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The regularity of powers of edge ideals

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Abstract In this paper, we prove the existence of a special order on the set of minimal monomial generators of powers of edge ideals of arbitrary graphs. Using this order, we find new upper bounds on the regularity of powers of edge ideals of graphs whose complement does not have any induced four cycles.

Keywords Castelnuovo–Mumford regularity · Powers of edge ideals · Cricket free

Mathematics Subject Classification (2000) Primary 13-02 · 13F55 · 05C10

1 Introduction

In this work, we find new upper bounds for the regularity of some classes of monomial ideals associated to graphs. Our original motivation is the following question, which is the base case of the Open Problem 1.11(2) in [13]:

Question 1.1 *Let $I(G)$ be the edge ideal of a graph G which does not have any induced four cycle in its complement. If* $\operatorname{reg}(I(G)) \leq 3$*, then is it true that for all $s \geq 2$, $I(G)^s$ has linear minimal free resolution?*

Bounds on the regularity of edge ideals have been studied by a number of researchers [1–9, 11–13]. For example, Fröberg [3] has shown that if when $I(G)$ is the edge ideal of a graph whose complement does not have any induced cycle of size greater than or equal to 4, then $I(G)$ has linear minimal free resolution.

We are interested in finding upper bounds on the regularities of the higher powers of $I(G)$. Herzog et al. have shown in [6] that if $I(G)$ is the edge ideal of a graph G which

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has no induced cycle of length ≥ 4 in its complement (that is $I(G)$ has linear minimal free resolution), then for all $s \geq 2$, $I(G)^s$ has linear minimal free resolution. Fransisco, Hà, and Van-Tuyl have further shown that if $I(G)^s$ has linear minimal free resolution for some s, then G has no induced four cycle in its complement [13, Proposition 1.8]. These two results lead us to study bounds on the regularity of powers of $I(G)$ when G has no induced four cycle in its complement. Our main result is Theorem 6.17 where we prove that all higher powers of edge ideals of a gap-free (equivalently, no induced four cycle in complement, as observed in Sect. 2) and cricket-free (defined in Sect. 2) graph have linear minimal free resolution. More precisely:

Theorem 1.2 *For every gap-free and cricket-free graph G and for all $s \geq 2$, $\mathrm{reg}(I(G)^s) = 2s$. As a consequence, $I(G)^s$ has a linear minimal free resolution.*

This partially answers Question 1.1, as we prove in Sect. 3 that edge ideals of gap-free and cricket-free graphs have regularity less than or equal to 3 (Theorem 3.4). As claw free graphs (defined in Sect. 2) are automatically cricket free, our results generalize a previous result by Nevo [12, Theorem 1.2] which states that the edge ideals of gap-free and claw-free graphs have regularity less than or equal to 3, and their squares have linear minimal free resolutions.

In order to prove Theorem 6.17, we first show that the minimal monomial generators of powers of the edge ideal $I(G)$ of any finite simple graph G have a specific order with a nice property (Lemma 4.11, Theorem 4.12). More precisely:

Theorem 1.3 *For each $n \geq 1$, there exists an ordered list $L^{(n)}$ of minimal monomial generators of $I(G)^n$ which has the following property:*
For all $k \geq 1$ and for all $j \leq k$, if $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in $(I(G)^{n+1} : L_{k+1}^{(n)})$; then there exists $i \leq k$ such that $(L_i^{(n)} : L_{k+1}^{(n)})$ is generated by a variable and $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (L_i^{(n)} : L_{k+1}^{(n)})$. For monomials m and n, $(m : n)$ stands for $((m) : (n))$.

Using this ordering, we shall prove that $\mathrm{reg}(I(G)^n)$ is bounded above by the maximum of $\mathrm{reg}(I(G)^n : e_1 \dots e_{n-1}) + 2n - 2$ for all possible $(n-1)$-fold products of edges $e_1 \dots e_{n-1}$ and $\mathrm{reg}(I(G)^{n-1})$ (see Theorem 5.2). Next, we prove that the ideals $(I(G)^n : e_1 \dots e_{n-1})$ are quadratic monomial ideals with generators satisfying certain conditions (see Theorems 6.1, 6.5, 6.7). Finally, using polarization, we get edge ideals corresponding to these quadratic monomial ideals with some regularity (see [8], Sect. 3.2 and Exercise 3.15 of [10] for details) and using Fröberg's theorem (see Theorem 1 of [3] and Theorem 1.1 of [13]) we get bounds on them. As a consequence, we also get a different proof of the Herzog et al.'s result mentioned above (Theorem 6.16).

2 Preliminaries

Throughout this paper, we let G be a finite simple graph with vertex set $V(G)$. For $u, v \in V(G)$, we let $d(u, v)$ denote the *distance* between u and v, the fewest number of edges that must be traversed to travel from u to v.

A subgraph $G' \subseteq G$ is called *induced* if uv is an edge of G' whenever u and v are vertices of G', and uv is an edge of G.

The *complement* of a graph G, for which we write G^c, is the graph on the same vertex set in which uv is an edge of G^c if and only if it is not an edge of G.

Finally, we denote by C_k the cycle on k vertices and by $K_{m,n}$ the complete bipartite graph with m vertices on one side, and n on the other.

Definition 2.1 Let G be a graph. We say that two disjoint edges uv and xy form a *gap* in G if G does not have an edge with one endpoint in $\{u, v\}$ and the other in $\{x, y\}$. A graph without gaps is called *gap free*. Equivalently, G is gap free if and only if G^c contains no induced C_4.

Thus, G is gap free if and only if it does not contain two vertex-disjoint edges as an induced subgraph.

Definition 2.2 Any graph isomorphic to $K_{1,3}$ is called a *claw*. Any graph isomorphic to $K_{1,n}$ is called an *n-claw*. If $n > 1$, then the vertex with degree n is called the root in $K_{1,n}$. A graph without an induced claw is called *claw free*. A graph without an induced *n-claw* is called *n-claw free*.

Definition 2.3 Any graph isomorphic to the graph with set of vertices $\{w_1, w_2, w_3, w_4, w_5\}$ and set of edges $\{w_1w_3, w_2w_3, w_3w_4, w_3w_5, w_4w_5\}$ is called a cricket. A graph without an induced cricket is called *cricket free*.

Definition 2.4 An edge in a graph is called a whisker if each of its vertices has degree one.

Definition 2.5 A graph is called an anticycle if its complement is a cycle.

Observation 2.6 *A claw free graph is cricket free.*

If G is a graph without isolated vertices then let S denote the polynomial ring on the vertices of G over some fixed field K. Recall that the *edge ideal* of G is

$$I(G) = (xy : xy \text{ is an edge of } G).$$

Definition 2.7 Let S be a standard graded polynomial ring over a field K. The Castelnuovo–Mumford regularity of a finitely generated graded S module M, written $\operatorname{reg}(M)$, is given by

$$\operatorname{reg}(M) := \max\{j - i \,|\, \operatorname{Tor}_i(M, K)_j \neq 0\}.$$

Definition 2.8 We say that $I(G)^s$ is *k-steps linear* whenever the minimal free resolution of $I(G)^s$ over the polynomial ring is linear for k steps, i.e., $\operatorname{Tor}_i^S(I(G)^s, K)_j = 0$ for all $1 \leq i \leq k$ and all $j \neq i + 2s$. We say that $I(G)$ has linear minimal free resolution if the minimal free resolution is k-steps linear for all $k \geq 1$.

We end this section by recalling a few well-known results; see [1] and [13].

Observation 2.9 *Let $I(G)$ be the edge ideal of a graph G. Then, $I(G)^s$ has linear minimal free resolution if and only if* $\operatorname{reg}(I(G)^s) = 2s$.

Lemma 2.10 *Let $I \subseteq S$ be a monomial ideal. Then, for any variable x,* $\mathrm{reg}(I, x) \leq \mathrm{reg}(I)$. *In particular, if v is a vertex in a graph G, then* $\mathrm{reg}(I(G - v)) \leq \mathrm{reg}((I(G))$.

The next statement follows from Lemma 2.10 of [1]:

Lemma 2.11 *Let $I \subseteq S$ be a monomial ideal, and let m be a monomial of degree d. Then*

$$\mathrm{reg}(I) \leq \max\{\mathrm{reg}(I : m) + d, \mathrm{reg}(I, m)\}.$$

Moreover, if m is a variable x appearing in I, then $\mathrm{reg}(I)$ *is equal to one of these terms.*

Finally, the following theorem due to Fröberg (see Theorem 1 of [3] and Theorem 1.1 of [13]) is used repeatedly throughout this paper:

Theorem 2.12 *The minimal free resolution of $I(G)$ is linear if and only if the complement graph G^c is chordal, that is, no induced cycle in G^c has length greater than three.*

3 Gap-free graphs

In this section we observe some basic results concerning gap-free graphs and their regularity. We prove that a cricket free and gap free graph has regularity at most 3, generalizing Nevo's result [1, Theorem 3.3] that a gap-free and claw-free graph has regularity at most 3. We generalize Nevo's result in another direction by proving that n-claw free and gap free graphs have regularity at most n.

Definition 3.1 For any graph G, we write $\mathrm{reg}(G)$ as a shorthand for $\mathrm{reg}(I(G))$.

Recall that the *star* of a vertex x of G, which we denote by stx, is defined as

$$stx = \{y \in V(G) : xy \text{ is an edge of } G\} \cup \{x\}.$$

The following lemma, which is Lemma 3.1 of [1], will be used repeatedly in this work.

Lemma 3.2 *Let x be a vertex of G with neighbors $y_1, y_2, \ldots, y_m$. Then*

$$(I(G) : x) = (I(G - stx), y_1, \ldots, y_m) \text{ and } (I(G), x) = (I(G - x), x).$$

Thus, $\mathrm{reg}(G) \leq \max\{\mathrm{reg}(G - stx) + 1, \mathrm{reg}(G - x)\}$. *Moreover,* $\mathrm{reg}(G)$ *is equal to one of these terms.*

The next proposition is Proposition 3.2 of [1].

Proposition 3.3 *Let G be gap free, and let x be a vertex of G of highest degree. Then, $d(x, y) \leq 2$ for all vertices y of G.*

We prove the next two theorems using Proposition 3.3. Our proof is motivated by the proof of Theorem 3.3 of [1].

Theorem 3.4 *Suppose G is both cricket free and gap free. Then,* $\operatorname{reg}(G) \leq 3$.

Proof Let x be a vertex of maximum degree. As G is gap free and cricket free, so is $G - x$. By induction, $G - x$ has regularity less than or equal to 3. Because of Lemma 3.2 and Theorem 2.12, it is enough to show that $(G - \operatorname{st} x)^c$ has no induced cycle of length greater than or equal to 4. As G is gap free, so is $(G - \operatorname{st} x)$; hence, $(G - \operatorname{st} x)^c$ has no induced 4−cycle. So, it is enough to show that it does not have an induced cycle of length ≥ 5.

Let $\{y_1, y_2, y_3, y_4, \ldots, y_n\}$ be an induced cycle ($n \geq 5$) in $(G - \operatorname{st} x)^c$; because of Proposition 3.3, there is a w such that xw and wy_1 are edges in G. As $y_2 y_n$ is an edge in G, and neither $y_1 y_2$ nor $y_1 y_n$ is an edge in G, at least one of wy_2 and wy_n is an edge in G. If both are edges, then $\{x, w, y_1, y_2, y_n\}$ forms an induced cricket.

Suppose only one of them is an edge. Without loss of generality, we may assume that wy_2 is an edge. As $y_3 y_n$ is an edge in G, and G is gap free, wy_3 is an edge in G; otherwise $\{x, w, y_3, y_n\}$ forms a gap in G. This makes $\{x, w, y_1, y_2, y_3\}$ an induced cricket. □

Theorem 3.5 *The edge ideal of a graph, which is gap free and n-claw free, has regularity less than or equal to n.*

Proof For $n = 3$, this was proved by Nevo and this is Theorem 3.3 of [1]. So we may assume $n \geq 4$. Let x be a vertex with maximum degree. Because of Lemma 3.2, it is enough to show that $G - \operatorname{st} x$ has regularity less than or equal to $n - 1$; as $G - x$ has regularity less than or equal to n by induction on number of vertices. Hence, it is enough to show that $G - \operatorname{st} x$ is $(n - 1)$-claw free.

If $a_1, a_2, a_3, \ldots, a_n$ is a $(n - 1)$-claw with root a_1 in $G - \operatorname{st} x$, then any w in the neighborhood of x is either connected to a_1 or all of $a_2, a_3, \ldots, a_n$; otherwise, if w is not connected to a_1 and a_i, then xw and $a_1 a_i$ will form a gap. If a_1 is connected to all neighbors of x, then it has degree larger than that of x, a fact which contradicts the assumption x is a vertex with maximum degree. Hence, there is a neighbor w which is not connected to a_1 but is connected to all of $a_2, a_3, \ldots, a_n$. As x is not connected to any of the a_is, $\{x, w, a_2, a_3, \ldots, a_n\}$ forms an n-claw with root w, which contradicts the hypothesis. □

4 Ordering the minimal monomial generators of powers of edge ideals

Discussion 4.1 *We will denote by* Mingens*(J) the set of minimal monomial generators of an ideal* $J \subset S$. *Let I be an arbitrary edge ideal and set* $\operatorname{Mingens}(I) = \{L_1, L_2, \ldots, L_k\}$. *We consider the order* $L_1 > L_2 > \cdots > L_k$ *on* Mingens*(I) and for every integer* $n \geq 2$ *we endow the set* $\operatorname{Mingens}(I^n)$ *with the following order: we set* $M > N$ *for* $M, N \in \operatorname{Mingens}(I^n)$ *if there exists an expression* $L_1^{a_1} L_2^{a_2} \ldots L_k^{a_k} = M$ *such that for all expressions* $L_1^{b_1} \ldots L_k^{b_k} = N$, *we have* $(a_1, \ldots, a_k) >_{lex} (b_1, \ldots, b_k)$. *If* $(a_1, \ldots, a_k) \geq_{lex} (c_1, \ldots, c_k)$ *for all* $(c_1, \ldots, c_k)$ *such that* $L_1^{c_1} \ldots L_k^{c_k} = M$, *then*

$L_1^{a_1} L_2^{a_2} \dots L_k^{a_k}$ is called a maximal expression of M. Let $L^{(n)}$ be the totally ordered set of minimal monomial generators of I^n, ordered in the way discussed above.

Definition 4.2 If m_1 is a minimal monomial generator of I^k and m_2 is a minimal monomial generator of I^n where $n > k$, then we say m_1 divides m_2 as an edge and use the notation $m_1|^{\text{edge}} m_2$; if there exists m_3, a minimal monomial generator of I^{n-k} with $m_2 = m_1 m_3$.

Example 4.3 If $I = (ab, bc, ad, bd)$, then $ab|^{\text{edge}} ab^2 d$ as $bd = \frac{ab^2 d}{ab}$ is a minimal monomial generator of I but $ab \nmid^{\text{edge}} abcd$ as $cd = \frac{abcd}{ab}$ is not a minimal monomial generator of I.

Discussion 4.4 *We have the following for the list $L^{(n)}$ created above:*

1. *$L^{(1)} = L := \{L_1 > \cdots > L_k\}$*
2. *For any minimal monomial generator m of I^n, $n \geq 2$, the maximal expression of m is an expression of m as a product of n elements of L, $m = L_{i_1} L_{i_2} \dots L_{i_n}$, where:*
   a. *i_1 is the minimum integer such that $L_{i_1}|^{edge} m$*
   b. *For all $l \geq 1$, i_{l+1} is the minimal integer such that $L_{i_{l+1}}|^{edge} \frac{m}{L_{i_1} \dots L_{i_l}}$. For any edge cd, we say that cd is a part of the maximal expression of m if $cd = L_{i_k}$ for some k.*

   This expression is unique by the construction.
3. *For two minimal monomial generators m_1, m_2 with maximal expressions $m_1 = L_{i_1} \dots L_{i_n}$ and $m_2 = L_{j_1} \dots L_{j_n}$, we have $m_1 >_{lex} m_2$ if for the minimum integer l such that $i_l \neq j_l$, $i_l < j_l$.*
4. *If L_i and L_j are two generators of I with $i < j$, then we say "L_j comes after L_i" or "L_i comes before L_j."*

Example 4.5 Let $I = (ab, bc, ad, bd)$. Let $L^{(1)} = \{ab > bc > ad > bd\}$. Then $L^{(2)} = \{a^2b^2 > ab^2c > a^2bd > ab^2d > b^2c^2 > abcd > b^2cd > a^2d^2 > abd^2 > b^2d^2\}$.

Definition 4.6 If $L_i = ab$ is an edge, that is a minimal monomial generator of I, and m is a minimal monomial generator of I^n, $n \geq 2$, then we say that m belongs to ab, or m belongs to L_i, if i is the least integer such that $L_i|^{\text{edge}} m$.

Example 4.7 Let $I = (ab, bc, ad, bd)$ with $L = L^{(1)} = \{ab > bc > ad > bd\}$. Then, $abcd$ belongs to $L_2 = bc$ as $ab \nmid^{\text{edge}} abcd$ and $bc|^{\text{edge}} abcd$ and ab^2d belongs to $L_1 = ab$ as $ab|^{\text{edge}} ab^2 d$.

We record several easy observations that we need in the sequel.

Observation 4.8 *For two minimal monomial generators m_1, m_2, if m_1 belongs to an edge L_i and m_2 belongs to another edge L_j with $i < j$, then $m_1 >_{lex} m_2$.*

Observation 4.9 *For two minimal monomial generators m_1, m_2 of I^n which both belong to an edge L_i, we see that $m_1 >_{lex} m_2$ if and only if $\frac{m_1}{L_i} >_{lex} \frac{m_2}{L_i}$.*

Observation 4.10 *Suppose m is a minimal monomial generator of I^n, $n \geq 2$, and gh is an edge which is a part of the maximal expression of m. Write $m = ghm'$. For any minimal monomial generator m'' of I^{n-1} such that $m'' >_{lex} m'$, then $ghm'' >_{lex} m$.*

Proof Let $L = \{L_1 > L_2 > \cdots > L_k\}$. Let $gh = L_j$ for some j. Let $m'' = L_1^{a_1} L_2^{a_2} \ldots L_k^{a_k}$ be the maximal expression of m'' and $m' = L_1^{b_1} L_2^{b_2} \ldots L_k^{b_k}$ be the maximal expression of m'. As gh is a part of the maximal expression of m, the maximal expression of m is $L_1^{b_1} \ldots L_j^{b_j+1} \ldots L_k^{b_k}$. As by assumption $(a_1, \ldots, a_j, \ldots, a_k) >_{\text{lex}} (b_1, \ldots, b_j, \ldots, b_k)$, we have $(a_1, \ldots, a_j+1, \ldots, a_k) >_{\text{lex}} (b_1, \ldots, b_j+1, \ldots, b_k)$. Now $L_1^{a_1} \ldots L_j^{a_j+1} \ldots L_k^{a_k}$ is an expression for ghm''. Hence, $ghm'' >_{\text{lex}} ghm' = m$. □

The next lemma is the most important technical result of this paper as it allows us to build the framework of Sect. 5. Using the framework of Sect. 5, we obtain our bounds in Sect. 6.

Lemma 4.11 *For all $k \geq 1$ and for all $j \leq k$, if $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in $(I^{n+1} : L_{k+1}^{(n)})$ and $L_j^{(n)}$ belongs to an edge that comes before the edge $L_{k+1}^{(n)}$ belongs to, then there exists $i \leq k$, such that $(L_i^{(n)} : L_{k+1}^{(n)})$ is generated by a variable, $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (L_i^{(n)} : L_{k+1}^{(n)})$ and $L_i^{(n)}$ belongs to an edge that comes before or equal to the edge $L_j^{(n)}$ belongs to.*

Proof We proceed by induction on n. We recall that for two monomials m_1 and m_2, $(m_1 : m_2) = (\frac{m_1}{\gcd(m_1,m_2)})$. This is going to be used in several places.

If $n = 1$, $(L_j : L_{k+1})$ is either (L_j), in which case $(L_j : L_{k+1}) \subseteq (I^2 : L_{k+1})$ or it is generated by a variable in which case we take $L_i = L_j$. Hence, the lemma is true for $n = 1$.

Suppose the result is true for $n-1$. Let $L_j^{(n)}$ belong to ab, so that $L_j^{(n)} = abM_1$ where $M_1 \in L^{(n-1)}$. By assumption, $L_{k+1}^{(n)}$ belongs to an edge which comes after ab in L. If neither a nor b divide $L_{k+1}^{(n)}$, then $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (ab) \subseteq (I^{n+1} : L_{k+1}^{(n)})$ which is contrary to our assumption.

Without loss of generality, we assume $a | L_{k+1}^{(n)}$. As $L_{k+1}^{(n)}$ is a product of edges, there exists an edge ac with $ac|^{\text{edge}} L_{k+1}$, where ac is a part of the maximal expression of $L_{k+1}^{(n)}$. So, $L_{k+1}^{(n)} = acM_2$ for some $M_2 \in L^{(n-1)}$ which is the remaining part of the maximal expression. Now $ab \nmid^{\text{edge}} L_{k+1}^{(n)}$ as $L_{k+1}^{(n)}$ belongs to an edge that comes after ab. Hence, $b \neq c$.

If $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (b)$, then we take $L_i^{(n)} = abM_2$. Clearly, $L_i^{(n)}$ belongs to ab or some edge that comes before ab. Also, $(L_i^{(n)} : L_{k+1}^{(n)}) = (abM_2 : acM_2) = (b)$. Hence, $L_i^{(n)}$ has all the required properties.

If $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in (b), then there is a variable d such that bd is an edge and $bd|^{\text{edge}} M_2$ and bd is a part of maximal expression of M_2. Let $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (f)$ where f is a variable. If $(L_j^{(n)} : L_{k+1}^{(n)}) = (f)$, then we take $L_i^{(n)} = L_j^{(n)}$. This has all the required properties.

So let us assume $(L_j^{(n)} : L_{k+1}^{(n)}) = (M_1 b : M_2 c) \subsetneq (f)$. Let $(L_j^{(n)} : L_{k+1}^{(n)}) = (fm)$ where m is a monomial which is not 1. So there is an edge fg such that $fg|^{\text{edge}} M_1$ and fg is part of the maximal expression of M_1. If $g \nmid M_2 c$, then $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (fg) \subseteq (I^{n+1} : L_{k+1}^{(n)})$ which contradicts our assumption. So $g|M_2 c$.

If $g = c$, then either $f = d$, that is $fcab = bdac$ or $(fcab : bdac) = (f)$. In the first case, $L_{k+1} = acM_2 = acbd\frac{M_2}{bd} = fcab\frac{M_2}{bd}$. Now $bd|^{\text{edge}} M_2$, so $ab|^{\text{edge}} L_{k+1}^{(n)}$ which is a contradiction. In the second case, we take $L_i^{(n)} = (fc)(ab)\frac{L_{k+1}^{(n)}}{bdac}$. Clearly, $L_i^{(n)}$ belongs to ab or a some edge that comes before ab and $(L_i^{(n)} : L_{k+1}^{(n)}) = (f)$, which contains $(L_j^{(n)} : L_{k+1}^{(n)})$. Hence, $L_i^{(n)}$ has the required properties.

Now let us assume $g \neq c$. So, there is an edge gh such that $gh|^{\text{edge}} M_2$, such that gh is a part of the maximal expression of M_2. Let $\frac{M_1}{fg} = N_1$ and $\frac{M_2}{gh} = N_2$. As $(L_j^{(n)} : L_{k+1}^{(n)}) = (fm)$, $fgabN_1 | fmghacN_2$. So $abN_1 | hmacN_2$. So $(hm) \subset (abN_1 : acN_2)$. We observe that $(abN_1 : acN_2)$ is either (m) or (hm). For if $m'|m$, then $abN_1 | hm'acN_2$ implies $fgabN_1 | fm'ghacN_2$ implies $fm|fm'$ implies $m = m'$.

If $(N_1 ab : N_2 ac) = (m)$, then $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (m) = (abN_1 : acN_2)$. Now both abN_1 and acN_2 are in $L^{(n-1)}$. As abN_1 belongs to ab and acN_2 belongs to some edge which comes after ab, $abN_1 >_{\text{lex}} acN_2$. By induction, either $(abN_1 : acN_2) \subseteq (I^n : acN_2)$ or there exists M_0 in $L^{(n-1)}$, $M_0 >_{\text{lex}} acN_2$, $(abN_1 : acN_2) \subseteq (M_0 : acN_2)$, $(M_0 : acN_2)$ is generated by a variable, and M_0 belongs to an edge that comes before or equal to ab. In the first case, $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (abN_1 : acN_2) \subseteq (I^n : acN_2) \subset (I^{n+1} : ghacN_2) = (I^{n+1} : L_{k+1}^{(n)})$, which is a contradiction. In the second case, write $L_i^{(n)} = ghM_0$. We know that $L_i^{(n)} >_{\text{lex}} L_{k+1}^{(n)}$ as M_0 belongs to an edge that comes before or equal to ab. Also, $(L_i^{(n)} : L_{k+1}^{(n)}) = (M_0 : acN_2)$, $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (m) = (abN_1 : acN_2) \subseteq (M_0 : acN_2)$ and $(M_0 : acN_2)$ is generated by a variable.

Now let us assume $(abN_1 : acN_2) = (hm)$. As $abN_1 >_{\text{lex}} acN_2$, by induction either $(abN_1 : acN_2) \subseteq (I^n : acN_2)$ or there exists M_0' in $L^{(n-1)}$, $M_0' >_{\text{lex}} acN_2$, with $(abN_1 : acN_2) \subseteq (M_0' : acN_2)$, $(M_0' : acN_2)$ is generated by a variable, and M_0' belongs to an edge that comes before or equal to ab. In the first case $hmacN_2 \in I^n$, so $fmghacN_2 = fgmhacN_2 \in I^{n+1}$. So $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (I^{n+1} : L_{k+1}^{(n)})$, which is a contradiction. In the second case, if $(M_0' : acN_2) \neq (h)$, then let $L_i^{(n)} = ghM_0'$. As M_0' belongs to an edge that comes before or equal to ab, $L_i^{(n)} >_{\text{lex}} L_{k+1}^{(n)}$. Also $(L_i^{(n)} : L_{k+1}^{(n)}) = (M_0' : acN_2)$ which contains $(L_j^{(n)} : L_{k+1}^{(n)})$ and is generated by a variable. If $(M_0' : acN_2) = (h)$ we take $L_i^{(n)} = fgM_0'$. By same reasoning, $L_i^{(n)} >_{\text{lex}} L_{k+1}^{(n)}$. As $L_i^{(n)}$ cannot be same as $L_{k+1}^{(n)}$, we observe $(L_i^{(n)} : L_{k+1}^{(n)}) = (f)$. So, this $L_i^{(n)}$ has all the required properties. This completes the proof. □

Theorem 4.12 *For all $k \geq 1$ and for all $j \leq k$, if $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in $(I^{n+1} : L_{k+1}^{(n)})$, then there exists $i \leq k$, such that $(L_i^{(n)} : L_{k+1}^{(n)})$ is generated by a variable and $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (L_i^{(n)} : L_{k+1}^{(n)})$.*

Proof We have $L_j^{(n)} = mm_1$ and $L_{k+1}^{(n)} = mm_2$ where $m \in \text{Mingens}(I^k)$ and $m_1, m_2 \in \text{Mingens}(I^{n-k})$ with m_1 belong to an edge that comes strictly before the edge m_2 belongs. We observe $(L_j^{(n)} : L_{k+1}^{(n)}) = (m_1 : m_2)$ and $(I^{n-k+1} : m_2) \subseteq (I^{n+1} : mm_2)$. With these two observations, the theorem follows from Lemma 4.11. This finishes the proof. □

5 Bounding the regularity: the framework

In this section, we create the framework from which we shall prove our bounds. The framework is created by repeated use of Lemma 2.11. Let I and J be two homogeneous square-free monomial ideals in S generated in degrees n_1 and n_2 respectively. Assume $J \subset I$ and that $n_2 > n_1$. If the unique set of minimal monomial generators of I is $\{m_1, m_2, \dots, m_k\}$, then repeated use of Lemma 2.11 gives us the following lemma:

Lemma 5.1 *Let*

$$\begin{aligned} A &= \max\{\text{reg}(J : m_1) + n_1\} \\ B &= \max\{\text{reg}((J, m_1, \dots, m_l) : m_{l+1}) + n_1 | 1 \leq l \leq k-1\} \\ C &= \text{reg}(I). \end{aligned}$$

Then, $\text{reg}\, J \leq \max\{A, B, C\}$.

Proof We consider the follwing short exact sequence:

$$0 \longrightarrow \frac{S}{(J : m_1)}(-n_1) \overset{.m_1}{\longrightarrow} \frac{S}{J} \longrightarrow \frac{S}{(J, m_1)} \longrightarrow 0$$

This gives us $\text{reg}(J) \leq \max\{\text{reg}(J : m_1) + n_1 = A, \text{reg}(J, m_1)\}$. Let $J_l := ((J, m_1, \dots, m_{l-1}) : m_l)$ for all $l \geq 2$. For all $1 \leq l \leq k-1$, we can consider the exact sequence

$$0 \longrightarrow \frac{S}{(J_{l+1})}(-n_1) \overset{.m_{l+1}}{\longrightarrow} \frac{S}{(J, m_1, \dots, m_l)} \longrightarrow \frac{S}{(J, m_1, \dots, m_{l+1})} \longrightarrow 0,$$

This gives us

$$\text{reg}(J, m_1, \dots, m_l) \leq \max\{\text{reg}(J_{l+1}) + n_1, \text{reg}(J, m_1, \dots, m_{l+1})\}$$

from which $\text{reg}(J) \leq \max\{A, B, C\}$ follows. □

This lemma together with Theorem 4.12 gives the next theorem which is the main result we use for finding bounds on regularity of higher powers of edge ideals.

Theorem 5.2 *For any finite simple graph G and any $s \geq 1$, let the set of minimal monomial generators of $I(G)^s$ be $\{m_1, \dots, m_k\}$. Then*

$$\text{reg}(I(G)^{s+1}) \leq \max\{\text{reg}(I(G)^{s+1} : m_l) + 2s, 1 \leq l \leq k, \text{reg}(I(G)^s)\}.$$

Proof Minimal monomial generators of $I(G)^s$ forms the ordered list $L^{(s)}$ from Sect. 4. So, by Lemma 5.1,

$$\operatorname{reg}(I(G)^{s+1}) \leq \max\{A, B, C\},$$

where

$$\begin{aligned} A &= \max\{\operatorname{reg}(I(G)^{s+1} : L_1^{(s)}) + 2s\} \\ B &= \max\{\operatorname{reg}(((I(G)^{s+1}, L_1^{(s)}, \ldots, L_l^{(s)}) : L_{l+1}^{(s)}) + 2s | 1 \leq l \leq k-1\} \\ C &= \operatorname{reg}(I(G)^s). \end{aligned}$$

But in light of Theorem 4.12, $((I(G)^{s+1}, L_1^{(s)}, \ldots, L_l^{(s)}) : L_{l+1}^{(s)})$ is the same as $((I(G)^{s+1} : L_{l+1}^{(s)})$, some variables). So, by Lemma 2.10

$$\operatorname{reg}((I(G)^{s+1}, L_1^{(s)}, \ldots, L_l^{(s)}) : L_{l+1}^{(s)}) \leq \operatorname{reg}((I(G)^{s+1} : L_{l+1}^{(s)}),$$

and the theorem follows. □

As a corollary to the above theorem, we get the following important result:

Corollary 5.3 *If for all $s \geq 1$ and for all minimal monomial generators m of $I(G)^s$,* $\operatorname{reg}(I(G)^{s+1} : m) \leq 2$ *and* $\operatorname{reg}(I(G)) \leq 4$, *then for all* $s \geq 1$, $\operatorname{reg}(I(G)^{s+1}) = 2s+2$*; as a consequence, $I(G)^{s+1}$ has a linear minimal free resolution.*

Proof We observe that under the condition if $\operatorname{reg}(I(G)^s) \leq 2s + 2$, then $\operatorname{reg}(I(G)^{s+1}) \leq 2s + 2$ too. Now $\operatorname{reg}(I(G)) \leq 4$ implies $\operatorname{reg}(I(G)^2) \leq 4$. By induction, assume $\operatorname{reg} I(G)^k \leq 2k$. As $2k < 2k + 2$, $\operatorname{reg} I(G)^k \leq 2k + 2$. Hence, $\operatorname{reg} I(G)^{k+1} \leq 2k + 2$. This proves the corollary. □

6 Bounding the regularity: the results

In this section, we give some new bounds on $\operatorname{reg}(I(G)^s)$ for certain classes of gap-free graphs G. The main idea is to carefully analyze the ideal $(I(G)^{s+1} : e_1 \ldots e_s)$ for an arbitrary s-fold product of edges, i.e., for $i \neq j$, $e_i = e_j$ is a possibility. Now, any s-fold product can be written as product of s edges in various ways. In this section, we fix a presentation and work with respect to that. We first prove that these ideals are generated in degree two for any graph G.

Theorem 6.1 *For any graph G and for any s-fold product $e_1 \ldots e_s$ of edges in G (with the possibility of e_i being same as e_j as an edge for $i \neq j$), the ideal $(I(G)^{s+1} : e_1 \ldots e_s)$ is generated by monomials of degree two.*

Proof We prove this using induction on s. For $s = 0$, the result is clear as $(I(G) : (1)) = I(G)$, which is generated by monomials of degree two. Now, let us assume that the theorem is true till $s - 1$.

Let m be a minimal monomial generator of $(I(G)^{s+1} : e_1 \dots e_s)$. Then, $e_1 \dots e_s m$ is divisible by an $(s+1)$-fold product of edges. By degree consideration, m cannot have degree 1. If m has degree greater than or equal to 3, then again by a degree consideration for some i, $e_i = pq$ such that $e_1 \dots e_{i-1} q e_{i+1} \dots e_s m$ is divisible by an $(s+1)$-fold product of edges. Without loss of generality, we may assume $e_1 = pq$ and there is an $(s+1)$-fold product $f_1 \dots f_{s+1}$ such that $f_1 \dots f_{s+1} | q e_2 \dots e_s m$.

If $q | f_1 \dots f_{s+1}$, without loss of generality, we may assume $f_1 = p'q$. So, $p'q f_2 \dots f_{s+1} | q e_2 \dots e_s m$. Hence, $f_2 \dots f_{s+1} | e_2 \dots e_s m$. If q does not divide $f_1 \dots f_{s+1}$, then $f_1 \dots f_{s+1} | e_2 \dots e_s m$, and hence $f_2 \dots f_{s+1} | e_2 \dots e_s m$. In both cases, $m \in (I(G)^s : e_2 \dots e_s)$.

Now $(I(G)^s : e_2 \dots e_s) \subset (I(G)^{s+1} : e_1 \dots e_s)$ and m is a minimal monomial generator of $(I(G)^{s+1} : e_1 \dots e_s)$. So m has to be a minimal monomial generator of $(I(G)^s : e_2 \dots e_s)$. Hence, by induction, m has degree two, which is a contradiction to the assumption that m has degree greater than or equal to three. Hence, m has to have degree two. □

To analyze the generators of $(I(G)^{s+1} : e_1 \dots e_s)$, we introduce the notion of *even-connectedness* with respect to s-fold products.

Definition 6.2 Two vertices u and v (u may be equal to v) are said to be even-connected with respect to an s-fold product $e_1 \dots e_s$ if there is a path $p_0 p_1 \dots p_{2k+1}$, $k \geq 1$ in G such that

1. $p_0 = u$, $p_{2k+1} = v$.
2. For all $0 \leq l \leq k-1$, $p_{2l+1} p_{2l+2} = e_i$ for some i.
3. For all i,

$$|\{l \geq 0 | p_{2l+1} p_{2l+2} = e_i\}| \leq |\{j | e_j = e_i\}|$$

4. For all $0 \leq r \leq 2k$, $p_r p_{r+1}$ is an edge in G.

If these properties are satisfied, then $p_0, \dots, p_{2k+1}$ is said to be an even-connection between u and v with respect to $e_1 \dots e_s$.

Example 6.3 Let $I(G) = (xy, xu, yv, yw, wz, zv)$ and $e_1 = xy$, $e_2 = wz$, then u, x, y, w, z, v is an even-connection between u and v with respect to $e_1 e_2$.

The following observation is an immediate consequence of the definition:

Observation 6.4 *If $u = p_0, \dots, p_{2k+1} = v$ is an even-connection with respect to some s-fold product $e_1 \dots e_s$, then for any $j' \geq j \geq 0$, any neighbor x of p_{2j+1} and any neighbor y of $p_{2j'+2}$ are even-connected with respect to $e_1 \dots e_s$.*

The next theorem also easily follows from the definition.

Theorem 6.5 *If $u = p_0, \dots, p_{2k+1} = v$ is an even-connection with respect to some s-fold product $e_1 \dots e_s$, then $uv \in (I(G)^{s+1} : e_1 \dots e_s)$.*

Proof By conditions 2 and 3 of the definition, $e_1 \dots e_s = p_1 \dots p_{2k} \cdot e_{j_1} \dots e_{j_{s-k}}$, for some $\{j_1, j_2, \dots, j_{s-k}\} \subset \{1, \dots, s\}$ and by conditions 1 and 4, $up_1 \dots p_{2k}v$ is a $(k+1)$-fold product of edges in G. Hence, $uve_1 \dots e_s$ is an $(s+1)$-fold product of edges in G and the result follows. □

Although we fix a representation for all s-fold product and work with respect to that representation, it is worth noting that our definition of even-connectedness is independent of the representation we choose in the following sense:

Theorem 6.6 *If $f_1 \dots f_s = e_1 \dots e_s$ are two different representations of same s-fold product as product of edges and u and v are even-connected with respect to $e_1 \dots e_s$, then u and v are even-connected with respect to $f_1 \dots f_s$.*

Proof Let $u = p_0, \dots, p_{2k+1} = v$ be an even-connection between u and v with respect to $e_1 \dots e_s$. We shall construct an even-connection $q_0, \dots, q_{2r+1}$ between u and v with respect to $f_1 \dots f_s$.

Let i be minimal such that $p_{2i+1}p_{2i+2}$ is not equal to any edge $f_1, \dots, f_s$. Let $q_0 = p_0, \dots, q_{2i+1} = p_{2i+1}$. We have $(up_1)(p_2p_3)\dots(p_{2k}v)e_{t_1}\dots e_{t_{s-k}} = (uv)f_1 \dots f_s$. Then $p_{2i+1}(p_{2i+2}p_{2i+3})\dots(p_{2k}v)e_{t_1}\dots e_{t_{s-k}} = vf_{j_1}\dots f_{j_{s-i}}$. If $v = p_{2i+1}$, then we are done. Otherwise, p_{2i+1} divides one of the fs; without loss of generality let $f_{j_1} = p_{2i+1}q_{2i+2}$. If vq_{2i+2} is an edge in G, then we are done by taking $q_{2i+3} = v$. Otherwise, we have $vq_{2i+2}f_{j_2}\dots f_{s-i}$ is an $(s-i)$-fold product of edges $g_1 \dots g_{s-i}$, where without loss of generality $g_1 = q_{2i+2}q_{2i+3}$ and $f_{j_2} = q_{2i+3}q_{2i+4}$. After selecting (without loss of generality) $g_l = q_{2i+2l}q_{2i+2l+1}$ and $f_{j_{l+1}} = q_{2i+2l+1}q_{2i+2l+2}$, we select $q_{2i+2l+3}$ inductively. If $vq_{2i+2l+2}$ is an edge in G, then we are done by choosing $q_{2i+2l+3} = v$. Otherwise, $g_{l+1}\dots g_{s-i} = vq_{2i+2l+2}f_{j_{l+2}}\dots f_{j_{s-i}}$. If v is connected to $q_{2i+2l+2k}$ for some k in G then we are done by choosing $q_{2i+2l+2k+1} = v$. If not, then $g_1 \dots g_{s-i} = vg_1g_2\dots g_{s-i-1}q_{2i+2s-2}$; but this will force $g_{s-i} = q_{2i+2s-2}v$, contradicting the fact that v is not connected to $q_{2i+2l+2k}$ for any k.

The conditions 1, 2, 4 of the definition are automatically satisfied by our construction. Condition 3 is satisfied because each $q_{2i+1}q_{2i+2}$ is f_{r_i} for some integer r_i and $q_{2i+3}q_{2i+4}$ is some $f_{r_{i+1}}$ where $r_{i+1} \notin \{r_1, \dots, r_i\}$. □

We now observe that all edges of G belong to $(I(G)^{s+1} : e_1 \dots e_s)$. If uv (u may be equal to v) belongs to $(I(G)^{s+1} : e_1 \dots e_s)$ and uv is not an edge, then we prove that u and v have to be even-connected with respect to the s-fold product $e_1 \dots e_s$. The conditions 1, 2, 3, 4 are satisfied by the way of construction.

Theorem 6.7 *Every generator uv (u may be equal to v) of $(I(G)^{s+1} : e_1 \dots e_s)$ is either an edge of G or even-connected with respect to $e_1 \dots e_s$, for $s \geq 1$.*

Proof Suppose uv is not an edge, and u and v are not even-connected. Now $uve_1 \dots e_s = f_0 \dots f_s$ is an $(s+1)$-fold product of edges, where $f_0 = up_0$ such that there is an edge $e_{i_0} = p_0q_1$, $1 \leq i_0 \leq s$. After selecting $f_j = q_jp_j$ and $e_{i_j} = p_jq_{j+1}$, $1 \leq i_j \leq s$ and all i_j are different, we select f_{j+1} and $e_{i_{j+1}}$ inductively. q_{j+1} is part of an edge $q_{j+1}p_{j+1}$ in the $(s+1)$-fold product $f_0 \dots f_s$. We choose $f_{j+1} = q_{j+1}p_{j+1}$. Now as u and v are not even-connected, p_{j+1} is not v. So it is part of an edge amongst the remaining e_i s. So, there exists $e_{i_{j+1}} = p_{j+1}q_{j+2}$,

$i_{j+1} \in \{1, \ldots, s\} \setminus \{i_1 \ldots i_j\}$. Now, as u and v are not even-connected, $v \neq p_k$ for any k. We observe $f_0 \ldots f_s = u(p_0q_1)(p_1q_2)\ldots(p_{s-1}q_s)p_s = uve_1 \ldots e_s$. By construction, $(p_0q_1)(p_1q_2)\ldots(p_{s-1}q_s) = e_1 \ldots e_s$. This forces $p_s = v$, which is a contradiction. □

Example 6.8 Let $I(G) = (xy, xu, xv, xz, yz, yw)$. Then $(I(G)^2 : xy) = I(G) + (z^2, uz, vz, wz, uw, vw)$. Here, z is even-connected to itself and u, v, w with respect to xy; also u, w and v, w are even-connected with respect to xy.

We observe that $(I(G)^{s+1} : e_1 \ldots e_s)$ need not be square free as there is a possibility that some vertex u is even-connected to itself with respect to $e_1 \ldots e_s$. So we polarize $(I(G)^{s+1} : e_1 \ldots e_s)$ to get a square-free quadratic monomial ideal (i.e., an edge ideal) $(I(G)^{s+1} : e_1 \ldots e_s)^{\text{pol}}$. For details of polarization, we refer to [8], Sect. 3.2 of [10] and exercise 3.15 of [10]. Here, we just recall the definition and one theorem which states a quadratic monomial ideal and its polarization have same regularity.

Definition 6.9 For any quadratic monomial ideal I in $K[x_1, \ldots, x_n]$, I^{pol} is a square-free quadratic monomial ideal in $K[x_1, \ldots, x_n, x_1', \ldots, x_n']$ where $I^{\text{pol}} = \langle x_i x_j, x_k x_k' | x_i x_j \in I, x_k^2 \in I \rangle$.

The following theorem, which we state without proof, is a special case of Proposition 1.3.4 of [8], we also refer to Sect. 3.2 and Exercise 3.15 of [10].

Theorem 6.10 $\operatorname{reg}(I^{\text{pol}}) = \operatorname{reg}(I)$.

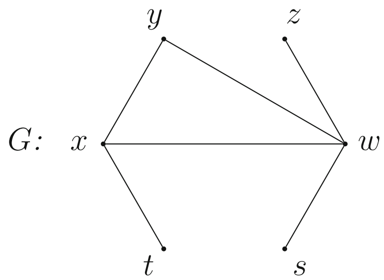
Clearly by Theorems 6.1, 6.5, and 6.7, $(I(G)^{s+1} : e_1 \ldots e_s)^{\text{pol}}$ is an edge ideal with the same regularity as $\operatorname{reg}(I(G)^{s+1} : e_1 \ldots e_s)$. We describe the graph associated to this edge ideal in the following Lemma:

Lemma 6.11 $(I(G)^{s+1} : e_1 \ldots e_s)^{\text{pol}}$ *is the edge ideal of a new graph G' which has*

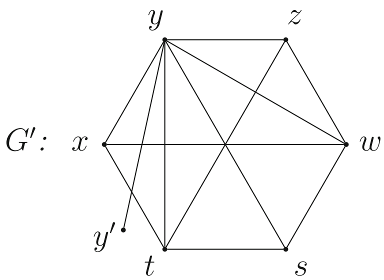
1. *All vertices and edges of G.*
2. *Any two vertices $u, v, u \neq v$ of G that are even-connected with respect to $e_1 \ldots e_s$ are connected by an edge in G'.*
3. *For every vertex u which is even-connected to itself with respect to $e_1 \ldots e_s$, there is a new vertex u' which is connected to u by an edge and not connected to any other vertex (so uu' is a whisker).*

Proof By Theorem 6.7, every generator uv (u may be equal to v) of $(I(G)^{s+1} : e_1 \ldots e_s)$ is either an edge of G or even-connected with respect to $e_1 \ldots e_s$, for $s \geq 1$. If it is an edge in G, then it satisfies condition 1; if it is an even-connection with $u \neq v$, then it satisfies condition 2; if it is an even-connection with $u = v$, then by definition of polarization, there will be a whisker u' on u in G', and hence it will satisfy condition 3. Conversely, edges described by the conditions 1,2 and 3 belong to G' by Theorems 6.5 and 6.7.

Example 6.12 Let G be the following graph:



Then, the graph G' associated to $(I(G)^2 : xw)^{\text{pol}}$ is the following:



Next, we prove several lemmas that will be useful to get our main results.

Lemma 6.13 *Suppose $u = p_0, \dots, p_{2k+1} = v$ is an even-connection between u and v and $z = q_0, \dots, q_{2l+1} = w$ is an even-connection between z and w, both with respect to $e_1 \dots e_s$. If for some i and j, $p_{2i+1}p_{2i+2}$ and $q_{2j+1}q_{2j+2}$ has a common vertex in G, then u is even-connected to either z or w with respect to $e_1 \dots e_s$, and v is even-connected to either z or w with respect to $e_1 \dots e_s$.*

Proof We prove it for u, and the proof for v follows by symmetry. Let i be the smallest integer such that there is j with the required property. If $p_{2i+1} = q_{2j+1}$, then $u = p_0, \dots, p_{2i+1} = q_{2j+1}, q_{2j+2}, q_{2j+3}, \dots, q_{2l+1} = w$ gives an even-connection between u and w with respect to $e_1 \dots e_s$ (conditions 1,2, and 4 are automatically satisfied, and condition 3 is satisfied as i is the smallest integer such that there is a j). Similarly, if $p_{2i+1} = q_{2j+2}$, then $u = p_0, \dots, p_{2i+1} = q_{2j+2}, q_{2j+1}, q_{2j}, \dots, q_0 = z$ gives an even-connection between u and z with respect to $e_1 \dots e_s$; if p_{2i+1} is not equal to either q_{2j+1} or q_{2j+2} and $p_{2j+2} = q_{2j+1}$, then $u = p_0, \dots, p_{2i+1}, p_{2j+2} = q_{2j+1}, q_{2j+2}, q_{2j+1}, q_{2j}, \dots, q_0 = z$ gives an even-connection between u and z with respect to $e_1 \dots e_s$; if p_{2i+1} is not equal to either q_{2j+1} or q_{2j+2} and $p_{2j+2} = q_{2j+2}$, then $u = p_0, \dots, p_{2i+1}, p_{2j+2} = q_{2j+2}, q_{2j+1}, q_{2j+2}, , \dots, q_{2l+1} = w$ gives an even-connection between u and w with respect to $e_1 \dots e_s$; in each of these cases, conditions 1,2, and 4 are satisfied automatically, and condition 3 is satisfied as i is the smallest integer with the property. This covers all the cases. □

The next two lemmas are results about gap-free graphs:

Lemma 6.14 *If G is gap free, then so is the graph G' associated to $(I(G)^{s+1} : e_1 \dots e_s)^{\text{pol}}$, for every s-fold product $e_1 \dots e_s$.*

Proof There are three possibilities of gap formation in G' :

1. Between two edges from G.
2. Between two edges that are not edges in G.
3. Between two edges where one of them is an edge in G another is not.

No two edges in G can form a gap in G as G is gap free. So they cannot form an edge in G' as in G', no edge of G is being deleted.

For the second case, suppose uv and zw are even-connected with respect to $e_1 \dots e_s$, and neither uv nor zw is an edge in G. Without loss of generality, we may assume $\gcd(uv, zw) = 1$ as there is no question of gap formation otherwise. Let $u = p_0, \dots, p_{2k+1} = v$ be an even-connection between u, v with respect to $e_1 \dots e_s$, and let $z = q_0, \dots, q_{2l+1} = w$ be an even-connection between z, w with respect to $e_1 \dots e_s$. In light of Lemma 6.13, we may assume for no i, j, $p_i = q_j$. If $u = q_1$, then $zu = zq_1$ is an edge in G and if $z = p_1$, then $uz = up_1$ is an edge in G, so there is nothing to prove. Otherwise, as up_1 and zq_1 are edges in G and G is gap free, there are four possibilities:

a. u is connected to z in G, in which case uv (or uu' in case $u = v$) and zw (or zz' in case $z = w$) cannot form a gap, as in that case, uz is an edge in G' too.
b. p_1 is connected to z, in which case $z, p_1, \dots, p_{2k+1} = v$ is an even-connection between z and v in G so zv is an edge in G'; hence, uv (or uu' if $u = v$) and zw (or zz' if $z = w$) cannot form a gap.
c. p_1 is connected to q_1, in which case $v = p_{2k+1}, p_{2k}, \dots, p_1, q_1, q_2, \dots, q_{2l+1} = w$ gives an even-connection between v and w, and vw is an edge in G'.
d. q_1 is connected to u, in which case $u, q_1, \dots, q_{2l+1} = w$ is an even-connection between u and w in G so uw is an edge in G'; hence, uv (or uu' if $u = v$) and zw (or zz' if $z = w$) cannot form a gap.

In the third case, u and v are even-connected with respect to $e_1 \dots e_s$, and zw is an edge in G, whereas uv is not an edge in G. Like before, we may assume $\gcd(uv, zw) = 1$. Let $u = p_0, \dots, p_{2k+1} = v$ be an even-connection between u, v with respect to $e_1 \dots e_s$. If $z = p_1$, then $uz = up_1$ is an edge in G and if $w = p_1$, then $uw = up_1$ is an edge in G, so there is nothing to prove in these cases. Otherwise, as up_1 and zw are edges in G and G is gap free, there are four choices:

a. u is connected to z, in which case uv (or uu' in case $u = v$) and zw cannot form a gap as in that case, uz is an edge G' too.
b. p_1 is connected to z, in which case $z, p_1, \dots, p_{2k+1} = v$ is an even-connection between z and v in G, so zv is an edge in G'; hence, uv (or uu' if $u = v$) and zw cannot form a gap.
c. p_1 is connected to w, in which case $v = p_{2k+1}, p_{2k}, \dots, p_1, w$ is an even-connection; hence, uv and zw can not form a gap.
d. w is connected to u, in which case uw is an edge in G, and hence in G'.

This finishes the proof. □

Lemma 6.15 *Suppose G is gap free. If $w_1, \dots, w_n$ is an anticycle in the graph G' defined by $(I(G)^{s+1} : e_1 \dots e_s)$ for some $s \geq 1$ and for $n \geq 5$, then $w_1, \dots, w_n$ is an anticycle in G.*

Proof First of all, whiskers on any vertex cannot be part of any anticycle of length ≥ 5 as they only have degree 1. Observe that it is enough to prove that for all i and j, w_i and w_{i+j} are never even-connected with respect to $e_1 \dots e_s$. Suppose on the contrary such i, j exists. Without loss of generality, we may choose j to be minimal such that for some i, w_i and w_{i+j} are even-connected with respect to $e_1 \dots e_s$. Observe that $j \geq 2$ as $w_i w_{i+1}$ cannot be connected in an anticycle. Without loss of generality, we may further assume that w_1 and w_{1+j} are even-connected with respect to $e_1 \dots e_s$ via $w_1 = p_0, p_1, \dots, p_{2k+1} = w_{1+j}$. Now, observe that w_{2+j} is not connected to p_1 by an edge in G as that will force w_{1+j} and w_{2+j} to be connected in G' by Observation 6.4 leading to a contradiction. So, there exists a smallest $l \geq 0, 2+j \leq n-l \leq n$ such that w_{n-l} is not connected to p_1 by an edge in G. If $l = 0$, then w_n is not connected to p_1 by an edge in G and if $l > 0$, then w_{n-l} is not connected to p_1 by an edge to p_1 in G, and $w_n, w_{n-1}, \dots, w_{n-l+1}$ are connected to p_1 by an edge in G.

Next, we look at the edge $w_2 w_{n-l}$ in G'. If w_2 is connected to p_1 in G, then $w_2, p_1, \dots, p_{2k+1} = w_{1+j}$ will be an even-connection that will violate the minimality of j. If w_2 is connected to p_2 in G, then by Observation 6.4, $w_1 w_2$ has to be an edge in G', which will contradict the fact that $w_1 \dots w_n$ is an anticycle. We observe w_{n-l} cannot be connected to p_1 by selection. If w_{n-l} is connected to p_2 and $l = 0$, then by Observation 6.4, w_1 and w_n have to be connected to each other in G'. If w_{n-l} is connected to p_2 and $l > 0$, then by Observation 6.4, w_{n-l+1} and w_{n-l} have to be connected to each other in G'. Both cases lead to a contradiction as $w_1 \dots w_n$ is an anticycle, so w_2 and w_{n-l} are not connected to each other in G, and neither of them is connected to p_1 or p_2 (and hence w_2, w_{n-l}, p_1, p_2 are four distinct vertices). As $p_1 p_2$ is an edge in G, $w_2 w_{n-l}$ cannot be an edge in G; otherwise, they will form a gap. So w_2 and w_{n-l} are even-connected with respect to $e_1 \dots e_s$. Let $w_2 = q_0, \dots, q_{2r+1} = w_{n-l}$ be an even-connection between w_2 and w_{n-l} with respect to $e_1 \dots e_s$.

If for some $t_1, t_2 \geq 0$, $p_{2t_1+1} p_{2t_1+2}$ and $q_{2t_2+1} q_{2t_2+2}$ are the same edges of G, then by Lemma 6.13, w_2 has to be even-connected to either w_1 or w_{1+j}. The first case is not possible as $w_1 \dots w_n$ is an anticycle and the second case is not possible by the minimality of j. So for no $t_1, t_2 \geq 0$, $p_{2t_1+1} p_{2t_1+2}$ and $q_{2t_2+1} q_{2t_2+2}$ are the same edges of G. So we look at $w_{n-l} q_{2r}$ and $p_1 p_2$. Observe that p_1 is not connected to w_{n-l} because of the selection. If w_{n-l} is connected to p_2 and $l = 0$, then by Observation 6.4, w_1 and w_n have to be connected to each other in G'. If w_{n-l} is connected to p_2 and $l > 0$, then by Observation 6.4, w_{n-l+1} and w_{n-l} have to be connected to each other in G'. Both cases lead to a contradiction as $w_1 \dots w_n$ is an anticycle. So p_2 is not connected to w_{n-l} in G. If p_1 is connected to q_{2r}, then w_2 and w_{1+j} will be even-connected with respect to $e_1 \dots e_s$ violating the minimality of j. If p_2 is connected to q_{2r}, then w_1 and w_2 will be even-connected and, hence, connected in G'.

Hence, for no i, j are w_i and w_{i+j} even-connected with respect to $e_1 \dots e_s$. So $w_1 \dots w_n$ is an anticycle in G. □

Using this lemma, we get the following theorem of Herzog et al. [13, Theorem 1.2] as a corollary:

Theorem 6.16 *If $I(G)$ has linear resolution, then for all $s \geq 2$, $I(G)^s$ has regularity $2s$. In other words, $I(G)^s$ has a linear minimal free resolution.*

Proof As $I(G)$ has a linear resolution, it is gap free, and hence the polarizations of all $(I(G)^{s+1} : e_1 \dots e_s)$ are gap free by Lemma 6.14, and any anticycle of length ≥ 5 in the polarization of $(I(G)^{s+1} : e_1 \dots e_s)$ is an anticycle of G by Lemma 6.15. But as $I(G)$ has linear resolution, G does not have any anticycle. By Theorem 2.12, $\operatorname{reg}(I(G)^{s+1} : e_1 \dots e_s)^{\text{pol}} = 2$ for all $e_1 \dots e_s$. Hence, $\operatorname{reg}(I(G)^{s+1}) = 2s + 2$ by Theorems 1.2 and 6.10. □

Next, we prove that for any gap free and cricket free graph G, and for all $s \geq 2$, $\operatorname{reg}(I(G)^s) = 2s$. This result is our main new result in this paper. This answers Question 1.1 partially. This also generalizes Nevo's result (Theorem 1.2 of [12]) that for any gap-free and claw-free graph G, $\operatorname{reg} I(G)^2 = 4$.

Theorem 6.17 *For any gap free and cricket free graph G and for all $s \geq 2$, $\operatorname{reg}(I(G)^s) = 2s$.*

Proof In light of Theorem 2.12, Theorem 3.4, Corollary 5.3, Theorem 6.10, and Lemma 6.14, it is enough to show that the polarization of $(I(G)^{s+1} : e_1 \dots e_s)$ does not have any anticycle $w_1 \dots w_n$ for $n \geq 5$, $s \geq 1$, for every s-fold product $e_1 \dots e_s$.

Suppose $w_1 \dots w_n$, $n \geq 5$, is an anticycle in the polarization of $(I^{s+1} : e_1 \dots e_s)$ and $e_1 = xy$. By Lemma 6.15, $w_1 \dots w_n$ is also an anticycle of G. Either w_1 or w_3 is a neighbor of x or neighbor of y else $w_1 w_3$ and e_1 forms a gap in G, a contradiction. Without loss of generality, we may assume that w_1 is a neighbor of x. Now neither w_2 nor w_n can be x as they are not connected to w_1; also, neither of them is y as if say $y = w_2$, then $w_n x y w_1$ is an even-connection; hence, $w_1 w_n$ is an edge in G', a contradiction to the assumption on anticycle; similar thing happens if $y = w_n$. By Observation 6.4 every neighbor of y is connected to every neighbor of x in G'. As neither $w_1 w_n$, nor $w_1 w_2$ is an edge in G', neither of w_2 and w_n are neighbors of y in G. So, one of them has to be neighbor of x, as G is gap free. Again, without loss of generality, we may assume that w_2 is a neighbor of x. Next, we consider $w_3 w_n$. As w_1 and w_2 are neighbors of x and neither $w_1 w_n$ nor $w_2 w_3$ are edges in G', by Observation 6.4, neither w_3 nor w_n can be neighbor of y. Neither w_3 nor w_n can be x as they are $w_2 w_3$ and $w_1 w_n$ are not edges in G'. If $w_3 = y$, as $w_1 w_3$ is an edge in G, then w_1, being a neighbor of y, has to be connected to w_2, which is a neighbor of x in G' by Observation 6.4. That will force $w_1 w_2$ to be an edge in G', which is a contradiction. Similarly, if $w_n = y$, w_3 being a neighbor of y has to be connected to w_2 in G' leading to a contradiction. Then, either w_3 or w_n has to be a neighbor of x. Without loss of generality, we may assume that w_3 is a neighbor of x. Notice that y is not connected to w_1 in G as that will force w_2, a neighbor of x to be connected to w_1 in G' leading to a contradiction. Hence, $\{y, w_2, x, w_1, w_3\}$ forms a cricket. □

Next, we prove that for any gap-free graph G with $\operatorname{reg}(I(G)) = r$, the $\operatorname{reg}(I(G)^s)$ is bounded above by $2s + r - 1$. But to do that, we need a lemma about "longest" connections. Observe that if G' is the graph associated to the polarization of $(I(G)^{s+1} : e_1 \dots e_s)$, for some s-fold product, and u and v are even-connected with respect to $u = p_0, \dots, p_{2k+1} = v$, then uv is not only an edge in G' but also an edge in the graph $(G' - \{y_1, \dots, y_l\})$ for any set of points $y_1, \dots, y_l$ as long as $u, v \notin \{y_1, \dots, y_l\}$. We further emphasize that some of the p_is can also belong to $\{y_1, \dots, y_l\}$ as long as they are not same as u or v.

Lemma 6.18 *Let G' be the graph associated to the polarization of $(I(G)^{s+1} : e_1 \dots e_s)$ for some s-fold product. Let us assume that u and v are even-connected with respect to $u = p_0, \dots, p_{2k+1} = v$. Suppose for some set of vertices $\{y_1, \dots, y_l\}$, we have $u, v \notin \{y_1, \dots, y_l\}$. Let us also assume for any other even-connection $u' = p'_0, \dots, p'_{2k'+1} = v'$ such that $u', v' \notin \{y_1, \dots, y_l\}$, we have $k' \leq k$. Then, $(G' - \{y_1, \dots, y_l\} - st\, u)$ is $G'' \cup$ {isolated whisker vertices}, where G'' is a subgraph of G obtained by deleting vertices.*

Proof For the set of points $\{y_1, \dots, y_l\}$, uv is an edge in $(G' - \{y_1, \dots, y_l\})$ such that $u, v \notin \{y_1, \dots, y_l\}$ are even-connected with respect to $e_1 \dots e_s$ via $u = p_0, p_1, p_2, \dots, p_{2k+1} = v$. We also have that k is maximum over all such even-connected edges in $(G' - \{y_1, \dots, y_l\})$. Let $u'v'$ be any edge in $(G' - \{y_1, \dots, y_l\})$ such that $u', v' \notin \{y_1, \dots, y_l\}$ and they are even-connected with respect to $e_1 \dots e_s$ via $u' = x_0, x_1, x_2, \dots, x_{2k'+1} = v'$. If for any j, j', $p_{2j+1}p_{2j+2}$ and $x_{2j'+1}x_{2j'+2}$ form the same edge in G then by Lemma 6.13, either u' or v' will be not a vertex in $(G' - \{y_1, \dots, y_l\} - st\, u)$. Now observe, if for any j, j', $p_{2j+1}p_{2j+2}$ and $x_{2j'+1}x_{2j'+2}$ do not form same edge in G then either x_1 or x_2 has to be connected to p_1 or p_2 to avoid x_1x_2 and p_1p_2 forming a gap. If any of them (for example x_1) is connected to p_1 in G that will make $\{v' = x_{2k'+1}, x_{2k'}, \dots, x_1, p_1, \dots, p_{2k+1}\}$ a longer connection violating the maximality of k. A similar thing happens if x_2 is connected to p_1 in G. So either of them has to be connected to p_2. If x_1 is connected to p_2 in G, then u is connected to v' in G' as $u, p_1, p_2, x_1, \dots, x_{2k'+1} = v'$ will be an even-connection. Similarly, if x_2 is connected to p_2, then u is connected to u' in G' as $u, p_1, p_2, x_2, x_1, u'$ will be an even-connection. In both the cases, either u' or v' will not be a vertex in $(G' - \{y_1, \dots, y_l\} - st\, u)$. This proves that any edge in $(G' - \{y_1, \dots, y_l\} - st\, u)$ is an edge in G. Hence, the Lemma follows. □

Using Lemma 6.18, we prove the next theorem which guarantees that the gap between the regularity of powers of edge ideals of gap-free graphs and the regularity of monomial ideals generated in the same degree and having a linear resolution cannot be arbitrarily large.

Theorem 6.19 *For any gap-free graph G with $\operatorname{reg}(I(G)) = r$ and any $s \geq 2$, the $\operatorname{reg}(I(G)^s)$ is bounded above by $2s + r - 1$.*

Proof Let G' be the graph associated to the polarization of $(I(G)^{s+1} : e_1 \dots e_s)$. We have $\operatorname{reg}(G') \leq \max\{\operatorname{reg}(G' - st x) + 1, \operatorname{reg}(G' - x)\}$ by Lemma 3.2 for each vertex x. We choose u_1 and v_1 even-connected by $u_1 = p_0, \dots, p_{2k_1+1} = v_1$ such that k_1 is maximum. By Lemma 6.18, $(G' - st\, u_1)$ is a subgraph of G obtained by vertex deletion along with some isolated whisker vertices. As isolated vertices do not affect the regularity of edge ideal, $\operatorname{reg}((G' - st\, u_1) \leq r$ by Lemma 2.10.

Next, we apply Lemma 3.2 on $(G' - u_1)$, from which we delete a vertex u_2 which is even-connected to another vertex v_2 via $u_2 = q_0, \dots, q_{2k_2+1} = v_2$ with k_2 maximum. Again, by Lemma 6.18, $(G' - u_1 - st\, u_2)$ is a subgraph obtained from $G - u_1$ by deletion of vertices along with some whisker vertices. Hence, $\operatorname{reg}(G' - u_1 - st\, u_2) \leq r$. We keep selecting $u_1, u_2, \dots$ and apply Lemmas 3.2 and 6.18. As we are in a finite setup, for some l, $(G' - u_1, \dots, u_l)$ itself is a subgraph of G obtained by repeated vertex deletion along with some isolated whisker vertices and $reg(G') \leq r + 1$. Therefore, by Theorem 1.2 and induction, the result follows. □

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