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The Regularized Low Pass Filter

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ABSTRACT

In this paper the low pass filter is discussed in the noisy case. And a regularized low pass filter is presented. The convergence property of the regularized low pass filtering algorithm is proved in theory and tested by numerical results.

KEYWORDS

Noise; Low Pass Filter; Regularization

1. Introduction

Filtering is widely applied in engineering [1-5]. In this paper, the problem of the low pass filtering is analyzed in theory and by examples in detail. A regularized low pass filtering algorithm is presented with the proof of the convergence property and numerical results.

First, we describe the band-limited signals $f \in L^1(\mathbf{R})$. The details can be seen in [6].

Definition: A function $f \in L^1(\mathbf{R})$ is said to be Ω band-limited if

$$\hat{f}(\omega) = 0,\ \forall \omega \notin [-\Omega, \Omega].$$

Here $\hat{f}$ is the *Fourier transform* of f:

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t, \omega \in \mathbf{R}. \quad (1)$$

We then have the inversion formula:

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega, \text{ a.e. } t \in \mathbf{R}.$$

In many practical problems, the signal $f(t)$ is noisy:

$$f(t) = f_E(t) + \eta(t) \quad (2)$$

where $\eta(t)$ is the noise

$$|\eta(t)| \le \delta \quad (3)$$

and $f_E \in L^1$ is the exact band-limited signal.

In this paper, we will consider the problem low pass filtering:

$$f_L(t) = \int_{-\infty}^{\infty} f(u) \frac{\sin \Omega(t-u)}{\pi(t-u)} \mathrm{d}u, \text{ a.e. } t \in \mathbf{R}. \quad (4)$$

If the signal is noisy however, the filter is not reliable. We will give an example to show that the noise can become very large after the low pass filtering process. So this filter is not reliable in the noisy case. And a regularized low pass filtering algorithm will be presented.

In Section 2 we give the property of the low pass filter. A regularized filtering algorithm and the proof of its convergence are in Section 3. The numerical results of some examples are given in Section 4. Finally, the conclusion is given in Section 5.

2. The Property of the Low Pass Filter

In this section, we discuss the property of the low pass filter.

Example. Assume the noise is $\eta(t) = \epsilon \operatorname{sgn}\{\sin \Omega(t_0 - t) / \Omega(t_0 - t)\}$ where t_0 is a given point in the time domain and ϵ is close to zero. Then the noise signal after the filtering is

$$\eta_L(t) = \int_{-\infty}^{\infty} \eta(u) \frac{\sin \Omega(t-u)}{\Omega(t-u)} \mathrm{d}u$$
$$= \epsilon \int_{-\infty}^{\infty} \frac{\sin \Omega(t-u)}{\Omega(t-u)} \operatorname{sgn}\left[\frac{\sin \Omega(t_0-u)}{\Omega(t_0-u)}\right] \mathrm{d}u.$$

We can see that $|\eta(t)| \le \epsilon$. However, the noise at $t = t_0$ after the filtering is

$$\eta_L(t_0) = \epsilon \int_{-\infty}^{\infty} \left| \frac{\sin\Omega(t_0 - u)}{\Omega(t_0 - u)} \right| du = \infty.$$

Also at any point $t = t_0 + k\pi/\Omega$, $k \in \mathbf{Z}$

$$\begin{aligned} &\eta_L(t_0 + k\pi/\Omega) \\ &= \epsilon \int_{n=-\infty}^{\infty} (-1)^k \frac{\sin\Omega(t_0 - u)}{\Omega(t_0 + k\pi/\Omega - u)} \operatorname{sgn}\left[\frac{\sin\Omega(t_0 - u)}{\Omega(t_0 - u)}\right] du \\ &= (-1)^k \epsilon \int_{-\infty}^{\infty} \frac{\Omega(t_0 - u)}{\Omega(t_0 + k\pi/\Omega - u)} \left| \frac{\sin\Omega(t_0 - u)}{\Omega(t_0 - u)} \right| du = \pm\infty. \end{aligned}$$

So the error after the filtering becomes ∞.

Remark. This is only an example for analysis. In the section of numerical results we will show that the low pass filter (4) is not very effective for white noise.

3. The Regularized Filtering Algorithm

First, we consider the regularized Fourier transform [7]:

$$\hat{f}_\alpha(\omega) = \int_{-\infty}^{\infty} \frac{f(t) e^{i\omega t} dt}{1 + 2\pi\alpha + 2\pi\alpha t^2},$$

where $\alpha > 0$ is the regularization parameter. Here $\hat{f}_\alpha(\omega)$ is the minimizer of a smoothing functional. We have proved $\hat{f}_\alpha(\omega)$ converges to the exact Fourier transform as the error of $f(t)$ approaches to zero. In [7], we have successfully used the regularized Fourier transform in extrapolation. So the weight function

$$\frac{1}{1 + 2\pi\alpha + 2\pi\alpha t^2}$$

is helpful to solve ill-posed problems.

Based on the regularized Fourier transform we present the regularized filtering formula:

$$f_\alpha(t) = \int_{-\infty}^{\infty} \frac{\sin\Omega(t - u)}{\pi(t - u)} \frac{f(u) du}{1 + 2\pi\alpha + 2\pi\alpha u^2}, \text{ a.e. } t \in \mathbf{R} \quad (5)$$

where $f(t)$ is given in (2).

The convergence property of this regularized filtering formula is given in the theorem below.

Theorem 3.1. For $f_E \in L^1$, if $\alpha \to 0$ and $\delta/\sqrt{\alpha} \to 0$ as $\delta \to 0$, then $f_\alpha \to f_E$ according to the maximum norm as $\delta \to 0$.

Proof.

$$\begin{aligned} &\hat{f}_\alpha(t) - f_E(t) \\ &= -\int_{-\infty}^{\infty} \frac{f_E(u)(2\pi\alpha + 2\pi\alpha u^2)}{1 + 2\pi\alpha + 2\pi\alpha u^2} \frac{\sin\Omega(t - u)}{\pi(t - u)} du \\ &+ \int_{-\infty}^{\infty} \frac{\eta(u)}{1 + 2\pi\alpha + 2\pi\alpha u^2} \frac{\sin\Omega(t - u)}{\pi(t - u)} du \end{aligned}$$

where

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{\eta(u)}{1 + 2\pi\alpha + 2\pi\alpha u^2} \frac{\sin\Omega(t - u)}{\pi(t - u)} du \right| \\ &\le \delta \int_{-\infty}^{\infty} \frac{1}{1 + 2\pi\alpha + 2\pi\alpha u^2} du = \frac{\pi\delta}{\sqrt{2\pi\alpha(1 + 2\pi\alpha)}}. \end{aligned}$$

For each $\epsilon > 0$, there exists $T > 0$ such that

$$\int_{|u|>T} |f(u)| du < \epsilon.$$

Then

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{f_E(u)(2\pi\alpha + 2\pi\alpha u^2)}{1 + 2\pi\alpha + 2\pi\alpha u^2} \frac{\sin\Omega(t - u)}{\pi(t - u)} du \right| \\ &\le \int_{-T}^{T} \frac{|f_E(u)|(2\pi\alpha + 2\pi\alpha u^2)}{1 + 2\pi\alpha + 2\pi\alpha u^2} du + \int_{|u|>T} |f_E(u)| du \end{aligned}$$

where

$$\int_{-T}^{T} \frac{|f_E(u)|(2\pi\alpha + 2\pi\alpha u^2)}{1 + 2\pi\alpha + 2\pi\alpha u^2} du \to 0$$

as $\alpha \to 0$.

4. Numerical Results

In this section, we give some examples to show that the regularized filtering algorithm (5) is more effective in reducing the noise than the convolution (4).

Suppose the exact signal in example 1 and 2 is

$$f_E(t) = \frac{1 - \cos t}{\pi t^2}.$$

Then construct

$$\hat{f}_E(\omega) = \begin{cases} 1 - |\omega|, & \omega \in [-\Omega, \Omega] \\ 0, & \omega \notin [-\Omega, \Omega] \end{cases}$$

where $\Omega = 1$.

Example 1. We consider the noise

$$\eta(nh) = \epsilon \operatorname{sgn}\{\sin\Omega(t_0 - nh)/\Omega(t_0 - nh)\}$$

where $h = \pi/\Omega = \pi$, $t_0 = 0$, and $\epsilon = 0.05$. This noise is used in the analysis of the stability in Section 2.

The result of (4) and the result of the regularized filtering algorithm with $\alpha = 0.02$ are in **Figure 1**.

Example 2. We consider the noise to be white noise that is Gauss distribution whose variance is 0.01. The result of (4) and the result of the regularized sampling algorithm with $\boldsymbol{\alpha = 0.02}$ are in **Figure 2**.

5. Conclusion

The filter of convolution with sinc function is not stable. For some noises the results of the filtering are not reliable.

Regularized filtering algorithm is more effective in reducing the noise.

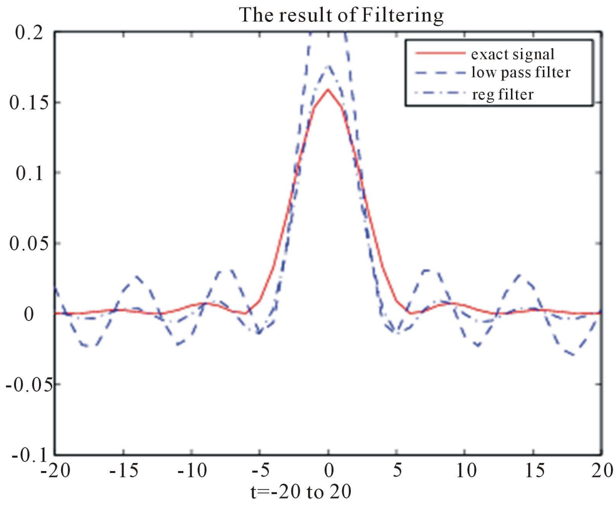

Figure 1. The numerical results of example 1.

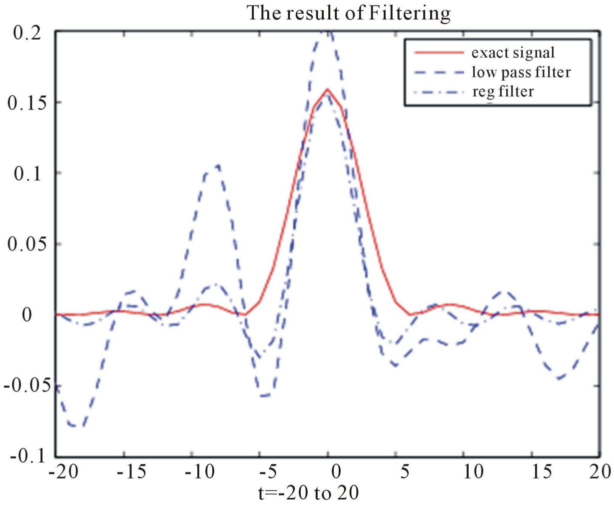

Figure 2. The numerical results of example 2.

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