5. Recall $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{N_1}$ are the connected component of $\mathfrak{A}$ (see (5.61)). Let $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{N_2}$ denote the connected components of

$$\left(\Omega_{\epsilon^{-\frac{1}{2}}} \backslash \mathbb{L}_1^{-1}(\sigma_1)\right) \setminus \left(\bigcup_{i=1}^{N_1} \mathcal{R}_i\right).$$

Note that each $\mathcal{D}_i$ forms a polygon. As before for simplicity we will assume none of the sides of $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$ is parallel to w_1. Let c_Ω denote the length of the shortest side of $\partial\Omega$, we can assume without loss of generality $\sqrt{\epsilon} < c_\Omega$, so we have that any $\overline{\mathcal{D}_i}$ will intersect at most two sides of $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$. Let $E_1 := \{i \in \{1,2,\ldots,N_2\} : \partial\mathcal{D}_i$ has 4 sides$\}$. So any $i \in \{1,2,\ldots,N_2\} \backslash E_1$ is such that $\partial\mathcal{D}_i$ has 5 or 3 sides.

Let $E_2 := \{i \in \{1,2,\ldots,N_2\} : \partial\mathcal{D}_i$ has 5 sides$\}$. For any $i \in E_2$ let a_i, b_i be the endpoints of $\partial\Omega_{\epsilon^{-\frac{1}{2}}} \cap \overline{\mathcal{D}_i}$ and let c_i, d_i denote the corners of the polytope $\mathcal{D}_i$ that do not intersect $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$.

Define $\widetilde{\mathcal{D}_i} = \operatorname{conv}(a_i, b_i, c_i, d_i)$ for $i \in E_2$ and define $\widetilde{\mathcal{D}_i} = \mathcal{D}_i$ for $i \in E_1$. Finally define $T_i := \mathcal{D}_i \backslash \widetilde{\mathcal{D}_i}$ for $i \in E_2$, note each T_i forms a triangle.

For each $i \in E_1 \cup E_2$ we can split each $\widetilde{\mathcal{D}_i}$ into two triangles τ_i^1, τ_i^2, each of which has a side parallel to w_1 (*i.e.* $\overline{\widetilde{\mathcal{D}_i}} = \overline{\tau_i^1 \cup \tau_i^2}$). Let $\{\tau_{2N_1+1}, \tau_{2N_1+2}, \ldots, \tau_{N_3}\}$ denote the additional set of triangles that are formed by

$$\{\tau_i^q : i \in E_1 \cup E_2,\ q \in \{1,2\}\}, \{\mathcal{D}_i : i \in \{1,2,\ldots,N_2\} \setminus (E_1 \cup E_2)\} \text{ and } \{T_i : i \in E_2\}$$

and let

$$\mathbb{B}_d := \left\{ i \in \{1,2,\ldots,N_3\} : \tau_i \subset N_{64\sigma^{-2}}\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)\right\}. \tag{5.95}$$

Firstly we will show that

$$N_3 - 2N_1 \le c\epsilon^{-\frac{1}{2}} \text{ and } \operatorname{Card}(\mathbb{B}_d) \le c\epsilon^{-\frac{1}{2}}. \tag{5.96}$$

Secondly let l_i be the affine interpolation of v on the corners of τ_i for $i \in \mathbb{B}_d$ we will also show

$$\sum_{i \in \mathbb{B}_d} |Dl_i|^2 \le c\epsilon^{-1} m_\epsilon^p. \tag{5.97}$$

Proof of Step 5. To start with since $\bigcup_{i \in \mathbb{B}_d} \tau_i \subset N_{64\sigma^{-2}}\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)$ and since $L^2(\tau_i) > c$ for any $i \in \mathbb{B}_d$. So

$$\begin{aligned} \operatorname{Card}(\mathbb{B}_d) &\le cL^2\left(N_{64\sigma^{-2}}\left(\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)\right) \\ &\le c\epsilon^{-\frac{1}{2}} \end{aligned}$$

note also $\{2N_1+1, \ldots, N_3\} \subset \mathbb{B}_d$ which gives (5.96).

For any $i \in E_1 \cup E_2$ we will order the triangles τ_i^1, τ_i^2 so that two of the corners of τ_i^2 intersects $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$ and two of the corners of τ_i^1 intersects $\bigcup_{i \in \{1,2,\ldots,2N_1\}} \overline{\mathcal{R}_i}$.

So let $\{a_i, b_i, c_i\}$ denote the corners of τ_i^1 we can order them so that $\frac{a_i - b_i}{|a_i - b_i|} = w_1$ and $\frac{c_i - b_i}{|c_i - b_i|} = w_2$. So $[a_i, b_i] \subset \mathbb{L}_1^{-1}(\sigma_1)$, $[c_i, b_i] \subset \mathbb{L}_2^{-1}(\sigma_2)$. So by definition of $\mathbb{L}_1^{-1}(\sigma_1)$ we have that

$$[a_i, b_i] \subset (\mathbb{R}_+ w_1 + (t + k_1) w_2) \cup (\mathbb{R}_- w_1 + (t + k_1) w_2)$$

for some $k_1 \in \{Q_1^1, Q_1^1 + 1, \ldots, Q_2^1 - 1\}$, $\sigma_1 \in \mathsf{P}_1^+ \cap \mathsf{P}_1^-$. By definition (5.43) and by (5.42) we have that $[a_i, b_i] \cap \mathsf{A}_0 \ne \emptyset$. So there exists $x_i \in [a_i, b_i]$ such that $d(Dv(x_i), K) \le 1$. Thus

$$\sup\{|Dv(z)| : z \in [a_i, b_i] \cup [b_i.c_i]\} \le c + \int_{[a_i,b_i] \cup [b_i,c_i]} \left|D^2 v(z)\right| \mathrm{d}H^1 z. \tag{5.98}$$

Let L_i^1 be the affine function we obtain from the interpolation of v on the corners of τ_i^1. We have

$$\begin{aligned}\left|DL_i^1 w_1\right| &\leq c\left|v(a_i)-v(b_i)\right|\\ &\leq c\int_{[a_i,b_i]}\left|Dv(z)\right|\mathrm{d}H^1 z\\ &\overset{(5.98)}{\leq} c+c\int_{[a_i,b_i]\cup[b_i,c_i]}\left|D^2 v(z)\right|\mathrm{d}H^1 z.\end{aligned}$$

And in exactly the same way we have

$$\left|DL_i^1 w_2\right| = \left|\frac{L_i^1(c_i)-L_i^1(b_i)}{|c_i-b_i|}\right| \leq +c\int_{[a_i,b_i]\cup[b_i,c_i]}\left|D^2 v(z)\right|\mathrm{d}H^1 z.$$

Thus

$$\begin{aligned}\left|DL_i^1\right|^2 &= c\left(\left|DL_i^1 w_1\right|^2+\left|DL_i^1 w_2\right|^2\right)\\ &\leq c+c\left(\int_{[a_i,b_i]\cup[b_i,c_i]}\left|D^2 v(z)\right|\mathrm{d}H^1 z\right)^2\\ &\leq c+c\int_{\partial\tau_i^1\cap\left(\mathbb{L}_1^{-1}(\sigma_1)\cup\mathbb{L}_2^{-1}(\sigma_2)\right)}\left|D^2 v(z)\right|^2\mathrm{d}H^1 z. \qquad (5.99)\end{aligned}$$

Now let us consider the triangle τ_i^2. Let $\{a_i,b_i,c_i\}$ denote the corners of τ_i^2 where we have ordered a_i, b_i, c_i such that $\frac{a_i-b_i}{|a_i-b_i|}=w_1$ and $b_i,c_i\in\partial\Omega_{\epsilon^{-\frac{1}{2}}}$. Let L_i^2 denote the affine map we get from interpolation of v on the corners of τ_i^2. Arguing exactly as we have before we can show that

$$\left|DL_i^2 w_1\right|^2 \leq c+c\int_{\partial\tau_i^2\cap\left(\mathbb{L}_1^{-1}(\sigma_1)\cup(\mathbb{L}_2^{-1}(\sigma_2)\right)}\left|D^2 v(z)\right|^2\mathrm{d}H^1 z.$$

Now $\left|DL_i^2\left(\frac{b_i-c_i}{|b_i-c_i|}\right)\right|^2 \leq c\left|l_F(b_i)-l_F(c_i)\right|^2\leq c$. Since w_1 and $\frac{b_i-c_i}{|b_i-c_i|}$ are not parallel this implies

$$\left|DL_i^2\right|^2 \leq c+c\int_{\partial\tau_i^2\cap\left(\mathbb{L}_1^{-1}(\sigma_1)\cup\mathbb{L}_2^{-1}(\sigma_2)\right)}\left|D^2 v(z)\right|^2\mathrm{d}H^1 z. \qquad (5.100)$$

Now for any $i\in\{1,2,\ldots,N_2\}\setminus(E_1\cup E_2)$, $\mathcal{D}_i$ forms a triangle with the corners in $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$, let I_i be the affine map we obtain by interpolation of v on the corners of $\mathcal{D}_i$, then I_i has the property that

$$|DI_i|\leq c \text{ for any } i\in\{1,2,\ldots,N_2\}\setminus(E_1\cup E_2). \qquad (5.101)$$

For any $i\in E_2$ let J_i be the affine function we get from interpolating v on T_i, since again the corners of τ_i belong to $\partial\Omega_{\epsilon^{-\frac{1}{2}}}$ we have

$$|DJ_i|\leq c \text{ for any } i\in E_2. \qquad (5.102)$$

Let l_i be the affine map we obtain from interpolating v on τ_i for $i\in\mathbb{B}_d$. For any $i\in\mathbb{B}_d\setminus\{2N_1+1,\ldots,N_3\}$ let $\{a_i,b_i,c_i\}$ denote the corners of τ_i where $\frac{a_i-b_i}{|a_i-b_i|}=w_1$ and $\frac{c_i-b_i}{|c_i-b_i|}=w_2$. Exactly as in the case where we considered triangle τ_i^1 for $i\in E_1\cup E_2$ we must have that $[a_i,b_i]\subset\mathbb{L}_1^{-1}(\sigma_1)$ and $[c_i,b_i]\subset\mathbb{L}_2^{-1}(\sigma_2)$. We will assume a_i, b_i are ordered so that $\mathrm{d}\left(a_i,\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)<d\left(b_i,\partial\Omega_{\epsilon^{-\frac{1}{2}}}\right)$. Let $d_i\in\partial\Omega_{\epsilon^{-\frac{1}{2}}}$ be such that $[a_i,b_i]\subset[d_i,b_i]$.

By definition of $\mathbb{B}_d$ we know $|d_i - b_i| < 32\sigma^{-2}$. Let $\Gamma_i := [d_i, b_i] \cup [b_i, c_i]$, by arguing exactly the same way as we did to show (5.99) we have

$$|Dl_i|^2 \leq c + \int_{\Gamma_i} \left|D^2 v(z)\right|^2 \mathrm{d}H^1 z. \tag{5.103}$$

So let l_i be the affine map we obtain from interpolating v on τ_i for $i \in \mathbb{B}_d$ we have by (5.99), (5.100), (5.101), (5.102) and (5.103)

$$\begin{aligned}\sum_{i \in \mathbb{B}_d} |Dl_i|^2 &\overset{(5.96)}{\leq} c\epsilon^{-\frac{1}{2}} + c\int_{\mathbb{L}_1^{-1}(\sigma_1) \cup \mathbb{L}_2^{-1}(\sigma_2)} \left|D^2 v(z)\right|^2 \mathrm{d}H^1 z \\ &\overset{(4.2),(5.59)}{\leq} c\epsilon^{-1} m_\epsilon^p. \end{aligned} \tag{5.104}$$

Step 6. Let $w \in \mathcal{F}_F^{\sqrt{\epsilon},\varsigma}$ be defined by $w(z) = l_i(z)$ for $z \in \tau_i$, $i = 1, 2, \ldots, N_3$. We will show that

$$\sum_{i \in J(w)} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 \leq c\epsilon^{-1} m_\epsilon^p. \tag{5.105}$$

Proof of Step 6. Let

$$V_1(i) = \left\{ j \in \{1, 2, \ldots, N_3\} : H^1\left(\overline{\tau_i} \cap \overline{\tau_j}\right) > 0 \right\}. \tag{5.106}$$

Let

$$\mathbb{I}_0 := \left\{ i \in \{1, 2, \ldots, N_3\} : \tau_i \subset \Omega \backslash N_{32\sigma^{-2}}(\partial\Omega) \right\}. \tag{5.107}$$

Note that for any $i \in \{1, 2, \ldots, N_3\} \backslash \mathbb{I}_0$, $V_1(i) \subset \mathbb{B}_d$. So

$$\begin{aligned}\sum_{J(w) \backslash \mathbb{I}_0} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 &\leq \sum_{i \in J(w) \backslash \mathbb{I}_0} \left(\sum_{j \in V_1(i)} |Dl_i - Dl_j|^2 + |Dl_i - F|^2 \right) \\ &\leq c \sum_{i \in \mathbb{B}_d} |Dl_i|^2 + c\,\mathrm{Card}(\mathbb{B}_d) \\ &\overset{(5.96),(5.97),(4.2)}{\leq} c\epsilon^{-1} m_\epsilon^p. \end{aligned} \tag{5.108}$$

Also note that if $i \in \mathbb{I}_0$ then $V_1(i) \subset \{1, 2, \ldots, 2N_1\}$ and $V_1(i) = V_0(i)$ (see definition (5.84)) in addition we know $\partial\tau_i \cap \partial\Omega = \emptyset$ so $N_i(w) = V_0(i)$ and $J(w) \cap \mathbb{I}_0 = \Upsilon_0$ (see (5.85)). So

$$\begin{aligned}\sum_{i \in J(w) \cap \mathbb{I}_0} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 &= \sum_{i \in \Upsilon_0} \sum_{j \in V_0(i)} |Dl_i - Dl_j|^2 \\ &\overset{(5.86)}{\leq} c\epsilon^{-1} m_\epsilon^p. \end{aligned} \tag{5.109}$$

Now

$$\begin{aligned}\sum_{i \in J(w)} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 &= \sum_{i \in J(w) \cap \mathbb{I}_0} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 \\ &\quad + \sum_{i \in J(w) \backslash \mathbb{I}_0} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 \\ &\overset{(5.108),(5.109)}{\leq} c\epsilon^{-1} m_\epsilon^p. \end{aligned}$$

Step 7. We will show

$$\sum_{j=1}^{N_3} d^p\left(Dw_{\lfloor \tau_i}, K\right) \leq c\epsilon^{-1} m_\epsilon^p. \tag{5.110}$$

Proof of Step 7. Since for any $j \in \{2N_1+1, \ldots, N_3\}$ we have

$$d^p\left(Dw_{\lfloor \tau_j}, K\right) \leq c + \left|Dw_{\lfloor \tau_j}\right|^p \leq c + \left|Dw_{\lfloor \tau_j}\right|^2 \tag{5.111}$$

so using the fact $\{2N_1+1, \ldots, N_3\} \subset \mathbb{B}_d$ for the last inequality

$$\begin{aligned} \sum_{j=1}^{N_3} d^p\left(Dw_{\lfloor \tau_i}, K\right) &\overset{(5.82),(5.111)}{\leq} c\epsilon^{-1} m_\epsilon^p + c\left(N_3 - 2N_1 + 1\right) + \sum_{j=2N_1+1}^{N_3} \left|Dw_{\lfloor \tau_j}\right|^2 \\ &\overset{(4.2),(5.96),(5.97)}{\leq} c\epsilon^{-1} m_\epsilon^p. \end{aligned}$$

Step 8. We will show that (for small enough ς) there exists a function $\tilde{u} \in \mathcal{D}_F^{\varsigma,h}$ such that

$$\int_\Omega d^p\left(D\tilde{u}\left(z\right), K\right) \mathrm{d}L^2 z \leq c m_\epsilon^p. \tag{5.112}$$

Proof of Step 8. Recall definition of d_0, see (4.3). Let

$$\mathsf{G}_g := \left\{i \in \{1, 2, \ldots, N_3\} : d\left(Dw_{\lfloor \tau_i}, K\right) \leq d_0\right\}.$$

Recall $V_1(i)$ is defined by (5.106). Let $\mathbb{V}(i) := \bigcup_{k \in V_1(i)} V_1(k)$ and (recall the definition of $\mathbb{I}_0$, see (5.107)) let $\mathsf{G}_{gi} := \{i \in \mathbb{I}_0 : \mathbb{V}(i) \subset \mathsf{G}_g\}$. Note $\mathrm{Card}\left(\mathbb{V}(i)\right) \leq 12$. Let $\mathbb{A}_0 := \bigcup_{i \in \mathbb{I}_0 \backslash \mathsf{G}_{gi}} \overline{\tau_i}$, so

$$L^2\left(\mathbb{A}_0\right) \geq c\,\mathrm{Card}\left(\mathbb{I}_0 \backslash \mathsf{G}_{gi}\right). \tag{5.113}$$

Let $\mathsf{O}_i := \bigcup_{j \in \mathbb{V}(i)} \overline{\tau_j}$, to by applying the $5r$ Covering Theorem (see Thm. 2.1 [25]) we can find a subset $\{i_1, i_2, \ldots, i_{P_1}\} \subset \mathbb{I}_0 \backslash \mathsf{G}_{gi}$ such that

$$\mathbb{A}_0 \subset \bigcup_{k=1}^{P_1} N_{60}\left(\mathsf{O}_{i_k}\right) \tag{5.114}$$

and $\left\{\mathsf{O}_{i_1}, \mathsf{O}_{i_2}, \ldots, \mathsf{O}_{i_{P_1}}\right\}$ are disjoint. Note (5.113), (5.114) imply $P_1 \geq c\,\mathrm{Card}\left(\mathbb{I}_0 \backslash \mathsf{G}_{gi}\right)$ and since for every $k \in \{1, 2, \ldots, P_1\}$ since $\mathbb{V}(i_k) \not\subset \mathsf{G}_{gi}$ (by definition of G_{gi}) we can find $q_k \in \{1, 2, \ldots, N_3\}$ such that $\tau_{q_k} \subset \mathsf{O}_{i_k}$ and $d\left(Dw_{\lfloor \tau_{q_k}}, K\right) > d_0$. We also know that $\left\{\tau_{q_1}, \tau_{q_2}, \ldots, \tau_{q_{P_1}}\right\}$ are disjoint. So

$$d_0^p P_1 \leq \sum_{k=1}^{P_1} d^p\left(Dw_{\lfloor \tau_{q_k}}, K\right) \overset{(5.110)}{\leq} c\epsilon^{-1} m_\epsilon^p.$$

Thus $\mathrm{Card}\left(\mathbb{I}_0 \backslash \mathsf{G}_{gi}\right) \leq c\epsilon^{-1} m_\epsilon^p \overset{(5.33)}{\leq} c\,\mathcal{C}_0 \epsilon^{-1}$. Now $\mathrm{Card}\left(\mathbb{I}_0\right) \geq c\epsilon^{-1}$ so

$$\mathrm{Card}\left(\mathbb{I}_0 \cap \mathsf{G}_{gi}\right) \geq c\epsilon^{-1} - c\,\mathcal{C}_0 \epsilon^{-1}.$$

Assuming constant $\mathcal{C}_0$ at the start of Proposition 5.2 was chosen small enough we have

$$\mathrm{Card}\left(\mathbb{I}_0 \cap \mathsf{G}_{gi}\right) \geq c\epsilon^{-1}. \tag{5.115}$$

Note that again by applying the $5r$ covering Theorem we can find subset $\{j_1, j_2, \ldots, j_{P_2}\} \subset \mathbb{I}_0 \cap \mathsf{G}_{gi}$ such that

$$\bigcup_{i\in\mathbb{I}_0\cap\mathsf{G}_{gi}} \tau_i \subset \bigcup_{k=1}^{P_2} N_{60}\left(\mathsf{O}_{j_k}\right) \tag{5.116}$$

and $\left\{\mathsf{O}_{j_1}, \mathsf{O}_{j_2}, \ldots, \mathsf{O}_{j_{P_2}}\right\}$ are disjoint. Inequalities (5.115) and (5.116) imply that

$$P_2 \geq c\epsilon^{-1}. \tag{5.117}$$

We denote the corners of τ_i by $\left\{\omega_i^1, \omega_i^2, \omega_i^3\right\}$ for any $i = 1, 2, \ldots, N_3$. Let $q \in \{1, 2, \ldots, P_2\}$ and pick $c_q \in \left\{\omega_{j_q}^1, \omega_{j_q}^2, \omega_{j_q}^3\right\}$. Let $\mathbb{W}\left(j_q\right) \subset \mathbb{V}\left(j_q\right)$ be defined by $\mathbb{W}\left(j_q\right) := \{k \in \mathbb{V}\left(j_q\right) : \overline{\tau_k} \cap c_q \neq \emptyset\}$. Note that for any $k \in \mathbb{W}\left(j_q\right)$, since $\mathbb{V}\left(j_q\right) \subset \mathsf{G}_g$ we have

$$\left|w\left(\omega_k^a\right) - w\left(c_q\right)\right| \leq 4\sigma^{-1} \text{ for any } a \in \{1, 2, 3\}. \tag{5.118}$$

For each $k \in \mathbb{W}\left(j_q\right)$ define the affine map $\tilde{l}_k : \tau_k \to \mathbb{R}^2$ by

$$\tilde{l}_k\left(b\right) = \begin{cases} w\left(b\right) & \text{for } b \in \left\{\omega_k^1, \omega_k^2, \omega_k^3\right\} \setminus \{c_q\} \\ w\left(c_q\right) + 30\sigma^{-1}e_1 & \text{for } b = c_q. \end{cases}$$

For simplicity we order the corners $\left\{\omega_k^1, \omega_k^2, \omega_k^3\right\}$ so that $\omega_k^1 = c_q$. Note

$$\begin{aligned} \left|D\tilde{l}_k\left(\frac{\omega_k^1 - \omega_k^2}{\left|\omega_k^1 - \omega_k^2\right|}\right)\right| &= \left|\omega_k^1 - \omega_k^2\right|^{-1}\left|w\left(\omega_k^1\right) - w\left(\omega_k^2\right) + 30\sigma^{-1}e_1\right| \\ &\overset{(5.118)}{\geq} 10\sigma^{-1}. \end{aligned}$$

In exactly the same way we have $\left|D\tilde{l}_k\left(\frac{\omega_k^1-\omega_k^3}{\left|\omega_k^1-\omega_k^3\right|}\right)\right| \geq 10\sigma^{-1}$ which implies

$$\left|D\tilde{l}_k\right| \geq 10\sigma^{-1}. \tag{5.119}$$

In a very similar way we can show $\left|D\tilde{l}_k\left(\frac{\omega_k^1-\omega_k^2}{\left|\omega_k^1-\omega_k^2\right|}\right)\right| \leq 60\sigma^{-1}$, $\left|D\tilde{l}_k\left(\frac{\omega_k^1-\omega_k^3}{\left|\omega_k^1-\omega_k^3\right|}\right)\right| \leq 60\sigma^{-1}$ and thus

$$\left|D\tilde{l}_k\right| \leq 60\sigma^{-1}. \tag{5.120}$$

From (5.119) we know

$$\begin{aligned} \sum_{k\in\mathbb{W}(j_q)} d^p\left(D\tilde{l}_k, K\right) L^2\left(\tau_k\right) &\geq d^p\left(D\tilde{l}_{j_q}, K\right) L^2\left(\tau_{j_q}\right) \\ &\overset{(5.119)}{\geq} 9\sigma^{-p} L^2\left(\tau_{j_q}\right) \end{aligned} \tag{5.121}$$

and

$$\sum_{k\in\mathbb{W}(j_q)} d^p\left(D\tilde{l}_k, K\right) L^2\left(\tau_k\right) \overset{(5.120)}{\leq} 120^2\sigma^{-2} \times 100\varsigma^{-2}. \tag{5.122}$$

Note recall from (5.117) $P_2 \geq c\epsilon^{-1} \overset{(5.33)}{>} \frac{m_\epsilon^p}{\epsilon}$ so we can define piecewise affine function $\tilde{v} : \Omega_{\epsilon^{-\frac{1}{2}}} \to \mathbb{R}^2$ by

$$\tilde{v}(z) = \begin{cases} w(z) & \text{for } z \in \tau_i,\ i \in \{1,2,\dots,N_3\} \setminus \left(\bigcup_{q=1}^{[\epsilon^{-1}m_\epsilon^p]} \mathbb{W}(j_q)\right) \\ \tilde{l}_i(z) & \text{for } z \in \tau_i,\ i \in \left(\bigcup_{q=1}^{[\epsilon^{-1}m_\epsilon^p]} \mathbb{W}(j_q)\right). \end{cases}$$

So

$$\begin{aligned} \int_{\Omega_{\epsilon^{-\frac{1}{2}}}} d^p(D\tilde{v}(z), K)\, dL^2 z &= \sum_{i \in \{1,2,\dots,N_3\} \setminus \left(\bigcup_{q=1}^{[\epsilon^{-1}m_\epsilon^p]} \mathbb{W}(j_q)\right)} d^p\left(Dw_{\lfloor \tau_i}, K\right) L^2(\tau_i) \\ &\quad + \sum_{i \in \left(\bigcup_{q=1}^{[\epsilon^{-1}m_\epsilon^p]} \mathbb{W}(j_q)\right)} d^p\left(D\tilde{l}_i, K\right) L^2(\tau_i) \\ &\overset{(5.110),(5.122)}{\leq} c\epsilon^{-1} m_\epsilon^p + c\left[\epsilon^{-1} m_\epsilon^p\right] \\ &\leq c\epsilon^{-1} m_\epsilon^p \end{aligned} \tag{5.123}$$

and

$$\begin{aligned} \int_{\Omega_{\epsilon^{-\frac{1}{2}}}} d^p(D\tilde{v}(z), K)\, dL^2 z &\geq \sum_{q=1}^{[\epsilon^{-1}m_\epsilon^p]} \int_{\mathsf{O}_{j_q}} d^p(D\tilde{v}(z), K)\, dL^2 z \\ &\overset{(5.121)}{\geq} c\left[\epsilon^{-1} m_\epsilon^p\right]. \end{aligned} \tag{5.124}$$

Let $\mathbb{Y} := \left\{ i \in \{1,2,\dots,N_3\} : V_1(i) \cap \left(\bigcup_{q=1}^{[\epsilon^{-1}m_\epsilon^p]} \mathbb{W}(j_q)\right) = \emptyset \right\}$. Note

$$\mathrm{Card}\left(\{1,2,\dots,N_3\} \setminus \mathbb{Y}\right) \leq c\epsilon^{-1} m_\epsilon^p \tag{5.125}$$

and note

$$\sum_{M \in N_i(\tilde{v})} \left|D\tilde{v}_{\lfloor \tau_i} - M\right|^2 \leq \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 + c \text{ for any } i \in J(\tilde{v}) \setminus \mathbb{Y} \tag{5.126}$$

so as $J(\tilde{v}) \cap \mathbb{Y} = J(w) \cap \mathbb{Y}$ and $D\tilde{v}_{\lfloor \tau_j} = Dw_{\lfloor \tau_j}$ for every $j \in \bigcup_{i \in J(\tilde{v}) \cap \mathbb{Y}} V_1(i)$ we have

$$\begin{aligned} \sum_{i \in J(\tilde{v})} \sum_{N_i(\tilde{v})} \left|D\tilde{v}_{\lfloor \tau_i} - M\right|^2 &= \sum_{i \in J(w) \cap \mathbb{Y}} \sum_{M \in N_i(w)} \left|Dw_{\lfloor \tau_i} - M\right|^2 \\ &\quad + \sum_{i \in J(\tilde{v}) \setminus \mathbb{Y}} \sum_{M \in N_i(\tilde{v})} \left|D\tilde{v}_{\lfloor \tau_i} - M\right|^2 \\ &\overset{(5.105),(5.106)}{\leq} c\epsilon^{-1} m_\epsilon^p + c\,\mathrm{Card}\left(J(\tilde{v}) \setminus \mathbb{Y}\right) \\ &\overset{(5.125)}{\leq} c\epsilon^{-1} m_\epsilon^p. \end{aligned} \tag{5.127}$$

Thus

$$\int_{\Omega_{\epsilon^{-\frac{1}{2}}}} d^p(D\tilde{v}(z), K)\, dL^2 z \overset{(5.124),(5.127)}{\geq} c \sum_{i \in J(\tilde{v})} \sum_{M \in N_i(\tilde{v})} \left|D\tilde{v}_{\lfloor \tau_i} - M\right|^2. \tag{5.128}$$

Define $\tilde{u}(z) = \tilde{v}(\sqrt{\epsilon} z)\,\epsilon^{-\frac{1}{2}}$. We have that

$$\int_{\Omega} d^p (D\tilde{u}(z), K)\, \mathrm{d}L^2 z = \epsilon \int_{\Omega_{\epsilon^{-\frac{1}{2}}}} d^p (D\tilde{v}(z), K)\, \mathrm{d}L^2 z \tag{5.129}$$

and thus

$$\int_{\Omega} d^p (D\tilde{u}(z), K)\, \mathrm{d}L^2 z \overset{(5.128)}{\geq} c\epsilon \sum_{i \in J(\tilde{v})} \sum_{M \in N_i(\tilde{v})} \left| D\tilde{v}_{\lfloor \tau_i} - M \right|^2. \tag{5.130}$$

Now (for small enough ς) $\{\sqrt{\epsilon}\tau_i\}$ forms a (h, ς) triangulation of Ω and it is easy to see that

$$\sum_{i \in J(\tilde{u})} \sum_{M \in N_i(\tilde{u})} \left| D\tilde{u}_{\lfloor \sqrt{\epsilon}\tau_i} - M \right|^2 = \sum_{i \in J(\tilde{v})} \sum_{M \in N_i(\tilde{v})} \left| D\tilde{v}_{\lfloor \tau_i} - M \right|^2.$$

Thus (again assuming ς is small enough) we have from (5.130)

$$\sum_{i \in J(\tilde{u})} \sum_{M \in N_i(\tilde{u})} \epsilon \left| D\tilde{u}_{\lfloor \sqrt{\epsilon}\tau_i} - M \right|^2 \leq \frac{\varsigma^{-1}}{2} \int_{\Omega} d^p (D\tilde{u}(z), K)\, \mathrm{d}L^2 z. \tag{5.131}$$

Thus we have that $u \in \mathcal{D}_F^{\varsigma, \sqrt{\epsilon}}$. We also know from (5.129) and (5.123) that $\tilde{u}$ satisfies (5.112). □

Proposition 5.3. *Let $w_1 \in S^1$ be such that $w_2 \in w_1^{\perp}$ we have that w_1, w_2 and $\frac{w_1 - w_2}{|w_1 - w_2|}$ are not in the set of rank-1 connections between $SO(2)\,A_i$ and $SO(2)\,A_j$ for any $i \neq j$. Let $p \in [1, 2]$. Let $F \notin K$, given function $u \in \mathcal{D}_F^{\varsigma, \sqrt{\epsilon}}$ we define $w : \Omega_2 \to \mathbb{R}^2$ by*

$$\tilde{w}(z) = \begin{cases} u(z) & \text{if } z \in \Omega \\ l_F(z) & \text{if } z \in \Omega_2 \backslash \Omega. \end{cases} \tag{5.132}$$

We will show there exists a small positive constant $\eta = \eta(w_1, A_1, \ldots, A_N)$ such that for $\tilde{w} = w * \rho_{\eta\sqrt{\epsilon}}$ and

$$w(z) = \tilde{w}\left(\frac{z}{1 + \eta\sqrt{\epsilon}}\right)\left(1 + \eta\sqrt{\epsilon}\right) \tag{5.133}$$

then $w \in A_F$ and w satisfies

$$\int_{\Omega} d^p (Dw(z), K) + \epsilon \left| D^2 w(z) \right|^2 \mathrm{d}L^2 z \leq c \int_{\Omega} d^p (Du(z), K)\, \mathrm{d}L^2 z. \tag{5.134}$$

Proof. Firstly note u is piecewise affine on a triangulation which we will label $\{\tau_1, \tau_2, \ldots, \tau_{N_3}\}$. Given triangle τ_i we define the *neighbouring gradients* $N_i(u)$ by (1.3) and we define the *jump triangles* $J_i(u)$ by (1.4). Now since $u \in \mathcal{D}_F^{\varsigma, \sqrt{\epsilon}}$ we have

$$\sum_{i \in J(u)} \sum_{M \in N_i(u)} \left| Du_{\lfloor \tau_i} - M \right|^2 \leq \varsigma^{-1} \epsilon^{-1} \int_{\Omega} d^p (Du(z), K)\, \mathrm{d}L^2 z. \tag{5.135}$$

Let $v(z) = u(\sqrt{\epsilon} z)\,\epsilon^{-\frac{1}{2}}$. Let

$$\alpha_0 = \int_{\Omega_{\epsilon^{-\frac{1}{2}}}} d^p (Dv(z), K)\, \mathrm{d}L^2 z. \tag{5.136}$$

Let $V(j) := \left\{k : H^1\left(\overline{\tau_k} \cap \overline{\tau_j}\right) > 0\right\}$. Define $\mathbb{V}_0(i) := \bigcup_{j \in V(i)} V(j)$ and $\mathbb{V}_1(i) := \bigcup_{j \in \mathbb{V}_0(i)} V(j)$.

Let $G_0 := \left\{i : \mathrm{d}\left(Dv_{\lfloor\tau_i}, K\right) \leq \eta\right\}$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{N_1}$ denote the connected components of $\bigcup_{i\in G_0} \overline{\tau_i}$. Let

$$\mathcal{G}_k := \{i : \tau_i \subset \mathcal{A}_k\} \text{ and define } \widetilde{\mathcal{A}}_k := \bigcup_{\{i:\mathbb{V}_1(i)\subset\mathcal{G}_k\}} \overline{\tau_i}. \tag{5.137}$$

Define

$$E(z) = \{i : \overline{\tau_i} \cap B_\eta(z) \neq \emptyset\} \text{ for any } z \in Q_{\epsilon^{-\frac{1}{2}}+\eta}(0). \tag{5.138}$$

Note Card $(E(z)) \leq c$ and note

$$E(z) \subset \mathbb{V}_1(i) \text{ for any } z \text{ such that } B_{\frac{3\eta}{2}}(z) \cap \overline{\tau_i} \neq \emptyset. \tag{5.139}$$

Step 1. Given $k \in \{1, 2, \ldots, N_1\}$ we will show there exists $k_0 \in \{1, 2, \ldots, N\}$ such that

$$d\left(Dv_{\lfloor\tau_i}, SO(2)\, A_{k_0}\right) = d\left(Dv_{\lfloor\tau_i}, K\right) \text{ for every } i \in \mathcal{G}_k. \tag{5.140}$$

Proof of Step 1. Suppose this is not true. So we can find $k_0 \in \{1, 2, \ldots, N_1\}$ and some $N_0 \in \{2, 3, \ldots, N\}$ for which we have disjoint subsets $\Omega_1, \Omega_2, \ldots, \Omega_{N_0} \subset \mathcal{G}_{k_0}$ with $\bigcup_{i=1}^{N_0} \Omega_i = \mathcal{G}_{k_0}$ and for each $k \in \{1, 2, \ldots, N_0\}$ there exists $p_k \in \{1, 2, \ldots, N\}$ such that

$$d\left(Dv_{\lfloor\tau_i}, SO(2)\, A_{p_k}\right) = d\left(Dv_{\lfloor\tau_i}, K\right) \text{ for all } i \in \Omega_k \text{ for } k = 1, 2, \ldots, N_0.$$

Since $\bigcup_{i\in\mathcal{G}_{k_0}} \tau_i = \mathcal{A}_{k_0}$ and $\mathcal{A}_{k_0}$ is connected we must be able to find $i_1 \in \Omega_1$ and $i_2 \in \Omega_2$ such that $H^1(\partial\tau_{i_1} \cap \partial\tau_{i_2}) \geq \varsigma$. Let a, b be the endpoints of $\partial\tau_{i_1} \cap \partial\tau_{i_2}$, since (by definition of G_0) $d\left(Dv_{\lfloor\tau_{i_1}}, SO(2)\, A_{p_1}\right) \leq \eta$, $d\left(Dv_{\lfloor\tau_{i_2}}, SO(2)\, A_{p_2}\right) \leq \eta$ and $Dv_{\lfloor\tau_{i_1}}(a-b) = Dv_{\lfloor\tau_{i_2}}(a-b)$ we must have that for some $R_1, R_2 \in SO(2)$,

$$|R_1 A_{p_1}(a-b) - R_2 A_{p_2}(a-b)| \leq 3\eta \tag{5.141}$$

since $u \in \mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}$ the edges of the triangles are parallel to w_1, w_2 and $\frac{w_1-w_2}{|w_1-w_2|}$. Thus (assuming a, b are ordered correctly) $\frac{a-b}{|a-b|} \in \left\{w_1, w_2, \frac{w_1-w_2}{|w_1-w_2|}\right\}$. Recall we chose w_1, w_2 so that $\left\{w_1, w_2, \frac{w_1-w_2}{|w_1-w_2|}\right\}$ are not in the set of rank-1 connections between $SO(2)\, A_{p_1}$ and $SO(2)\, A_{p_2}$. So $\left|A_{p_1}\left(\frac{a-b}{|a-b|}\right)\right| \neq \left|A_{p_2}\left(\frac{a-b}{|a-b|}\right)\right|$, we can assume without loss of generality there is a constant $c_4 = c_4(w_1, w_2) > 1$ such that $\left|A_{p_1}\left(\frac{a-b}{|a-b|}\right)\right| > c_4\left|A_{p_2}\left(\frac{a-b}{|a-b|}\right)\right|$. Assuming we chose η small enough this contradicts (5.141) this completes the proof of Step 1.

Step 2. Given $k_0 \in \{1, 2, \ldots, N_1\}$ and $x \in \widetilde{\mathcal{A}}_{k_0}$ we will show that

$$\max\left\{\left|Dv_{\lfloor\tau_i} - Dv_{\lfloor\tau_l}\right| : i, l \in E(x)\right\} \leq c \max\left\{d\left(Dv_{\lfloor\tau_j}, K\right) : j \in E(x)\right\}. \tag{5.142}$$

Proof Step 2. Firstly by change of variables we can assume k_0 is such that $Dv_{\lfloor\tau_i} \in N_\eta(SO(2))$ for any $i \in G_0$. We introduce some notation, let $j \in \{1, 2, \ldots, N_3\}$ for any $p \in V(j)$ define

$$a(j,p) := \max\left\{d\left(Dv_{\lfloor\tau_j}, SO(2)\right), d\left(Dv_{\lfloor\tau_p}, SO(2)\right)\right\}$$

so there exists $R_j \in SO(2)$, $R_p \in SO(2)$ such that

$$\left|Dv_{\lfloor\tau_p} - R_p\right| \leq 2a(j,p), \left|Dv_{\lfloor\tau_j} - R_j\right| \leq 2a(j,p). \tag{5.143}$$

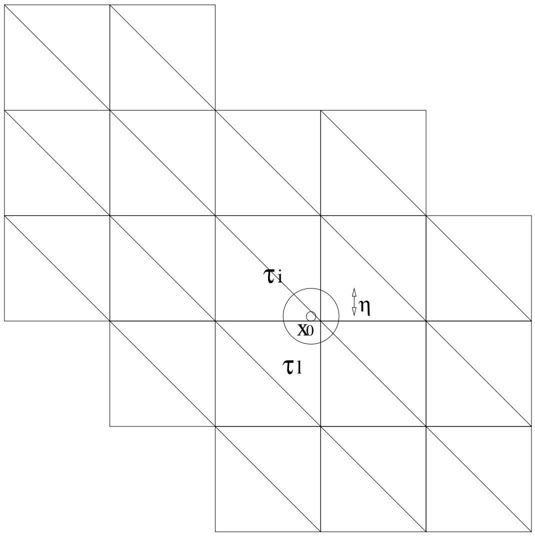

FIGURE 6. A typical ball in the triangulation.

Since $H^1(\overline{\tau_p}\cap\overline{\tau_j})\geq\varsigma$, let a,b denote the endpoints of $\overline{\tau_p}\cap\overline{\tau_j}$, so as $Dv_{\lfloor\tau_p}(a-b)=Dv_{\lfloor\tau_j}(a-b)$ we have $|R_p(a-b)-R_j(a-b)|\leq 4a(j,p)$ which implies $|R_p-R_j|\leq 4\varsigma^{-1}a(j,p)$. Putting this together with (5.143) gives

$$\left|Dv_{\lfloor\tau_p}-Dv_{\lfloor\tau_j}\right|\leq ca(j,p). \tag{5.144}$$

Pick $i,l\in E(x)$, now (see Fig. 6) we must be able to find[7] $i_1,i_2,\dots,i_{M_1}\in E(x_0)$ with the following properties

(1) $i_0=i$, $i_{M_1}=l$;
(2) $i_{r+1}\in V(i_r)$ for $r=0,1,\dots,M_1-1$;
(3) $i_{r_1}\neq i_{r_2}$ for $r_1\neq r_2$;
(4) $E(x_0)\subset\bigcup_{r=0}^{M_1}V(i_r)$.

We have

$$\begin{aligned}\left|Dv_{\lfloor\tau_{i_0}}-Dv_{\lfloor\tau_{i_{M_1}}}\right| &\overset{(5.144)}{\leq} \sum_{r=0}^{M_1-1}ca(i_r,i_{r+1})\\ &\leq cM_1\max\left\{d\left(Dv_{\lfloor\tau_r},SO(2)\right):r\in E(x)\right\}.\end{aligned}$$

Since from property (3) we know $M_1\leq c\,\mathrm{Card}(E(x_0))\leq c$ this gives (5.142).

Step 3. Let $\tilde{v}:=v*\rho_\eta$ we will show

$$\sum_{k=1}^{N_1}\int_{\tilde{\mathcal{A}}_k}d^p(D\tilde{v}(z),K)\,\mathrm{d}L^2z\leq c\alpha_0. \tag{5.145}$$

[7]Since $B_\eta(x)$ is open and $\tau_i\cap B_\eta(x)\neq\emptyset$, $\tau_l\cap B_\eta(x)\neq\emptyset$ we have $H^1(\partial B_\eta(x)\cap\tau_i)>0$ and $H^1(\partial B_\eta(x)\cap\tau_l)>0$. Pick point $s_0\in\tau_i\cap\partial B_\eta(x)$ and a point $s_{M_1}\in\tau_l\cap\partial B_\eta(x)$, since all but finitely many points on $\partial B_\eta(x)$ are contained in $\bigcup_j\tau_j$ we can go clockwise from s_1 to s_{M_1}, the first triangle τ_j we encounter after τ_i with $H^1(\tau_j\cap\partial B_\eta(x))>0$ will have the property that $\tau_j\cap B_\eta(x)\neq\emptyset$ (and hence $j\in E(x_0)$) and $j\in V_1(i)$ so define $i_1=j$. We can then define i_2 to be the first τ_l we encounter going clockwise on $\partial B_\eta(x)$ after $\tau_{i_1}\cap\partial B_\eta(x)$, continuing in this way gives us the sequence $i_1,i_2,\dots,i_{M_1}$ with the properties we want.

Proof of Step 3. Let $\mathbb{D} := \{i : \partial\tau_i \cap \partial\Omega \neq \emptyset\}$. We define $p : Q_{\epsilon^{-\frac{1}{2}}+\eta}(0) \to \{1, 2, \ldots, N_3\}$ by

$$p(z) := \begin{cases} \min\{i : z \in \overline{\tau_i}\} & \text{for } z \in \overline{\Omega_{\epsilon^{-\frac{1}{2}}}} \\ \min\left\{i \in \mathbb{D} : B_{\frac{3\eta}{2}}(z) \cap \overline{\tau_i} \neq \emptyset\right\} & \text{for } z \in \Omega_{\epsilon^{-\frac{1}{2}}+\eta} \backslash \overline{\Omega_{\epsilon^{-\frac{1}{2}}}}. \end{cases} \tag{5.146}$$

Fix $k_0 \in \{1, 2, \ldots, N_1\}$, assume $\widetilde{\mathcal{A}}_{k_0} \neq \emptyset$. Let $y \in \widetilde{\mathcal{A}}_{k_0}$. Pick $i_0 \in E(y)$ and let $R_0 \in K$ be such that $d\left(Dv_{\lfloor \tau_{i_0}}, K\right) = \left|Dv_{\lfloor \tau_{i_0}} - R_0\right|$. Now

$$\begin{aligned} |D\tilde{v}(y) - R_0| &= \left|\sum_{j \in E(y)} \int_{\tau_j} \left(Dv_{\lfloor \tau_j}(x) - R_0\right) \rho_\eta(x-y)\, \mathrm{d}L^2 x\right| \\ &\leq c \sum_{j \in E(y)} \left|Dv_{\lfloor \tau_j} - Dv_{\lfloor \tau_{i_0}}\right| + \left|Dv_{\lfloor \tau_{i_0}} - R_0\right| \\ &\overset{(5.142)}{\leq} c \max\left\{d\left(Dv_{\lfloor \tau_j}, K\right) : j \in E(y)\right\}. \end{aligned} \tag{5.147}$$

Define $c(i) \in \mathbb{V}_1(i)$ to be such that

$$d\left(Dv_{\lfloor \tau_{c(i)}}, K\right) = \max\left\{d\left(Dv_{\lfloor \tau_j}, K\right) : j \in \mathbb{V}_1(i)\right\}. \tag{5.148}$$

Note for any $z \in Q_{\epsilon^{-\frac{1}{2}}+\eta}(0)$ from (5.139) we know (recall definition (5.138)) that $E(y) \subset \mathbb{V}_1(p(y))$, so

$$d^p(D\tilde{v}(y), K) \overset{(5.147),(5.148)}{\leq} c\, d^p\left(Dv_{\lfloor \tau_{c(p(y))}}, K\right). \tag{5.149}$$

Now

$$\begin{aligned} \int_{\widetilde{\mathcal{A}}_{k_0}} d^p(D\tilde{v}(z), K)\, \mathrm{d}L^2 z &\leq \sum_{\{i : \mathbb{V}_1(i) \subset \mathcal{G}_{k_0}\}} L^2(\tau_i) \sup\{d^p(D\tilde{v}(z), K) : z \in \tau_i\} \\ &\overset{(5.149)}{\leq} \sum_{\{i : \mathbb{V}_1(i) \subset \mathcal{G}_{k_0}\}} c\, d^p\left(Dv_{\tau_{c(i)}}, K\right). \end{aligned}$$

Note $\max\left\{\mathrm{Card}\left(c^{-1}(i)\right) : i \in \mathcal{G}_{k_0}\right\} \leq c$ and so $\int_{\widetilde{\mathcal{A}}_{k_0}} d^p(D\tilde{v}(z), K)\, \mathrm{d}L^2 z \leq c \sum_{i \in \mathcal{G}_{k_0}} d^p\left(Dv_{\lfloor \tau_i}, K\right)$. Thus summing over $k_0 = 1, 2, \ldots, N_1$ gives (5.145).

Step 4. We will show that

$$\int_{Q_{\epsilon^{-\frac{1}{2}}+\eta}(0)} d^p(D\tilde{v}(z), K)\, \mathrm{d}L^2 z \leq c\alpha_0 + c\eta\epsilon^{-\frac{1}{2}}. \tag{5.150}$$

Proof of Step 4. Let $\mathbb{D} := \{i : \partial\tau_i \cap \partial\Omega \neq \emptyset\}$. Note (recalling definition (5.146), (5.138))

$$p(z) \in E(z) \text{ for any } z \in \Omega_{\epsilon^{-\frac{1}{2}}} \tag{5.151}$$

so

$$\begin{aligned}\left|D\tilde{v}\left(z\right)\right| &= \left|F\int_{B_\eta(z)\backslash\Omega_{\epsilon^{-\frac{1}{2}}}}\rho_\eta\left(a-z\right)\mathrm{d}L^2a+\sum_{i\in E(z)}Dv_{\lfloor\tau_i}\int_{\tau_i}\rho_\eta\left(a-z\right)\mathrm{d}L^2a\right|\\ &\overset{(5.139)}{\leq} c+c\sum_{i\in\mathbb{V}_1(p(z))}d\left(Dv_{\lfloor\tau_i},K\right).\end{aligned} \tag{5.152}$$

Thus

$$\begin{aligned}d^p\left(D\tilde{v}\left(z\right),K\right) &\overset{(5.152)}{\leq} \left(c+c\sum_{i\in\mathbb{V}_1(p(z))}d\left(Dv_{\lfloor\tau_i},K\right)\right)^p+c\\ &\leq c+c\sum_{i\in\mathbb{V}_1(p(z))}d^p\left(Dv_{\lfloor\tau_i},K\right).\end{aligned} \tag{5.153}$$

Let $\mathbb{B}:=\{i:\mathbb{V}_1(i)\not\subset G_0\}$. Note that if i is such that $\mathbb{V}_1(i)\subset G_0$ then $\mathbb{V}_1(i)\subset\mathcal{G}_k$ for some $k\in\{1,2,\ldots,N_1\}$ (and recall definition (5.137)) and hence $\tau_i\subset\widetilde{\mathcal{A}}_k$, thus

$$\bigcup_{i\in\mathbb{B}}\overline{\tau_i}=\overline{\Omega_{\epsilon^{-\frac{1}{2}}}\backslash\left(\bigcup_{k=1}^{N_1}\widetilde{\mathcal{A}}_k\right)}. \tag{5.154}$$

So

$$\begin{aligned}\int_{\bigcup_{i\in\mathbb{B}}\overline{\tau_i}}d^p\left(D\tilde{v}\left(z\right),K\right)\mathrm{d}L^2z &\overset{(5.153)}{\leq} \sum_{i\in\mathbb{B}}L^2\left(\tau_i\right)\left(c+c\sum_{j\in\mathbb{V}_1(i)}d^p\left(Dv_{\lfloor\tau_j},K\right)\right)\\ &\overset{(5.136)}{\leq} c\alpha_0+c\,\mathrm{Card}\left(\mathbb{B}\right).\end{aligned} \tag{5.155}$$

By an easy application of the $5r$ Covering Theorem (Thm. 2.1. [25]) we know

$$\mathrm{Card}\left(\mathbb{B}\right)\leq c\left(\{1,2,\ldots,N_3\}\backslash G_0\right)\leq c\alpha_0. \tag{5.156}$$

Now

$$\tau_{p(z)}\subset\Omega_{\epsilon^{-\frac{1}{2}}}\backslash\Omega_{\epsilon^{-\frac{1}{2}}-10\varsigma^{-1}}\ \text{ for any } z\in\Omega_{\epsilon^{-\frac{1}{2}}+\eta}\backslash\Omega_{\epsilon^{-\frac{1}{2}}}. \tag{5.157}$$

Let $\{l_1,l_2,\ldots,l_{X_1}\}$ be an ordering of the set $\left\{p\left(z\right):z\in\Omega_{\epsilon^{-\frac{1}{2}}+\eta}\backslash\Omega_{\epsilon^{-\frac{1}{2}}}\right\}$ we have that $X_1\leq c\epsilon^{-\frac{1}{2}}$. And thus

$$\begin{aligned}\int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}\backslash\Omega_{\epsilon^{-\frac{1}{2}}}}d^p\left(D\tilde{v}\left(z\right),K\right)\mathrm{d}L^2z &\overset{(5.153)}{\leq} c\sum_{k=1}^{X_1}L^2\left(p^{-1}\left(l_k\right)\backslash\Omega_{\epsilon^{-\frac{1}{2}}}\right)+\sum_{k=1}^{X_1}\sum_{i\in\mathbb{V}_1(l_k)}c\,d^p\left(Dv_{\lfloor\tau_i},K\right)\\ &\overset{(5.136)}{\leq} c\eta\epsilon^{-\frac{1}{2}}+c\alpha_0.\end{aligned} \tag{5.158}$$

So putting things together, by (5.145), (5.154), (5.155), (5.156) and (5.158) we have

$$\begin{aligned}\int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}} d^p\left(D\tilde{v}\left(z\right),K\right)\mathrm{d}L^2z &= \int_{\bigcup_{i\in\mathbb{B}}\tau_i} d^p\left(D\tilde{v}\left(z\right),K\right)\mathrm{d}L^2z \\ &\quad+\int_{\bigcup_k^{N_1}\widetilde{\mathcal{A}}_k} d^p\left(D\tilde{v}\left(z\right),K\right)\mathrm{d}L^2z \\ &\quad+\int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}(0)\backslash\Omega_{\epsilon^{-\frac{1}{2}}}} d^p\left(D\tilde{v}\left(z\right),K\right)\mathrm{d}L^2z \\ &\leq c\alpha_0+c\eta\epsilon^{-\frac{1}{2}},\end{aligned} \tag{5.159}$$

which completes the proof of (5.150).

Step 5. We will show

$$\sum_{k=1}^{N_1}\int_{\widetilde{A}_k}\left|D^2\tilde{v}\left(y\right)\right|^2\mathrm{d}L^2y\leq c\alpha_0. \tag{5.160}$$

Proof of Step 5. Let $y\in\bigcup_{k=1}^{N_1}\widetilde{A}_k$, for each $j\in E\left(y\right)$ define $A_j:=\int_{\tau_j}D\rho_\eta\left(x-y\right)\mathrm{d}L^2x$, note $\sum_{j\in E(y)}A_j=0$. So $D^2\tilde{v}\left(y\right)=\sum_{j\in E(y)}\int_{\tau_j}-Dv_{\lfloor\tau_j}\otimes D\rho_\eta\left(x-y\right)\mathrm{d}L^2x=\sum_{j\in E(y)}-Dv_{\lfloor\tau_j}\otimes A_j$. So we have $D^2\tilde{v}\left(y\right)=\sum_{j\in E(y)}-\left(Dv_{\lfloor\tau_j}-Dv_{\lfloor\tau_{p(y)}}\right)\otimes A_j$ and so

$$\begin{aligned}\left|D^2\tilde{v}\left(y\right)\right|^2 &\leq c\sum_{j\in E(y)}\left|Dv_{\lfloor\tau_j}-Dv_{\lfloor\tau_{p(y)}}\right|^2 \\ &\overset{(5.142),(5.151)}{\leq} c\left(\max\left\{d\left(Dv_{\lfloor\tau_l},K\right):l\in E\left(y\right)\right\}\right)^2.\end{aligned} \tag{5.161}$$

Thus (recall the definition $c\left(i\right)$, (5.148)) we have

$$\begin{aligned}\int_{\widetilde{A}_k}\left|D^2\tilde{v}\left(y\right)\right|^2\mathrm{d}L^2y &\overset{(5.139),(5.161)}{\leq} \sum_{\{i:\mathbb{V}_1(i)\subset\mathcal{G}_k\}}c\left(\max\left\{d\left(Dv_{\lfloor\tau_l},K\right):l\in\mathbb{V}_1\left(i\right)\right\}\right)^2 \\ &= \sum_{\{i:\mathbb{V}_1(i)\subset\mathcal{G}_k\}}c\,d^2\left(Dv_{\lfloor\tau_{c(i)}},K\right) \\ &\leq c\sum_{i\in\mathcal{G}_k}d^p\left(Dv_{\lfloor\tau_i},K\right).\end{aligned}$$

Thus summing over $k=1,2,\ldots,N_1$ gives (5.160).

Step 6. We will show

$$\int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}\backslash\left(\bigcup_{k=1}^{N_1}\widetilde{A}_k\right)}\left|D^2\tilde{v}\left(z\right)\right|^2\mathrm{d}L^2z\leq c\alpha_0+c\eta\epsilon^{-\frac{1}{2}}. \tag{5.162}$$

Proof of Step 6. Now let $y\in\Omega_{\epsilon^{-\frac{1}{2}}+\eta}$. Note that if $B_\eta\left(y\right)\not\subset\Omega_{\epsilon^{-\frac{1}{2}}}$ then define $A_y:=\int_{B_\eta(y)\backslash\Omega_{\epsilon^{-\frac{1}{2}}}}D\rho_\eta\left(x-y\right)\mathrm{d}L^2x$ otherwise define $A_y=0$.

As in Step 5 for each $j\in E\left(y\right)$ define $A_j=\int_{\tau_j}D\rho_\eta\left(x-y\right)\mathrm{d}L^2x$. So we have $\sum_{j\in E(y)}A_j+A_y=0$. So as in Step 5 $-D^2\tilde{v}\left(y\right)=F\otimes A_y+\sum_{j\in E(y)}Dv_{\lfloor\tau_j}\otimes A_j=\left(F-Dv_{\lfloor\tau_{p(y)}}\right)\otimes A_y+\sum_{j\in E(y)}\left(Dv_{\lfloor\tau_j}-Dv_{\lfloor\tau_{p(y)}}\right)\otimes A_j$.

Thus for any $y \in Q_{\epsilon^{-\frac{1}{2}}+\eta}(0)$

$$\begin{aligned} \left|D^2\tilde{v}(y)\right|^2 &\leq c\left|F - Dv_{\lfloor\tau_{p(y)}}\right|^2 |A_y|^2 + c\sum_{j\in E(y)}\left|Dv_{\lfloor\tau_j} - Dv_{\lfloor\tau_{p(y)}}\right|^2 \\ &\overset{(5.139)}{\leq} c\left|F - Dv_{\lfloor\tau_{p(y)}}\right|^2 |A_y|^2 + c\sum_{j\in \mathbb{V}_1(p(y))}\left|Dv_{\lfloor\tau_j} - Dv_{\lfloor\tau_{p(y)}}\right|^2. \end{aligned} \tag{5.163}$$

Now as in Step 1 for any $i,j \in \mathbb{V}_1(p(y))$ we can find a finite sequence $l_1, l_2, \dots, l_{N_j} \in \mathbb{V}_1(p(y))$ such that $l_1 = i$, $l_{a+1} \in V(l_a)$ for $a = 1, 2, \dots, N_j - 1$ and $l_{N_j} = j$ so

$$\begin{aligned} \left|Dv_{\lfloor\tau_i} - Dv_{\lfloor\tau_j}\right|^2 &\leq c\sum_{a=1}^{N_j-1}\left|Dv_{\lfloor\tau_{l_{a+1}}} - Dv_{\lfloor\tau_{l_a}}\right|^2 \\ &\leq c\sum_{l\in\left\{l_1,l_2,\dots,l_{N_j-1}\right\}}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 \\ &\leq c\sum_{l\in\mathbb{V}_1(p(y))}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2. \end{aligned}$$

So from (5.163) for any $y \in Q_{\epsilon^{-\frac{1}{2}}+\eta}(0)$ we have

$$\begin{aligned} \left|D^2\tilde{v}(y)\right|^2 &\leq c\left|F - Dv_{\lfloor\tau_{p(y)}}\right|^2 |A_y|^2 + c\sum_{l\in\mathbb{V}_1(p(y))}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 \\ &\leq c\left|F - Dv_{\lfloor\tau_{p(y)}}\right|^2 |A_y|^2 + c\sum_{l\in\mathbb{V}_1(p(y))\cap J(v)}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 + c. \end{aligned} \tag{5.164}$$

Recall $\mathbb{D} = \left\{i : \partial\tau_i \cap \partial\Omega_{\epsilon^{-\frac{1}{2}}} \neq \emptyset\right\}$. Note if $y \in \bigcup_{i\notin\mathbb{D}}\overline{\tau_i}$ then $B_\eta(y) \subset \Omega_{\epsilon^{-\frac{1}{2}}}$ and so $A_y = 0$. For $i \in \mathbb{B}$ let $y_i \in \overline{\tau_i}$ be such that $\left|D^2\tilde{v}(y_i)\right| = \sup\left\{\left|D^2\tilde{v}(y)\right| : y \in \tau_i\right\}$, thus

$$\begin{aligned} \int_{\Omega_{\epsilon^{-\frac{1}{2}}}\setminus\left(\bigcup_{k=1}^{N_1}\tilde{\mathcal{A}}_k\right)} \left|D^2\tilde{v}(y)\right|^2 dL^2 y &\overset{(5.154)}{\leq} \sum_{i\in\mathbb{B}} L^2(\tau_i)\left|D^2\tilde{v}(y_i)\right|^2 \\ &\overset{(5.164)}{\leq} c\sum_{i\in\mathbb{B}\setminus\mathbb{D}}\sum_{l\in\mathbb{V}_1(i)\cap J(v)}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 \\ &\quad + c\sum_{i\in\mathbb{B}\cap\mathbb{D}}\left(\left|F - Dv_{\lfloor\tau_i}\right|^2 |A_{y_i}|^2 + \sum_{l\in\mathbb{V}_1(i)\cap J(v)}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2\right) + c\,\mathrm{Card}(\mathbb{B}) \\ &\leq c\sum_{i\in\mathbb{B}}\sum_{l\in\mathbb{V}_1(i)\cap J(v)}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 + c\sum_{i\in\mathbb{B}\cap\mathbb{D}}\left|F - Dv_{\lfloor\tau_i}\right|^2 + c\,\mathrm{Card}(\mathbb{B}) \\ &\leq c\sum_{l\in J(v)}\sum_{k\in V(l)}\left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 + c\sum_{i\in\mathbb{D}}\left|F - Dv_{\lfloor\tau_i}\right|^2 + c\,\mathrm{Card}(\mathbb{B}) \\ &\overset{(5.156)}{\leq} c\sum_{l\in J(v)}\sum_{M\in N(l)}\left|Dv_{\lfloor\tau_l} - M\right|^2 + c\alpha_0 \\ &\overset{(5.135),(5.136)}{\leq} c\alpha_0. \end{aligned} \tag{5.165}$$

Now to estimate $\int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}\backslash\Omega_{\epsilon^{-\frac{1}{2}}}} \left|D^2\tilde{v}(z)\right|^2 dL^2 z$ we argue as in Step 3, let $\{l_1, l_2, \ldots, l_{X_1}\}$ be an ordering of the set $\left\{p(z) : z \in \Omega_{\epsilon^{-\frac{1}{2}}+\eta}\backslash\Omega_{\epsilon^{-\frac{1}{2}}}\right\}$, recall we have $X_1 \leq c\epsilon^{-\frac{1}{2}}$. And of course, from (5.146) we have $\{l_1, l_2, \ldots, l_{X_1}\} \subset \mathbb{D}$. So

$$\begin{aligned}
\int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}(0)\backslash\Omega_{\epsilon^{-\frac{1}{2}}}} \left|D^2\tilde{v}(z)\right|^2 dL^2 z &\overset{(5.164)}{\leq} \sum_{a=1}^{X_1} c\left|F - Dv_{\lfloor\tau_{l_a}}\right|^2 + c\sum_{l\in\mathbb{V}_1(l_a)\cap J(v)}\sum_{k\in V(l)} \left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 \\
&\quad + c\sum_{b=1}^{X_1} cL^2\left(p^{-1}(l_b)\right) \\
&\leq c\sum_{l=1}^{N_3}\sum_{k\in V(l)} \left|Dv_{\lfloor\tau_l} - Dv_{\lfloor\tau_k}\right|^2 + c\sum_{i\in\mathbb{D}} \left|F - Dv_{\lfloor\tau_i}\right|^2 + c\eta\epsilon^{-\frac{1}{2}} \\
&\overset{(5.135)}{\leq} c\int_\Omega d^p\left(Dv(z), K\right) dL^2 z + c\eta\epsilon^{-\frac{1}{2}} \\
&\leq c\alpha_0 + c\eta\epsilon^{-\frac{1}{2}}.
\end{aligned}$$

Putting this together with (5.165) gives (5.162).

Proof of Proposition 5.2. Let $w(z) := \frac{\tilde{v}\left(\left(\epsilon^{-\frac{1}{2}}+\eta\right)z\right)}{\epsilon^{-\frac{1}{2}}+\eta}$, it is clear w can also be defined by equation (5.133). So from (5.162) and (5.160) we have

$$\int_\Omega \left|D^2 w(z)\right|^2 dL^2 z \leq c\alpha_0 + c\eta\epsilon^{-\frac{1}{2}} \tag{5.166}$$

and

$$\begin{aligned}
\int_\Omega d^p\left(Dw(z), K\right) dL^2 z &= \int_{\Omega_{\epsilon^{-\frac{1}{2}}+\eta}} d^p\left(D\tilde{v}(y), K\right)\left(\epsilon^{-\frac{1}{2}}+\eta\right)^{-2} dL^2 y \\
&\overset{(5.150)}{\leq} c\epsilon\alpha_0 + c\eta\epsilon^{\frac{1}{2}}.
\end{aligned} \tag{5.167}$$

Putting this together with (5.166) gives

$$\int_\Omega d^p\left(Dw(z), K\right) + \epsilon\left|D^2 w(z)\right|^2 dL^2 z \leq c\epsilon\alpha_0 + c\eta\epsilon^{\frac{1}{2}}. \tag{5.168}$$

Now by (4.2) we have that there exists some small constant $c_1 = c_1(\sigma)$ such that

$$c_1\epsilon^{\frac{1}{2}} \leq \int_\Omega d^p\left(Dw(z), K\right) + \epsilon\left|D^2 w(z)\right|^2 dL^2 z$$

so assuming we have chosen η small enough we have that

$$\int_\Omega d^p\left(Dw(z), K\right) + \epsilon\left|D^2 w(z)\right|^2 dL^2 z - c\eta\epsilon^{\frac{1}{2}} \geq \frac{1}{2}\int_\Omega d^p\left(Dw(z), K\right) + \epsilon\left|D^2 w(z)\right|^2 dL^2 z$$

hence from (5.168) we have

$$\begin{aligned}\int_{\Omega} d^p\left(Dw\left(z\right),K\right)+\epsilon\left|D^2w\left(z\right)\right|^2 \mathrm{d}L^2z &\leq c\epsilon\alpha_0\\ &\overset{(5.136)}{=} c\int_{\Omega} d^p\left(Dw\left(z\right),K\right)\mathrm{d}L^2z\end{aligned}$$

which completes the proof of (5.134). □

5.1. The proof of Theorem 1.1 completed

By Proposition 5.2 for any $\epsilon>0$ we can find $u\in\mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}$ such that $\int_{\Omega} d^p\left(Du\left(z\right),K\right)\mathrm{d}L^2z\leq cm_\epsilon^p$ which obviously implies there must exist constant $\mathcal{C}_1<1$ such that $\mathcal{C}_1\alpha\left(\sqrt{\epsilon}\right)\leq m_\epsilon^p$.

Let $u\in\mathcal{D}_F^{\varsigma,\sqrt{\epsilon}}$ be such that $\int_{\Omega} d^p\left(Du\left(z\right),K\right)\mathrm{d}L^2z\leq c\alpha_p\left(\sqrt{\epsilon}\right)$. By Proposition 5.3 function w defined by (5.132) and (5.133) has the property that

$$I_\epsilon\left(w\right)\leq c\int_{\Omega} d^p\left(Du\left(z\right),K\right)\mathrm{d}L^2z\leq c\alpha_p\left(\sqrt{\epsilon}\right)$$

which implies there exists a constant $\mathcal{C}_2>1$ such that $m_\epsilon^p\leq\mathcal{C}_2\alpha_p\left(\sqrt{\epsilon}\right)$. □

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