THE REGULARITY OF FREE BOUNDARIES IN HIGHER DIMENSIONS

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Introduction

The problem of studying the regularity of the free boundary that arises when considering the energy minimizing function over the set of those functions bigger than a given "obstacle" has been the subject of intensive research in the last decade. Let me mention H. Lewy and G. Stampacchia [14], D. Kinderlehrer [11], J. C. Nitsche [15] and N. M. Riviere and the author [5] among others. In two dimensions, by the use of analytic reflection techniques due mainly to H. Lewy [13], much was achieved.

Recently, the author was able to prove, in a three dimensional filtration problem [4], that the resulting free surface is of class C^1 and all the second derivatives of the variational solution are continuous up to the free boundary, on the non-coincidence set. This fact has not only the virtue of proving that the variational solution is a classical one, but also verifies the hypothesis necessary to apply a recent result due to D. Kinderlehrer and L. Nirenberg, [12] to conclude that the free boundary is as smooth as the obstacle. Nevertheless, in that paper ([4]), strong use was made of the geometry of the problem: this implied that the free boundary was Lipschitz. Also it was apparently essential that the Laplacian of the obstacle was constant.

In the first part of this paper we plan to treat the general non-linear free boundary problem as presented in H. Brezis–D. Kinderlehrer [2]. Our main purpose is to prove that if X_0 is a point of density for the coincidence set, in a neighborhood of X_0 the free boundary is a C^1 surface and all the second derivatives of the solution are continuous up to it. In the second part we will study the parabolic case (one phase Stefan problem) as presented by G. Duvaut [7] or A. Friedman and D. Kinderlehrer [9]. There we prove that if for a fixed time, t_0, the point X_0 is a density point for the coincidence set (the ice) then in a

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neighborhood in space and time of (X_0, t_0) the free boundary is a surface of class C^1 in space and time and all the second derivatives (in space and time) of the solution are continuous up to the free boundary. The solution is hence, a classical one in that neighborhood. The appendix contains some Harnack type inequalities and some geometrical lemmas. I would like to thank D. Kinderlehrer for helping me clarify the presentation of this work.

1. The elliptic case

1.1. The case considered by H. Brezis and D. Kinderlehrer in [2] is the following: given are Ω, a bounded, open, connected subset of $\mathbf{R}^n$, a locally coercive C^2-vector field $a_i(P)$; $(P=(p_1, ..., p_n))$ and a function ψ (the obstacle) satisfying, $\psi \in C^2(\Omega)$, $\psi \leqslant 0$ on $\partial\Omega$.

It is proven there that a solution, u, to the problem

$$u \in K; \int_\Omega a_j(Du)\, D_j(v-u)\, dX \geqslant 0 \quad \text{for all} \quad v \in K$$

with $K=\{v: v \text{ is Lipschitz}, v \geqslant \varphi, v|_{\partial\Omega}=0\}$ is of class $C^{1,1}(M)$ for any $M \subset \Omega$, compact. On Ω one distinguishes the subsets $D=\{X: u=\varphi\}$ and $\Omega \setminus D$. On $(\partial D) \cap \Omega$, $u=\varphi$ and $\nabla u = \nabla\varphi$ and on $\Omega \setminus D = W$, $A(u) = -\partial_i(a_i(\nabla u)) = -\sum a_{ij}(\nabla u)\, u_{ij} = 0$ where A is elliptic.

If φ is assumed to be of class $C^4(\Omega)$ and $A\varphi$, $\nabla(A\varphi)$ do not vanish simultaneously, it was observed in [5], that if $X_0 \in (\partial D) \cap \Omega$, then $A(\varphi) < 0$ in a neighborhood of X_0. Hence locally we have the following situation: Given are an open set W, a ball $B_\varrho(X_0)$ and two functions:

$$\begin{aligned}
&\varphi \in C^4(B_\varrho(X_0)),\ A(\varphi) < \lambda_0 < 0 \quad \text{on } B_\varrho(X_0),\\
&u \in C^{1,1}(B_\varrho(X_0)),\ u|_{B_\varrho(X_0)\setminus W} = \varphi,\\
&u \geqslant \varphi \quad \text{on } B_\varrho(X_0),\ Au = 0 \quad \text{on } W \cap B_\varrho(X_0).
\end{aligned}$$

(In particular $\nabla u = \nabla\varphi$ on $(\partial W) \cap B_\varrho(X_0)$.)

Finally, if we subtract φ from u, $v = u - \varphi$ satisfies, in a new subneighborhood $B_{\varrho'}(X_0)$,

$$a_{ij}(\nabla\varphi)\, D_{ij}(u-\varphi) = f > 0 \quad \text{on } W \cap B_{\varrho'}(X_0).$$

As observed in [5], f can be extended by $-a_{ij}(\nabla\varphi)\, D_{ij}(\varphi)$ to be a $C^{1/2}(B_{\varrho'}(X_0))$ function, since near $(\partial W) \cap B_{\varrho'}$

$$a_{ij}(\nabla\varphi)\, D_{ij}(u) = a_{ij}(\nabla u)\, D_{ij}(u) + O(d(X, \partial W)) = O(d(X, \partial W))$$

and in the interior the growth of the Hölder norm of $D_{ij}(u)$ is controlled by the Schauder estimates.

Therefore, if we assume $a_{ij}(P)$ to be C^3 functions, the general problem reduces to the one treated next.

1.2. We are given an open set W, a linear elliptic operator $Au = \sum a_{ij}(X)\partial_i\partial_j u$ with coefficients $a_{ij} \in C^3$ in a neighborhood of W and a function $v \in C^{1,1}(W)$ and satisfying:

$(H1)$ $v \geqslant 0$, $A(v) = f$ where f has a $C^{1/2}$ extension f^*, to a neighborhood of $\overline{W}$, $f^* > \lambda_0 > 0$. ($f^* \in C^\varepsilon$ is all we will use.)

(H2) The boundary of W, ∂W, may be decomposed into $\partial_1 W$ and $\partial W \setminus \partial_1 W$, where $\partial_1 W$ is an open set of ∂W, and $v|_{\partial_1 W} = 0$, $\nabla v|_{\partial_1 W} = 0$.

The investigation of the regularity properties of $\partial_1 W$ are the purpose of this work for which it suffices to restrict ourselves to an open subset F of $\partial_1 W$ satisfying $d(F, \partial W \setminus \partial_1 W) > \mu_0 > 0$. The dependence of the estimates obtained for F, on μ_0, $\|v\|_{C^{1,1}}$, $\|f^*\|_{C^{1/2}}$ and λ_0 is not going to be taken into account. The letter C will denote a constant. When different constants appear at different steps in a proof we will keep the same letter C unless we want to stress the dependence of that constant on some new variable appearing in the context.

1.3. We begin with the following observation, necessary in the proof of Theorem 1:

LEMMA 1. *Let u be a non-negative $C^{1,1}(\bar{B}_\varrho(X_0))$ function with norm*

$$\|u\|_{C^{1,1}} = \sup(u) + \sup(|\nabla u|) + \sup(|u_{ij}|) = \lambda$$

and assume that for some point $Y_0 \in \partial B_\varrho(X_0)$, $u(Y_0) = 0$ and $\nabla u(Y_0) = 0$.

Then given δ, $0 < \delta < 1/2$, and a pure second derivative u_{ii} there is a point Y_δ such that $|Y_\delta - X_0| < (1 - \delta/2)\varrho$ and

$$u_{ii}(Y_\delta) \geqslant -C\lambda\delta^{1/2}.$$

Proof. If $Y_1 = (1-\delta)(Y_0 - X_0) + X_0$ then $u(Y_1) \leqslant \frac{1}{2}\lambda(\delta\varrho)^2$ and $|\nabla u(Y_1)| < \lambda(\delta\varrho)$. Now, in the i^{th} or $-i^{th}$ direction the segment $I = [Y_1, Y_2]$, with origin Y_1 and length $\frac{1}{2}\delta^{1/2}\varrho$ is contained in $B_{\varrho(1-\delta/2)}(X_0)$ (the worst case of all takes place when I is perpendicular to the radius $[Y_1, X_0]$). Then

$$0 \leqslant u(Y_2) = u(Y_1) \pm u_i(Y_1)(|Y_2 - Y_1|) + \int\!\!\int u_{ii} \leqslant \tfrac{1}{2}\lambda(\delta\varrho)^2 + \frac{\lambda}{2}\delta^{3/2}\varrho^2 + (\sup_I u_{ii})\frac{\delta}{4}\varrho^2.$$

That is,

$$\sup_I u_{ii} \geqslant -C\lambda\delta^{1/2}. \qquad \|$$

The next theorem is perhaps the most fruitful observation of the work.

THEOREM 1. *Let v, W and F be as in (H1), (H2) of paragraph 1.2, and v_{tt} a pure second derivative of v. Then there exist positive constants C and ε depending only on f_0, the smoothness of the data, and ellipticity of the operator A, such that for any $X \in W$, $v_{tt}(X) > -C|(\log \varrho)|^{-\varepsilon}$ where $\varrho = d(X, F)$.*

Remark. In the proof of this theorem the fact that $f > 0$ is not used, only its C^α character.

Proof. Let $0 < \varrho$, $|\log \varrho|^{-1} < M < 1$. We will prove, by means of the Harnack inequality that, if $v_{tt}(X) > -M$ for any X such that $d(X, F) < \varrho$, $v_{tt}(X) > -M + CM^{2n-1}$ whenever $d(X, F) < \varrho/2$. A simple iteration then shows that if for $d(X, F) < \varrho_0$, $v_{tt}(X) > -1$, then for $d(X, F) < 2^{-k}\varrho_0$, $v_{tt}(X) > -Ck^{-1/(2n-2)}$. Assume, therefore, that $d(X_0, F) < \varrho/2$ and let us consider the biggest ball $B_{\varrho'}(X_0) \subset W$, $(\varrho' < \varrho/2)$. To this ball, Lemma 1 applies and therefore, given δ, there is a Y_δ, as in Lemma 1, such that $v_{tt}(Y_\delta) \geq -C\sqrt{\delta}$ and $d(Y_\delta, \partial B_{\varrho'}) \geq \frac{1}{2}\delta\varrho'$.

We now apply the Harnack estimate in Lemma A1, choosing $\delta = CM$ (C small enough) and we get

$$v_{tt}(X_0) + M \geq CM^{2(n-1)}(M - C\varrho^{1/2}) - C\varrho^{1/2}.$$

Since we are willing to assume $M > |\log \varrho|^{-1}$, choosing initially ϱ_0 small enough,

$$v_{tt}(X_0) + M \geq CM^{2n-1}.$$

1.4. We will now study the geometric implications of Theorem 1. In order to do so, let us introduce some notation.

Notation 1. We will systematically make use of half balls with inner normal η

$$HB_\varrho(Y, \eta) = B_\varrho(Y) \cap \{X: \langle X - Y, \eta\rangle > 0\}.$$

If η is of no interest for us, we will delete it.

Notation 2. We will also consider the angle between two vectors Y and Z, which we will denote by $\alpha(Y, Z)$.

Notation 3. We will use functions $(\varepsilon > 0)$

$$\gamma_\varepsilon(\varrho) = C\varrho\,|\log \varrho|^\varepsilon,$$
$$\gamma_{-\varepsilon}(\varrho) = C\varrho\,|\log \varrho|^{-\varepsilon}.$$

The constant C may vary from step to step. Obviously $\gamma_\varepsilon(\gamma_{-\varepsilon}(\varrho)) \sim \varrho$. The first geometric consequence of Theorem 1 is

COROLLARY 1. *Let* $X \in \overline{W}$ *and assume that*

(a) $v(X) = \varrho_0^2$

(b) $d(X, F) \leqslant \varrho_0^{\varepsilon'}$, $(0 < \varepsilon')$.

Then, if we choose the constant $C = C(\varepsilon, \varepsilon')$ *in* γ *sufficiently small*

$$HB_{\gamma_{\varepsilon/2}(\varrho_0)}(X, \nabla v) \subset W.$$

Proof. Let us consider a ray, $I = [X, X^*]$ of HB, and see how far it may be traversed before v becomes negative. Since $v_i(X)$, the directional derivative in the I direction is positive (because $\langle i, \alpha \rangle > 0$),

$$0 = v(X^*) = v(X) + v_i(X)|I| + \iint_I v_{ii} \geqslant \varrho_0^2 - \tfrac{1}{2}C|I|^2 |\log(|I| + \varrho_0^{\varepsilon'})|^{-\varepsilon}.$$

From this inequality, it is easy to see that $|I|$ is at least of size $\gamma_{\varepsilon/2}(\varrho_0)$. ||

A further consequence of Theorem 1 is obtained by using the following lemma.

LEMMA 2. *Let* $X \in \overline{W}$, $d(X, F) < \varrho_2/2$, *and* $0 < \varrho < \varrho_0/2$ *be given. Then*

$$\sup_{W \cap \partial B_\varrho(X)} v \geqslant C\varrho^2.$$

Proof. If $X \in F$, this fact was proven in [5], Lemma 1.1; if $\varrho > 2d(X, F)$, $B_\varrho(X) \supset B_{\varrho/2}(X')$ for some $X' \in F$ and this case follows from the preceeding one. Finally, if $\varrho < 2\,d(X, \partial\Omega)$, using that for an appropriate selection of δ, $A(v(Y) - \delta(|Y - X|^2)) \geqslant 0$ and the maximum principle, we get

$$\sup_{\partial B_{\varrho/2}(X)} (v - \delta(\varrho/2)^2) \geqslant 0. \quad ||$$

We are now ready to prove the next corollary, suggesting that the set of coincidence $\complement W$ is "almost convex".

COROLLARY 2. *Let* $S \subset \complement W$ *have diameter* ϱ_0 *and let* $\Gamma(S)$ *denote its convex envelope. If* $d(X, \complement(\Gamma(S))) > \gamma_{-\varepsilon/4}(\varrho_0)$, *then* $X \in \complement W$.

Proof. Assume that $X \in W$. By Lemma 2, there exists a $Y \in B_{\gamma_{-\varepsilon/4}(\varrho_0)}(X)$ verifying $v(Y) > C[\gamma_{-\varepsilon/4}(\varrho_0)]^2$. Therefore, by Corollary 1 there is a direction η for which

$$HB_{\gamma_{\varepsilon/2}(\gamma_{-\varepsilon/4}(\varrho_0))}(Y) \subset W.$$

Since $\gamma_{\varepsilon/2}(\gamma_{-\varepsilon/4}(\varrho_0)) \gg \varrho_0$, and $Y \in \Gamma(S)$, $HB \cap \Gamma(S) \neq \varnothing$ and therefore $HB \cap S \neq \varnothing$, a contradiction. ||

1.5. Let us introduce the following notation for the "spherical" disk

$$D(X, \varepsilon_0, \varrho, \eta) = \partial B_\varrho(X) \cap \{Y: \alpha(Y - X, \eta) < \varepsilon_0\}.$$

The next two lemmas are designed to prove the following fact. Consider for an $X \in F$ and an ε_0 (small enough) the disks $D(X, \varepsilon_0, \varrho, \eta)$. The fact is that once ε_0 has been fixed, if for some η, $D(X, \varepsilon_0, \varrho, \eta)$ happens to be contained in $\complement W$ for a small enough ϱ, then for any $\varrho_k = 2^{-k}\varrho$ we are going to be able to find again a disk $D(X, \varepsilon_0, \varrho_k, \eta_k)$ contained in $\complement W$. That is, if $\complement W$ "thins" when approaching X, it must do so in a uniform way. (Of course for smaller ε_0's the initial values of $\varrho(\varepsilon_0)$ will be smaller.)

The first lemma asserts that if that were not the case, the set $\complement W$ would have a special geometric configuration.

Notation 4. The symbol $\sigma(\varrho)$ as used below will denote an increasing function of the positive real variable ϱ verifying $\lim_{\varrho \to 0} \sigma(\varrho) = 0$. For instance, in the case below, $\sigma(\varrho)$ is in fact some small power of $|\log \varrho|^{-1}$.

LEMMA 3. *Fix an angle $0 < \alpha_0 < \pi/2$ and an $\varepsilon_0 \leqslant C\alpha_0$ (C small enough). Then if $X_0 \in F$, there exists a $\varrho_0 = \varrho(\varepsilon_0) > 0$ such that, if for some $\varrho < \varrho_0$ and some η, $D(X_0, \varepsilon_0, \varrho, \eta) \subset \complement W$ and $D(X_0, \varepsilon_0, \varrho/2\eta') \not\subset \complement W$ for any η', then $(\complement W) \cap B_\varrho$ is contained in the acute angle between two planes, π_1 and π_2, verifying*

(a) $d(\pi_1 \cap \pi_2, X_0) < \sigma(\varrho)\varrho$

(b) $\alpha(\pi_1, \pi_2) < \alpha_0$.

Proof. In order to understand better the idea of the lemma, let us first present the proof in the two-dimensional case, for which the geometry involved is much simpler:

Since $D_1 = D(X_0, \varepsilon_0, \varrho, \eta) \subset \complement W$ and $X_0 \in \partial W$, by Corollary 2

$$D_2 = D(X_0, \varepsilon_0 - C\,|\log \varrho|^{-\varepsilon/4}, \varrho/2, \eta) \subset \complement W.$$

By hypothesis, if η_i ($i = 1, 2$) denote the two directions such that

$$\alpha(\eta_i, \eta) = C\,|\log \varrho|^{-\varepsilon/8},$$

there exists points X_i verifying

$$X_i \in \overline{W} \cap D(X_0, \varepsilon_0, \varrho/2, \eta_i) \subset D(X_0, \varepsilon_0, \varrho/2, \eta_i) \setminus D_2,$$

which is for $i = 1$ or 2 an arc of circle of apperture smaller than $C\,|\log \varrho|^{-\varepsilon/8}$ tangent to D_2 at each one of its endpoints. Therefore, again by Corollary 2 we can associate to each X_i

a line L_i, tangent to $\Gamma(\complement W \cap B_\varrho(X_0))$ and such that $d(L_i, X_i) < \gamma_{-\varepsilon/4}(\varrho)$. The L_i's are the required π_i.

In the n-dimensional case, employing the same notations as before, we conclude again that $D_2 \subset \complement W$, but there are now infinitely many directions η', for which $\alpha(\eta, \eta') = C\,|\log \varrho|^{-\varepsilon/8}$ and for each one of them we may find an

$$X_{\eta'} \in D(X_0, \varepsilon_0, \varrho/2, \eta') \cap W = D_{\eta'} \cap W \subset D_{\eta'} \setminus D_2.$$

The set $D_{\eta'} \setminus D_2$ is no longer an arc, but it is located "on the same side of η as η'" in the sense that, if φ denotes projection onto the normal plane to η

$$\alpha(\varphi(X_{\eta'} - X_0), \varphi(\eta' - \eta)) < \pi/2 + C\,|\log \varrho|^{-\varepsilon/8}$$

(See Lemma A5, (a)). In turn, by Corollary 2 to each $X_{\eta'}$ we can associate a plane of support for $\Gamma(\complement W \cap B_\varrho(X_0))$, $\pi_{\eta'}$ with $d(X_{\eta'}, \pi_{\eta'}) < \gamma_{\varepsilon/2}(\varrho)$. Therefore, by a small motion of the $\pi_{\eta'}$, we can find planes $\bar{\pi}_{\eta'}$, satisfying the conditions of Lemma A5 (b). But any point X, of $\complement W \cap B_\varrho(X_0)$ verifyes $d(X, \Sigma) < C\,|\log \varrho|^{-\varepsilon/8}$, and therefore a further translation of the planes H_1 and H_2 determined in Lemma A5 (b) by $C\,|\log \varrho|^{-\varepsilon/8}$ gives us the required planes. $\|$

It is interesting to notice that in the preceeding lemma we only made use of the "almost convexity" of our coincidence set. To rule out the possibility of the situation considered in it, we have to make use again of the properties of the free boundary.

LEMMA 4. *Given ε_0, there exists a $\varrho_0(\varepsilon_0)$ such that, if $\varrho < \varrho_0(\varepsilon_0)$ and $D(X_0, \varepsilon_0, \varrho, \eta) \subset \complement W$, then $D(X_0, \varepsilon_0, \varrho/2, \eta') \subset \complement W$ for some η'.*

Note. ε_0 is chosen to be smaller than a fixed multiple of $\pi/2$ depending on the ellipticity of A.

Proof. After an affine transformation we may assume that $A(X_0) = \Delta$ (that is $a_{ij}(X_0) = \delta_{ij}$). Assume for contradiction that the conclusion is false and that $\complement W \cap B_\varrho(X)$, according to the preceeding lemma, is contained between two perpendicular planes π_1, π_2 and also that $d(X_0, X_1) < C\varrho\,|\log \varrho|^{\varepsilon''}$ for some $X_1 \in \pi_1 \cap \pi_2$.

The function $V = (x_1 x_2)/|X|^{n+2}$ is harmonic outside the origin, and after a rotation we may assume that $V(X - X_1)$ is positive in the angle between π_1 and π_2 enclosing $\complement W$.

Let us apply the Green formula to the domain $D_\varrho \cap W$, where

$$D_\varrho = B_{\varrho/2}(X_1) \setminus B_{\gamma_{-\varepsilon/8}(\varrho)}(X_1).$$

From the formula

$$\int_{D_\varrho \cap W} \{V(X-X_1)\,\Delta v - v\Delta V(X-X_1)\}\,dX = \int_{\partial(D_\varrho \cap W)} \{V(X-X_1)\,\partial_\nu v - v\partial_\nu V(X-X_1)\}\,d\sigma,$$

which is easy to verify (cf. [5], Lemma 3), we deduce that

$$\left|\int_{D_\varrho \cap W} V(X-X_1)\right| < C$$

and hence

$$|I| = \left|\int_{D_\varrho \cap \complement W} V(X-X_1)\right| < C.$$

But, the application of Corollary 2 to $D(X_0, \varepsilon_0, \varrho, \eta) \cup \{X_0\}$ implies that the truncated cone of inner radius $\gamma_{-\varepsilon/8}(\varrho)$, outer radius $\varrho - \gamma_{-\varepsilon/8}(\varrho)$ and aperture $\varepsilon_0 - |\log \varrho|^{-\varepsilon/8}$ is contained in $D_\varrho \cap \complement(W)$. Therefore:

$$|I| \geqslant C\varepsilon_0^{n-1}\,|\log(|\log \varrho|)|$$

which contradicts that $|I| < C$. ‖

1.6. Let us reevaluate the situation in geometric terms. To do that, let us introduce the

Definition 1. Given a set $S \subset \mathbf{R}^n$, the minimum diameter of S is the infimum among the distances between pairs of parallel planes enclosing S.

Obviously, min diam (S) = min diam $\Gamma(S)$ and it is proven in the appendix in the last part of Lemma A5, (b) that, if S is convex min diam (S) is proportional to the radius of the largest ball that we can inscribe in S. Therefore, in our particular case of "almost convexity" we may prove that

LEMMA 5. *Let λ be the* min diam. $(\complement W \cap B_\varrho(X_0))$, $(X_0 \in (\overline{\complement W}))$, *and assume that* $\lambda \gg \gamma_{\varepsilon/8}(\varrho)$. *Then we can inscribe a ball in $\complement W \cap B_\varrho(X_0)$ of radius proportional to λ.*

Proof. Since we can inscribe in $\Gamma(\complement W \cap B_\varrho(X_0))$ a ball B of radius proportional to λ, by diminishing the radius of B by $\gamma_{-\varepsilon/8}(\varrho)$ we obtain, according to Corollary 2, the required ball. Hence the two preceding lemmas say that

COROLLARY 3. *There exists a modulus of continuity $\sigma_1(\varrho)$ (as in Notation 4), such that, given a point $X_0 \in F$ either* min diam $(\complement W \cap B_\varrho(X_0) < \sigma_1(\varrho)\varrho$ *or if for some* ϱ_0, min diam $(\complement W \cap B_{\varrho_0}(X_0)) > \sigma_1(\varrho_0)\varrho_0$, *for any* $\varrho < \varrho_0$, *then*

$$\text{min diam}\,(\complement W \cap B_\varrho(X_0)) > C\sigma_1(\varrho_0)\varrho.$$

Furthermore, in the second instance, that is, if min diam $(\complement W \cap B_\varrho(X_0)) > \sigma_1(\varrho_0)\varrho$, then for any point X_1 verifying $d(X_1, X_0) \leqslant \tau$,

$$\min \operatorname{diam} (\complement W \cap B_{\varrho+\tau}(X_1)) > \sigma_1(\varrho_0)\varrho,$$

since

$$B_\varrho(X_0) \subset B_{\varrho+\tau}(X_1)$$

But for a small enough ϱ_1

$$C\sigma_1(\varrho_0)\varrho_1 > 2\sigma_1(2\varrho_1)\varrho_1$$

From which we conclude the

COROLLARY 4. *For any* $X_1 \in B_{\varrho_1}(X_0) \cap F$

$$\min \operatorname{diam} (\complement W \cap B_{2\varrho_1}(X_1)) > \sigma_1(2\varrho_1) 2\varrho_1$$

and hence

$$\min \operatorname{diam} (\complement W \cap B_\varrho(X_1)) > \sigma_1(2\varrho_1)\varrho, \quad \forall \varrho < 2\varrho_1.$$

In particular, if $|A|$ *denotes Lebesgue measure of* A, *then*

$$\frac{|(\complement W \cap B_\varrho(X_1)|}{|B_\varrho(X_1)|} > C\sigma_1(2\varrho_1) > 0$$

$$\forall \varrho < 2\varrho_1 \quad \text{and } \forall X_1 \in B_{\varrho_1}(X_0) \cap F. \qquad (\text{See Lemma } 5.)$$

A particular consequence is that whenever $X_0 \in F$ has positive Lebesgue density δ with respect to $\complement W$, there is a neighborhood of X_0 for which each point $X \in F$ has density $C\delta$ with respect to $\complement W$, uniformly on B_ϱ. (That is, $|\complement W \cap B_\varrho(X)|/|B_\varrho(X)| > C\delta \; \forall \varrho < \varrho_0(X_0)$.)

1.7. The scope of this section is to prove that, if a point, X_0, of F, has positive Lebesgue density for $\complement W$, then F admits a representation as a Lipschitz function in a neighborhood of X_0.

In order to prove this, our first step will be to improve the "almost convexity" of $\complement W$ near X_0, by obtaining new estimates for the v_{tt}. If v_{tt} were harmonic, well known theorems would say that $v_{tt}(X) > -C\varrho^\varepsilon$, $\forall X \in W \cap B_{\varrho_0(X_0)}(X_0)$, where $\varrho = d(X, F)$.

It is not difficult to adapt those theorems to our situation, as it is shown in the lemma below.

LEMMA 6. *If there exists a* ϱ_0 *and a* $K > 0$ *such that*

$$\frac{|\complement W \cap B_\varrho(X)|}{|B_\varrho(X)|} > K > 0 \quad \forall \varrho < \varrho_0$$

then, $v_{tt}(Y) > -C\varrho^\varepsilon$ *for any* Y *such that* $|Y - X| = \varrho < \varrho_0$ *for some* $\varepsilon > 0$ *depending only on* K *and the operator* A.

Proof. We assume that for $|Y-X|<3^{-k}\varrho_0$, $v_{tt}>-M$, and we show that for $|Y-X|<3^{-(k+1)}\varrho_0$, $v_{tt}>\lambda M$ $(\lambda<1)$ (provided that we also assume $M>C(3^{-k+1}\varrho_0)^{1/2}$).

To do so, we first notice that

$$B_{3^{-(k+1)}\varrho_0}(X)\subset B_{2\cdot 3^{-(k+1)}\varrho_0}(Y)\subset B_{3^{-(k+1)}\varrho_0}(X);$$

then we add a correcting factor u:

$$Au=Av_{tt},\ u|_{\partial B_{2\cdot 3^{-(k+1)}\varrho_0}}=0.$$

$A(v_{tt})$ can be prolonged as the second derivative of a $C^{1/2}$ function to the whole ball, see also Lemma A1

$$\sup|u|<C\varrho^{1/2},\quad (\varrho=3^{-(k+1)}\varrho_0).$$

We therefore have

(1) $A(v_{tt}-u)=0$, and $v_{tt}-u>-M-C\varrho^{1/2}$ on $W\cap B_{2\varrho}(Y)$.

(2) $v_{tt}-u>-C\varrho^{1/2}$ on $F\cap B_{2\varrho}(Y)$.

Hence, if we consider

$$h=\min(v_{tt}-u,\ -2C\varrho^{1/2})$$

Lemma A2 applies and we obtain

$$h(Y)>-\alpha(M+C\varrho^{1/2})-(1-\alpha)(C\varrho^{1/2})>-\lambda M,$$

since we are willing to assume $M\gg\varrho^{1/2}$. That is

$$v_{tt}-u>-\lambda M$$

or $v_{tt}>-\lambda' M$ (since $u<C\varrho^{1/2}$).

Now, a standard iterative argument completes the proof. ||

From now on we restrict ourselves to the subball $B_{\varrho_1}(X_0)$, where the hypothesis of Lemma 6 hold.

There, we immediately obtain an improvement of Corollary 2,

COROLLARY 5. *If S is a subset of $\complement W$ with diameter,* diam $(S)<\varrho$, *and X_1 verifies*

$$d(X_1,\ \complement(\Gamma(S)))>C\varrho^{1+\varepsilon/2}$$

then $X_1\in\complement W$. (ε from Lemma 6.)

Notation 5. Similar to our previous notation $\gamma_\alpha(\varrho)$, we use now the notation $\xi_\alpha(\varrho)$ for $C\varrho^{1+\alpha}$, where the constant C may assume different values.

The new estimate for $\complement W$ gives us an interesting asymptotic behavior for $\complement W$ along lines, that says

LEMMA 7. *Assume that $X_0 \in \complement W$ and $X_1 = (X_0 + t_0\eta) \in \overline{W}$ then, for any $t > 2t_0 > 0$,*

$$W \cap D(X_0, K\varrho^{\varepsilon/4}, \varrho, \eta) \neq \emptyset \quad \textit{for a suitable } K.$$

Proof. Suppose the contrary. Applying Corollary 5, inductively on k, if

$$D(X_0, K(2^{-k}\varrho)^{\varepsilon/4}, 2^{-k}\varrho, \eta) \subset \complement W$$

then

$$D(X_0, K(2^{-k}\varrho)^{\varepsilon/4} - C(2^{-k}\varrho)^{\varepsilon/2}, 2^{-k+1}\varrho, \eta) \subset \complement W.$$

If we choose K such that

$$K - C > K2^{-\varepsilon/2}$$

we obtain

$$D(X_0, K(2^{-(k+1)}\varrho)^{\varepsilon/4}, 2^{-(k+1)}\varrho, \eta) \subset \complement W.$$

But then, Corollary 5, gives us a contradiction if we choose k verifying

$$2^{-(k+2)}\varrho \leqslant t_0 \leqslant 2^{-(k+1)}.$$

Now we are in conditions to prove the desired theorem.

THEOREM 2. *If X_0 is a point of positive Lebesgue density for $\complement W$, then in a neighborhood of X_0, F can be represented as the graph of a Lipschitz function.*

Proof. We are going to prove that there exist constants γ and ϱ_2 such that in an appropriate system of coordinates, for all $X \in F \cap B_{\varrho_2}(X_0)$ the truncated cones

$$\Gamma_1 = \{Y: \alpha(Y - X, e_n) < \gamma, \ |Y - X| < \varrho_2\}$$

$$\Gamma_2 = \{Y: \alpha(Y - X, e_n) > \pi - \gamma, \ |Y - X| < \varrho_2\}$$

verify $\Gamma_1 \subset \complement W$, $\Gamma_2 \subset W$. ($e_n = (0, 0, ..., 1)$).

For any $\varrho < \varrho_1$, $\complement W \cap B_\varrho(X_0)$ contains a ball of radius proportional to ϱ, $B_{\theta\varrho}(X_\varrho)$, according to Lemma 5 and Corollary 3.

Let us choose a ϱ_2 small enough as to make $K\varrho_2^{1+\varepsilon/4} \ll \theta\varrho_2$ and a $\varrho_3 \ll \varrho_2$.

In an appropriate system of coordinates, where $X_{\varrho_2} - X_0$ is parallel to e_n assume that the points X and $Y = X + t\eta$, $t > 0$, verify

(a) $X, Y \in B_{\varrho_3}(X_0)$

(b) $\alpha(\eta, e_n) \ll \theta$

(c) $X \in (\overline{\complement W})$, $Y \in \overline{W}$.

Then, according to Lemma 7,

$$D \cap W = D(X, K|X_{\varrho_2} - X_0|^{\varepsilon/4}, |X_{\varrho_2} - X_0|, \eta) \cap W \neq \emptyset,$$

but, because of (a), (b) and the way we choose ϱ_2 and ϱ_3, $D \subset B_{\theta\varrho_3}(X_{\varrho_2}) \subset \complement W$. This is a contradiction and completes the proof.

1.8. Our next step is to prove that if the free boundary is Lipschitz, it is really a C^1 surface and the second derivatives of v are continuous up to it. More precisely, let us fix a system of coordinates $(x_1, ..., x_{n-1}, y)$. We will, from now on, denote by X a point in $\mathbf{R}^{n-1}$ and (X, y) a point in $\mathbf{R}^n$. The techniques here employed were used by the author in [4].

THEOREM 3. *Assume that on a cylinder $C(\varrho_0, \delta_0) = \{X, y: |X| < \varrho_0, |y| < \delta_0\}$, the free boundary, may be expressed $F \cap C(\varrho_0, \delta_0) = \{X, y: y = g(X)\}$ where g is a Lipschitz function, and $W = \{X, y: y < g(X)\}$. Without loss of generality, assume also $g(0) = 0$, $|g(X)| < \delta_0/2$. Then, for any subcylinder $C(\varrho_1, \delta_0)$, g is a function of class C^1, its modulus of continuity being independent of X_0 and any second derivative of v, v_{ij} is continuous up to $F \cap C(\varrho_1, \delta_0)$.*

Proof. The proof of the theorem is basically divided in two steps: First we bound g below and then above:

LEMMA 8. *At any point $X \in B_{\varrho_1}(0)$, $g(X)$ has a convex cone of tangent rays, $C(X)$, which g approaches by below faster than $\varrho^{1+\varepsilon'}$ (for some ε').*

Proof. Our candidate for a tangent ray is the lim sup over all possible chords in a given direction. In order to show that such a ray is tangent, and that the estimate from below holds, we notice that if $X_t = t\eta$ (η a unit vector in $\mathbf{R}^{n-1}$), whenever $0 < t_1 < t_2/2$,

$$\frac{g(X_{t_1}) - g(0)}{t_1} < \frac{g(X_{t_2}) - g(0)}{t_2} + Ct_2^{\varepsilon/2} \tag{1.8.1}$$

(we are replacing X by 0 for simplicity). In fact, consider the ball

$$B = B_{C''t_2^{1+\varepsilon'}}(X_{t_2}, g(X_{t_2}) + C't_2^{1+\varepsilon'}).$$

Since g is Lipschitz, for C'/C'' big enough, $B \subset \complement W$. On the other hand, if

$$\frac{g(X_{t_1}) - g(0)}{t_1}$$

does not satisfies the inequality for a big enough C, the set of points X_ϱ of Lemma 7 applied to $(0, g(0))$ and $(X_{t_1}, g(X_{t_1}))$ would hit B or pass over it. In any case a contradiction. A similar argument shows that such a cone is convex (see [4], lemma). ||

Proof of Theorem 3. The proof of the theorem follows now from an argument similar to that of Lemma 4. Let us present the proof for the point $(0, g(0)) = (0, 0)$ supposing that

$A(0)u=\Delta u$. If π is a plane of support for the convex cone C at (0,0), with normal ν and $V_{\nu y}$ is the second derivative of $V=1/|X|^{n-2}$, it is easily seen from the properties of v that

$$\left|\int_{C_\varepsilon \cap W} V_{\nu y}\,\Delta v\right| < K,$$

where $C_\varepsilon = C(\varrho_1, \delta_0) - C(\varepsilon_0, \delta_0)$ or also

$$\left|\int_{C_\varepsilon \cap W} V_{\nu y}\right| < K.$$

(Since $|\Delta v(X)-\Delta v(0)| \leqslant |\Delta v(X)-Av(X)| + |Av(X)-Av(0)| \leqslant C|X|$). Then $|\int_{C_\varepsilon \cap CW} V_{\nu y}|$ $<K$ and if $C'_\varepsilon = C_\varepsilon \cap \{(X, y) \text{ below } \pi\}$, since

$$\left|\int_{C'_\varepsilon \cap CW} V_{\nu y}\right| < K'$$

(because of the uniform approximation by below to the convex cone proved in Lemma 8) we obtain

$$\left|\int_{(C_\varepsilon - C'_\varepsilon)\cap W} V_{\nu y}\right| < K.$$

Integrating in y, since V_ν vanishes on $\pi-(0, 0)$ we obtain

$$\left|\int_{\delta<|X|<\varrho_0} \frac{d((X, g(X)), \pi)}{|X, g(X)|^n}\, dX\right| < C.$$

Let us show that for ϱ small enough we must have $d(X, g(X), \pi) < C|X|\,|\log|X||^{-\varepsilon}$. In fact, if it wasn't so, and we wrote $d(X, g(X))$ in polar coordinates $\delta(\varrho, \sigma)$ we would obtain

$$|I| = \left|\int_\Sigma \int_{\varepsilon<\varrho<\varrho_0} \frac{\delta(\varrho, \sigma)}{\varrho^n}\, \varrho^{n-2} d\varrho\, d\sigma\right| < C.$$

If $\delta(\varrho_1, \sigma) > C\varrho_1\, |\log \varrho_1|^{-\varepsilon}$, using that g is Lipschitz $\delta(\varrho_1, \sigma) > C\varrho_1\,|\log \varrho_1|^{-\varepsilon}$ for σ in a solid angle of apperture $C|\log \varrho_1|^{-\varepsilon}$, and applying (1.8.1) of Lemma 8, once again for any ray in that solid angle, any $\varrho > 2\varrho_1$, $\delta(\varrho, \sigma') > C''\varrho\,|\log \varrho|^{-\varepsilon} - C\varrho^{1+\varepsilon}$. But if ϱ_1 is taken small enough, the integral I will surpass any constant C, a contradiction.

This proves that g is C^1. To show that v_{ij} is continuous up to G we will prove that if $(X_0, y_0) \in F$ and ν is the inner normal to F at $(X_0, y_0)\, v_{ij}$ converges to

$$\frac{(Av)(X_0, y_0)\langle i, \nu\rangle\langle j, \nu\rangle}{a_{ljm}\nu_l \nu_{jm}}.$$

Since this last function is continuous on $\partial\Omega$, our result will follow. To prove this, on the other hand, it is enough to prove that if $\langle i, \nu\rangle = 0$, $v_{ij} \to 0$. This we prove by means of the Harnack inequality. Assume that we have been able to prove that $|v_{ij}| < M$ for any (X, y)

such that $d((X, y), (X_0, g(X_0))) < \varrho$ and assume that for any $(X, g(X))$, such that $d((X, g(X)), (X_0, g(X_0))) < \varrho$, $\alpha(\nu_{X, g(X)}, \nu_0) < \varepsilon$. Take (X, y) verifying

$$|(X, y) - (X_0, g(X_0))| < \varrho/2$$

and $B_\varrho(X, y)$ the biggest ball contained in W. Chose X_1 such that $(X_1, g(X_1)) \in \partial B_\varrho(X, y) \cap F$. Then $\alpha((X_1, g(X_1) - (X, y)), \nu_0) < \varepsilon$. We integrate v_{ij} along a segment I, in the i-direction, with length $\delta^{1/2}\varrho'$ and such that, $d(I, (X_1, g(X_1))) < C\delta\varrho'$ and $d(I, \partial B_{\varrho'}) > C'\delta\varrho'$. (If $\varepsilon \ll \delta$ this is possible) and if $\delta^{1/2} = CM$ we obtain as in Theorem 1, $|w_{ij}(X, y)| < M - CM^{2m}$. (Note that $|\int_I w_{ij}| < |w_j(X_1)| + |w_j(X_2)| < C\delta\varrho$ if ε is much smaller than δ.) ‖

Remark. To this situation, (free boundary of class C^1 and continuous second derivatives), applies the result of David Kinderlehrer and Louis Nirenberg [12] asserting that the free boundary is as smooth as the data in a neighborhood of the point under consideration (linear or nonlinear case), and analytic if the data are analytic.

Comment. It is our opinion that a much more accurate description of the exceptional set should be possible. For instance, improvements can be done when some topological information is available, as in the work by Friedman [8], or the elasto plasticity problem as treated by Brezis–Stampacchia in [3] (see also [6]).

2. The parabolic case

In the parabolic case, we will limit ourselves to a localized version of the one phase Stefan problem, as treated by G. Duvaut [7] and A. Friedman and D. Kinderlehrer [9]. They found, by the methods of variational inequalities a solution to the problem in different geometric settings. The local properties of interest to us possessed by the solution in question, in a neighborhood of the free boundary could be summarized as follows:

2.1. Given are:

(PH1) *A domain* $W \subset R^n \times [t_0, t_1]$, *$W$ is known to be increasing in time (that is if* $(X, t) \in W$, *then* $X, t' \in W$, $\forall t < t' < t_1$).

(PH2) *A function v, with bounded second spatial derivatives* ($v \in C_X^{1,1}(W)$) *and bounded time derivative* ($v \in \Lambda_t^1(W)$), $v_t \geq 0$ *on W, satisfying* $Hv = \Delta v - v_t = 1$.

Remark. The $C_X^{1,1}$ character of v was pointed out to me by D. Kinderlehrer. The proof follows the lines of the work by H. Brezis and D. Kinderlehrer [2]. A penalization function as in [9] is used.

(PH3) *On an open portion $\partial_1 W$ of ∂W, v and $\nabla_X v$ vanish and if we prolong v_t to $\complement W$ by zero, across $\partial_1 W$, $Hv_t = \Delta v_t - v_{tt} \geqslant 0$ in the sense of distributions.* (Although this is not explicit in [9], v_t is proven, there, to be the limit of a sequence of functions v_{ε_t} with that property.)

As in the elliptic case, we will restrict ourselves to a portion F of $\partial_1 W$ that stays far from $\partial W \setminus \partial_1 W$. This part of the work can clearly be divided into two parts: First we prove spatial regularity by using the elliptic techniques. For temporal regularity a further effort is required.

2.2. The equivalent of Lemma 1, would be

LEMMA 9. *Let u be a non-negative function on the cylinder*

$$\Gamma_\varrho = \{|X| \leqslant \varrho, \quad 0 < t \leqslant C\varrho^2\},$$

$u \in C_X^{1,1}(\Gamma_\varrho) \cap \Lambda_t^1(\Gamma_\varrho)$, and assume that for some X_0, $|X_0| \leqslant \varrho$, $u(X_0, 0) = 0$, $\nabla_X u(X_0, 0) = 0$ then given a pure second spatial derivative u_{ii} and a $\delta < 1/2$, $\exists X_1$, with $|X_1| < \varrho(1-\delta/2)$, such that

$$u_{ii}(X_1, (\delta\varrho)^2) \geqslant -C\delta^{1/2}.$$

Proof. We first go inside the ball a distance $\delta\varrho$ (if necessary), where u satisfies $u(X_1, 0) \leqslant C(\varrho\delta)^2$, then we go up a distance $(\delta\varrho)^2$ and there u still satisfies

$$u(X_1, (\delta\varrho)^2) \leqslant C^1(\varrho\delta)^2.$$

Finally, we observe that, since u remains positive in a ball (in space) of radius $(\varrho\delta)$ around $(X_1, (\delta\varrho)^2)$, $|\nabla u| < (\delta\varrho)$. The proof follows now that of Lemma 1.

THEOREM 4. *If v_{ii} denotes a pure second spatial derivative of v, $v_{ii}(X, t) > -C|\log \varrho|^{-\varepsilon}$, where ϱ denotes the parabolic distance to F.*

Proof. The proof is the same as that of Theorem 1, using now the Harnack inequality of Lemma A3.

2.3. Let us denote by W_{t_0} the restriction of W to a fixed time t_0; since on W_{t_0}, $\Delta v = 1 + v_t > 1$, Lemma 2 is still valid (replacing W by W_{t_0} and F by F_{t_0}) (see also the last section of [5]):

LEMMA 10. *Let (X, t) be given and assume that $d((X,t), F_t) < \varrho_0/2$. Consider a ϱ such that $0 < \varrho < \varrho_0/2$. Then*

$$\sup_{W_t \cap \partial B_\varrho(X)} v \geqslant C\varrho^2. \qquad \|$$

It follows then that Corollaries 1 and 2 can also be transplanted to W_{t_0} and F_{t_0}, since they depended only on the conclusions of Theorem 1 and Lemma 2.

We restate them for completeness.

COROLLARY 6. *Let $X \in \overline{W}_{t_0}$ and assume that*

(a) $v(X, t_0) = \varrho_0^2$

(b) $d(X, F_{t_0}) \leqslant \varrho_0^{\varepsilon'}$, $(0 < \varepsilon')$.

Then if we choose the constant $C(\varepsilon, \varepsilon')$ in γ sufficiently small (ε now as in Theorem 4).

$$HB_{\gamma_{\varepsilon/2}}(X, \nabla v) \subset W_{t_0}.$$

COROLLARY 7. *Let $S \subset (\complement W)_{t_0}$ have diameter ϱ_0 and let $\Gamma(S)$ denote its convex envelope. If $d(X, \complement\Gamma(S)) > \gamma_{-\varepsilon/4}(\varrho_0)$, then $X \in (\complement W)_{t_0}$.*

2.4. Of course, we could also reproduce Lemma 3, but since in Lemma 4 we must now replace V by a parabolic singular integral and to infer that this integral diverges we must go backwards in time, we need not only to force W_{t_0} to stay between π_1 and π_2 but we also need, in an appropriate system of coordinates

$$(\complement W)_t \subset \{t - t_0 \leqslant Ax_n - B\varrho^2,\ 0 < t_0 - t < \varepsilon_0\}.$$

To accomplish this we employ these basic ideas: recalling the geometric configuration from Lemma 3 (with W_{t_0} instead of W), the disc $D_1 = D(X_0, \varepsilon_0, \varrho, \eta)$ as well as the point X_0 are contained in $(\complement(W))_t$ for $t < t_0$ due to the increasing nature of W. Therefore, to show that $(\complement W)_t$ does not grow too much as t decreases from the time t_0 it would suffice to assess its behavior half way between D_1 and X_0, in particular, near X_η (cf. Lemma 3).

For that we first need an estimate in v_t.

LEMMA 11. *Assume that*

$$D(X_0, \varepsilon_0, \varrho, \eta) \subset (\complement W)_{t_0}$$

and $(X_0, t_0) \in F$. Then, there exists a δ such that if $(X, t) \in W$ $(t < t_0)$ and

$$d(X, D(X_0, \varepsilon_0, \varrho/2, \eta)) < \nu\varrho$$

(with $C\,|\log \varrho|^{-\varepsilon/8} < \nu \ll 1$).

Then $0 \leqslant v_t(X, t) < C\nu^\delta$.

Proof. For any X as above

$$\frac{|B_{4\varepsilon\varrho}(X) \cap \complement W|}{|B_{4\nu\varrho}(X)|} > \lambda_0 > 0$$

because of the "almost convexity" of $\complement W$ in Corollary 7.

Hence Lemma A4 can be applied inductively to v_t for those ν of the form $\nu = 8^{-k}$ and the conclusion follows. ||

COROLLARY 8. *Assume that* $X \in W_{t_0}$ *and*

$$d(X, D(X_0, \varepsilon_0, \varrho/2, \eta)) < C\varrho\, |\log \varrho|^{-\varepsilon/8}.$$

Then

$$B_{\nu\varrho}(X) \cap W_t \neq \emptyset$$

for any $0 < t_0 - t < C(\nu\varrho)^2/\nu^\delta$.

Proof. There exists a $Y \in B_{\nu\varrho}(X) \cap W_{t_0}$ such that $v(Y, t_0) > C(\nu\varrho)^2$. Since $v_t(Y, t) < \nu^\delta$ for $t < t_0$ the corollary follows. ||

Now we reformulate Lemma 3.

LEMMA 12. *Let* A, B, ε_0 *be positive numbers,* $1/2 > B/A \gg \varepsilon_0$, *and assume that* $X_0 \in F_{t_0}$, $D(X_0, \varepsilon_0, \varrho, \eta) \subset (\complement W)_{t_0}$ *and that for any* η'

$$D(X_0, \varepsilon_0, \varrho/2, \eta') \cap \overline{W}_{t_0} \neq \emptyset.$$

Then, there exists a $\varrho_0 = \varrho_0(\varepsilon_0, A, B)$ *such that for any* $\varrho < \varrho_0$, *in an appropriate system of coordinates*

(a) $|X_0| < \varrho\, |\log \varrho|^{-\varepsilon/8}$

(b) *For* $0 < t_0 - t < K\varrho^2$, $K = K(\varepsilon_0, B, A) \ll 1$ *a constant*

$$(\complement W)_t \cap B_\varrho(0) \subset \{t - t_0 < Ax_n^2 - Bx_{n-1}^2\}.$$

Proof. Let us begin the proof of this lemma, for $(\complement W)_{t_0}$ at the point of Lemma 3 where we construct the points $X_{\eta'}$, and let us consider a t such that $|t_0 - t| < \nu^{2-\delta}\varrho^2$ with ν to be choosen. According to Corollary 8, there is a $Y_{\eta'}$ such that $(Y_{\eta'}, t) \in \overline{W}_t$ and $|Y_{\eta'} - X_{\eta'}| < [\nu^\delta(t_0 - t)]^{1/2}$.

Therefore there is a plane of support $\pi^*_{\eta'}$ for $\Gamma((\complement W)_t)$, verifying $d(\pi^*_{\eta'}, X_{\eta'}) < C[\nu^\delta(t_0 - t)]^{1/2} < \nu\varrho$.

Since $D(X_0, \varepsilon_0, \varrho, \eta)$ and X_0 must stay on the same side of $\pi^*_{\eta'}$,

$$B(X_0, \varrho) \cap \pi^*_{\eta'} \subset \{X : d(X, \bar{\pi}_{\eta'}) < C[\nu^\delta(t_0 - T)]^{1/2}\}$$

where $C = C(\varepsilon_0)$.

Therefore

$$B_\varrho(X_0) \cap (\complement W)_t \subset \{X : d(X, \Gamma(H_1^+ \cup H_2^+)) < C[\nu^\delta(t_0 - t)]^{1/2}\}.$$

(where H_i^+ denotes the half plane bounding the acute angle of Lemma 3). If ν is taken small enough the proof is complete. ||

2.6. We are now in conditions to prove

LEMMA 13. *Hypothesis as in Lemma* 12, $\varepsilon_0 \ll \pi/2$ *then there exists a* $\varrho_0 = \varrho_0(\varepsilon_0)$ *such that if* $\varrho < \varrho_0$, *then for some* η'
$$D(X_0, \varepsilon_0, \varrho/2, \eta') \subset [\complement(W)]_{t_0}.$$

Proof. As in Lemma 4 we argue by contradiction about the convergence of a certain integral. We consider a second derivature of the fundamental solution

$$H = F_{22} - 2F_{11} = \left[\frac{4A^2(x_2^2 - 2x_1^2) + 2At}{t^2}\right]\frac{1}{t^{n/2}}\,e^{A|X|^2/t}.$$

In the system of coordinates of Lemma 13, after an eventual rotation, we obtain that for any $-K\varrho^2 < t < 0$

$$H(X, t_0 - t)\big|_{(CW)_t \cap B_\varrho(0) \setminus B_{\gamma_{-\varepsilon/8}(\varrho)}(0)} \geqslant \frac{C|X|^2}{(t_0 - t)^{n/2+2}}\,e^{-B|X|^2/(t_0 - t)},$$

(because of Lemma 12), and also $(CW)_t$ contains a truncated cone of exterior radius ϱ, interior radius $\gamma_{-\varepsilon/8}(\varrho)$ and aperture $\varepsilon_0 - |\log \varrho|^{-\varepsilon/8}$.

Hence as, in Lemma 4, we obtain on one hand, that

$$|I| = \left|\int_0^{K\varrho^2}\int_{W_t \cap B_\varrho(0) \setminus B_{\gamma_{-\varepsilon/8}}(0)} H(X, t_0 - t)\,dX\,dt\right| < C$$

because of Green's formula applied to H and v and on the other hand that $|I|$ surpasses any constant when ϱ is taken small enough. ||

From Lemmas 12 and 13 we are able to obtain the same conclusions that we discussed in Corollary 3 and 4 for the elliptic case, i.e., that if $X_0 \in F_{t_0}$ either

$$\frac{\min \operatorname{diam}((CW)_{t_0} \cap B_\varrho(X_0))}{\varrho} < \sigma(\varrho)$$

for any ϱ or, in case X_0 is a point of positive Lebesgue density for F_{t_0}, any other point in $F_{t_0} \cap B_{\varrho_1}(X_0)$ also has that property. But a simple observation will allow us to extend this property to a neighborhood in time of (X_0, t_0).

Remark. If $B_\varrho(X_0) \subset [\complement(W)]_{t_0}$ then for an appropriate constant C

$$B_{\varrho - Ch} \subset [\complement(W)]_{t_0 + h^2}.$$

Proof. Assume that $Y_0 \in (B_{\varrho - Ch}(X_0)) \cap (\overline{W})_{t_0 + h^2}$ then

$$\sup_{B_\varrho(X_0)} v(Y, t_0 + h^2) > \bar{C}h^2$$

(by Lemma 10).

Since v_t is bounded,

$$\sup_{B_\varrho(X_0)} v(Y, t_0) > 0$$

if the constants are properly chosen. ||

COROLLARY 9 (see Corollary 3). *There exists a modulus of continuity $\sigma_1(\varrho)$ (as in Notation 4) such that given a point $X_0 \in F_{t_0}$ either* min diam $(\complement W)_{t_0} \cap B_\varrho(X_0) < \sigma_1(\varrho)\varrho$ *or, if for some ϱ_0,* min diam $(\complement W)_{t_0} \cap B_{\varrho_0}(X_0) > \sigma_1(\varrho_0)_{\varrho_0}$, *then there exist constants $C_i = C_i(\varrho_0)$ such that for any $t \geq t_0$, for any $\delta \leq \varrho_0/2$*

$$\text{min diam } ((\complement W)_t \cap B_{\varrho_0}(X_0) \geq C_1[\sigma_1(\varrho_0) - C_2(t-t_0)^{1/2}]\varrho.$$

2.7. In particular, Corollary 9 tells us that

COROLLARY 10. *If X_0 is a point of positive Lebesgue density for $(\complement W)_{t_0}$ then there exist constants λ, ϱ_2 and a neighborhood*

$$Q = B_{\varrho_1}(X_0) \times [t_0 - \varepsilon, t_0 + \varepsilon]$$

such that for any $(X, t) \in Q \cap F$, for any $\varrho < \varrho_2$

$$\frac{|C(W)_t \cap B_\varrho(X)|}{|B_\varrho(X)|} > \lambda > 0. \qquad ||$$

This corollary allows us to easily reduce the problem of space regularity for $F_t \cap Q$ to the elliptic theory as soon as we make the following remark.

Remark. (a) There exists a δ, $0 < \delta < 1$, such that for any $(X, t) \in Q$, $0 \leq v_t(X, t) \leq [d(X, F_t)]^\delta$.

(b) In particular, $\Delta v|_{W_t}$ is of class $C^{\delta/4}$.

Proof of remark. To prove (a) we apply iteratively Lemma A4 to the points (X, t) verifying

$$d(X, F_t) < 2^{-k}\varrho_1.$$

To show (b) we use Schauder's estimates to prove that v_t is Hölder continuous. ||

Furthermore, Theorem 2 and 3, which, according to the remark, hold now for each $F_t \cap B_{\varrho_2}(X_0)$ ($\varrho_2 \ll \varrho_1$, ϱ_1 as defined in Corollary 10, $|t-t_0| < \varepsilon$) can be done uniform in time.

THEOREM 5. *Assume that X_0 is a point of F_{t_0} of positive Lebesgue density for $(\complement W)_{t_0}$ then, there exist constants $\bar{\varepsilon}$, ϱ_2, k, and a system of coordinates $X=(x_1, ..., x_n)$ on which*

$$F \cap \Phi = F \cap \{X: |(x_1, ..., x_{n-1})| < \varrho_2, |x_n| < k, |t-t_0| < \bar{\varepsilon}\}$$

can be represented as the graph of a function

$$x_n = g(x_1, ..., x_{n-1}, t)$$

where g is (uniformly in t) of class C^1 on the space variables $x_1, ..., x_{n-1}$ and $C^{1/2}$ on the t-variable and $W \cap \Phi = \{x_n \leqslant g(x_1, ..., x_{n-1}, t)\} \cap \Phi$.

Proof. The only new assertion in this theorem is the fact that the system of coordinates can be chosen to be the same for a whole interval of t's around t_0 and the t-regularity. To verify that, we must simply go back to the proof of Theorem 2 and notice that the ball $B_{\theta\varrho_2}(X_0)$ can be taken to be the same for those values of t close enough to t_0 because of the remark after Lemma 13. The Hölder continuity in t follows also from the remark after Lemma 13. ||

2.8. From now on we restrict ourselves to a subneighborhood of Φ (as determined in Theorem 5). We denote a point in $\mathbf{R}^n$ by (X, y) where $X=(x_1, ..., x_{n-1})$ and $y=x_n$.

We want now to obtain further regularity in time.

The idea is to use v_y as a barrier for v_t and obtain the boundedness of v_{ti} (i a direction in $\mathbf{R}^n$) from that of v_{yi} and then, that of v_{tt} from that of v_{yt}.

First we make the following observation.

LEMMA 14. *There exists a neighborhood of*

$$(X, g(X, t), t)$$

on which $v_{yy} > \lambda_0 > 0$. In particular, $-v_y(X, g(X, t)-\delta, t) > \lambda_0 \delta$.

Proof. On F_t, according to Theorem 3, $v_{yy} = (y, \nu)^2 > \lambda > 0$ where ν is the normal vector to the surface F_t. ||

Notation 6. $\Gamma_{\varrho,\varepsilon}(X_0, t_0)$ will denote the cylinder

$$\{(Y, s): |Y - X_0| < \varrho, t_0 - \varepsilon < s < t_0\}.$$

We are now in the position to prove

LEMMA 15. *There exist constants C_i such that in a neighborhood of F*

$$v_t(X, y, t) < -C_1 v_y(X, y, t) + C_2 d((X, y), F_t)^2.$$

Proof. Fix $(X_0, y_0) \in F_{t_0}$ and for $t_0 - \varepsilon < t < t_0$, $|(X, y) - (X_0, y_0)| < \varrho$, compare the two solutions of the heat equation, v_t and $w = -Av_y + [(2n+1)^{-1}(|X - X_0|^2 + (y - y_0)^2 + (t_0 - t) - v]$ in the boundary of the set

$$[W \cap \Gamma_{\varrho,\varepsilon}((X_0, y_0), t_0)].$$

Clearly v_t is bounded in $(\partial\Gamma_{\varrho,\varepsilon}) \cap W$ and vanishes in $F \cap \Gamma_{\varrho,\varepsilon}$, and w is non-negative in $F \cap \Gamma_{\varrho,\varepsilon}$, and strictly positive in $(\partial\Gamma_{\varrho,\varepsilon}) \cap W$ if A is chosen big enough. Therefore, multiplying w by a suitable constant we obtain the desired result. ‖

The preceeding lemma shows that when considered in the appropriate domain of definition D_h, the space h-incremental quotients $\Delta_h v_t$ of v_t remain bounded by those of w in that part of ∂D_h that arises from F. A localization argument will give us the following theorem.

THEOREM 6. *The derivatives $v_{t,i}$ and v_{tt} are all bounded.*

Proof. Let φ be a C_0^∞ function with support

$$S \subset \Gamma_{\varrho,\varepsilon+1}(X_0, y_0, t_0 + 1)$$

and $\varphi \equiv 1$ in a neighborhood of $(X_0, y_0, t_0) \in F$. Then

$$H(\varphi v_t) = v_t(H\varphi) + (\nabla\varphi)\nabla v_t.$$

Since $v_t \in C^\varepsilon$, we can construct, just by convolving with the fundamental solution, a function $w \in C^{1,\varepsilon}_{(X,y)}$ in $[\Gamma_{\varrho,\varepsilon+1}(X_0, y_0, t_0 + 1)]$ and satisfying

$$Hw = H(\varphi v_t).$$

Hence the incremental quotients (in space), $\Delta_h(w - \varphi v_t)$ are uniformly bounded at the boundary of $\Gamma_{\varrho,\varepsilon}(X_0, y_0, t_0)$ and in particular ∇v_t is bounded in a neighborhood of (X_0, y_0, t_0).

Now we look back at $H(\varphi v_t)$, with a φ having smaller support, and notice that it is bounded in a neighborhood, Γ' of (X_0, y_0, t_0). Hence

$$w \in (C^{1,\alpha}_{X,y} \cap C^\alpha_t)(\Gamma') \quad \forall \alpha < 1$$

which tells us that $v_t \in C^\alpha_t(W \cap \Gamma')$, since the incremental quotients in t, $\Delta_h v_t$ are uniformly bounded on $F \cap \Gamma_1$ by $\Delta_h v_y$.

Therefore, for any $\varepsilon > 0$, they grow at most like

$$\frac{C_\varepsilon}{[d((X, y), \Gamma_1)]^\varepsilon}$$

on $(\partial\Gamma_1)$ when approaching F because of Schauder's inequality. If we represent $\Delta_h v_t$ by their boundary values on $W \cap \Gamma_1$ we obtain that in a subneighborhood of (X_0, y_0, t_0), $v_{t,t}$ is bounded. $\|$

COROLLARY 11. *If $B_\varrho(X, y) \subset (\complement W)$, then for a suitable C, $B_{\varrho-Ch} \subset (\complement W)_{t+h}$. In particular $g \in \Lambda_t^1$ (is Lipschitz in time).*

Proof. See the proof of the remark after Lemma 13.

2.9. We need now to use the a.e. existence of non-tangential limits to prove the next lemma. For that, we refer to J. Kemper [10].

LEMMA 16. *Let v_{jj} denote any pure second directional derivative in space and time (that is j is a unit vector in $\mathbf{R}^{n+1}$). Then $v_{jj}(X, y, t) > -C\varrho^\varepsilon$ for some $\varepsilon > 0$, where $\varrho = d((X, y, t), F)$.*

Proof. Since v_{jj} is bounded, and $Hv_{jj} = 0$, it would be enough to show that if f_{jj} denote the $L^\infty(\mu)$ boundary values of v_{jj} on F (μ = caloric measure), $f_{jj} \geqslant 0$ a.e., (see [10]).

In order to do that, let $-M$ be the essential infimum of f_{jj} on F, and K any compact subset of F. Let

$$D_h = \{(X, y, t) \colon (X, y, t) + \lambda j \in W, \ \forall 0 < \lambda < h\}$$

and on D_h, consider the two functions

$$F_h^1 = \frac{2}{h^2} \int_0^h \int_0^s v_{jj}((X, y, t) + rj)\, dr\, ds$$

$$F_h^2 = \frac{2}{h^2} \int_0^h \int_s^h v_{jj}(X, y, t) + rj)\, dr\, ds.$$

Clearly $HF_h^i = 0$, and F_h^i converges to v_{jj} on W when $h \to 0$. Since

$$v_{jj}(X, y, t) > -M - C_K \varrho_K^\varepsilon$$

(where $\varrho_K = d((X, y, t), K)$.

$$F_h^i > -M - C_K \varrho_K^\varepsilon \quad (i = 0, 1).$$

But if we call

$$\partial_1 D_h = \{(X, y, t) \in \partial D_h, \ (X, y, t) + \lambda j \in K \quad \text{for some } \lambda, \ 0 \leqslant \lambda \leqslant h\},$$

and for $i = 0$ or 1, $F_h^i > -\frac{3}{4}M - C_K h^\varepsilon$, because, if, for instance, $\lambda \leqslant h/2$ in the definition of $\partial_1 D_h$,

$$\int_\lambda^h \int_\lambda^s v_{jj}((X, y, t) + rj)\, dr\, ds = v((X, y, t) + hj) \geqslant 0.$$

Therefore

$$(F_h^1+F_h^2)\,|_{\partial_1 D_h} \geqslant -\tfrac{7}{4}M - C_K h^\varepsilon$$

that is,

$$F_h^1+F_h^2 \geqslant -\tfrac{7}{4}M - C_K(\varrho_K^\varepsilon + h^\varepsilon)$$

But $F_h^1+F_h^2$ converges to $2v_{jj}$, and hence,

$$\operatorname*{ess\,inf}_K v_{jj} \geqslant -\tfrac{7}{8}M,$$

a contradiction. ||

Corollary. *At any point X, t, $g(X,t)$ has a convex cone of tangents $C(X,t)$, which g approaches by below faster than $\varrho^{1+\varepsilon}$.*

Since g is C_X^1, $C(X,t)$ is composed of two hyperplanes.

Proof. That of Lemma 8 (see also Lemma 7). ||

We have now all the necessary tools to prove

Theorem 7.

(a) *$g(X,t)$ is of class $C^1_{X,t}$*

(b) *all second derivatives of v are continuous up to F*

(c) $\langle(\nabla_{X,y}v_t)\cdot\nabla_{X,y}(y-g(X,t)\rangle = -D_t(y-g(X,t))$ *on F.*

Remark. Part (c) asserts that the solution is a classical one near F.

Proof. We begin by proving that the tangent cone $C(X_0,t_0)$ is really a tangent plane

$$\pi = \{y = g(X_0,t_0) + \langle A\cdot X - X_0\rangle + C(t-t_0)\}.$$

Let us suppose that for $t=t_0$ the X-tangent plane to $g(X_0,t_0)$ is horizontal and hence the two tangent half planes are given by

$$\{y = a(t-t_0) + g(X_0,t_0),\ t\geqslant t_0\}$$
$$\{y = b(t-t_0) + g(X_0,t_0),\ t\leqslant t_0\},\quad (b\leqslant a)$$

and let σ be the (uniform in t) modulus of continuity of $\nabla_X g$ as a function of X and of all second spatial derivatives v_{ij} as functions of X, y. That is, for any $|t-t_0|<\delta$, $|(X_i,y_i)-(X_0,g(X_0,t_0))|<\delta$, these inequalities hold:

(i) $|\nabla g(X_1,t)-\nabla g(X_2,t)|<\sigma(|X_1-X_2|)$

(ii) $|v_{ij}(X_1,y_1,t)-v_{ij}(X_2,y_2,t)|\leqslant\sigma(|(X_1,y_1)-(X_2,y_2)|)$.

We look now at three points in a vertical segment

$$A = (X_0, g(X_0, t_0) - \alpha, t_0)$$
$$B = (X_0, g(X_0, t_0) - \alpha, t_0 + \beta)$$
$$C = (X_0, g(X_0, t_0) - \alpha, t_0 - \beta).$$

We are going to select $\beta = \sigma^{1/2}(\alpha^{1/2})\,\alpha$. Then we have

$$|v(A) - \tfrac{1}{2}\alpha^2| < \sigma(\alpha)\,\alpha^2.$$

Also, let us decompose

$$g(X_0, t_0 + \beta) = g(X_0, t_0) + a\beta + \varepsilon_1(\beta)$$
$$g(X_0, t_0 - \beta) = g(X_0, t_0) - b\beta + \varepsilon_2(\beta) \quad (\varepsilon_i(\beta) > -C\beta^{1+\varepsilon}).$$

We want to show that $a = b$ and that ε_i are uniformly bounded above by some $\sigma(\beta)$.

We do it as follows: First we notice that

$$|v(B) - 1/2(a\beta + \varepsilon_1(\beta) + \alpha)^2| < C[\sigma(M\alpha) + \sigma^2(\alpha^{1/2}) + \sigma(\alpha^{1/2})\,\alpha]\,\alpha^2$$

(M an absolute constant).

To obtain this estimate we notice that

$$v_{yy}(X_0, g(X_0, t_0 + \beta), t_0 + \beta) = (\cos\theta)^2$$

where θ is the angle between the y-axis and the spatial norm to g at $(X_0, t_0 + \beta)$ and we estimate θ. Since g is increasing in t, we obtain for θ the estimate (D any real number, $-\delta_0 < D < \delta_0$, for some δ_0 depending on δ in the definition of σ)

$$(\operatorname{tg}\theta)\,D - \sigma(|D|)\,|D| + a\beta + \varepsilon_1(\beta) \geqslant \sigma(|D|)\,|D|.$$

In particular, if we make $D = -\beta^{1/2}$, we obtain

$$|\sin\theta| < \sigma(\beta^{1/2}) + \frac{a\beta + \varepsilon_1(\beta)}{\beta^{1/2}}.$$

Using that $a\beta + \varepsilon_1(\beta) < K\sigma^{1/2}(\alpha^{1/2})\,\alpha$ by the Lipschitz character of g, we get

$$v_{yy}(X_0, g(X_0, t_0 + \beta), t_0 + \beta) \geqslant 1 - C[\sigma^2(\beta^{1/2}) + \sigma(\beta^{1/2})^\beta]$$

and the desired estimate on $v(B)$.

A similar estimate may be obtained for $v(C)$. Hence, if we consider the second difference

$$v(C) + v(B) - 2v(A) \geqslant (a - b)\,\alpha^2\sigma^{1/2}(\alpha^{1/2}) + (\varepsilon_1 + \varepsilon_2)\,\alpha - C[\sigma(\alpha^{1/2})]\,\alpha^2.$$

On the other hand,

$$v(B)-v(A)-[v(A)-v(C)] \leqslant [\sup_{I_1} v_t - \inf_{I_2} v_t]\,\sigma^{1/2}(\alpha^{1/2})\,\alpha \leqslant C\sigma(\alpha^{1/2})\,\alpha^2$$

(v_{tt} being bounded). This is possible only if $a=b$ and

$$\varepsilon_i < C\sigma(\alpha^{1/2})\,\alpha = o(\beta).$$

That completes the proof of part (a). Part (b) and (c) follow from the fact that if v_{ij} is a second derivative in X, y and t, and i or j is tangential to g at (X_0, t_0), v_{ij} converges to zero at $(X_0, g(X_0, t_0), t_0)$. This can be done by approximating v_{ij} by

$$F_h = \frac{1}{h}[v_j((X, y, t)+hi) - v_j((X, y, t))$$

as in Lemma 15 and that, if ν is the spatial normal to g, $v_{\nu\nu}$ converges to one. ||

Appendix

In this appendix we collect several lemmas related to the Harnack inequalities. Although probably some of them can be found in the literature, perhaps in more general form, we have been unable to find them.

LEMMA A1. *Let* $A(u)=\sum a_{ij}(X)\partial_i\partial_j u$ *be a second order uniformly elliptic operator, with* $a_{ij}\in C^3(\bar{B}_\varrho(X))$ $(\varrho<1)$ *and* f *a* $C^{1/2}(B_\varrho(X))$ *function, then, there is a* $C^{1/2}(\bar{B}_\varrho(X))$ *function* v *such that*

$$A(v) = f_{ii} = D_{ii} f \quad \text{on } B_\varrho(X)$$

(in the sense of distributions) and

$$v\,|_{\partial B_\varrho(X)} \equiv 0.$$

Furthermore

$$\|v\|_{C^{1/2}(B_\varrho(X))} \leqslant C\|f\|_{C^{1/2}}$$

where C *depends only on the ellipticity and smoothness of* A.

In particular, the following Harnack inequality holds. If w *is a non-negative solution of* $A(w)=f_{ii}$ *on* $B_\varrho(X)$ *and* $Y\in B_\varrho(X)$, $w(X)+C^1\|f\|_{C^{1/2}}\varrho^{1/2} \geqslant \{w(Y)-C^1\|f\|_{C^{1/2}}\varrho^{1/2}\}\times (d(Y,\partial B)/\varrho)^{n-1}$.

Proof. We first solve

$$Au^0 = f,\ u^0\,|_{\partial B} = 0.$$

Such a u^0 is of class $C^{2,1/2}(\bar{B})$:

$$\|u^0\|_{C^{2,1/2}} \leqslant C\|f\|_{C^{1/2}}.$$

(See S. Agmon, A. Douglis, L. Nirenberg [1].) Formally, $Au^0_{ii} = f_{ii} - 2\sum \partial_i(a_{jk})u^0_{jki} - \sum \partial_{ii}(a_{jk})u^0_{jk}$. We now solve

$$Au^1 = 2\sum \partial_i(a_{jk})u^0_{jk}, \quad u^1|_{\partial B} = 0.$$

Formally again

$$Au^1_i = 2\sum \partial_i(a_{jk})u^0_{jki} - \sum \partial_{ii}a_{jk}u^1_{jk}$$

and $u^1 \in C^{2,1/2}$

$$\|u^1\|_{C^{2,1/2}} \leqslant C\|f\|_{C^{1/2}}.$$

Finally we solve

$$A(u^2) = \sum \partial_{ii}(a_{jk})(u^0_{jk} + u^1_{jk})$$

and again

$$\|u^2\|_{C^{2,1/2}} \leqslant C\|f\|_{C^{1/2}}.$$

To obtain the correct boundary value we solve

$$Aw = 0$$

$$w|_{\partial B_\varrho(X)} = u^0_{ii} + u^1_i + u^2$$

and $u^0_{ii} + u^1_i + u^2 - w$ is our solution.

About the Harnack inequality, given w, we consider

$$h = w + v + C\varrho^{1/2}$$

that satisfies $A(h) = 0$, $h \geqslant 0$.

The necessary estimates of the Poisson kernel to obtain the usual Harnack inequality can be found in Serrin [15], section 2 (The Parametrix). ||

The next lemma asserts some super-mean value properties:

Lemma A2. *Let A be as in Lemma A1, and assume that $u \in C(\bar{B}_\varrho(X))$ satisfies $A(u) \leqslant 0$ (in the sense of distributions), then, there exists a continuous function $P_X(Y)$, with*

(a) $0 < \alpha < P_X(Y) < \beta$,

(b) $\int_{B_\varrho(X)} P_X(Y) = |B_\varrho(X)|$

such that

$$u(X) \geqslant \frac{1}{|B_\varrho(X)|}\int_{B_\varrho(X)} u(Y)P_X(Y)\,dY.$$

In particular, if $u \geqslant -M$ on $B_\varrho(X)$ and

$$\frac{|\{u > -M/2 \cap B_\varrho(X)\}|}{|B_\varrho(X)|} > \alpha > 0,$$

$\exists\lambda=\lambda(\alpha)<1$ *such that*

$$u(X)\geqslant -\lambda M.$$

Proof. $P_X(Y)$, for $|X-Y|=\varrho'$, is of course, the normalized Poisson kernel on $\partial B_{\varrho'}(X)$, the estimate (a) can be found in Serrin [15], and the estimate (b) is standard.

The next two lemmas are dedicated to the parabolic case and they are, as Lemmas A1 and A2, a Harnack inequality and a mean value property.

LEMMA A3. *Let u be a positive solution of the equation $Hu=\Delta u-u_t=0$ in the cylinder $\Gamma=\{|X|<1\ 0<t<C\}$ (C depending on the dimension) and let (Y,t) verify $|Y|=1-\delta$ $t=\delta^2$ with $\delta<\min(C/2, 1/2)$. Then $u(0,C)>C^1\delta^{n+2}u(Y,t)$.*

Remark. Estimates for t close to C can also be obtained.

Proof. Let $G_{X,t}(Y,s)$ denote the Green function of the cylinder $\{|Y|\leqslant 1\}$ with pole at X, t. As usual, we must simply estimate $\partial_\nu G_{0,0}$ by below and $\partial_\nu G_{Y,C-t}$ by above on the sides of Γ and $G_{0,0}$ by below and $G_{Y,C-t}$ by above on the top of Γ.

In order to simplify the proof we will avoid normalization factors and assume that the fundamental solution takes the form

$$W(|X|,t)=\frac{1}{t^{n/2}}e^{-|X|^2/t}.$$

Then, for $|X|=1$

$$W_t(1,t)=\frac{-n/2(t+1)}{t^{n/2+2}}e^{-1/t}$$

and if we choose $C<2/n$, W is increasing on t along the sides of Γ. Therefore the function

$$W(|X|,t)-W(1,t_0)<G_{0,0}(|X|,t)$$

for any $|X|<1$, $t\leqslant t_0$ and hence

$$\text{(a)}\qquad \partial_\nu G_{0,0}(1,t_0)>|\partial_\nu W(1,t_0)|=\frac{1}{t_0^{n/2+1}}e^{-1/t_0}.$$

$$\text{(b)}\qquad \frac{1}{C^{n/2}}(e^{-|X|/C}-e^{-1/C})<G_{0,0}(|X|,C).$$

To bound $G_{Y,t}$, we use the Green function of the half space Σ, tangent to Γ along the line $\{X=(1,0,0\ldots)=e_1\}$

$$W^*_{Y,t}(X,s)=\frac{1}{(s-t)^{n/2}}\left[\exp\left(-\frac{|X-Y|^2}{s-t}\right)-\exp\left(-\frac{|X-Y^*|^2}{s-t}\right)\right]$$

(Y^*, the reflection of Y respect to $\partial\Sigma$.)

Since $G_{Y,t} \leqslant W^*_{Y,t}$ we have

(a) On the side of Γ, for $s \geqslant t$

$$\partial_\nu G_{Y,t}(e_1, s) \leqslant \frac{2y_1}{(s-t)^{n/2+1}} \exp\left(-\frac{|e_1 - Y|^2}{(s-t)}\right).$$

Since $|e_1 - Y| > \delta$ and $(s-t) < \delta^2$ we get

$$\partial_\nu G_{Y,t} \leqslant \frac{C}{\delta^{n+1}} e^{-C}.$$

(b) On the top, if $y_n x_n$ are coordinates respect to Σ

$$G_{Y,t}(X, C) < \frac{1}{\delta^n}\left[\exp\left(-\frac{2y_n x_n}{\delta^2}\right) - \exp\left(-\frac{2y_n x_n}{\delta^2}\right)\right] < \frac{C}{d^n}\frac{y_n x_n}{\delta^2} \leqslant \frac{C}{\delta^{n+2}}(1 - |X|)$$

and the proof is complete. ||

LEMMA A4. *Let u be a bounded semicontinuous subcaloric function on*

$$\Gamma = \{X, t\colon |X| < 1, \quad 0 < t < 1\}.$$

That is, assume that for any subcylinder $\Gamma^1 \subset \Gamma$, and any $Y, s \in (\Gamma^1)$,

$$u(Y, s) < \int_{\partial_1 \Gamma^1} G_\nu(Y - X, s - t)\, u(X, t)\, d\sigma_X\, dt + \int_{\partial_2 \Gamma^1} G(Y - X, s - t_0)\, u(X, t)\, dX.$$

Then, if $u \leqslant M$ on Γ and

$$\frac{|\Gamma \cap \{u < M/2\}|}{|\Gamma|} > \lambda_0 > 0,$$

$\exists \gamma = \gamma(\lambda_0) < 1$ such that

$$u(0, 1) < \gamma M.$$

Proof. The proof is an application of Green formula over an appropriate subcylinder of Γ. ||

LEMMA A5. (a) *Assume that $\alpha(\eta, \eta') = \varepsilon$. Then if $X \in D(0, \varepsilon_0, 1, \eta') \setminus D(0, \varepsilon_0 - \varepsilon^2, 1, \eta)$, the following inequality holds*

$$\alpha(\varphi(X), \varphi(\eta')) < \pi/2 + C(\varepsilon_0)\varepsilon \quad (\varepsilon_0 < \pi/20,\ \varepsilon \text{ small}).$$

(b) *Let $\Gamma(\varepsilon_0) \leqslant \{X\colon \alpha(x, \eta) < \varepsilon_0\}$ and assume that, for each η' we have a plane $\pi_{\eta'}$, tangent to $\Gamma(\varepsilon_0)$, with normal ν verifying $\alpha(\varphi(\eta'), \varphi(\nu)) < \pi/2$, then, if $H(\pi_{\eta'})$ denotes the half space containing Γ and $\Sigma = \cap H\pi_{\eta'}$, Σ is contained between two half planes H_1, H_2 forming an angle $\alpha(H_1, H_2) < C\varepsilon_0$ where C depends only on the dimension.*

Proof. (a) The critical case would take place when $X \in [\partial D(0, \varepsilon_0, 1, \eta')] \cap [\partial D(0, \varepsilon_0 - \varepsilon^2, 1, \eta)]$.

In that case, in a suitable system of coordinates

$$\eta = (1, 0, \ldots, 0),$$
$$\eta' = (\cos \varepsilon, \sin \varepsilon, 0, \ldots, 0),$$
$$X = (\cos (\varepsilon_0 - \varepsilon^2), \sin (\varepsilon_0 - \varepsilon^2) \cos \sigma, \sin (\varepsilon_0 - \varepsilon^2) \sin \sigma),$$

because $X \in \partial D(0, \varepsilon_0 - \varepsilon^2, 1, \eta)$, and also, since $X \in \partial D(0, \varepsilon_0, 1, \eta')$,

$$d(X, \eta') = [\cos (\varepsilon_0 - \varepsilon^2) - \cos \varepsilon]^2 + [\sin (\varepsilon_0 - \varepsilon^2) \cos \sigma - \sin \varepsilon]^2 + \sin (\varepsilon_0 - \varepsilon^2) \sin \sigma = (2 \sin \tfrac{1}{2}\varepsilon_0)^2.$$

That is

$$2 - 2 \cos \varepsilon \cos (\varepsilon_0 - \varepsilon^2) - 2 \sin (\varepsilon_0 - \varepsilon^2) \cos \sigma \sin \varepsilon = 2 - 2 \cos \varepsilon_0.$$

Hence $\cos \sigma > -C\varepsilon / \sin (\varepsilon_0 - \varepsilon^2) > -C(\varepsilon_0)\varepsilon$.

(b) Let us first notice that $\Gamma(\varepsilon_0)$ is the cone of largest aperture contained by Σ.

For contradiction assume there is a cone $\Gamma(\eta', \sigma) \subset \Sigma$, with $\sigma > \varepsilon_0$. Fix

$$\eta = (1, 0, \ldots, 0)$$
$$\eta' = (\cos \theta, \sin \theta, \ldots, 0) \quad (0 < \theta < \pi)$$

and let $\delta = (-\sin \varepsilon_0, (\cos \varepsilon_0)\nu)$ be the exterior normal to $\pi_{\eta'}$ (ν a unit vector in $\mathbf{R}^{n-1}$), $\nu = (\nu_1, \nu_2, 0, \ldots, 0)$. Then $\nu_1 \geqslant 0$ and since the vector

$$\gamma = \eta' + (\sin \sigma)\delta \in \Gamma(\eta', \sigma) \subset \Sigma, \quad \text{then } \langle \gamma, \delta \rangle < 0.$$

That is

$$-(\cos \theta) \sin \varepsilon_0 + [\sin \theta \cos \varepsilon_0]\nu_1 + \sin \sigma < 0$$

or

$$\sin \sigma < \sin \varepsilon_0,$$

which proves our observation.

Therefore, if we slice Σ with a plane π passing through η and perpendicular to it we are left with a convex set $\Sigma^* = \Sigma \cap \pi$ such that the maximum ball that it inscribes, $\Gamma(\eta, \sigma) \cap \pi$ has radius $\alpha_0 = \operatorname{tg} \sigma$. We will prove under these circumstances, Σ^* is contained between two parallel planes π_1 and π_2 with $d(\pi_1, \pi_2) < C(n)\alpha_0$.

For $n = 1$, the sphere and the convex set are both a segment and hence $C(1) = 2$. For a general n, assume that Σ^* is bounded, and suppose its diameter D is realized by $(-D/2,$

0, ..., 0) and $(D/2, 0, ..., 0)$. Let E_1 be the coordinate plane $E_1=\{X=x_1=0\}$ and suppose that the closest planes on E_1 that contain $E_1 \cap \Sigma^*$ have the form

$$\{X: x_2=a\} \cap E_1 \quad \text{and } \{X: x_2=b\} \quad (b \leqslant 0 \leqslant a,\ a-b=h).$$

Then, one one hand $\Sigma^* \subset \{X: -2h<x_2<2h\}$ and on the other, by inductive hypothesis, $\Sigma^* \cap E_1$ contains a sphere of radius $h/C(n-1)$ which means that Σ^* contains a sphere of radius $K(h/C(n+1))$, $(0<K<1)$ and that completes the proof.

References

[1]. Agmon, S., Douglis, A. & Nirenberg, L., Estimates near the boundary of solutions of elliptic partial differential equations satisfying general boundary conditions. *Comm. Pure Appl. Math.*, 12 (1959), 623–727.

[2]. Brezis, H. & Kinderlehrer, D., The Smoothness of Solutions to Nonlinear Variational Inequalities. *Indiana Univ. Math. J.*, 23, 9 (March 1974), 831–844.

[3]. Brezis, H. & Stampacchia, C., Sur la regularite de la solution d'inequations elliptiques. *Gull. Soc. Math. France*, 96 (1968), 153–180.

[4]. Caffarelli, L., The smoothness of the free surface in a filtration problem. *Arch. Rational Mech. Anal.* (1977).

[5]. Caffarelli, L. & Riviere, N. M., On the smoothness and analyticity of free boundaries in variational inequalities. *Ann. Scuola Norm. sup. Pisa, Sci. Fis. Mat.*, (*Ser. IV*), 3 (1976), 289–310.

[6]. —— The smoothness of the elastic-plastic free boundary of a twisted bar. *Proc. Amer. Math. Soc.*, 63 (1977), 56–58.

[7]. Duvaut, G., Résolution d'un probleme de Stefan (Fusion d'un bloc de glace a zero degreé). *C.R. Acad. Sci., Paris, sèr A.–B*, 276 (1973), 1461–1463.

[8]. Friedman, A., The shape and smoothness of the free boundary for some elliptic variational inequalities. To appear.

[9]. Friedman, A. & Kinderlehrer, D., A one phase Stefan problem. *Indiana Univ. Math. J.*, 24 (1975), 1005–1035.

[10]. Kemper, J., Temperatures in several variables: Kernel functions, representations and parabolic boundary values. *Trans. Amer. Math. Soc.*, 167 (1972) 243–261.

[11]. Kinderlehrer, D., How a minimal surface leaves an obstacle. *Acta Math.*, 130 (1973), 221–242.

[12]. Kinderlehrer, D. & Nirenberg, L., Regularity in Free Boundary Problems. *Ann. Scuola Norm. sup. Pisa Sci. Fis. Mat.* To appear.

[13]. Lewy, H., On the reflection laws of second order differential equations in two independent variables. *Bull. Amer. Math. Soc.*, 65 (1959), 37–58.

[14]. Lewy, H. & Stampacchia, G., On the regularity of the solution of a variational inequality. *Comm. Pure Appl. Math.*, 22 (1969), 153–188.

[15]. Nitsche, J. C. C., Variational problems with inequalities as boundary conditions or How to fashion a cheap hat for Giacometti's brother. *Arch Rational Mech. Anal.*, 35 (1969), 83–113.

[16]. Serrin, J., On the Harnack inequality for linear elliptic equations. *J. Analyse*, 4 (1954–56), 292–308.

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