

The Reidemeister spectrum of 2-step nilpotent groups determined by graphs

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Abstract

In this paper we study the Reidemeister spectrum of 2-step nilpotent groups associated to graphs. We develop three methods, based on the structure of the graph, that can be used to determine the Reidemeister spectrum of the associated group in terms of the Reidemeister spectra of groups associated to smaller graphs. We illustrate our methods for several families of graphs, including all the groups associated to a graph with at most four vertices. We also apply our results in the context of topological fixed point theory for nilmanifolds.

1 Introduction

In this paper we will be studying Reidemeister numbers (this is the number of so called twisted conjugacy classes) of automorphisms of a wide class of 2-step nilpotent groups. Twisted conjugacy finds its origin in topological fixed point theory (see below) but pops up in several branches of mathematics, such as representation theory ([OV90], [Spr06]), Galois cohomology ([Ser02]), cryptography ([GK16]), ...

Our own main motivation comes from topological fixed point theory, more specifically from Reidemeister-Nielsen fixed point theory. We give a short overview of the main aspects of this theory and refer the reader to [JM06, Jia83, Th89] for more details.

Let $f : X \rightarrow X$ be a map on a closed manifold X and denote with $\text{Fix}(f) = \{x \in X \mid f(x) = x\}$ the set of fixed points of f . The main objective of Reidemeister-Nielsen fixed point theory is to find a good estimate for the minimal value of $\#\text{Fix}(g)$ where g is a map which is homotopic to f . Let us call this value $\text{MF}(f)$.

To study the fixed points of f one considers the universal covering space $p : \tilde{X} \rightarrow X$ of X . Then f can be lifted to a map $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ (with $p \circ \tilde{f} = f \circ p$) and it is easy to see that $p(\text{Fix}(\tilde{f})) \subseteq \text{Fix}(f)$. In fact, $\text{Fix}(f)$ is the union of all $p(\text{Fix}(\tilde{f}))$ where the union is taken over all possible lifts \tilde{f} of f . The group of covering transformations of the universal covering is isomorphic to the fundamental group of X and so we denote the group of covering transformations by $\pi(X)$. For any $\alpha, \beta \in \pi(X)$ and any lift \tilde{f} of f it holds that $\alpha \circ \tilde{f} \circ \beta$ is again a lift of f . It follows that $\pi(X)$ acts on the set of all lifts \tilde{f} of f via conjugation, so $\gamma \cdot \tilde{f} = \gamma \circ \tilde{f} \circ \gamma^{-1}$. We denote the orbit of \tilde{f} by $[\tilde{f}]$ and call this the lifting class of \tilde{f} . Then we have that for all $\tilde{f}' \in [\tilde{f}]$ it holds that $p(\text{Fix}(\tilde{f}')) = p(\text{Fix}(\tilde{f}))$ while $p(\text{Fix}(\tilde{f}')) \cap p(\text{Fix}(\tilde{f})) = \emptyset$ in case $[\tilde{f}'] \neq [\tilde{f}]$. From this we can conclude that

$$\text{Fix}(f) = \bigcup_{[\tilde{f}]} p(\text{Fix}(\tilde{f})),$$

which is a disjoint union. So in this union we consider one subset $p(\text{Fix}(\tilde{f}))$ for each lifting class $[\tilde{f}]$ and we call this the fixed point class of f determined by the lifting class $[\tilde{f}]$. Note that a fixed point class can be empty, but we still consider two empty fixed point classes different in case they are determined by a different lifting class. Hence, the number of fixed point classes is the same as the number of lifting classes and this number is called the Reidemeister number of f and is denoted by $R(f)$.

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There is an algebraic way to count the fixed point classes of a map f and this goes as follows. Fix one lifting \tilde{f}_0 of f . Then any other lift of f can be written uniquely as a composition $\alpha \circ \tilde{f}_0$ for some $\alpha \in \pi(X)$. So the set of liftings of f is in one-to-one correspondence with the fundamental group $\pi(X)$. The lift \tilde{f}_0 determines an endomorphism f_* of $\pi(X)$ by the relation $f_*(\alpha) \circ \tilde{f}_0 = \tilde{f}_0 \circ \alpha$ (for all $\alpha \in \pi(X)$). Note that under the right identification of $\pi(X)$ with the fundamental group of X , f_* is just the usual induced endomorphism of f on the fundamental group of X .

Now $\alpha \circ \tilde{f}_0 = \gamma \circ (\beta \circ \tilde{f}_0) \circ \gamma^{-1}$ if and only if $\alpha = \gamma \circ \beta \circ f_*(\gamma^{-1})$. In this case, we will say that α and β are twisted conjugate with respect to f_* . Being twisted conjugate is an equivalence relation on $\pi(X)$ and the number of equivalence classes is called the Reidemeister number of the morphism f_* and is denoted by $R(f_*)$. From the above we have that $R(f) = R(f_*)$ and so counting twisted conjugacy classes is an algebraic way of counting fixed point classes (or lifting classes).

Although the Reidemeister number of a map gives already some information about the fixed point classes of f , in general this number does not really give information on $\text{MF}(f)$, the minimal number of fixed points in the homotopy class of f . There is a second number, the Nielsen number of f which does provide more information, but unfortunately is much more difficult to compute in general. To define the Nielsen number, there is a way to attach to each fixed point class an index, which is an integer. It would lead us too far to explain this index in more detail, but the idea is that a fixed point class has index 0 if it can disappear (become empty) via a homotopy. A fixed point class is called essential (resp. non essential) if it has an index $\neq 0$ (resp. = 0). The Nielsen number of f , denoted by $N(f)$, is then the number of essential fixed point classes of f . This Nielsen number (and also the Reidemeister number) is a homotopy invariant and by a result of Wecken ([Wec42]) it is known that $\text{MF}(f) = N(f)$ for all manifolds of dimension at least 3.

The focus of this paper lies on the class of nilmanifolds, these are obtained as quotient spaces $X = N \backslash G$, where G is a simply connected nilpotent Lie group and N is a uniform lattice of G . Such a uniform lattice N is a finitely generated torsion-free nilpotent group and $\pi(X) = N$ completely determines the nilmanifold $X = N \backslash G$ up to diffeomorphism (see e.g. [Rag72, OV93]).

For the class of nilmanifolds, there is a very strong relation between the Reidemeister number and the Nielsen number of a map f on such a manifold. Indeed, we have that (see [HK97]):

$$\begin{cases} N(f) = R(f) & \iff R(f) < \infty \\ N(f) = 0 & \iff R(f) = \infty \end{cases} .$$

As a conclusion we see that for nilmanifolds, we obtain a full understanding of the minimal number of fixed points ($\text{MF}(f) = N(f)$) in the homotopy class of a map f by studying the Reidemeister number $R(f)$ of that map and hence by studying the number $R(f_*)$ of twisted conjugacy classes of the induced endomorphism f_* .

It is not so difficult to see that for any nilmanifold $N \backslash G$ and any non negative integer n there is a self map f of $N \backslash G$ with $N(f) = n$ ([DTV20, Theorem 6.1]). The situation for self homeomorphisms (or self homotopy equivalences) is much more subtle and corresponds to the case where f_* is an automorphism of N . The set of all possible Reidemeister numbers one can obtain for these self homotopy equivalences f (automorphisms f_*) is called the Reidemeister spectrum of the manifold (or of the fundamental group N). In this paper we study this Reidemeister spectrum for groups N which are 2-step nilpotent and are associated to a graph. The nilmanifolds with such a 2-step nilpotent fundamental group have been the object of study in many geometric contexts and form a rich family of interesting examples. We refer to [CdBR21, Ova20, Nik20, DDM18] for some recent examples in this direction. Moreover, results on 2-step nilpotent groups can be used to study general nilpotent groups by considering their 2-step nilpotent quotient.

In the next section we recall some preliminaries on nilpotent groups and twisted conjugacy. Thereafter, we describe the class of 2-step nilpotent groups associated to a graph. In the next three sections we develop each time a general method, based on the structure of the graph, that can be used to determine the Reidemeister spectrum of the associated group, by reducing it to the situation of smaller graphs. We end by illustrating our methods in some general examples and give a full list of Reidemeister spectra for all graphs with at most 4 vertices.

2 Preliminaries on nilpotent groups and twisted conjugacy

2.1 Nilpotent groups

For any group G we denote with $\gamma_i(G)$ (for $i \in \mathbb{N}_0$) the *lower central series* of G , i.e. the nested series of subgroups of G defined by $\gamma_1(G) := G$ and $\gamma_{i+1}(G) := [\gamma_i(G), G]$ (for $i \in \mathbb{N}_0$). The group G is said to be *c-step nilpotent* if $\gamma_c(G) \neq 1$ and $\gamma_{c+1}(G) = 1$. It is generally known (see e.g. [KM79, Theorem 17.2.2]) that any finitely generated nilpotent group G has a series $1 = G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_s = G$ with cyclic factors, i.e. G_{i+1}/G_i is cyclic for any $i = 1, 2, \dots, s-1$. The *Hirsch number* $h(G)$ of G is the number of infinite cyclic factors in such a series. The next lemma describes some properties of the Hirsch number.

Lemma 2.1 ([Seg83, page 16]). *If G is a finitely generated nilpotent group, then the Hirsch number is well-defined (meaning that it is independent of the choice of series of G with cyclic factors). If $H \subseteq G$ is a subgroup of G and $N \triangleleft G$ a normal subgroup, then the following holds:*

- (i) $h(H) \leq h(G)$
- (ii) $h(H) = h(G) \iff [G : H] < \infty$
- (iii) $h(G) = h(N) + h(G/N)$
- (iv) $h(G) = 0 \iff |G| < \infty$

In section 3 we describe how to associate a finitely generated torsion-free 2-step nilpotent group to any finite undirected simple graph. The next two lemmas will be used frequently.

Lemma 2.2 ([KMS76]). *If G is a 2-step nilpotent group, then $[\cdot, \cdot] : G \times G \rightarrow G$ is bilinear, i.e. for all $g_1, g_2, g'_1, g'_2 \in G$ it holds that*

$$[g_1 g_2, g'_1 g'_2] = [g_1, g'_2] \cdot [g_2, g'_1] \cdot [g_1, g'_1] \cdot [g_2, g'_2].$$

Lemma 2.3. *If G is a finitely generated torsion-free nilpotent group and $\varphi : G \rightarrow G$ is a surjective morphism, then φ is also injective. In particular, φ is an automorphism of G .*

Proof. Since φ is surjective, it follows by the first isomorphism theorem and Lemma 2.1 (iii) that

$$h(\text{Ker}(\varphi)) = h(G) - h(G/\text{Ker}(\varphi)) = h(G) - h(\text{Im}(\varphi)) = 0.$$

So Lemma 2.1 (iv) implies that $\text{Ker}(\varphi)$ is finite. However, since G is torsion-free we obtain that $\text{Ker}(\varphi)$ is a finite torsion-free group. Hence, $\text{Ker}(\varphi)$ is trivial. \square

Fix any endomorphism $\varphi \in \text{End}(G)$. We denote with $\varphi_i \in \text{End}(\gamma_i(G)/\gamma_{i+1}(G))$ (for $i \in \mathbb{N}_0$) the induced morphisms on the factors of the lower central series of G . Since the terms of the lower central series are characteristic subgroups of G , these induced morphisms are well-defined. Moreover, if G is a finitely generated c -step nilpotent group, then these factors are finitely generated abelian groups (see for example Lemma 17.2.1 in [KM79]).

In case $\gamma_i(G)/\gamma_{i+1}(G)$ is a free abelian group, so isomorphic to \mathbb{Z}^k for some k , we can describe φ_i by a $k \times k$ matrix over \mathbb{Z} and in this way we can talk about the eigenvalues of φ_i and the determinant of φ_i . In case $\gamma_i(G)/\gamma_{i+1}(G)$ has torsion, we can still talk about the eigenvalues of φ_i by which we mean the eigenvalues of the induced endomorphism on the torsion-free quotient $\frac{\gamma_i(G)/\gamma_{i+1}(G)}{\tau(\gamma_i(G)/\gamma_{i+1}(G))}$ (where $\tau(\gamma_i(G)/\gamma_{i+1}(G))$ denotes the torsion subgroup of $\gamma_i(G)/\gamma_{i+1}(G)$).

The next result will be needed later on.

Lemma 2.4. *Let G be a finitely generated torsion-free nilpotent group and $\varphi : G \rightarrow G$ a morphism such that $\varphi_1 : G/\gamma_2(G) \rightarrow G/\gamma_2(G)$ is an automorphism. Then φ is an automorphism.*

Proof. Since φ_1 is surjective, it follows that

$$\frac{G}{\gamma_2(G)} = \text{Im}(\varphi_1) = \frac{\text{Im}(\varphi) \gamma_2(G)}{\gamma_2(G)}.$$

So it holds that $\text{Im}(\varphi) \gamma_2(G) = G$. Since G is nilpotent, this implies that $\text{Im}(\varphi) = G$ (see e.g. [KM79, Theorem 16.2.5]). Hence, φ is a surjective morphism and thus Lemma 2.3 yields the result. \square

2.2 Twisted conjugacy

In the introduction we already considered the notion of twisted conjugacy. Let us now define this concept in some more detail. Let G be a group and $\varphi \in \text{End}(G)$ a group endomorphism. Two elements $a, b \in G$ are called φ -conjugate or *twisted conjugate* (denoted by $a \sim_\varphi b$) if there exists some third element $c \in G$ such that $a = cb\varphi(c)^{-1}$. This induces an equivalence relation \sim_φ on G . The equivalence classes are called the *Reidemeister classes* of φ and the number of equivalence classes is called the *Reidemeister number* $R(\varphi)$ of φ . The collection of all Reidemeister numbers, by only considering automorphisms of G , is called the *Reidemeister spectrum* of G and is denoted by $\text{Spec}_R(G)$. Formally, we define $\text{Spec}_R(G) := \{R(\varphi) \mid \varphi \in \text{Aut}(G)\} \subseteq \mathbb{N}_0 \cup \{\infty\}$. If $\text{Spec}_R(G) = \{\infty\}$, then G is said to have the R_∞ -property. If $\text{Spec}_R(G) = \mathbb{N}_0 \cup \{\infty\}$, then G has a *full Reidemeister spectrum*.

The Reidemeister spectrum of a group is in general difficult to compute. However, when working with finitely generated nilpotent groups, there are some well-known techniques to study their Reidemeister spectrum.

We list two theorems without a proof.

Proposition 2.5 ([DG14, Lemma 2.2] and [Rom11, Corollary 4.2]). *Let G be a finitely generated c -step nilpotent group and φ an automorphism of G . Then the following are equivalent:*

- (i) $R(\varphi) = \infty$
- (ii) *There exists some $i = 1, 2, \dots, c$ such that φ_i has 1 as an eigenvalue.*
- (iii) *There exists some $i = 1, 2, \dots, c$ such that $R(\varphi_i) = \infty$.*

Theorem 2.6 ([DGO21, Proposition 5] and [Rom11, Lemma 2.7]). *Let G be a finitely generated nilpotent group. Let*

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots \supseteq G_c \supseteq G_{c+1} = 1$$

be a central series of G and $\varphi \in \text{Aut}(G)$ such that the following holds:

1. *All the factors G_i/G_{i+1} (with $i = 1, 2, \dots, c$) are torsion-free.*
2. *For all terms G_i (with $i = 1, 2, \dots, c+1$) it holds that $\varphi(G_i) = G_i$.*

Then it holds that:

$$R(\varphi) = \prod_{i=1}^c R(\varphi_i)$$

where $\varphi_i : G_i/G_{i+1} \rightarrow G_i/G_{i+1}$ (with $i = 1, 2, \dots, c$) are the induced automorphisms on the factor groups G_i/G_{i+1} .

Remark 2.7. Note that in Theorem 2.6 we used the notation φ_i to denote the induced automorphisms on the factor groups of the given central series of G . However, whenever we do not mention a central series, we reserve this notation for the induced automorphisms on the factors of the lower central series (as we introduced above Lemma 2.4).

When describing the Reidemeister spectrum of a group we will frequently use the map $|\cdot|_\infty$ which is defined by

$$|\cdot|_\infty : \mathbb{Z} \rightarrow \mathbb{N}_0 \cup \{\infty\} : x \mapsto |x|_\infty := \begin{cases} |x| & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

where $|\cdot|$ denotes the absolute value.

3 2-step nilpotent groups associated to graphs

To any undirected finite simple graph $\Gamma(V, E)$, we can associate a finitely generated 2-step nilpotent group. We do this by considering the 2-step nilpotent quotient of the *right angled Artin group* associated to Γ . More precisely, the idea is that we take the vertices (which we denote with x_i) as generators of our group and require that two of these generators commute if the corresponding vertices are connected via an edge. We denote with $y_{i,j} = [x_j, x_i]$ the commutators of two vertices that are not connected via an edge. At last, we make the group 2-step nilpotent by adding the constraints that the $y_{i,j}$ commute with all the vertices. The formal definition is given below.

Definition 3.1. Let $\Gamma(\{x_1, x_2, \dots, x_n\}, E)$ be an undirected finite simple graph. We define the group G_Γ by setting

$$G_\Gamma = \left\langle \begin{array}{l} x_1, x_2, \dots, x_n, \\ y_{i,j} \text{ if } x_i x_j \notin E \text{ and } i < j \end{array} \left| \begin{array}{ll} [x_j, x_i] = 1 & \text{if } x_i x_j \in E \\ [x_j, x_i] = y_{i,j} & \text{if } x_i x_j \notin E \text{ and } i < j \\ [x_l, y_{i,j}] = 1 & l = 1, 2, \dots, n; x_i x_j \notin E \text{ and } i < j \end{array} \right. \right\rangle$$

From now on, we will use Γ to denote an undirected finite simple graph. Note that if Γ is the complete graph on $n \in \mathbb{N}_0$ vertices, then the associated group G_Γ is isomorphic with \mathbb{Z}^n . The Reidemeister spectrum of \mathbb{Z}^n is well-known (see for example [Rom11, Section 3]) and is given by

$$\text{Spec}_R(\mathbb{Z}^n) = \begin{cases} \{2, \infty\} & \text{if } n = 1 \\ \mathbb{N}_0 \cup \{\infty\} & \text{if } n \geq 2 \end{cases}. \quad (3.1)$$

If on the other hand Γ is the graph on n vertices without any edges, then the associated group G_Γ is isomorphic with $F_n/\gamma_3(F_n)$ (where F_n denotes the free group of rank n). These groups are also known as the *free nilpotent groups of rank n and nilpotency class 2* and are frequently denoted by $N_{n,2}$. K. Dekimpe, S. Tertooy and A.R. Vargas extended in [DTV20, Section 4] the result from V. Roman'kov (see [Rom11, Section 3]) to

$$\text{Spec}_R(N_{n,2}) = \begin{cases} 2\mathbb{N}_0 \cup \{\infty\} & \text{if } n = 2 \\ (2\mathbb{N}_0 - 1) \cup 4\mathbb{N}_0 \cup \{\infty\} & \text{if } n = 3 \\ \mathbb{N}_0 \cup \{\infty\} & \text{if } n \geq 4 \end{cases}. \quad (3.2)$$

Denote with $N := |\{(i, j) \mid x_i x_j \notin E \text{ and } i < j\}|$. To simplify notation, we fix an order y_1, y_2, \dots, y_N to denote the elements $y_{i,j}$ where we define $y_l := y_{i_l, j_l}$. Using the definition, it follows that any element of G_Γ can be uniquely written as $x_1^{z_1} x_2^{z_2} \dots x_n^{z_n} y_1^{t_1} y_2^{t_2} \dots y_N^{t_N}$ with $z_i, t_l \in \mathbb{Z}$ and that the multiplication in G_Γ is given by

$$(x_1^{z_1} \dots x_n^{z_n} y_1^{t_1} \dots y_N^{t_N})(x_1^{v_1} \dots x_n^{v_n} y_1^{s_1} \dots y_N^{s_N}) = x_1^{z_1+v_1} \dots x_n^{z_n+v_n} y_1^{t_1+s_1+v_{i_1} z_{j_1}} \dots y_N^{t_N+s_N+v_{i_N} z_{j_N}}$$

for $z_i, v_i, t_l, s_l \in \mathbb{Z}$.

Using the operation in G_Γ , we obtain expressions for the center and commutator subgroup of G_Γ .

Lemma 3.2. *With the notations from above, we have that:*

$$Z(G_\Gamma) = \prod_{i=1}^N \langle y_i \rangle \times \prod_{\substack{i=1,2,\dots,n; \\ \deg(x_i)=n-1}} \langle x_i \rangle \cong \mathbb{Z}^{N+(\text{the number of vertices of degree } n-1)}$$

$$\text{and } \gamma_2(G_\Gamma) = \prod_{i=1}^N \langle y_i \rangle \cong \mathbb{Z}^N$$

Note that Lemma 3.2 implies that if Γ is not a complete graph, then the associated group G_Γ is a finitely generated torsion-free 2-step nilpotent group. Hence, we can apply Theorem 2.6 to the lower central series of G_Γ . Combined with the well-known description of the Reidemeister spectrum for finitely generated torsion-free abelian groups (see e.g. [GW09]) we obtain the next result.

Lemma 3.3. *For any $\varphi \in \text{Aut}(G_\Gamma)$ we have that:*

$$R(\varphi) = R(\varphi_1)R(\varphi_2) = |\det(\text{Id} - \varphi_1)|_\infty |\det(\text{Id} - \varphi_2)|_\infty$$

where $\varphi_1 : G_\Gamma/\gamma_2(G_\Gamma) \rightarrow G_\Gamma/\gamma_2(G_\Gamma)$ and $\varphi_2 : \gamma_2(G_\Gamma) \rightarrow \gamma_2(G_\Gamma)$ are the induced automorphisms on the factors of the lower central series.

We abuse notation and also denote with φ_1 (respectively φ_2) the matrix corresponding to the map φ_1 (respectively φ_2).

It is clear that if two graphs are isomorphic, then the associated groups are isomorphic. Also the converse is true. This can be proven by using the argument for the associated *right-angled Artin groups* in [Dro87]. In his argument Droms actually shows that when the 2-step nilpotent quotients of the right-angled Artin groups associated to the two graphs are isomorphic, that it follows that the graphs are isomorphic. However, since G_Γ is precisely the 2-step nilpotent quotient of the right-angled Artin group associated to Γ , this argument suffices to conclude the proof of the following Lemma.

Lemma 3.4. *Let Γ_1 and Γ_2 be two undirected simple graphs. The graphs Γ_1 and Γ_2 are isomorphic if and only if $G_{\Gamma_1} \cong G_{\Gamma_2}$.*

4 The first method: degree of the vertices

Recall that we want to determine the Reidemeister spectrum of the finitely generated 2-step nilpotent groups associated to graphs. To do so we develop three methods. For the first method we describe characteristic subgroups based on the degree of the vertices. The other two methods allow us to partition the graph by using the *simplicial join* or the *disjoint union*.

For any graph $\Gamma(\{x_1, x_2, \dots, x_n\}, E)$ and any $d \in \{1, 2, \dots, n-1\}$ we define the vertex set

$$V_d := \{x_i \mid i = 1, 2, \dots, n \text{ and } \deg x_i \geq d\}$$

and the subgroup $H_d \subseteq G_\Gamma$ by

$$H_d := \left\{ \prod_{x_i \in V_d} x_i^{z_i} \prod_{l=1}^N y_l^{t_l} \mid z_i, t_l \in \mathbb{Z} \right\}.$$

We argue that these subgroups H_d are characteristic subgroups of G_Γ . For this, we need two lemmas that describe the Hirsch number of the centralizers of elements of G_Γ .

Lemma 4.1. *For any $z_{i_0}, t_l \in \mathbb{Z}$ (for some $i_0 = 1, 2, \dots, n$ and any $l = 1, 2, \dots, N$) with $z_{i_0} \neq 0$ it holds that*

$$h \left(Z_{G_\Gamma} \left(x_{i_0}^{z_{i_0}} \prod_{l=1}^N y_l^{t_l} \right) \right) = \deg(x_{i_0}) + N + 1.$$

Proof. By the operation in G_Γ , it follows that the centralizer of $x := x_{i_0}^{z_{i_0}} \prod_{l=1}^N y_l^{t_l}$ has the following form

$$Z_{G_\Gamma}(x) = \{g \in G_\Gamma \mid [g, x] = 1\} = \left\{ \prod_{i=1}^n x_i^{v_i} \prod_{l=1}^N y_l^{s_l} \mid \begin{array}{l} v_i, s_l \in \mathbb{Z} \\ z_{i_0} v_i = 0 \text{ if } x_{i_0} x_i \notin E \text{ and } i_0 \neq i \end{array} \right\}.$$

Since $z_{i_0} \neq 0$, this precisely means that

$$Z_{G_\Gamma}(x) = \left\{ \prod_{\substack{i=1,2,\dots,n; \\ x_{i_0} x_i \in E \text{ or } i=i_0}} x_i^{v_i} \prod_{l=1}^N y_l^{s_l} \mid v_i, s_l \in \mathbb{Z} \right\}.$$

Using Lemma 2.1 it follows that

$$h(Z_{G_\Gamma}(x)) = N + |\{i = 1, 2, \dots, n \mid x_{i_0} x_i \in E \text{ or } i = i_0\}| = \deg(x_{i_0}) + N + 1.$$

□

Lemma 4.2. *For any $z_i, t_l \in \mathbb{Z}$ (for $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, N$) it holds that*

$$h \left(Z_{G_\Gamma} \left(\prod_{i=1}^n x_i^{z_i} \prod_{l=1}^N y_l^{t_l} \right) \right) \leq \min\{n-1, \min_{\substack{i=1,2,\dots,n; \\ z_i \neq 0}} \deg(x_i)\} + N + 1.$$

Proof. Denote $x := \prod_{i=1}^n x_i^{z_i} \prod_{l=1}^N y_l^{t_l}$. If all z_i are equal to zero, then $Z_{G_\Gamma}(x) = G_\Gamma$ and thus $h(Z_{G_\Gamma}(x)) = n + N$ and the result follows. So suppose that not all z_i are zero. We can assume without loss of generality that

$$z_i \neq 0 \iff i \in \{1, 2, \dots, k\}$$

for some $k \in \{1, 2, \dots, n\}$ and that

$$\deg(x_1) = \min_{i=1,2,\dots,k} \deg(x_i).$$

We denote with m the number of x_j 's (with $j = 2, 3, \dots, k$) such that $x_1 x_j \in E$. Without loss of generality, we can assume that

$$x_1 x_j \notin E \text{ (with } j \in \{2, 3, \dots, k\}) \iff j = 2, 3, \dots, k - m.$$

By the operation in G_Γ , we obtain that

$$Z_{G_\Gamma}(x) = \left\{ \prod_{i=1}^n x_i^{v_i} \prod_{l=1}^N y_l^{s_l} \left| \begin{array}{l} v_i, s_l \in \mathbb{Z} \\ z_1 v_j = v_1 z_j \quad (\forall j = 2, 3, \dots, k - m) \\ z_i v_j = v_i z_j \quad (\forall i, j = 2, 3, \dots, k \text{ with } x_i x_j \notin E \text{ and } i \neq j) \\ z_i v_j = 0 \quad (\forall i = 1, 2, \dots, k; j = k + 1, \dots, n \text{ with } x_i x_j \notin E) \end{array} \right. \right\}.$$

We define the subgroup $H \subseteq G_\Gamma$ (that contains $Z_{G_\Gamma}(x)$) by setting

$$H := \left\{ \prod_{i=1}^n x_i^{v_i} \prod_{l=1}^N y_l^{s_l} \left| \begin{array}{l} v_i, s_l \in \mathbb{Z} \\ z_1 v_j = v_1 z_j \quad (\forall j = 2, 3, \dots, k - m) \\ z_i v_j = 0 \quad (\forall i = 1, 2, \dots, k; j = k + 1, \dots, n \text{ with } x_i x_j \notin E) \end{array} \right. \right\}.$$

By Lemma 2.1 (i) it suffices to argue that $h(H) \leq \deg(x_1) + N + 1$. Since $z_1 \neq 0$, the equations $z_1 v_j = v_1 z_j$ (with $j = 2, 3, \dots, k - m$) can only be satisfied if

$$(v_1, v_2, \dots, v_{k-m}) = \lambda \left(\frac{z_1}{d}, \frac{z_2}{d}, \dots, \frac{z_{k-m}}{d} \right) \quad \text{for some } \lambda \in \mathbb{Z}$$

where $d := \gcd(z_1, z_2, \dots, z_{k-m})$. Since all the $z_l \neq 0$ (for $l = 1, 2, \dots, k$), it follows that $v_j = 0$ for all $j = k + 1, k + 2, \dots, n$ with $x_i x_j \notin E$ for some $i = 1, 2, \dots, k$. Hence, we obtain that

$$H = \left\{ \left(\prod_{i=1}^{k-m} x_i^{z_i/d} \right)^{v_0} \prod_{i=k-m+1}^k x_i^{v_i} \prod_{j \in J} x_j^{v_j} \prod_{l=1}^N y_l^{s_l} \left| v_0, v_i, v_j, s_l \in \mathbb{Z} \right. \right\}$$

where we defined

$$J := \{j = k + 1, k + 2, \dots, n \mid x_i x_j \in E \text{ for all } i = 1, 2, \dots, k\}.$$

By using Lemma 2.1 we can indeed conclude that

$$\begin{aligned} h(H) &= 1 + m + |J| + N \leq 1 + m + N + |\{j = k + 1, k + 2, \dots, n \mid x_1 x_j \in E\}| \\ &= 1 + m + N + |\{j = 2, 3, \dots, n \mid x_1 x_j \in E\}| - m \\ &= \deg(x_1) + N + 1. \end{aligned}$$

□

Theorem 4.3. *The subgroups H_d (for any $d = 1, 2, \dots, n - 1$) are characteristic subgroups of G_Γ .*

Proof. Fix any $d = 1, 2, \dots, n - 1$, an automorphism $\varphi \in \text{Aut}(G_\Gamma)$ and any $x_{i_0} \in V_d$. Suppose by contradiction that $\varphi(x_{i_0}) \notin H_d$. Hence, there exists some $x_{i_1} \notin V_d$ and $z_i, t_l \in \mathbb{Z}$ with $z_{i_1} \neq 0$ such that

$$\varphi(x_{i_0}) = \prod_{i=1}^n x_i^{z_i} \prod_{l=1}^N y_l^{t_l}.$$

By using Lemma 4.1 and Lemma 4.2 we now obtain that

$$d + N + 1 \leq \deg x_{i_0} + N + 1 = h(Z_{G_\Gamma}(x_{i_0})) = h(Z_{G_\Gamma}(\varphi(x_{i_0}))) \leq \deg x_{i_1} + N + 1 < d + N + 1$$

which is a contradiction and thus $\varphi(x_{i_0}) \in H_d$. By Lemma 3.2 and since $\gamma_2(G_\Gamma)$ is a characteristic subgroup of G_Γ , it now follows that $\varphi(H_d) \subseteq H_d$. We can use completely the same argument to argue that $\varphi^{-1}(H_d) \subseteq H_d$ and thus we can conclude that $\varphi(H_d) = H_d$. □

Theorem 4.3 provides extra information about the automorphisms of G_Γ . Therefore, we will use it frequently in the rest of the paper to determine the Reidemeister spectrum of groups associated to graphs and to develop new methods to do so. Moreover, Theorem 4.3 can be used to describe graphs for which the associated finitely generated 2-step nilpotent groups have the R_∞ -property.

Theorem 4.4. *Let $\Gamma(\{x_1, x_2, \dots, x_n\}, E)$ be an undirected simple graph which has maximal degree $n-2$ and for which this degree is attained only once, then G_Γ has the R_∞ -property.*

Proof. Take any $\varphi \in \text{Aut}(G_\Gamma)$. Assume without loss of generality that x_1 is the one vertex having degree $n-2$ and that $x_1x_2 \notin E$. Since H_{n-2} is a characteristic subgroup of G_Γ (by Theorem 4.3), it follows that $\varphi_1(x_1\gamma_2(G_\Gamma)) = x_1^{\pm 1}\gamma_2(G_\Gamma)$.

Fix any $j \in \{3, 4, \dots, n\}$ and denote $\varphi(x_j) = \prod_{i=1}^n x_i^{z_i} \prod_{l=1}^N y_l^{t_l}$. Since $x_1x_j \in E$, we obtain (by Lemma 2.2) that

$$1 = \varphi([x_j, x_1]) = \left[\prod_{i=1}^n x_i^{z_i}, x_1^{\pm 1} \right] = \prod_{i=1}^n [x_i, x_1]^{\pm z_i} = [x_2, x_1]^{\pm z_2}.$$

Hence, we obtain that $z_2 = 0$. Since this argument is valid for all $j \in \{3, 4, \dots, n\}$ and since φ is an automorphism, it follows that the matrix of φ_1 (with respect to $\{x_1\gamma_2(G_\Gamma), x_2\gamma_2(G_\Gamma), \dots, x_n\gamma_2(G_\Gamma)\}$) has the following form:

$$\begin{pmatrix} \pm 1 & b_1 & & C \\ 0 & \pm 1 & 0 & \dots & 0 \\ \hline 0 & b_3 & & & \\ \vdots & \vdots & & A & \\ 0 & b_n & & & \end{pmatrix}$$

where $B := (b_1 \ \pm 1 \ b_3 \ \dots \ b_n)^\top \in \mathbb{Z}^{n \times 1}$, $A \in \mathbb{Z}^{(n-2) \times (n-2)}$ and $C \in \mathbb{Z}^{1 \times (n-2)}$. Since φ_1 is an automorphism, it holds that $A \in \text{GL}_{n-2}(\mathbb{Z})$. Applying Lemma 3.3 yields

$$R(\varphi) = R(\varphi_1)R(\varphi_2) = |\pm 1 - 1|_\infty |\pm 1 - 1|_\infty |\det(A - \mathbf{1}_{n-2})|_\infty R(\varphi_2).$$

If one of the two ± 1 is equal to 1, then we get that $R(\varphi) = \infty$ and the result follows. Hence, we can assume without loss of generality that the two ± 1 in the matrix representation are both equal to -1 . However, by Lemma 2.2 we now obtain that

$$\begin{aligned} \varphi_2([x_2, x_1]) &= [x_1^{b_1} x_2^{-1} x_3^{b_3} \dots x_n^{b_n}, x_1^{-1}] = [x_1, x_1]^{-b_1} [x_2, x_1]^{(-1)(-1)} [x_3, x_1]^{-b_3} \dots [x_n, x_1]^{-b_n} \\ &= [x_2, x_1]. \end{aligned}$$

This implies that $[x_2, x_1]$ is an eigenvector of φ_2 with corresponding eigenvalue 1 and thus by Proposition 2.5 it follows that $R(\varphi) = \infty$. \square

Example 4.5. If Γ is the graph in Figure 1, then G_Γ has the R_∞ -property.

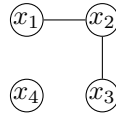


Figure 1: Graph for which the associated group has the R_∞ -property.

Corollary 4.6. *Let $\Gamma(\{x_1, x_2, \dots, x_n\}, E)$ be an undirected simple graph which has maximal degree $n-2$ and for which this degree is attained only once and let M be the nilmanifold with fundamental group G_Γ . Then any self-homotopy equivalence of M is homotopic to a fixed point free map.*

Remark 4.7. The nilmanifold M whose fundamental group is the group G_Γ of Example 4.5 is an 8-dimensional (since $h(G_\Gamma) = 8$) example of a 2-step nilmanifold for which every self-homotopy equivalence is homotopic to a fixed point free map. In Remark 7.8 we mention that the group associated to the cycle graph on four vertices where we remove one edge also has the R_∞ -property. Hence, this provides a 7-dimensional example. One can prove (see [Lat21, Example 8.2.2]) that this is a sharp bound when considering groups associated to graphs. Moreover, we will show in a forthcoming paper that this is a general lower bound in the sense that there do not exist 2-step nilmanifolds (so also not associated to a graph) in dimensions ≤ 6 having the R_∞ -property.

5 The second method: simplicial join

There are several ways to combine graphs. We discuss the *simplicial join* and the *disjoint union*.

Definition 5.1. Let $k \in \mathbb{N}_{>1}$ and $\Gamma_i(V_i, E_i)$ (with $i = 1, 2, \dots, k$) be graphs.

- The disjoint union $\sqcup_{i=1}^k \Gamma_i$ of the graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ is defined by

$$\bigsqcup_{i=1}^k \Gamma_i \left(\bigsqcup_{i=1}^k V_i, \bigsqcup_{i=1}^k E_i \right).$$

- The simplicial join $*_{i=1}^k \Gamma_i$ of the graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ is defined by

$$*_{i=1}^k \Gamma_i \left(\bigsqcup_{i=1}^k V_i, \bigsqcup_{i=1}^k E_i \cup \{x_i x_j \mid x_i \in V_i, x_j \in V_j \text{ and } 1 \leq i < j \leq k\} \right).$$

The group associated to the simplicial join of graphs can be written as a direct product. The next result follows almost immediately by using the definitions.

Lemma 5.2. *If $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are two undirected simple graphs, then the group $G_{\Gamma^{(1)} * \Gamma^{(2)}}$ associated to the simplicial join of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ is isomorphic with the direct product $G_{\Gamma^{(1)}} \times G_{\Gamma^{(2)}}$. This can be generalised to the simplicial join of any finite amount of undirected simple graphs.*

In order to study the endomorphisms of a direct product, we introduce some notation (which coincides with the notation from [Sen21]).

Notation 5.3. Let $G = \times_{i=1}^k G_i$ be the direct product of k groups G_i . For any $i = 1, 2, \dots, k$ we denote with $\pi_i : G \rightarrow G_i$ the canonical projection and with $e_i : G_i \rightarrow G$ the canonical inclusion using the direct product. For any endomorphism $\varphi \in \text{End}(G)$ and any $i, j = 1, 2, \dots, k$, we denote with $\varphi_{ij} : G_j \rightarrow G_i$ the morphisms defined by

$$\varphi_{ij} := \pi_i \circ \varphi \circ e_j : G_j \rightarrow G_i.$$

Note that if $\varphi \in \text{Aut}(G_{\Gamma^{(1)} * \Gamma^{(2)}})$ is an automorphism, then $\varphi_{ii} \in \text{End}(G_{\Gamma^{(i)}})$ (for $i = 1, 2$) is not necessarily an automorphism of $G_{\Gamma^{(i)}}$. However, the following lemma tells us something about the images of φ_{ij} . We refer to [Sen21, Lemma 2.2] for a proof.

Lemma 5.4. *Let $\varphi \in \text{Aut}(\times_{i=1}^k G_i)$ be an automorphism of $\times_{i=1}^k G_i$, then for all $i = 1, 2, \dots, k$ it holds that G_i is generated by $\text{Im}(\varphi_{i1}), \text{Im}(\varphi_{i2}), \dots, \text{Im}(\varphi_{ik})$.*

In order to describe the Reidemeister spectrum, we introduce some (obvious) notation.

Notation 5.5. Let $M, N \subseteq \mathbb{N}_0 \cup \{\infty\}$ be two subsets. We define the product set $M \cdot N$ by

$$M \cdot N := \{mn \mid m \in M, n \in N\}.$$

To avoid confusion, we only use this notation when we express the Reidemeister spectrum of a group.

If $\Gamma(V, E)$ is a graph and $V' \subset V$ a subset of the set of vertices, then we denote with $\Gamma(V')$ the *subgraph induced on V'* . This subgraph of Γ is defined by means of the vertex set V' and the edge set $\{vw \in E \mid v, w \in V'\}$.

A first step in studying the Reidemeister spectrum of the group associated to the simplicial join of graphs is to get rid of the vertices that are connected with all other vertices. Recall that these vertices are precisely contained in the vertex set V_{n-1} .

Theorem 5.6. *Let $\Gamma(V = \{x_1, x_2, \dots, x_n\}, E)$ be an undirected simple graph with $r := |V_{n-1}|$. If $r < n$, then it holds that*

$$\text{Spec}_R(G_\Gamma) = \text{Spec}_R(\mathbb{Z}^r) \cdot \text{Spec}_R(G_{\Gamma(V \setminus V_{n-1})}).$$

Proof. Note that $\Gamma = \Gamma(V_{n-1}) * \Gamma(V \setminus V_{n-1})$ and thus by Lemma 5.2

$$G \cong G_{\Gamma(V_{n-1})} \times G_{\Gamma(V \setminus V_{n-1})} \cong \mathbb{Z}^r \times G_{\Gamma(V \setminus V_{n-1})}.$$

Fix any automorphism $\varphi \in \text{Aut}(\mathbb{Z}^r \times G_{\Gamma(V \setminus V_{n-1})})$. Since none of the vertices of $\Gamma(V \setminus V_{n-1})$ is connected with all the other vertices of $\Gamma(V \setminus V_{n-1})$, it follows by Lemma 3.2 that

$$\varphi(\mathbb{Z}^r \times 1) \subseteq \varphi(\mathbb{Z}^r \times Z(G_{\Gamma(V \setminus V_{n-1})})) = Z(\mathbb{Z}^r \times G_{\Gamma(V \setminus V_{n-1})}) = \mathbb{Z}^r \times \gamma_2(G_{\Gamma(V \setminus V_{n-1})}).$$

Denote with $(\varphi_{ij})_1$ and $(\varphi_{ij})_2$ (with $i, j = 1, 2$) the morphisms induced on the first and second factor of the lower central series. Since $\text{Im}(\varphi_{21}) \subseteq \gamma_2(G_{\Gamma(V \setminus V_{n-1})})$ it holds that $\text{Im}((\varphi_{21})_1) = 1$. Hence, $(\varphi_{21})_1$ is the map sending everything to $1\gamma_2(G_{\Gamma(V \setminus V_{n-1})})$. So Lemma 5.4 implies that $(\varphi_{22})_1$ is surjective. Applying Lemma's 2.3 and 2.4 yields that φ_{22} is an automorphism of $G_{\Gamma(V \setminus V_{n-1})}$.

By taking a particular generating set of \mathbb{Z}^r and $G_{\Gamma(V \setminus V_{n-1})}$ it follows that the matrix of φ_1 (with respect to this generating set) is of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A, B and C are the matrices representing respectively $(\varphi_{11})_1, (\varphi_{12})_1$ and $(\varphi_{22})_1$. Since φ_1 and $(\varphi_{22})_1$ are automorphisms, we can conclude that also $(\varphi_{11})_1$ is an automorphism. Remark that since \mathbb{Z}^r is abelian, it holds that $(\varphi_{11})_1 = \varphi_{11}$.

Note that

$$\text{Im}((\varphi_{12})_2) = \text{Im}((\varphi_{21})_2) = 1$$

and thus $R(\varphi_2) = R((\varphi_{22})_2)$. By Lemma 3.3 we can conclude that

$$R(\varphi) = R(\varphi_{11})R((\varphi_{22})_1)R((\varphi_{22})_2) = R(\varphi_{11})R(\varphi_{22}).$$

Since $\varphi_{11} \in \text{Aut}(\mathbb{Z}^r)$ and $\varphi_{22} \in \text{Aut}(G_{\Gamma(V \setminus V_{n-1})})$ it follows that

$$\text{Spec}_{\mathbb{R}}(G_{\Gamma}) = \text{Spec}_{\mathbb{R}}(\mathbb{Z}^r \times G_{\Gamma(V \setminus V_{n-1})}) \subseteq \text{Spec}_{\mathbb{R}}(\mathbb{Z}^r) \cdot \text{Spec}_{\mathbb{R}}(G_{\Gamma(V \setminus V_{n-1})}).$$

The other inclusion is well-known (see e.g. [Sen21, Corollary 2.6]). \square

Notation 5.7. Using Theorem 5.6, we can restrict ourselves to look at graphs for which none of the vertices is connected (via an edge) with all other vertices. Let Γ be such a finite undirected simple graph. Assume that $\Gamma = *_{i=1}^k \Gamma^{(i)}$ and that Γ cannot be decomposed any further using the simplicial join. We use the superscript “ (i) ” to denote similar properties as before, but related to the graph $\Gamma^{(i)}$ (e.g. $V^{(i)}, x_1^{(i)}$ and $n^{(i)}$). Applying Lemma 5.2 yields that $G_{\Gamma} \cong \times_{i=1}^k G_{\Gamma^{(i)}}$. For any component $i = 1, 2, \dots, k$ we define the subgroup $H^{(i)} \subseteq G_{\Gamma}$ by

$$H^{(i)} := \times_{m=1}^{i-1} \gamma_2(G_{\Gamma^{(m)}}) \times G_{\Gamma^{(i)}} \times \times_{m=i+1}^k \gamma_2(G_{\Gamma^{(m)}}) = G_{\Gamma^{(i)}} \gamma_2(G_{\Gamma}).$$

Definition 5.8. Let $\Gamma(V, E)$ be an undirected simple graph. The complement Γ^c of the graph Γ is defined by

$$\Gamma^c(V, \{vw \mid v, w \in V, v \neq w \text{ and } vw \notin E\}).$$

The following lemma will be needed later on.

Lemma 5.9. *If $\Gamma = *_{i=1}^k \Gamma^{(i)}$ cannot be decomposed any further using the simplicial join, then $(\Gamma^{(i)})^c$ is connected (for all $i = 1, 2, \dots, k$).*

Proof. Fix any $i = 1, 2, \dots, k$ and denote for the sake of simplicity $\Gamma^{(i)} = X(V, E)$. Suppose by contradiction that X^c is not connected. So there exist two subgraphs $X_1(V_1, E_1)$ and $X_2(V_2, E_2)$ of X^c that are not connected with each other (with $V = V_1 \sqcup V_2$ and with $E_1 \cup E_2$ the edge set of X^c). This implies that $V_1, V_2 \subset V$ are non-empty sets of vertices such that

$$\{x_{i_1} x_{i_2} \mid x_{i_1} \in V_1 \text{ and } x_{i_2} \in V_2\} \subseteq E.$$

Thus, we obtain that

$$X = X(V_1) * X(V_2)$$

which contradicts the assumption. Hence, all the complements $(\Gamma^{(i)})^c$ are connected. \square

Lemma 5.9 allows us to describe the automorphisms of G_Γ .

Corollary 5.10. *Let $\Gamma(V = \{x_1, x_2, \dots, x_n\}, E)$ be an undirected simple graph such that $\Gamma = \ast_{i=1}^k \Gamma^{(i)}$, $|V_{n-1}| = 0$ and Γ cannot be decomposed any further using the simplicial join. Then, for any automorphism $\varphi \in \text{Aut}(G_\Gamma)$ there exists a unique permutation $\sigma \in S_k$ such that*

$$(i) \quad \varphi(H^{(i)}) = H^{(\sigma(i))} \text{ for any } i = 1, 2, \dots, k.$$

$$(ii) \quad \text{The corresponding components are isomorphic, i.e. } \Gamma^{(i)} \cong \Gamma^{(\sigma(i))} \text{ (for any } i = 1, 2, \dots, k).$$

Proof. Fix any component $i_0 \in \{1, 2, \dots, k\}$ and denote with $d := \deg_\Gamma(x_1^{(i_0)})$ the degree (in Γ) of $x_1^{(i_0)}$. Define for any $i = 1, 2, \dots, k$ the integer $d^{(i)}$ by

$$d^{(i)} := d - \sum_{\substack{j=1, \\ j \neq i}}^k n^{(j)} = d - n + n^{(i)}.$$

Note that $\deg_{\Gamma^{(i_0)}}(x_1^{(i_0)})$ equals $d^{(i_0)}$.

Since φ is an automorphism, it follows by Theorem 4.3 that $\varphi(x_1^{(i_0)}) \in H_d \setminus H_{d+1}$. Thus we can fix some component $i_1 \in \{1, 2, \dots, k\}$ such that $\varphi(x_1^{(i_0)})$ has a non-zero exponent for some vertex of $V^{(i_1)}$ of degree d in Γ . By Lemma 4.1, it holds that $h(Z_{G_\Gamma}(\varphi(x_1^{(i_0)}))) = d + N + 1$. However, by applying Lemma 4.2 we obtain (by also using Lemma 2.1 (i)) that

$$\begin{aligned} d + N + 1 &= h(Z_{G_\Gamma}(\varphi(x_1^{(i_0)}))) = \sum_{j=1}^k h(Z_{G_{\Gamma^{(j)}}}(\pi_j(\varphi(x_1^{(i_0)})))) \\ &\leq (d^{(i_1)} + 1 + N^{(i_1)}) + \sum_{\substack{j=1, \\ j \neq i_1}}^k (n^{(j)} + N^{(j)}) \\ &= d^{(i_1)} + 1 - n^{(i_1)} + \sum_{j=1}^k (n^{(j)} + N^{(j)}) = d^{(i_1)} + 1 - n^{(i_1)} + n + N \\ &= d + N + 1. \end{aligned}$$

So equality must hold throughout the calculations and thus for all $j = 1, 2, \dots, k$ with $j \neq i_1$ this implies that

$$h(Z_{G_{\Gamma^{(j)}}}(\pi_j(\varphi(x_1^{(i_0)})))) = n^{(j)} + N^{(j)}.$$

Since there are no vertices of degree $n^{(j)} - 1$ (in $\Gamma^{(j)}$), Lemma 4.2 implies that $\pi_j(\varphi(x_1^{(i_0)})) \in \gamma_2(G_{\Gamma^{(j)}})$ for all $j = 1, 2, \dots, k$ with $j \neq i_1$. So i_1 is the unique component such that $\varphi(x_1^{(i_0)}) \in H^{(i_1)}$ and thus we define $\sigma(i_0) := i_1$. Repeating this argument for any $i_0 \in \{1, 2, \dots, k\}$ yields a unique map $\sigma : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\}$ such that

$$\varphi(x_1^{(i)}) \in H^{(\sigma(i))} \quad \text{for all } i = 1, 2, \dots, k.$$

Fix any component i_0 and any vertex $x_j^{(i_0)}$ of that component (with $j \neq 1$) such that $x_1^{(i_0)} x_j^{(i_0)} \notin E$. Using the same argument as in the beginning of the proof, we can derive that $\varphi(x_j^{(i_0)}) \in H^{(\tilde{i}_0)}$ for some unique component \tilde{i}_0 . Since $[x_j^{(i_0)}, x_1^{(i_0)}] \neq 1$, it follows that

$$1 \neq [\varphi(x_j^{(i_0)}), \varphi(x_1^{(i_0)})] \in [H^{(\tilde{i}_0)}, H^{(\sigma(i_0))}] = \begin{cases} \gamma_2(G_{\Gamma^{(\sigma(i_0))}}) & \text{if } \tilde{i}_0 = \sigma(i_0) \\ 1 & \text{if } \tilde{i}_0 \neq \sigma(i_0) \end{cases}.$$

Thus we obtain that $\tilde{i}_0 = \sigma(i_0)$ and so $\varphi(x_j^{(i_0)}) \in H^{(\sigma(i_0))}$. If we fix any vertex $x_j^{(i_0)}$ (with $j \neq 1$), then by Lemma 5.9 there exists a path in $(\Gamma^{(i_0)})^c$ that connects $x_j^{(i_0)}$ and $x_1^{(i_0)}$. We can now use the previous argument inductively together with this path to conclude that $\varphi(x_j^{(i_0)}) \in H^{(\sigma(i_0))}$.

Hence, we can conclude that $\varphi(H^{(i)}) \subseteq H^{(\sigma(i))}$ for all components i . By construction of this unique map σ and since φ is an automorphism, one can derive that $\sigma \in S_k$ and $\varphi(H^{(i)}) = H^{(\sigma(i))}$ for all components i . The second item follows directly by applying Lemma 3.4. \square

Using this description, we are now able to describe the Reidemeister spectrum of G_Γ . We subdivide this description in two theorems, but we prove them at once.

Theorem 5.11. *Let $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(k)}$ be isomorphic finite undirected simple graphs with at least 2 vertices that cannot be decomposed using the simplicial join. Then*

$$\text{Spec}_R \left(G_{\ast_{i=1}^k \Gamma^{(i)}} \right) = \text{Spec}_R \left(\bigotimes_{i=1}^k G_{\Gamma^{(i)}} \right) = \bigcup_{i=1}^k \left\{ \prod_{m=1}^i R_m \mid R_m \in \text{Spec}_R(G_{\Gamma^{(1)}}) \right\}.$$

If we divide a graph using the simplicial join, then we say that two components *are of the same type* if they are isomorphic. If s is the amount of different types, then we fix some order to be able to address components of type j (with $j = 1, 2, \dots, s$).

Theorem 5.12. *Let $\Gamma(V = \{x_1, x_2, \dots, x_n\}, E)$ be a finite undirected simple graph. Assume that $\Gamma(V \setminus V_{n-1}) = \ast_{i=1}^k \Gamma^{(i)}$ and $\Gamma(V \setminus V_{n-1})$ cannot be decomposed any further using the simplicial join. Denote with s the number of types of components. Then, the Reidemeister spectrum of G_Γ is given by*

$$\text{Spec}_R(G_\Gamma) = \text{Spec}_R(\mathbb{Z}^{|V_{n-1}|}) \cdot \prod_{j=1}^s \text{Spec}_R \left(\bigotimes_{\substack{\text{components } i \\ \text{of type } j}} G_{\Gamma^{(i)}} \right).$$

Proof of Theorems 5.11 and 5.12. By Theorem 5.6 we can assume that $|V_{n-1}| = 0$. Fix any automorphism $\varphi \in \text{Aut}(G_\Gamma)$. Take $\sigma \in S_k$ as described in Corollary 5.10. Define the map $\bar{\varphi} \in \text{End}(G_\Gamma)$ by setting

$$\bar{\varphi} : G_\Gamma \rightarrow G_\Gamma : (g_1, g_2, \dots, g_k) \mapsto \prod_{i=1}^k (1, \dots, 1, \underbrace{\varphi_{\sigma(i)i}(g_i)}_{\text{position } \sigma(i)}, 1, \dots, 1).$$

Since the components $\Gamma^{(i)}$ and $\Gamma^{(\sigma(i))}$ are isomorphic, we view $\varphi_{\sigma(i)i} \in \text{End}(G_{\Gamma^{(i)}})$. Since $\varphi(H^{(i)}) = H^{(\sigma(i))}$ (for all $i = 1, 2, \dots, k$), it follows that if $i_2 \neq \sigma(i_1)$ then the induced map $(\varphi_{i_2 i_1})_1$ is the zero map. Hence, by using Lemma 5.4, Lemma 2.3 and Lemma 2.4 we obtain that $\varphi_{\sigma(i)i}$ is an automorphism of $G_{\Gamma^{(i)}}$ for all components $i = 1, 2, \dots, k$ and thus $\bar{\varphi}$ is an automorphism of G_Γ . By using the definition of the automorphism $\bar{\varphi}$, one can derive that $(\bar{\varphi})_1 = \varphi_1$ and $(\bar{\varphi})_2 = \varphi_2$. Applying Lemma 3.3 now yields that $R(\varphi) = R(\bar{\varphi})$ and thus it suffices to consider the automorphism $\bar{\varphi}$.

Since σ maps components to isomorphic components, we are able to write $\bar{\varphi} = \bar{\varphi}^{(1)} \times \bar{\varphi}^{(2)} \times \dots \times \bar{\varphi}^{(s)}$ for some automorphisms $\bar{\varphi}^{(j)}$ on the groups associated to the graph consisting of the simplicial join of the components of type j (with $j = 1, 2, \dots, s$). It is known that $R(\bar{\varphi}) = \prod_{j=1}^s R(\bar{\varphi}^{(j)})$ (see e.g. [Sen21, Corollary 2.6]). So it suffices to prove Theorem 5.11.

To limit notational complexity, we assume that there are only $k = 2$ components. The general case can be proven similarly. We refer the interested reader to [Lat21, Proposition 8.1.11].

Assume that $\Gamma = \Gamma^{(1)} \ast \Gamma^{(2)}$ (and Γ cannot be decomposed any further using the simplicial join) where $\Gamma^{(1)} \cong \Gamma^{(2)}$ are isomorphic finite undirected simple graphs with at least 2 vertices. We argue that $\text{Spec}_R(G_{\Gamma^{(1)} \ast \Gamma^{(2)}}) = \bigcup_{i=1}^2 \left\{ \prod_{m=1}^i R_m \mid R_m \in \text{Spec}_R(G_{\Gamma^{(1)}}) \right\}$. First we assume that $\varphi(g_1, 1) \in 1 \times G_{\Gamma^{(2)}}$ and that $\varphi(1, g_2) \in G_{\Gamma^{(1)}} \times 1$ for all $g_i \in G_{\Gamma^{(i)}}$ (in particular, $\sigma = (1\ 2) \in S_2$). We define a new set of generators for $G_{\Gamma^{(1)}}$ and $G_{\Gamma^{(2)}}$ by setting

$$\begin{aligned} \tilde{x}_j^{(i)} &:= \begin{cases} x_j^{(1)} & \text{if } i = 1 \\ \varphi_{21}(x_j^{(1)}) & \text{if } i = 2 \end{cases} & \text{for all } j = 1, 2, \dots, n^{(1)} \\ \tilde{y}_j^{(i)} &:= \begin{cases} y_j^{(1)} & \text{if } i = 1 \\ \varphi_{21}(y_j^{(1)}) & \text{if } i = 2 \end{cases} & \text{for all } j = 1, 2, \dots, N^{(1)}. \end{aligned}$$

One can check that this is well-defined since φ is an automorphism. Hence, there exists some automorphism $\psi \in \text{Aut}(G_{\Gamma^{(1)}})$ such that the matrix of φ_i (with $i = 1, 2$) with respect to this new set of generators has the following form

$$\begin{pmatrix} 0 & A_i \\ \mathbf{1} & 0 \end{pmatrix}$$

where A_i is the matrix of the induced automorphism ψ_i . By Lemma 3.3 we now obtain that

$$R(\varphi_i) = \left| \det \left(\mathbf{1} - \begin{pmatrix} 0 & A_i \\ \mathbf{1} & 0 \end{pmatrix} \right) \right|_{\infty} = |\det(\mathbf{1} - A_i)|_{\infty} = R(\psi_i)$$

and hence $R(\varphi) = R(\psi)$.

If on the other hand $\varphi = \varphi_{11} \times \varphi_{22}$ (and thus σ is the identity permutation), then $R(\varphi) = R(\varphi_{11})R(\varphi_{22})$ (see e.g. [Sen21, Corollary 2.6]). Combining these two cases, we obtain that $\text{Spec}_{\mathbb{R}}(G_{\Gamma^{(1)} * \Gamma^{(2)}}) \subseteq \bigcup_{i=1}^2 \left\{ \prod_{m=1}^i R_m \mid R_m \in \text{Spec}_{\mathbb{R}}(G_{\Gamma^{(1)}}) \right\}$. One can prove the other inclusion by using the same ideas for constructing the desired automorphisms. \square

Using Theorem 5.12, we can now construct more 2-step nilpotent groups having the R_{∞} -property.

Corollary 5.13. *Let $\Gamma(V, E)$ be a finite undirected simple graph such that $\Gamma = *_{i=1}^k \Gamma^{(i)}$ (where Γ cannot be decomposed any further). Then the following statements are equivalent*

- (i) *There exists some $i = 1, 2, \dots, k$ such that $G_{\Gamma^{(i)}}$ has the R_{∞} -property*
- (ii) *G_{Γ} has the R_{∞} -property*

As a direct consequence of this we now also find the following result.

Corollary 5.14. *For any $n \geq 7$ there exists a 2-step nilmanifold M_n of dimension n such that any self-homotopy equivalence of M_n is homotopic to a fixed point free map.*

Proof. Indeed, for M_7 we can take the 7-dimensional manifold of Remark 4.7. For any $n > 7$, let $k = n - 7$ and take $M_n = T^k \times M_7$, where T^k is the k -dimensional torus. Then, we have that the fundamental group of M_n is $\mathbb{Z}^k \times G_{\Gamma}$, where Γ is the cycle graph on four vertices with one edge removed. It follows that the fundamental group of M_n is then the group associated to the simplicial join of Γ and k graphs consisting of just one vertex. By the previous corollary, we know that $\mathbb{Z}^k \times G_{\Gamma}$ has the R_{∞} -property from which the result follows. \square

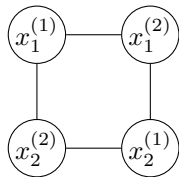
In order to illustrate Theorem 5.11 and Theorem 5.12 we give some examples.

Example 5.15. Consider the cycle graph C_4 on 4 vertices in Figure 2a. Note that C_4 is the simplicial join of twice the graph with 2 vertices and no edges. Hence, we can use Theorem 5.11 and equation (3.2) to conclude that

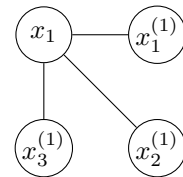
$$\begin{aligned} \text{Spec}_{\mathbb{R}}(G_{C_4}) &= \text{Spec}_{\mathbb{R}}(N_{2,2} \times N_{2,2}) = \bigcup_{i=1}^2 \left\{ \prod_{m=1}^i R_m \mid R_m \in \text{Spec}_{\mathbb{R}}(N_{2,2}) \right\} \\ &= (2\mathbb{N}_0 \cup \{\infty\}) \cup (4\mathbb{N}_0 \cup \{\infty\}) = 2\mathbb{N}_0 \cup \{\infty\}. \end{aligned}$$

Let us consider the graph Γ in Figure 2b. Hence, it follows that $V_{n-1} = \{x_1\}$ and thus Theorem 5.12 (together with equations (3.1) and (3.2)) yields that

$$\begin{aligned} \text{Spec}_{\mathbb{R}}(G_{\Gamma}) &= \text{Spec}_{\mathbb{R}}(\mathbb{Z}) \cdot \text{Spec}_{\mathbb{R}}(N_{3,2}) = (\{2, \infty\}) \cdot ((2\mathbb{N}_0 - 1) \cup 4\mathbb{N}_0 \cup \{\infty\}) \\ &= 2(2\mathbb{N}_0 - 1) \cup 8\mathbb{N}_0 \cup \{\infty\}. \end{aligned}$$



(a) Cycle graph



(b) Simplicial join of 1 vertex and 3 vertices

Figure 2: Illustration simplicial join on graphs with 4 vertices

6 The third method: connected components

Let $\Gamma(V, E)$ be a finite undirected simple graph. Denote with $\Gamma^{(0)}$ the induced subgraph $\Gamma(V_0 \setminus V_1)$ consisting of vertices of degree zero and with $\Gamma^{(i)}$ (for $i = 1, 2, \dots, k$) the connected components of $\Gamma(V_1)$. It follows that $\Gamma = \bigsqcup_{i=0}^k \Gamma^{(i)}$. We use the superscript “ (i) ” to denote similar properties as before, but related to the graph $\Gamma^{(i)}$. Define for all $i \in \{0, 1, \dots, k\}$ the subgroup $H^{(i)}$ of G_Γ by

$$H^{(i)} := \left\{ \prod_{j=1}^{n^{(i)}} \left(x_j^{(i)} \right)^{z_j^{(i)}} \prod_{l=1}^N y_l^{t_l} \mid z_j^{(i)}, t_l \in \mathbb{Z} \right\} = G_{\Gamma^{(i)}} \gamma_2(G_\Gamma).$$

As with the simplicial join, it turns out that any automorphism of G_Γ maps the subgroups $H^{(i)}$ (with $i = 1, 2, \dots, k$) to such a subgroup associated to an isomorphic component. In order to prove this, we need the following lemma, which studies the centralizers of elements of G_Γ .

Lemma 6.1. *If $\Gamma(V, E)$ is a finite undirected simple graph such that $\Gamma = \bigsqcup_{i=1}^k \Gamma^{(i)}$, then for any $z_j^{(i)}, t_l \in \mathbb{Z}$ with some $i_1 \neq i_2$ and j_1, j_2 such that $z_{j_1}^{(i_1)}, z_{j_2}^{(i_2)} \neq 0$, it holds that:*

$$Z_{G_\Gamma} \left(\prod_i \prod_j \left(x_j^{(i)} \right)^{z_j^{(i)}} \prod_{l=1}^N y_l^{t_l} \right) = \left\langle \prod_i \prod_j \left(x_j^{(i)} \right)^{z_j^{(i)}/d} \right\rangle \times \gamma_2(G_\Gamma) \cong \mathbb{Z}^{N+1}$$

where $d := \gcd_{i,j} z_j^{(i)}$.

Proof. Without loss of generality we assume that $i_1 = 1, i_2 = 2$ and $j_1 = j_2 = 1$ (and thus $z_1^{(1)}, z_1^{(2)} \neq 0$). To make notation more clear, we denote

$$x = \prod_i \prod_j \left(x_j^{(i)} \right)^{z_j^{(i)}} \prod_{l=1}^N y_l^{t_l}.$$

Using the operation in G_Γ , we obtain that

$$Z_{G_\Gamma}(x) = \left\{ \prod_i \prod_j \left(x_j^{(i)} \right)^{v_j^{(i)}} \prod_{l=1}^N y_l^{s_l} \mid \begin{array}{l} v_j^{(i)}, s_l \in \mathbb{Z} \\ z_{j_1}^{(i_1)} v_{j_2}^{(i_2)} = v_{j_1}^{(i_1)} z_{j_2}^{(i_2)} \quad \text{if } i_1 \neq i_2; \forall j_1, j_2 \\ z_{j_1}^{(i)} v_{j_2}^{(i)} = v_{j_1}^{(i)} z_{j_2}^{(i)} \quad \text{if } x_{j_1}^{(i)} x_{j_2}^{(i)} \notin E^{(i)} \end{array} \right\}.$$

Since $z_1^{(1)}, z_1^{(2)} \neq 0$, the equations of the form $z_1^{(1)} v_j^{(i)} = v_1^{(1)} z_j^{(i)}$ (with $i = 2, 3, \dots, k$) and the equations of the form $z_1^{(2)} v_j^{(1)} = v_1^{(2)} z_j^{(1)}$ can only be satisfied if there exists some $\lambda \in \mathbb{Z}$ such that

$$v_j^{(i)} = \lambda \frac{z_j^{(i)}}{d} \quad \text{for all } i = 1, 2, \dots, k \text{ and } j = 1, 2, \dots, n^{(i)}$$

where $d := \gcd_{i,j} z_j^{(i)}$. This solution satisfies all the other conditions in the expression of the centralizer of x and thus we can conclude that

$$Z_{G_\Gamma}(x) = \left\{ \left(\prod_i \prod_j \left(x_j^{(i)} \right)^{z_j^{(i)}/d} \right)^{v_0} \prod_{l=1}^N y_l^{s_l} \mid v_0, s_l \in \mathbb{Z} \right\} = \left\langle \prod_i \prod_j \left(x_j^{(i)} \right)^{z_j^{(i)}/d} \right\rangle \times \gamma_2(G_\Gamma) \cong \mathbb{Z}^{N+1}.$$

□

Lemma 6.1 allows us to study the automorphisms of G_Γ .

Corollary 6.2. *Let $\Gamma(V, E)$ be a finite undirected simple graph. Denote with $\Gamma^{(0)}$ the induced subgraph $\Gamma(V_0 \setminus V_1)$ and with $\Gamma^{(i)}$ the connected components of $\Gamma(V_1)$ (where $i = 1, 2, \dots, k$). Then for any automorphism $\varphi \in \text{Aut}(G_\Gamma)$ there exists a unique permutation $\sigma \in S_k$ such that:*

(i) $\varphi(H^{(i)}) = H^{(\sigma(i))}$ for any $i = 1, 2, \dots, k$.

(ii) The corresponding connected components are isomorphic, i.e. $\Gamma^{(i)} \cong \Gamma^{(\sigma(i))}$ (for any $i = 1, 2, \dots, k$).

Proof. Fix any component $i_0 \in \{1, 2, \dots, k\}$. Since $\Gamma^{(i_0)}$ is a subgraph of $\Gamma(V_1)$, it follows by Lemma 4.1 that

$$h(Z_{G_\Gamma}(\varphi(x_1^{(i_0)}))) = \deg(x_1^{(i_0)}) + N + 1 > N + 1.$$

Lemma 6.1 now implies that $\varphi(x_1^{(i_0)})$ can only have non-zero exponents for vertices from one component. Recall that H_1 is a characteristic subgroup (see Lemma 4.3). Hence, we obtain that there is a unique component $\sigma(i_0) \in \{1, 2, \dots, k\}$ such that $\varphi(x_1^{(i_0)}) \in H^{(\sigma(i_0))}$. Repeating this argument for any $i_0 \in \{1, 2, \dots, k\}$ yields a unique map $\sigma : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\}$ such that

$$\varphi(x_1^{(i)}) \in H^{(\sigma(i))} \quad \text{for all } i = 1, 2, \dots, k.$$

Fix any component $i_0 \in \{1, 2, \dots, k\}$ and some vertex $x_j^{(i_0)}$ of that component such that $x_1^{(i_0)} x_j^{(i_0)} \in E$. Using the same argument as in the beginning of the proof, we can derive that $\varphi(x_j^{(i_0)}) \in H^{(\tilde{i}_0)}$ for some unique component $\tilde{i}_0 \in \{1, 2, \dots, k\}$. Since $x_1^{(i_0)}$ and $x_j^{(i_0)}$ commute, it holds that also $\varphi(x_1^{(i_0)})$ and $\varphi(x_j^{(i_0)})$ commute. However, since $\varphi(x_1^{(i_0)}), \varphi(x_j^{(i_0)}) \notin \gamma_2(G_\Gamma)$ and vertices from different components are not connected via an edge (and thus the corresponding group elements do not commute), we obtain that $\tilde{i}_0 = \sigma(i_0)$ and thus $\varphi(x_j^{(i_0)}) \in H^{(\sigma(i_0))}$. If $x_j^{(i_0)}$ is any vertex of component i_0 , then there exists a path in $\Gamma^{(i_0)}$ connecting $x_1^{(i_0)}$ and $x_j^{(i_0)}$. Using the previous argument inductively with this path we can derive that $\varphi(x_j^{(i_0)}) \in H^{(\sigma(i_0))}$.

Hence, we can conclude that $\varphi(H^{(i)}) \subseteq H^{(\sigma(i))}$ for all components i . By construction of this unique map σ and since φ is an automorphism, one can derive that $\sigma \in S_k$ and $\varphi(H^{(i)}) = H^{(\sigma(i))}$ for all components $i \in \{1, 2, \dots, k\}$. The second item follows directly by applying Lemma 3.4. \square

In contrast to the situation for the simplicial join, it is not possible to give a nice general description of the Reidemeister spectrum of a group G_Γ in terms of the Reidemeister spectra of the groups associated to the connected components of Γ . Nevertheless, in practice Corollary 6.2 is very useful to determine the Reidemeister spectrum in concrete cases.

Indeed, fix any automorphism $\varphi \in \text{Aut}(G_\Gamma)$. Corollary 6.2 allows us to describe the matrices corresponding to φ_1 and φ_2 . These matrices will have the following form

$$\left(\begin{array}{cccc|c} A^{(1)} & 0 & \dots & 0 & * \\ 0 & A^{(2)} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & A^{(s)} & * \\ \hline 0 & \dots & \dots & 0 & A^{(0)} \end{array} \right) \quad \text{and} \quad \left(\begin{array}{cccc|ccc} (A^{(1)})_2 & 0 & \dots & 0 & 0 & * & * \\ 0 & (A^{(2)})_2 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & (A^{(s)})_2 & 0 & * & * \\ \hline 0 & \dots & \dots & 0 & T & * & * \\ \hline 0 & \dots & \dots & 0 & 0 & T' & * \\ \hline 0 & \dots & \dots & 0 & 0 & 0 & (A^{(0)})_2 \end{array} \right)$$

where (as with the simplicial join) we use s to denote the number of types of components. The matrices with a superscript correspond to automorphisms on the induced groups associated to the disjoint union of the components of a particular type.

The matrix T can be described by using blocks corresponding with the commutators between different connected components of $\Gamma(V_1)$. Each column of this block matrix consists of all zeros except at one position. The matrix at this spot can be described by means of the tensor product of two matrices. Similarly, the matrix T' can be described by using blocks corresponding with the commutators between $\Gamma(V_0 \setminus V_1)$ and the connected components of $\Gamma(V_1)$. Each column of this block matrix consists of all zeros except at one position which can be described by using the tensor product of some matrix with $A^{(0)}$. For the precise description of the matrices corresponding to φ_1 and φ_2 , we refer the interested reader to [Lat21, Application 7.1.6]. We illustrate this full description by considering two graphs with four vertices.

Example 6.3. Let us consider the graph Γ in Figure 3a. Hence, $\Gamma = \Gamma^{(0)} \sqcup \Gamma^{(1)}$ where $\Gamma^{(0)}$ and $\Gamma^{(1)}$ denote the induced subgraphs $\Gamma(V_0)$ and $\Gamma(V_1 \setminus V_0)$. Fix any automorphism $\varphi \in \text{Aut}(G_\Gamma)$. Using the full description it follows that the matrices of φ_1 and φ_2 have the following form

$$\begin{pmatrix} A^{(1)} & * \\ 0 & A^{(0)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^{(1)} \otimes A^{(0)} & * \\ 0 & \det(A^{(0)}) \end{pmatrix}$$

where $A^{(1)}, A^{(0)} \in \text{GL}_2(\mathbb{Z})$.

Consider the graph Γ in Figure 3b. Thus $\Gamma = \Gamma^{(1)} \sqcup \Gamma^{(2)}$ where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ denote the two connected components of $\Gamma(V_1 \setminus V_0)$. For any automorphism $\varphi \in \text{Aut}(G_\Gamma)$ we obtain by Corollary 6.2 that either $\varphi(H^{(1)}) = H^{(2)}$ and $\varphi(H^{(2)}) = H^{(1)}$ or that $\varphi(H^{(i)}) = H^{(i)}$ (for $i = 1, 2$). In the first case, the matrices of φ_1 and φ_2 have the following form

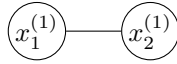
$$\begin{pmatrix} 0 & A_2^{(1)} \\ A_1^{(1)} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -(A_2^{(1)} \otimes A_1^{(1)}) \end{pmatrix}$$

where $A_1^{(1)}, A_2^{(1)} \in \text{GL}_2(\mathbb{Z})$. In the other case, the matrices of φ_1 and φ_2 have the following form

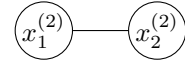
$$\begin{pmatrix} A_1^{(1)} & 0 \\ 0 & A_2^{(1)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_1^{(1)} \otimes A_2^{(1)} \end{pmatrix}$$

where $A_1^{(1)}, A_2^{(1)} \in \text{GL}_2(\mathbb{Z})$.

For the details of these two examples we refer the reader to [Lat21, Theorem 7.2.2 and 7.2.3].



(a) Only one edge



(b) 2 edges and 2 components

Figure 3: Illustration disjoint union on graphs with 4 vertices

7 Examples

We illustrate the results from the previous sections by determining the Reidemeister spectrum of some families of groups associated to graphs. We first introduce some notation.

Notation 7.1. Let $\Gamma(\{x_1, x_2, \dots, x_n\}, E)$ be a finite undirected simple graph and denote with $\lambda = (\lambda_1 \ \lambda_2 \ \dots \ \lambda_n)^T \in \mathbb{Z}^{n \times 1}$ some vector of integers. We introduce the notation $(x_1, \dots, x_n)^\lambda$ to denote $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. If $A \in \mathbb{Z}^{n \times n}$, then we denote by $A_{\cdot i}$ (respectively $A_{i \cdot}$) the i -th column (respectively row) of A (with $i = 1, 2, \dots, n$).

7.1 Disjoint union of a complete graph and an isolated vertex

Denote with Γ_n (for $n \in \mathbb{N}_{>2}$) the disjoint union of the complete graph on $n - 1$ vertices and an isolated vertex. We assume that x_n is the isolated vertex (and thus $\Gamma(\{x_1, x_2, \dots, x_{n-1}\})$ is a complete graph). Denote with $y_i := [x_n, x_i]$ (for $i = 1, 2, \dots, n - 1$) the commutators of G_{Γ_n} .

Theorem 7.2. *The Reidemeister spectrum of the groups associated to Γ_n (for $n \in \mathbb{N}_{>2}$) is given by*

$$\text{Spec}_R(G_{\Gamma_n}) = \begin{cases} 2\mathbb{N}_0^2 \cup 2|\mathbb{N}^2 - 4|_\infty \cup \{\infty\} & \text{if } n = 3 \\ 2(2\mathbb{N}_0 - 1) \cup 8\mathbb{N}_0 \cup \{\infty\} & \text{if } n \geq 4 \end{cases}$$

where \mathbb{N}^2 (respectively \mathbb{N}_0^2) denotes the squares (respectively non-zero squares) of integers.

Proof. We use a similar approach as in [DTV20, section 4] where they determine the Reidemeister spectrum of $N_{r,2}$ for $r \in \mathbb{N}_{\geq 2}$.

Fix any automorphism $\varphi \in \text{Aut}(G_{\Gamma_n})$. Corollary 6.2 implies that the matrix of φ_1 has the following form

$$\begin{pmatrix} A & * \\ 0 & \alpha \end{pmatrix}$$

where $* \in \mathbb{Z}^{(n-1) \times 1}$, $A \in \text{GL}_{n-1}(\mathbb{Z})$ and $\alpha \in \{-1, 1\}$. Note that by Lemma 2.2 for any $i = 1, 2, \dots, n-1$ it holds that

$$\varphi(y_i) = [\varphi(x_n), \varphi(x_i)] = [x_n^\alpha, (x_1, \dots, x_{n-1})^{A \cdot i}] = (y_1, \dots, y_{n-1})^{\alpha A \cdot i}$$

and thus the matrix of φ_2 is equal to αA . Using Lemma 3.3 it follows that

$$R(\varphi) = |\det(\mathbb{1}_{n-1} - A)|_\infty |1 - \alpha|_\infty |\det(\mathbb{1}_{n-1} - \alpha A)|_\infty.$$

If we assume that $R(\varphi) < \infty$, then we obtain that $\alpha = -1$. We denote with $p_A \in \mathbb{Z}[x]$ the characteristic polynomial of the matrix A . Hence, we get that $R(\varphi) = 2|p_A(1)p_A(-1)|_\infty$.

To any matrix $A \in \text{GL}_{n-1}(\mathbb{Z})$ we can associate an automorphism of G_{Γ_n} with Reidemeister number $2|p_A(1)p_A(-1)|_\infty$. Indeed, fix any matrix $A \in \text{GL}_{n-1}(\mathbb{Z})$. Define the map $\varphi : G_{\Gamma_n} \rightarrow G_{\Gamma_n}$ by setting (with $i = 1, 2, \dots, n-1$):

$$\begin{cases} \varphi(x_i) = (x_1, \dots, x_{n-1})^{A \cdot i} \\ \varphi(x_n) = x_n^{-1} \\ \varphi(y_i) = (y_1, \dots, y_{n-1})^{-A \cdot i} \end{cases}$$

and extending it to G_{Γ_n} . One can check that $\varphi \in \text{Aut}(G_{\Gamma_n})$ and that $R(\varphi) = 2|p_A(1)p_A(-1)|_\infty$. For any monic polynomial $p(x) = x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 \in \mathbb{Z}[x]$ (with $a_0 = \pm 1$) we consider the *companion matrix* C_p of the polynomial p , i.e. the matrix defined by

$$C_p = \left(\begin{array}{ccc|c} 0 & \dots & 0 & -a_0 \\ & & & -a_1 \\ & & & \vdots \\ & \mathbb{1}_{n-2} & & -a_{n-2} \end{array} \right).$$

Note that $C_p \in \text{GL}_{n-1}(\mathbb{Z})$ and its characteristic polynomial is equal to p . Thus using the matrix C_p and the above argument, it follows that there exists an automorphism of G_{Γ_n} with Reidemeister number $2|p(1)p(-1)|_\infty$. Hence, we obtain that

$$\text{Spec}_R(G_{\Gamma_n}) = \{2|p(1)p(-1)|_\infty \mid p(x) \in \mathbb{Z}[x] \text{ is monic, } \deg(p) = n-1 \text{ and } p(0) = \pm 1\}.$$

Fix any $p(x) = x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 \in \mathbb{Z}[x]$ (with $a_0 = \pm 1$). If $n = 3$, then

$$|p(1)p(-1)|_\infty = 2|(1 + a_1 + a_0)(1 - a_1 + a_0)|_\infty = \begin{cases} 2|a_1^2|_\infty & \text{if } a_0 = -1 \\ 2|4 - a_1^2|_\infty & \text{if } a_0 = 1 \end{cases}.$$

Using this, it indeed follows that

$$\text{Spec}_R(G_{\Gamma_3}) = 2\mathbb{N}_0^2 \cup 2|\mathbb{N}^2 - 4|_\infty \cup \{\infty\}.$$

If $n \geq 4$, then one can check that

$$2|p(1)p(-1)|_\infty = \begin{cases} 2 \left| \left(1 + \sum_{i=0}^{m-1} a_{2i} \right)^2 - \left(\sum_{i=1}^m a_{2i-1} \right)^2 \right|_\infty & \text{if } n = 2m + 1 \\ 2 \left| \left(\sum_{i=0}^{m-1} a_{2i} \right)^2 - \left(1 + \sum_{i=1}^{m-1} a_{2i-1} \right)^2 \right|_\infty & \text{if } n = 2m \end{cases}.$$

So in both cases the Reidemeister spectrum is two times the difference of two squares. Note that the difference of two squares is always a multiple of four or an odd number. Hence, for any $n \in \mathbb{N}_{\geq 4}$ it holds that

$$\text{Spec}_R(G_{\Gamma_n}) \subseteq 2(2\mathbb{N}_0 - 1) \cup 8\mathbb{N}_0 \cup \{\infty\}.$$

To prove equality, it suffices to define (for any $n \in \mathbb{N}_{\geq 4}$ and $k \in \mathbb{N}_0$) the polynomials

$$q_k(x) = \begin{cases} x^{2m} + (k-2)x^2 + (k-1)x + 1 & \text{if } n = 2m + 1 \\ x^{2m-1} + (k-1)x^2 + (k-2)x + 1 & \text{if } n = 2m \end{cases}$$

$$r_k(x) = \begin{cases} x^{2m} + (k-1)x^2 + (k-1)x + 1 & \text{if } n = 2m + 1 \\ x^{2m-1} + kx^2 + (k-2)x + 1 & \text{if } n = 2m \end{cases}$$

and note that

$$2|q_k(1)q_k(-1)|_\infty = 2(2k-1) \quad \text{and} \quad 2|r_k(1)r_k(-1)|_\infty = 8k.$$

□

7.2 Graphs with four vertices and one or two disjoint edges

In this section we again consider the two graphs from Example 6.3 (see Figure 3). We include these two graphs to illustrate that not all Reidemeister spectra can be described using short and easy expressions. In Example 6.3 we described the matrices of φ_1 and φ_2 (for any $\varphi \in \text{Aut}(G_\Gamma)$). Using these matrices, Lemma 3.3 gives us an expression of the Reidemeister number $R(\varphi)$. However, this expression depends on the tensor product of two invertible 2×2 matrices over \mathbb{Z} . The next lemma follows by some easy calculations. We refer to [Lat21, Lemma 7.2.1] for a detailed proof.

Lemma 7.3. *Let $A, B \in GL_2(\mathbb{Z})$ be invertible matrices and $\epsilon = \pm 1$ and let $t_A = \text{Tr}(A)$, $t_B = \text{Tr}(B)$, $d_A = \det(A)$ and $d_B = \det(B)$. Then*

$$\det(\mathbb{1}_2 - A) = 1 - t_A + d_A = \begin{cases} 2 - t_A & \text{if } d_A = 1 \\ -t_A & \text{if } d_A = -1 \end{cases}$$

$$\det(\mathbb{1}_4 - \epsilon A \otimes B) = \begin{cases} (t_B - \epsilon t_A)^2 & \text{if } d_A = d_B = 1 \\ -(t_B + \epsilon t_A)^2 & \text{if } d_A = d_B = -1 \\ -(t_B^2 - t_A^2 - 4) & \text{if } d_A = -1 \text{ and } d_B = 1 \\ t_B^2 - t_A^2 + 4 & \text{if } d_A = 1 \text{ and } d_B = -1. \end{cases}$$

The expressions from Lemma 7.3 together with the description in Example 6.3 allow us to express the Reidemeister number $R(\varphi)$ in terms of two integers. To prove that any such expression is contained in the spectrum, one can use matrices of the form $\begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \in GL_2(\mathbb{Z})$ (with $m \in \mathbb{Z}$) to construct automorphisms having these specified Reidemeister numbers. We omit the details and refer the interested reader to [Lat21, Theorems 7.2.2 and 7.2.3].

Theorem 7.4. *If Γ is the undirected simple graph with four vertices and one edge (see Figure 3a), then*

$$\text{Spec}_R(G_\Gamma) = \{2|nm(n+m)^2|_\infty, 2|nm(n^2 - m^2 - 4m)|_\infty \text{ with } m, n \in \mathbb{Z}\}.$$

If Γ is the undirected simple graph with four vertices and two disjoint edges (see Figure 3b), then

$$\text{Spec}_R(G_\Gamma) = \mathbb{N}_0^3 \cup \{|nm(n+m)^2|_\infty, |nm(n^2 - m^2 - 4m)|_\infty, |(n-2)(n+2)^2|_\infty \text{ with } m, n \in \mathbb{Z}\}.$$

7.3 Cycle graphs

In this section we consider the cycle graphs C_n . It turns out that for $n \in \mathbb{N}_{\geq 5}$ the associated group G_{C_n} has the R_∞ -property. We present the ideas behind this claim, but for the details and the proofs in this section, we refer to [Lat21, Section 9.1].

For any $n \in \mathbb{N}_{\geq 3}$ we define the cycle graph C_n by

$$C_n(\{x_1, x_2, \dots, x_n\}, \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}).$$

In order to describe the Reidemeister spectrum of G_{C_n} we need to understand the automorphisms of G_{C_n} . As before, one can study the Hirsch number of the centralizers of elements of G_{C_n} and derive the next lemma.

Lemma 7.5. *Let $n \geq 5$ be an integer. For any $z_i, t_l \in \mathbb{Z}$ (for $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, N$) it holds that*

$$h \left(Z_{G_{C_n}} \left(\prod_{i=1}^n x_i^{z_i} \prod_{l=1}^N y_l^{t_l} \right) \right) < N + 3$$

if there are two different indices $i_1, i_2 \in \{1, 2, \dots, n\}$ such that $z_{i_1}, z_{i_2} \neq 0$.

By Lemma 4.1 it holds that $h(Z_{G_{C_n}}(\varphi(x_i))) = N + 3$ for any $i = 1, 2, \dots, n$ and any $\varphi \in \text{Aut}(G_{C_n})$. Hence, Lemma 7.5 yields that $\varphi_1(x_i \gamma_2(G_{C_n})) \in \langle x_{\sigma(i)} \gamma_2(G_{C_n}) \rangle$ for some $\sigma(i) = 1, 2, \dots, n$. Based on this argument, one can proof the next corollary.

Corollary 7.6. *Let $n \geq 5$ be an integer and $\varphi \in \text{Aut}(G_{C_n})$ an automorphism of G_{C_n} . Then there exist a permutation $\sigma \in S_n$ and $\epsilon_i = \pm 1$ (for all $i = 1, 2, \dots, n$) such that*

$$\varphi_1(x_i \gamma_2(G_{C_n})) = x_{\sigma(i)}^{\epsilon_i} \gamma_2(G_{C_n}) \quad \text{for all } i = 1, 2, \dots, n.$$

Moreover, the permutation σ belongs to the dihedral group D_n of order $2n$. In particular, σ consists of a rotation or a reflection of the graph C_n .

Corollary 7.6 gives us enough information on the automorphisms to prove that G_{C_n} has the R_∞ -property for $n \geq 5$. This proof uses Proposition 2.5 by describing an eigenvector with eigenvalue 1 for φ_1 or φ_2 . Note that $G_{C_3} \cong \mathbb{Z}^3$ since C_3 is the complete graph on 3 vertices. We already discussed the cycle graph C_4 in Example 5.15. This leads to the next result.

Theorem 7.7. *The Reidemeister spectrum of the group G_{C_n} associated to the cycle graph C_n on n vertices is given by*

$$\text{Spec}_R(G_{C_n}) = \begin{cases} \mathbb{N}_0 \cup \{\infty\} & \text{if } n = 3 \\ 2\mathbb{N}_0 \cup \{\infty\} & \text{if } n = 4 \\ \{\infty\} & \text{if } n \geq 5 \end{cases}$$

Remark 7.8. Using similar ideas, one can consider the cycle graph P_n on n vertices with one edge removed (also known as the path graph on n vertices). For the details we refer to [Lat21, Section 9.2]. The Reidemeister spectrum of G_{P_n} is given by

$$\text{Spec}_R(G_{P_n}) = \begin{cases} 4\mathbb{N}_0 \cup \{\infty\} & \text{if } n = 3 \\ \{\infty\} & \text{if } n \geq 4 \end{cases}$$

In [GW09, Example 4.1] D. Gonçalves and P. Wong prove that G_{P_4} has the R_∞ -property. We linked the group to the path graph on four vertices to better understand its structure. The methods in this paper provide a more general framework to consider similar groups and allow for a more elegant proof of this result. Moreover, we presented new examples of finitely generated torsion-free 2-step nilpotent groups that are associated to a graph and have the R_∞ -property.

7.4 Reidemeister spectrum of groups associated to graphs with at most 4 vertices

Looking back at the different methods we developed, we are able to describe the Reidemeister spectrum of the groups associated to the graphs with at most four vertices. The result is summarised in Table 1 and Table 2.



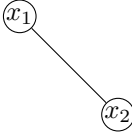
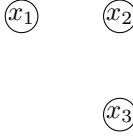
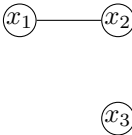
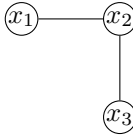
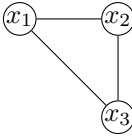
Graph Γ	Reidemeister spectrum $\text{Spec}_R(G_\Gamma)$	Graph Γ	Reidemeister spectrum $\text{Spec}_R(G_\Gamma)$
	$\{2, \infty\}$		
	$2\mathbb{N}_0 \cup \{\infty\}$		$\mathbb{N}_0 \cup \{\infty\}$
	$(2\mathbb{N}_0 - 1) \cup 4\mathbb{N}_0 \cup \{\infty\}$		$2\mathbb{N}_0^2 \cup 2 \mathbb{N}^2 - 4 _\infty \cup \{\infty\}$
	$4\mathbb{N}_0 \cup \{\infty\}$		$\mathbb{N}_0 \cup \{\infty\}$

Table 1: Reidemeister spectrum of groups associated to graphs with at most 3 vertices.

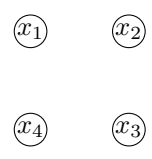
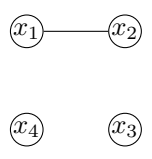
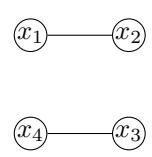
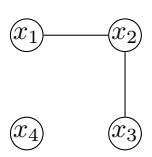
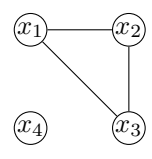
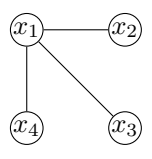
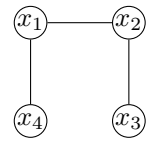
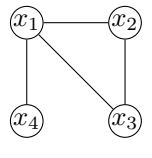
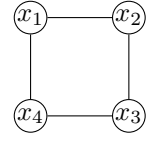
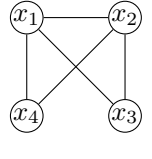
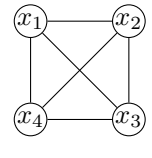
Graph Γ	Reidemeister spectrum $\text{Spec}_R(G_\Gamma)$	Graph Γ	Reidemeister spectrum $\text{Spec}_R(G_\Gamma)$
	$\mathbb{N}_0 \cup \{\infty\}$		$\left\{ \begin{array}{l} 2 nm(n+m)^2 _\infty, \\ 2 nm(n^2-m^2-4m) _\infty \\ \text{with } m, n \in \mathbb{Z} \end{array} \right\}$
	$\left\{ \begin{array}{l} \mathbb{N}_0^3 \cup \\ nm(n+m)^2 _\infty, \\ nm(n^2-m^2-4m) _\infty, \\ (n-2)(n+2)^2 _\infty \\ \text{with } m, n \in \mathbb{Z} \end{array} \right\}$		$\{\infty\}$
	$2(2\mathbb{N}_0 - 1) \cup 8\mathbb{N}_0 \cup \{\infty\}$		$2(2\mathbb{N}_0 - 1) \cup 8\mathbb{N}_0 \cup \{\infty\}$
	$\{\infty\}$		$4\mathbb{N}_0^2 \cup 4 \mathbb{N}^2 - 4 _\infty \cup \{\infty\}$
	$2\mathbb{N}_0 \cup \{\infty\}$		$2\mathbb{N}_0 \cup \{\infty\}$
	$\mathbb{N}_0 \cup \{\infty\}$		

Table 2: Reidemeister spectrum of groups associated to graphs with 4 vertices.

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