The Relative Weak Asymptotic Homomorphism Property for Inclusions of Finite von Neumann Algebras

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Abstract

A triple of finite von Neumann algebras $B \subseteq N \subseteq M$ is said to have the relative weak asymptotic homomorphism property if there exists a net of unitary operators $\{u_\lambda\}_{\lambda\in\Lambda}$ in B such that

$$\lim_{\lambda} \|\mathbb{E}_B(xu_\lambda y) - \mathbb{E}_B(\mathbb{E}_N(x)u_\lambda \mathbb{E}_N(y))\|_2 = 0$$

for all $x, y \in M$. We prove that a triple of finite von Neumann algebras $B \subseteq N \subseteq M$ has the relative weak asymptotic homomorphism property if and only if N contains the set of all $x \in M$ such that $Bx \subseteq \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \ldots, x_n$ in M. Such an x is called a one sided quasi-normalizer of B, and the von Neumann algebra generated by all one sided quasi-normalizers of B is called the one sided quasi-normalizer algebra of B. We characterize one sided quasi-normalizer algebras for inclusions of group von Neumann algebras and use this to show that one sided quasi-normalizer algebras and quasi-normalizer algebras are not equal in general. We also give some applications to inclusions $L(H) \subseteq L(G)$ arising from containments of groups. For example, when $L(H)$ is a masa we determine the unitary normalizer algebra as the von Neumann algebra generated by the normalizers of H in G.

Key Words: normalizer, quasi-normaliser, von Neumann algebra, discrete group, homomorphism
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1 Introduction

Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M. The algebra B has the *weak asymptotic homomorphism* *property* if there exists a net of unitary operators $\{u_\lambda\}_{\lambda\in\Lambda}$ in B such that

$$\lim_\lambda \|\mathbb{E}_B(xu_\lambda y) - \mathbb{E}_B(x)u_\lambda \mathbb{E}_B(y)\|_2 = 0, \quad x, y \in M. \tag{1.1}$$

This property was introduced by Robertson, Sinclair and the third author [19, 18, 20] for masas (maximal abelian subalgebras) of type II_1 factors. They showed that if B has the weak asymptotic homomorphism property, then B is singular in M, and the purpose of introducing this property was to have an easily verifiable criterion for singularity. In [21], Sinclair, White, Wiggins and the third author proved that the converse is also true: if B is a singular masa then B has the weak asymptotic homomorphism property (which is equivalent to the weakly mixing property of Jolissaint and Stalder [8] for masas). We note that the results of [21] are formulated for M a II_1 factor, but the proofs only require that M be a finite von Neumann algebra with a faithful normal trace. Thus the equivalence of the weak asymptotic homomorphism property and singularity for masas is still valid at this greater level of generality. However, this equivalence breaks down beyond the masa case; in [7], Grossman and Wiggins showed that if B is a singular factor then B does not necessarily have the weak asymptotic homomorphism property.

In [2], Chifan introduced a generalized version as follows. A triple of von Neumann algebras $B \subseteq N \subseteq M$ is said to have the *relative weak asymptotic homomorphism property* if there exists a net of unitary operators $\{u_\lambda\}_{\lambda\in\Lambda}$ in B such that

$$\lim_\lambda \|\mathbb{E}_B(xu_\lambda y) - \mathbb{E}_B(\mathbb{E}_N(x)u_\lambda \mathbb{E}_N(y))\|_2 = 0, \quad x, y \in M. \tag{1.2}$$

Let $\mathcal{N}_M(B) := \{u \text{ a unitary in } M : uBu^* = B\}$ denote the group of unitary normalizers of B in M. Chifan showed that if B is a masa in a separable type II_1 factor M, then

$$B \subseteq \mathcal{N}_M(B)'' \subseteq M \tag{1.3}$$

has the relative weak asymptotic homomorphism property (see [10] for a different proof).

A natural extension of Chifan's theorem is to consider a general triple of finite von Neumann algebras $B \subseteq N \subseteq M$ and to ask for conditions which ensure that the relative weak asymptotic homomorphism property holds. Our main purpose in this paper is to provide a characterization of this property and to consider some subsequent applications. Our characterization is based on certain operators that are closely related to the quasi-normalizers introduced by Popa in [14, 15]. Recall that he defined a *quasi-normalizer* for an inclusion $B \subseteq M$ to be an element $x \in M$ for which a finite set $\{x_1, \ldots, x_n\} \subseteq M$ can be found so that

$$Bx \subseteq \sum_{i=1}^n x_i B, \quad xB \subseteq \sum_{i=1}^n Bx_i, \tag{1.4}$$

and we denote the set of quasi-normalizers by $q\mathcal{N}_M(B)$. These are not quite the correct operators for our purposes, so we make a small adjustment by defining a *one sided quasi-normalizer* to be any element $x \in M$ satisfying only the first inclusion in (1.4), and we denote the set of such elements by $q\mathcal{N}_M^{(1)}(B)$. In Section 3, we prove that a triple of finite von Neumann algebras $B \subseteq N \subseteq M$ has the relative weak asymptotic homomorphism property if and only if N contains $q\mathcal{N}_M^{(1)}(B)$. The von Neumann algebra generated by $q\mathcal{N}_M^{(1)}(B)$ is called the *one sided quasi-normalizer algebra* of B in M, and is denoted by $W^*(q\mathcal{N}_M^{(1)}(B))$, a notation which reflects the fact that $q\mathcal{N}_M^{(1)}(B)$ is not necessarily self-adjoint. In the case that B is a masa this characterization, combined with Chifan's theorem, gives that $W^*(q\mathcal{N}_M^{(1)}(B)) = \mathcal{N}_M(B)''$. It had been shown earlier in [17] that $q\mathcal{N}_M(B)'' = \mathcal{N}_M(B)''$ when M is a separable II_1 factor, and so $W^*(q\mathcal{N}_M^{(1)}(B)) = q\mathcal{N}_M(B)''$ for masas, although these von Neumann algebras are different in general (see Example 5.3). The advantage of the one sided quasi-normalizers is that they seem to be easier to calculate in specific examples, as we will see below. We note that one sided objects of this type play a significant role in understanding normalizers. For example, one sided unitary normalizers were important in [22], and a one sided version of groupoid normalizers was a key technical tool in [6].

After our characterization has been established in Section 3, we devote Section 4 to the study of $W^*(q\mathcal{N}_M^{(1)}(B))$. Here we show, among other results, that the one sided quasi-normalizer algebra and the quasi-normalizer algebra of an atomic von Neumann subalgebra B of a finite von Neumann algebra M are equal to M.

In Section 5, we apply these results to inclusions of von Neumann algebras arising from inclusions $H \subseteq G$ of discrete groups. We characterize one sided quasi-normalizer algebras of such inclusions in terms of properties of the groups, and also show that one sided quasi-normalizer algebras and quasi-normalizer algebras are not equal in general. Making use of the one sided quasi-normalizers, we are able to study unitary normalizers and show, for example, that when $L(H)$ is a masa in $L(G)$, its unitary normalizer algebra is the von Neumann algebra of the group of normalizers of H in G. This leads to new characterizations of when $L(H)$ is either singular or Cartan. In section 6, we summarize the relationships between various types of normalizer algebras, and we show that one sided quasi-normalizer algebras have some special properties when compared to the other types. For example, we establish a tensor product formula in Proposition 6.1 which parallels similar formulas for groupoid normalizers and intertwiners proved in [6].

Sufficient background material on finite von Neumann algebras for this paper may be found in [20].

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2 Preliminaries

Throughout this paper, M is a finite von Neumann algebra with a given faithful normal trace τ. We use $L^2(M) = L^2(M,\tau)$ to denote the Hilbert space obtained by the GNS-construction of M with respect to τ. The image of $1 \in M$ via the GNS construction is denoted by ξ and the image of $x \in M$ is denoted by $x\xi$. Throughout this paper, we will reserve the letter ξ for this purpose. The trace norm of $x \in M$ is defined by $\|x\|_2 = \|x\|_{2,\tau} = \tau(x^*x)^{1/2}$. The letter J is reserved for the isometric conjugate linear operator on $L^2(M)$ defined on $M\xi$ by $J(x\xi) = x^*\xi$ and extended by continuity to $L^2(M)$ from the dense subspace $M\xi$.

Let $B \subseteq M$ be an inclusion of finite von Neumann algebras. Then there exists a unique faithful normal conditional expectation $\mathbb{E}_B$ from M onto B preserving τ. Let e_B be the projection of $L^2(M)$ onto $L^2(B)$. For $x \in M$, we have $e_B(x\xi) = \mathbb{E}_B(x)\xi$. The von Neumann algebra $\langle M, e_B\rangle$ generated by M and e_B is called *the basic construction*, which plays a crucial role in the study of von Neumann subalgebras of finite von Neumann algebras. The basic construction has many remarkable properties (see [9, 12, 20]). In particular, there exists a unique faithful tracial weight Tr on $\langle M, e_B\rangle$ such that

$$\mathrm{Tr}(xe_By) = \tau(xy), \quad x, y \in M. \tag{2.1}$$

Furthermore, we can choose a net of vectors $\{\xi_i\}_{i\in I}$ from $L^2(M)$ such that

$$\mathrm{Tr}(t) = \sum_{i\in I}\langle t\xi_i, \xi_i\rangle_{2,\tau}, \quad t \in \langle M, e_B\rangle^+ \tag{2.2}$$

(see [20, Lemma 4.3.4, Theorem 4.3.11]). An examination of the proof of [20, Lemma 4.3.4] shows that we may construct the index set I to have a minimal element $i = 1$ and we may take ξ_1 to be ξ. Letting $t = e_B$ in (2.2), we have $e_B\xi_i = 0$ for all $i \neq 1$, $i \in I$.

There is a well defined map $\Psi : Me_BM \to M$, given by

$$\Psi(xe_By) = xy, \quad x, y \in M, \tag{2.3}$$

and called the *pull down map*. It was shown in [12] (also see [20]) that the pull down map can be extended to a contraction from $L^1(\langle M, e_B\rangle, \mathrm{Tr})$ to $L^1(M,\tau)$, which is just the predual of the embedding $M \hookrightarrow \langle M, e_B\rangle$.

Let $w \in \langle M, e_B\rangle$, and let $\eta = w(\xi) \in L^2(M)$. Then

$$L_\eta(x\xi) = Jx^*J(\eta), \quad x \in M, \tag{2.4}$$

is a densely defined operator affilated with M. We may identify L_η with η in a canonical way so that $\|L_\eta\|_{2,\tau} = \tau(L_\eta^*L_\eta)^{1/2} = \|\eta\|_{2,\tau}$ is well defined (see [11]). Note that $we_B = L_\eta e_B$. Indeed, for $x \in M$, we have

$$L_\eta e_B(x\xi) = L_\eta(\mathbb{E}_B(x)\xi) = J\mathbb{E}_B(x^*)J(\eta) = J\mathbb{E}_B(x^*)Jw(\xi)$$

$$= wJ\mathbb{E}_B(x^*)J(\xi) = we_B(x\xi). \tag{2.5}$$

For $z \in \langle M, e_B\rangle$, define $\|z\|_{2,\mathrm{Tr}} = \mathrm{Tr}(z^*z)^{1/2}$. The following lemmas are well known to experts. For the reader's convenience, we include the proofs.

Lemma 2.1. *Suppose that $w \in \langle M, e_B\rangle$ and $\eta = w(\xi) \in L^2(M)$. Then $\|we_B\|_{2,\mathrm{Tr}} = \|\eta\|_{2,\tau}$.*

Proof. The equalities

$$\begin{aligned}\|we_B\|_{2,\mathrm{Tr}}^2 = \mathrm{Tr}(e_Bw^*we_B) &= \sum_{i\in I}\langle we_B\xi_i, we_B\xi_i\rangle_{2,\tau}\\ &= \langle w(\xi), w(\xi)\rangle_{2,\tau} = \langle \eta,\eta\rangle_{2,\tau} = \|\eta\|_{2,\tau}^2\end{aligned} \tag{2.6}$$

follow from (2.2) and the fact that $e_B\xi_i = 0$ for $i \neq 1$. □

The following lemma plays a key role in the proof of Lemma 3.5.

Lemma 2.2. *Suppose that $w \in \langle M, e_B\rangle$ and $\eta = w(\xi) \in L^2(M)$. Then*

$$\Psi(we_Bw^*) = L_\eta L_\eta^*, \tag{2.7}$$

where Ψ is the pull down map and L_η is the operator defined by (2.4).

Proof. Since $\eta \in L^2(M)$, there exists a sequence $\{x_n\}_{n=1}^\infty$ in M such that

$$\lim_{n\to\infty}\|L_\eta - x_n\|_{2,\tau} = \lim_{n\to\infty}\|\eta - x_n\xi\|_{2,\tau} = 0. \tag{2.8}$$

Therefore,

$$\begin{aligned}\|L_\eta L_\eta^* - x_nx_n^*\|_{1,\tau} &\le \|(L_\eta - x_n)L_\eta^*\|_{1,\tau} + \|x_n(L_\eta^* - x_n^*)\|_{1,\tau}\\ &\le \|L_\eta - x_n\|_{2,\tau}\|L_\eta\|_{2,\tau} + \|x_n\|_{2,\tau}\|L_\eta - x_n\|_{2,\tau} \to 0.\end{aligned} \tag{2.9}$$

By Lemma 2.1,

$$\lim_{n\to\infty}\|we_B - x_ne_B\|_{2,\mathrm{Tr}} = \lim_{n\to\infty}\|\eta - x_n\xi\|_{2,\tau} = 0, \tag{2.10}$$

and so $\lim_{n\to\infty}\|we_Bw^* - x_ne_Bx_n^*\|_{1,\mathrm{Tr}} = 0$. Noting that $\Psi(x_ne_Bx_n^*) = x_nx_n^*$, the equality $\Psi(we_Bw^*) = L_\eta L_\eta^*$ follows since Ψ is a continuous contraction from $L^1(\langle M, e_B\rangle, \mathrm{Tr})$ to $L^1(M,\tau)$. □

We have included here only the facts about the basic construction that we will need subsequently. Much more detailed coverage can be found in [3, 9, 12, 20].

3 Main result

This section is devoted to the main result of the paper, Theorem 3.1. We state it immediately, but defer the proof until Lemmas 3.2–3.5 have been established. We have included part (iii) for emphasis, but it is of course just a notational restatement of (ii).

Theorem 3.1. *The following conditions are equivalent for finite von Neumann algebras:*

(i) *The triple $B \subseteq N \subseteq M$ has the relative weak asymptotic homomorphism property;*

(ii) *If $x \in M$ satisfies $Bx \subseteq \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \ldots, x_n$ in M, then $x \in N$;*

(iii) *$q\mathcal{N}_M^{(1)}(B) \subseteq N$.*

To prove this theorem, we will need several lemmas. The following is essentially [2, Lemma 2.5] (see also [16, Corollary 2.3]), and so we omit the proof.

Lemma 3.2. *Suppose that a triple of finite von Neumann algebras $B \subseteq N \subseteq M$ does not have the relative weak asymptotic homomorphism property. Then there exists a nonzero projection $p \in B' \cap \langle M, e_B \rangle$ such that $0 < \mathrm{Tr}(p) < \infty$ and $p \leq 1 - e_N$.*

Lemma 3.3. *Let $B \subseteq N \subseteq M$ be a triple of finite von Neumann algebras, let $p \in \langle M, e_B \rangle$ be a finite projection satisfying $p \leq 1 - e_N$, and let $\varepsilon > 0$. Then there exists a finite number of elements $x_1, \ldots, x_n \in M$ such that $\mathbb{E}_N(x_i) = 0$ for $1 \leq i \leq n$, and*

$$\left\| p - \sum_{i=1}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} < \varepsilon. \tag{3.1}$$

Proof. By [13, Lemma 1.8], there are elements $y_1, \ldots, y_n \in M$ such that

$$\left\| p - \sum_{i=1}^{n} y_i e_B y_i^* \right\|_{2,\mathrm{Tr}} < \varepsilon/3. \tag{3.2}$$

For $1 \leq i \leq n$, let $x_i = y_i - \mathbb{E}_N(y_i)$, and note that $\mathbb{E}_N(x_i) = 0$. Since $pe_N = 0$, it follows from (3.2) that

$$\left\| \sum_{i=1}^{n} \mathbb{E}_N(y_i) e_B y_i^* \right\|_{2,\mathrm{Tr}} = \left\| e_N \left(p - \sum_{i=1}^{n} y_i e_B y_i^* \right) \right\|_{2,\mathrm{Tr}} < \varepsilon/3. \tag{3.3}$$

Also, the identity

$$(1 - e_N) \left(p - \sum_{i=1}^{n} y_i e_B y_i^* \right) e_N = - \sum_{i=1}^{n} (1 - e_N) y_i e_N e_B e_N y_i^* e_N$$

$$= -\sum_{i=1}^{n} (y_i - \mathbb{E}_N(y_i)) e_B \mathbb{E}_N(y_i^*)$$
$$= -\sum_{i=1}^{n} x_i e_B \mathbb{E}_N(y_i^*) \tag{3.4}$$

shows that

$$\left\| \sum_{i=1}^{n} x_i e_B \mathbb{E}_N(y_i^*) \right\|_{2,\mathrm{Tr}} = \left\| (1 - e_N) \left(p - \sum_{i=1}^{n} y_i e_B y_i^* \right) e_N \right\|_{2,\mathrm{Tr}} < \varepsilon/3, \tag{3.5}$$

from (3.2). Using the expansion

$$\begin{aligned} y_i e_B y_i^* &= (x_i + \mathbb{E}_N(y_i)) e_B y_i^* = x_i e_B y_i^* + \mathbb{E}_N(y_i) e_B y_i^* \\ &= x_i e_B x_i^* + x_i e_B \mathbb{E}_N(y_i^*) + \mathbb{E}_N(y_i) e_B y_i^* \end{aligned} \tag{3.6}$$

and the inequalities (3.2), (3.3), and (3.5), we see that

$$\begin{aligned} \left\| p - \sum_{i=1}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} &\leq \left\| p - \sum_{i=1}^{n} y_i e_B y_i^* \right\|_{2,\mathrm{Tr}} + \left\| \sum_{i=1}^{n} \mathbb{E}_N(y_i) e_B y_i^* \right\|_{2,\mathrm{Tr}} \\ &\quad + \left\| \sum_{i=1}^{n} x_i e_B \mathbb{E}_N(y_i^*) \right\|_{2,\mathrm{Tr}} < \varepsilon, \end{aligned} \tag{3.7}$$

proving the result. □

If $p \in B' \cap \langle M, e_B \rangle$, then $\mathcal{H} = pL^2(M)$ is a B-bimodule. Conversely, if $\mathcal{H} \subseteq L^2(M)$ is a B-bimodule and p is the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$, then $p \in B' \cap \langle M, e_B \rangle$. In the following we recall some basic facts about B-bimodules.

Suppose that a Hilbert subspace $\mathcal{H} \subseteq L^2(M)$ is a right B-module. Let $\mathcal{L}_B(L^2(B), \mathcal{H})$ be the set of bounded right B-modular operators from $L^2(B)$ into $\mathcal{H}$. For instance, if $\mathcal{H} = L^2(B)$, then $\mathcal{L}_B(L^2(B), L^2(B))$ consists of operators induced by the left action of B on $L^2(B)$.

Let B be a finite von Neumann algebra with a faithful normal trace τ. Suppose that B acts on the right on a Hilbert space $\mathcal{H}$. Then the dimension of $\mathcal{H}$ over B is defined as

$$\dim_B(\mathcal{H}) = \mathrm{Tr}(1), \tag{3.8}$$

where Tr is the unique tracial weight on B' satisfying the following condition

$$\mathrm{Tr}(xx^*) = \tau(x^*x), \quad x \in \mathcal{L}_B(L^2(B), \mathcal{H}). \tag{3.9}$$

We say that $\mathcal{H}$ is a *finite right B-module* if $\mathrm{Tr}(1) < \infty$. For details of finite right B-modules, we refer to [23, Appendix A].

Suppose that $\mathcal{H} \subseteq L^2(M)$ is a right B-module. Then $\mathcal{H}$ is called a *finitely generated right B-module* if there exists a finite set of elements $\{\eta_1, \dots, \eta_n\} \subseteq \mathcal{H}$ such that $\mathcal{H}$ is the closure of $\sum_{i=1}^n \eta_i B$. A set $\{\eta_i\}_{i=1}^n$ of elements in $\mathcal{H}$ is called an *orthonormal basis* of $\mathcal{H}$ if $\mathbb{E}_B(\eta_i^* \eta_j) = \delta_{ij} p_i \in B$, where each p_i is a projection and, for every $\eta \in \mathcal{H}$, we have

$$\eta = \sum_{i=1}^{n} \eta_i \mathbb{E}_B(\eta_i^* \eta). \tag{3.10}$$

Note that, by putting $\eta = \eta_j$ into (3.10), we have $\eta_j = \eta_j p_j$, $1 \le j \le n$. It might appear that the vectors on the right hand side of (3.10) are not in $\mathcal{H}$, since $\eta_i \in \mathcal{H} \subseteq L^2(M)$ and $\mathbb{E}_B(\eta_i^* \eta) \in L^1(B)$, but the construction of the orthonormal basis ensures that they do lie in this Hilbert space.

Let $p_{\mathcal{H}}$ be the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Following [15, Lemma 1.4.2], we have $p_{\mathcal{H}} = \sum_{i=1}^n L_{\eta_i} e_B L_{\eta_i}^*$, where L_η is defined as in (2.4). Let $w_i = L_{\eta_i} e_B$, a bounded operator since $w_i w_i^* \le p_{\mathcal{H}}$. For each $x \in M$ and $b \in B$,

$$\begin{aligned} w_i JbJ(x\xi) &= w_i(xb^*\xi) = L_{\eta_i} e_B(xb^*\xi) = L_{\eta_i}(\mathbb{E}_B(xb^*)\xi) \\ &= L_{\eta_i}(\mathbb{E}_B(x)b^*\xi) = Jb\mathbb{E}_B(x^*)J(\eta_i) \\ &= JbJJ\mathbb{E}_B(x^*)Jw_i(\xi) = JbJw_iJ\mathbb{E}_B(x^*)J(\xi) = JbJw_i(x\xi). \end{aligned} \tag{3.11}$$

Thus $w_i JbJ = JbJw_i$, which implies that $w_i \in \langle M, e_B \rangle$. Summarizing the above arguments, we have shown that

$$p_{\mathcal{H}} = \sum_{i=1}^{n} w_i e_B w_i^*, \tag{3.12}$$

where $w_i = L_{\eta_i} e_B \in \langle M, e_B \rangle$. We note that every finitely generated right B-module has an orthonormal basis, [15, 1.4.1].

The following lemma is proved by Vaes in [23, Lemma A.1] (see also [15, Lemma 1.4.2]). It is designed to circumvent the difficulty that finite right B-modules might not be finitely generated.

Lemma 3.4. *Suppose that $\mathcal{H}$ is a finite right B-module. Then there exists a sequence of projections $\{z_n\}_{n=1}^\infty$ in $Z(B) = B' \cap B$ and a sequence of integers $\{k_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and $\mathcal{H}z_n$ is unitarily equivalent to a left $p_n\mathbb{M}_{k_n}(B)p_n$ right B-module $p_n(L^2(B)^{(k_n)})$ for each n, where p_n is a projection in $\mathbb{M}_{k_n}(B)$. In particular, $\mathcal{H}z_n$ is a finitely generated right B-module.*

The following lemma is motivated by [15, Lemma 1.4.2].

Lemma 3.5. *Suppose that $\mathcal{H} \subseteq L^2(M)$ is a B-bimodule, and that $\mathcal{H}$ is a finitely generated right B-module with an orthonormal basis of length k. Let $p_{\mathcal{H}}$ be the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Then there exists a sequence of projections z_n in $B' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and for each n,*

$$z_n p_{\mathcal{H}} z_n(x\xi) = \sum_{i=1}^{k} x_{n,i} \mathbb{E}_B(x_{n,i}^* x)\xi, \quad x \in M, \tag{3.13}$$

for a finite number of elements $x_{n,1}, \ldots, x_{n,k} \in M$.

Proof. Let $\{\eta_i\}_{i=1}^k \subseteq \mathcal{H} \subseteq L^2(M,\tau)$ be an orthonormal basis for $\mathcal{H}$, in which case $\mathcal{H} = \oplus_{i=1}^k [\eta_i B]$. By (3.12), $p_{\mathcal{H}} = \sum_{i=1}^k w_i e_B w_i^* \in B' \cap \langle M, e_B\rangle$, where $w_i = L_{\eta_i} e_B \in \langle M, e_B\rangle$. For $b \in B$, we have

$$b \sum_{i=1}^{k} w_i e_B w_i^* = \sum_{i=1}^{k} w_i e_B w_i^* b. \tag{3.14}$$

Applying the pull down map to both sides and noting that $w_i(\xi) = \eta_i$, we obtain

$$b \left(\sum_{i=1}^{k} L_{\eta_i} L_{\eta_i}^* \right) = \left(\sum_{i=1}^{k} L_{\eta_i} L_{\eta_i}^* \right) b \tag{3.15}$$

by Lemma 2.2. Since $\sum_{i=1}^k L_{\eta_i} L_{\eta_i}^*$ is an operator affiliated with M, $q \in B' \cap M$ for all spectral projections q of $\sum_{i=1}^k L_{\eta_i} L_{\eta_i}^*$. Therefore, there exists a sequence of projections $z_n \in B' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and $\sum_{i=1}^k z_n L_{\eta_i} L_{\eta_i}^* z_n$ is a bounded operator for each n. Let $x_{n,i} = z_n L_{\eta_i}$, $1 \le i \le k$. Then $x_{n,i} \in M$ and

$$z_n p_{\mathcal{H}} z_n(x\xi) = \sum_{i=1}^{k} x_{n,i} \mathbb{E}_B(x_{n,i}^* x)\xi, \quad x \in M, \tag{3.16}$$

as required. $\square$

This completes the preparations for the proof of our main result, which we now give. We will establish only the equivalence of (i) and (ii) since, as already noted, (iii) is just a restatement of (ii).

Proof of Theorem 3.1. (ii)$\Rightarrow$(i). To derive a contradiction, suppose that (ii) holds but that the triple $B \subseteq N \subseteq M$ does not have the relative weak asymptotic homomorphism property. By Lemma 3.2, there exists a projection $p \in B' \cap \langle M, e_B\rangle$ such that $0 < \mathrm{Tr}(p) < \infty$ and $p \le 1 - e_N$.

Let $\mathcal{H} = pL^2(M)$. Then $\mathcal{H}$ is a B-bimodule and a finite right B-module. By Lemma 3.4, we may assume that $\mathcal{H}$ is a finitely generated right B-module. By Lemma 3.5, there exists a sequence of projections z_n in $B' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and for each n,

$$z_n p z_n(x\xi) = \sum_{i=1}^{k} x_{n,i}\mathbb{E}_B(x_{n,i}^* x)\xi \in M\xi, \quad x \in M, \tag{3.17}$$

for a finite number of elements $x_{n,1}, \ldots, x_{n,k} \in M$. Note that $z_n p z_n \in B' \cap \langle M, e_B\rangle$. Thus, for every $x \in M$,

$$B\left(z_n p z_n(x\xi)\right) = (z_n p z_n)(Bx\xi) \subseteq \sum_{i=1}^{k} x_{n,i} B\xi. \tag{3.18}$$

Thus $z_n p z_n(x\xi) \in N\xi \subseteq L^2(N)$ by the assumption (ii) of Theorem 3.1. Hence, for each $\eta \in L^2(M)$, $z_n p z_n(\eta) \in L^2(N)$. Since $\lim_{n\to\infty} z_n = 1$ in the strong operator topology,

$$p(\eta) = \lim_{n\to\infty} z_n p z_n(\eta) \in L^2(N), \tag{3.19}$$

and so $p \le e_N$. On the other hand, $p \le 1 - e_N$ and we arrive at the contradiction $p = 0$.

(i)$\Rightarrow$(ii). Suppose that $x \in M$ satisfies $Bx \subseteq \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \ldots, x_n$ in M, and let $\mathcal{H}$ be the closure of $BxB\xi$ in $L^2(M)$. Then $\mathcal{H}$ is a B-bimodule and $\mathcal{H} \subseteq L^2(\sum_{i=1}^{n} x_i B)$. Thus $\mathcal{H}$ is a finite right B-module. Let p be the projection of $L^2(M)$ onto $\mathcal{H}$. Then $p \in B' \cap \langle M, e_B\rangle$ and $0 < \mathrm{Tr}(p) < \infty$. We need only prove that $p \le e_N$ since if this is the case, then $x\xi = p(x\xi) = e_N(x\xi) \in L^2(N)$, implying that $x \in N$.

Suppose that $e_N p e_N = p$ is not true. Then $(1 - e_N)p \neq 0$. Replacing p by a nonzero spectral projection of $(1 - e_N)p(1 - e_N)$ corresponding to some interval $[c, 1]$ with $c > 0$, we may assume that p is a nonzero subprojection of $1 - e_N$.

Let $\varepsilon > 0$. By Lemma 3.3, there exists a finite set of elements $\{x_1, \ldots, x_n\} \subseteq M$ such that $\mathbb{E}_N(x_i) = 0$ and

$$\|p - \sum_{i=1}^{n} x_i e_B x_i^*\|_{2,\mathrm{Tr}} < \varepsilon/2. \tag{3.20}$$

Let $p_0 = \sum_{i=1}^{n} x_i e_B x_i^*$. Since $p \in B' \cap \langle M, e_B\rangle$, $upu^* = p$ for all unitary operators $u \in B$. Thus

$$\|up_0u^* - p_0\|_{2,\mathrm{Tr}} \le \|u(p_0 - p)u^*\|_{2,\mathrm{Tr}} + \|p_0 - p\|_{2,\mathrm{Tr}} < \varepsilon, \quad u \in \mathcal{U}(B). \tag{3.21}$$

Therefore,

$$\begin{aligned} 2\|p_0\|_{2,\mathrm{Tr}}^2 &= \|up_0u^* - p_0\|_{2,\mathrm{Tr}}^2 + 2\mathrm{Tr}(up_0u^*p_0) \\ &= \|up_0u^* - p_0\|_{2,\mathrm{Tr}}^2 + 2\sum_{1\le i,j\le n} \mathrm{Tr}(ux_ie_Bx_i^*u^*x_je_Bx_j^*) \end{aligned}$$

$$\leq \varepsilon^2 + 2 \sum_{1\leq i,j\leq n} \tau(\mathbb{E}_B(x_i^* u^* x_j) x_j^* u x_i)$$
$$\leq \varepsilon^2 + 2 \sum_{1\leq i,j\leq n} \|\mathbb{E}_B(x_j^* u x_i)\|_{2,\tau}^2 \tag{3.22}$$

for all unitary operators u in B. By the assumption of (i), there exists a sequence of unitary operators $\{u_k\}_{k=1}^\infty$ in B such that $\sum_{1\leq i,j\leq n} \|\mathbb{E}_B(x_j^* u_k x_i)\|_{2,\tau}^2 \to 0$ when $k \to \infty$. Hence, $\|p_0\|_{2,\mathrm{Tr}} < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows from (3.20) that $p = 0$, giving a contradiction and completing the proof. □

4 One sided quasi-normalizer algebras

Recall that an element $x \in M$ is said to be a *one sided quasi-normalizer* of B if there exists a finite set of elements $\{x_1, \ldots, x_n\} \subseteq M$ such that $Bx \subseteq \sum_{i=1}^n x_i B$. The set of one sided quasi-normalizers of B in M is denoted by $q\mathcal{N}_M^{(1)}(B)$ while the von Neumann algebra it generates is written $W^*(q\mathcal{N}_M^{(1)}(B))$ and called the one sided quasi-normalizer algebra of B. We now present some immediate consequences of Theorem 3.1.

Corollary 4.1. *The triple $B \subseteq W^*(q\mathcal{N}_M^{(1)}(B)) \subseteq M$ has the relative weak asymptotic homomorphism property.*

For the next corollary, we note that $B \subseteq M$ has the weak asymptotic homomorphism property precisely when the triple $B \subseteq B \subseteq M$ has the relative version.

Corollary 4.2. *A von Neumann subalgebra B of a finite von Neumann algebra M has the weak asymptotic homomorphism property if and only if $W^*(q\mathcal{N}_M^{(1)}(B)) = B$.*

Suppose that B is a subfactor of a factor M and $[M : N] < \infty$. Then $M = W^*(q\mathcal{N}_M^{(1)}(B))$ by [12, Proposition 1.3]. Thus we have the following corollary, which was first proved by Grossman and Wiggins [7].

Corollary 4.3. *If B is a finite index subfactor of a type II_1 factor M and $B \neq M$, then B does not have the weak asymptotic homomorphism property.*

In comparing $W^*(q\mathcal{N}_M^{(1)}(B))$ with the von Neumann algebra $q\mathcal{N}_M(B)''$ generated by the set of quasi-normalizers, it is clear that $W^*(q\mathcal{N}_M^{(1)}(B)) \supseteq q\mathcal{N}_M(B)''$. It is an interesting question to know under what conditions equality holds. In this direction, we have the following result.

Proposition 4.4. *If B is an atomic von Neumann subalgebra of M, then*

$$W^*(q\mathcal{N}_M^{(1)}(B)) = q\mathcal{N}_M(B)'' = M. \tag{4.1}$$

Proof. We need only show that $q\mathcal{N}_M(B)'' = M$. Since B is atomic, $B = \oplus_{n=1}^N B_n$, where each B_n is a full matrix algebra and $1 \le N \le \infty$. Let p_n be the central projections in B corresponding to B_n. In the following we will show that $p_n M p_m \subseteq q\mathcal{N}(B)$ for $n \ne m$, which implies that $q\mathcal{N}_M(B)'' = M$. Let $x \in p_n M p_m$. With respect to a choice of matrix units of $B_n = p_n B p_n \cong M_r(\mathbb{C})$ and $B_m = p_m B p_m \cong M_s(\mathbb{C})$, we can write $x = (x_{ij})_{1\le i\le r, 1\le j\le s}$. Let y_{ij} be the $r \times s$ matrix with the (i,j)-th entry x_{ij} and other entries 0 with respect to the same matrix units of B_n and B_m. Now

$$\begin{aligned} Bx = B_n x &= \{(\lambda_{ij} x_{ij})_{1\le i\le r, 1\le j\le s} : \lambda_{ij} \in \mathbb{C}\} \\ &= \sum_{1\le i\le r, 1\le j\le s} y_{ij} B_m = \sum_{1\le i\le r, 1\le j\le s} y_{ij} B. \end{aligned} \tag{4.2}$$

By symmetry, $xB \subseteq \sum_{i=1}^n Bx_i$ for a finite set of elements $\{x_1, \ldots, x_n\} \subseteq M$. Thus $x \in q\mathcal{N}_M(B)$, completing the proof. $\square$

Using Chifan's theorem in [2], we have the following corollary of Theorem 3.1. Note that the equality of the first and third algebras is already known by measure theoretic methods [17].

Corollary 4.5. *If B is a masa in a separable type* II_1 *factor M, then*

$$\mathcal{N}_M(B)'' = W^*(q\mathcal{N}_M^{(1)}(B)) = q\mathcal{N}_M(B)''. \tag{4.3}$$

In reference to Corollary 4.5, we do not know if the stronger equality $q\mathcal{N}_M^{(1)}(B) = q\mathcal{N}_M(B)$ holds for masas, even in the special cases considered in Section 5.

We end this section with the following observation.

Theorem 4.6. *Let $N = W^*(q\mathcal{N}_M^{(1)}(B))$. If $p \in B' \cap \langle M, e_B\rangle$ is a finite projection in $\langle M, e_B\rangle$, then $p \le e_N$. Furthermore, $W^*(q\mathcal{N}_M^{(1)}(B)) = q\mathcal{N}_M(B)''$ if and only if e_N is the supremum of all projections $p \in B' \cap \langle M, e_B\rangle$ such that p is finite in $\langle M, e_B\rangle$.*

Proof. The first statement is implied by the proof of Theorem 3.1. Suppose that e_N is the supremum of all projections $p \in B' \cap \langle M, e_B\rangle$ such that p is finite in $\langle M, e_B\rangle$. Then $e_N\,(B' \cap \langle M, e_B\rangle)\,e_N$ is a semi-finite von Neumann algebra. Let $Q = q\mathcal{N}_M(B)''$. Clearly, $e_Q \le e_N$, so suppose that $e_Q \ne e_N$. Then there is a nonzero finite projection $p \le e_N - e_Q$ such that $p \in B' \cap \langle M, e_B\rangle$ and p is finite in $\langle M, e_B\rangle$. By Lemma 1.4.2 of [15], any projection $p' \in B' \cap \langle M, e_B\rangle$ with $p' \le JpJ$ must be infinite. On the other hand, $JpJ \le Je_NJ = e_N$ and therefore $JpJ\,(B' \cap \langle M, e_N\rangle)\,JpJ$ is semifinite. This is a contradiction. If $W^*(q\mathcal{N}_M^{(1)}(B)) = q\mathcal{N}_M(B)''$, then e_N is the supremum of all projections $p \in B' \cap \langle M, e_B\rangle$ such that p is finite in $\langle M, e_B\rangle$ by [15, Lemma 1.4.2 (iii)]. $\square$

5 Group von Neumann algebras

In this section we will apply our previous results to the study of inclusions $L(H) \subseteq L(G)$ arising from inclusions $H \subseteq G$ of discrete groups. We will make the standard abuse of notation and write g for a unitary in $L(G)$ and for a vector in $\ell^2(G)$. Thus we denote the Fourier series of $x \in L(G)$ by $x = \sum_{g\in G} \alpha_g g$ where $\sum_{g\in G} |\alpha_g|^2 < \infty$. We do not assume that G is I.C.C., so that $L(G)$ may not be a factor. However, when using a trace, it will always be the standard one given by $\tau(e) = 1$ and $\tau(g) = 0$ for $g \in G \setminus \{e\}$.

The notion of one sided quasi-normalizers of von Neumann algebras has an obvious counterpart for group inclusions $H \subseteq G$. We say that $g \in G$ is a *one sided quasi-normalizer* of H if there exists a finite set $\{g_1, \dots, g_n\} \subseteq G$ such that

$$Hg \subseteq \cup_{i=1}^n g_i H. \tag{5.1}$$

It is immediate that these elements form a semigroup inside G, denoted $q\mathcal{N}_G^{(1)}(H)$. However, for ease of notation, we will also denote this by Γ throughout the section. There are two distinguished subgroups of G associated with Γ. We denote by H_1 the maximal subgroup $\Gamma \cap \Gamma^{-1}$ inside Γ (corresponding to the quasi-normalizers $q\mathcal{N}_G(H)$ defined by a two sided version of (5.1)). We let H_2 denote the subgroup of G generated by Γ, and we note that the containment $H_1 \subseteq H_2$ can be strict, as we show by a subsequent example. Many of the results in this section will depend on the following.

Theorem 5.1. *Let $H \subseteq G$ be an inclusion of discrete groups, let $x \in q\mathcal{N}_{L(G)}^{(1)}(L(H))$, and write $x = \sum_{g\in G} \alpha_g g$ for its Fourier series. If $g_0 \in G$ is such that $\alpha_{g_0} \neq 0$, then $g_0 \in \Gamma$.*

Proof. Let $M = L(G)$ and $B = L(H)$. We may assume that $\|x\| = 1$ and $Bx \subseteq \sum_{i=1}^r x_i B$ for a finite number of elements $x_1, \dots, x_r \in M$. Let $\mathcal{H}$ be the closure of $BxB\xi$ in $L^2(M)$ so that $\mathcal{H}$ is a B-bimodule. Since $\mathcal{H} \subseteq L^2(\sum_{i=1}^r x_i B)$, $\mathcal{H}$ is a finitely generated right B-module, so there exist vectors $\eta_1, \dots, \eta_k \in \mathcal{H} \subseteq L^2(M)$ such that

$$\eta = \sum_{i=1}^k \eta_i \mathbb{E}_B(\eta_i^* \eta), \quad \eta \in \mathcal{H}, \tag{5.2}$$

where η, η_i are viewed as unbounded operators affiliated with M. In particular, we have

$$bx = \sum_{i=1}^k \eta_i \mathbb{E}_B(\eta_i^* bx), \quad b \in B. \tag{5.3}$$

Set $C = \max\{\|\eta_i\|_2 : 1 \leq i \leq k\}$, and let $\eta_i = \sum_{g\in G} \alpha_g^i g$ be the Fourier series for η_i, $1 \leq i \leq k$. Since

$$\sum_{i=1}^k \sum_{g\in G} |\alpha_g^i|^2 = \sum_{i=1}^k \|\eta_i\|_2^2 < \infty, \tag{5.4}$$

there is a finite set $S = \{g_1, \ldots, g_n\} \subseteq G$ such that

$$k^2C^2 \left(\sum_{i=1}^{k} \sum_{g \in S^c} |\alpha_g^i|^2 \right) < |\alpha_{g_0}|^2. \tag{5.5}$$

For each $h \in H$, it follows from (5.3) that

$$L_h x = \sum_{i=1}^{k} \eta_i \mathbb{E}_B(\eta_i^* L_h x). \tag{5.6}$$

Since $\mathbb{E}_B(\eta_i^* L_h x) \in L^2(B)$ for $1 \leq i \leq k$, these elements have Fourier series which we write as $\mathbb{E}_B(\eta_i^* L_h x) = \sum_{h' \in H} \beta_{h'}^i h'$. Then

$$\sum_{g \in G} \alpha_g hg = \sum_{i=1}^{k} \left(\sum_{g' \in G} \alpha_{g'}^i g' \sum_{h' \in H} \beta_{h'}^i h' \right). \tag{5.7}$$

Comparing the coefficients of hg_0 on both sides of (5.7), we have

$$\alpha_{g_0} = \sum_{i=1}^{k} \left(\sum_{h' \in H} \alpha_{hg_0(h')^{-1}}^i \beta_{h'}^i \right). \tag{5.8}$$

Since $\|x\| = 1$,

$$\|\mathbb{E}_B(\eta_i^* L_h x)\|_2 \leq \|\eta_i^* L_h x\|_2 \leq \|\eta_i\|_2 \leq C, \tag{5.9}$$

and so $\sum_{h' \in H} |\beta_{h'}^i|^2 \leq C^2$ for $1 \leq i \leq k$. The Cauchy–Schwarz inequality gives

$$\begin{aligned} |\alpha_{g_0}|^2 &\leq k^2 \sum_{i=1}^{k} \left(\sum_{h' \in H} \alpha_{hg_0(h')^{-1}}^i \beta_{h'}^i \right)^2 \\ &\leq k^2 \sum_{i=1}^{k} \left(\sum_{h' \in H} |\alpha_{hg_0(h')^{-1}}^i|^2 \sum_{h' \in H} |\beta_{h'}^i|^2 \right) \\ &\leq k^2C^2 \sum_{i=1}^{k} \left(\sum_{h' \in H} |\alpha_{hg_0(h')^{-1}}^i|^2 \right). \end{aligned} \tag{5.10}$$

If $hg_0(h')^{-1} \in S^c$ for all $h' \in H$, then we have

$$|\alpha_{g_0}|^2 \leq k^2C^2 \left(\sum_{i=1}^{k} \sum_{g \in S^c} |\alpha_g^i|^2 \right), \tag{5.11}$$

and this contradicts (5.5). Thus there exists an $h' \in H$ such that $hg_0(h')^{-1} \in \{g_1, \ldots, g_n\}$, from which it follows that $hg_0 \in g_i H$ for some i, $1 \leq i \leq n$. Since $h \in H$ is arbitrary, we have shown that $Hg_0 \subseteq \cup_{i=1}^{n} g_i H$, and therefore $g_0 \in \Gamma$. $\square$

A consequence of Theorem 5.1 is that we can now describe both $q\mathcal{N}_{L(G)}(L(H))''$ and $W^*(q\mathcal{N}^{(1)}_{L(G)}(L(H)))$ in terms of groups.

Corollary 5.2. *Let $H \subseteq G$ be an inclusion of discrete groups. Then*

(i) $q\mathcal{N}_{L(G)}(L(H))'' = L(H_1)$;

(ii) $W^*(q\mathcal{N}^{(1)}_{L(G)}(L(H))) = L(H_2)$.

Proof. The inclusion "$\supseteq$" is obvious in both cases. If $x \in q\mathcal{N}^{(1)}_{L(G)}(L(H))$ with Fourier series $\sum_{g\in G} \alpha_g g$, then any $g \in G$ for which $\alpha_g \neq 0$ must lie in $\Gamma \subseteq H_2$ by Theorem 5.1. This establishes "$\subseteq$" in (ii).

Now assume that $x \in q\mathcal{N}_{L(G)}(L(H))$, which is equivalent to $x, x^* \in q\mathcal{N}^{(1)}_{L(G)}(L(H))$. If $x = \sum_{g\in G} \alpha_g g$ then $x^* = \sum_{g\in G} \overline{\alpha_g} g^{-1}$, so Theorem 5.1 gives $g, g^{-1} \in \Gamma$ whenever $\alpha_g \neq 0$. Then such elements g lie in $\Gamma \cap \Gamma^{-1} = H_1$ and this shows the containment "$\subseteq$" in (i). $\square$

Based on the above corollary, we can now present an example where the quasi-normalizers and the one sided quasi-normalizers are distinct.

Example 5.3. Consider the free group $\mathbb{F}_\infty$, where the generators are written $\{g_i : i \in \mathbb{Z}\}$, and for each $n \in \mathbb{Z}$, let K_n be the subgroup generated by $\{g_i : i \geq n\}$. The shift $i \to i+1$ on $\mathbb{Z}$ induces an automorphism ϕ of $\mathbb{F}_\infty$ defined on generators by $\phi(g_i) = g_{i+1}$, $i \in \mathbb{Z}$. Then $n \to \phi^n$ gives a homomorphism $\alpha : \mathbb{Z} \to \mathrm{Aut}(\mathbb{F}_\infty)$, and we let G be the semidirect product $\mathbb{F}_\infty \rtimes_\alpha \mathbb{Z}$. Let $H = K_0$. In the following we will show that $H_1 \neq H_2$. We denote by t the generator of $\mathbb{Z}$. Then every element of G can be written as wt^n, where $w \in \mathbb{F}_\infty$. Note that $tHt^{-1} = K_1 \subseteq H$. So $Ht^{-1} \subseteq t^{-1}H$ and $t^{-1} \in H_2$. Suppose that $t^{-1} \in H_1$. Then $t^{-1}H \subseteq \cup_{i=-N}^{N} Ha_i t^i$ for some large positive integer N and some $a_i \in \mathbb{F}_\infty$. Multiplying on the right by t gives $K_{-1} \subseteq \cup_{i=-N}^{N} K_0 a_i t^{i+1}$ and so $K_{-1} \subseteq K_0 a_{-1}$. If r is the total number of occurrences of g_{-1} in a_{-1}, then $g_{-1}^{r+1} \in K_{-1}$ but $g_{-1}^{r+1} \notin K_0 a_{-1}$ and we reach a contradiction. Thus $t^{-1} \notin H_1$ and so $H_1 \neq H_2$. $\square$

We now list some algebraic conditions on group inclusions $H \subseteq G$ that will be useful subsequently. The first two come from [4]. When H is abelian, (C1) below gives a necessary and sufficient condition for $L(H)$ to be a masa in $L(G)$, while (C1) and (C2) combined give a sufficient condition for $L(H)$ to be a singular masa [4]. Subsequently (C2) alone was shown to be a necessary and sufficient condition in [8] (see also the review of this paper, MR2465603 (2010b:46127), by Stuart White).

(C1) For each $g \in G \setminus H$, $\{hgh^{-1} : h \in H\}$ is infinite.

(C2) Given $g_1, \ldots, g_n \in G \setminus H$, there exists $h \in H$ such that

$$g_i h g_j \notin H, \quad 1 \leq i, j \leq n.$$

(C3) If $g \in G$ and there exists a finite set $\{g_1, \ldots, g_n\} \subseteq G$ such that

$$Hg \subseteq \cup_{i=1}^{n} g_i H,$$

then $g \in H$. ($\Gamma = q\mathcal{N}_G^{(1)}(H) = H$ in our notation).

We note that (C1) is a consequence of (C2) and also of (C3): if an element $g \in G \setminus H$ had only a finite number of H-conjugates $\{g_1, \ldots, g_n\}$, then (C2) would fail for the finite set $\{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\}$, while (C3) would fail since we would have $Hg \subseteq \cup_{i=1}^{n} g_i H$. No abelian hypothesis on H is required for this.

Combining Theorem 3.1 and Corollary 5.2, we obtain a purely algebraic characterization for the weak asymptotic homomorphism property. Note that we are not assuming H to be abelian.

Corollary 5.4. *Let $H \subseteq G$ be an inclusion of discrete groups. Then $L(H) \subseteq L(G)$ has the weak asymptotic homomorphism property if and only if condition* (C3) *is satisfied.*

As mentioned above, condition (C2) is necessary and sufficient to imply that $L(H)$ is a singular masa in $L(G)$ when H is abelian (see [18, 8]). The following gives a different necessary and sufficient condition for singularity of $L(H) \subseteq L(G)$ in terms of the group structure. After Corollary 5.5 has been proved, it will be apparent that conditions (C2) and (C3) are equivalent when H is abelian. The direction (C2) $\Rightarrow$ (C3) is routine, but we do not have a purely group theoretic argument for the reverse implication.

Corollary 5.5. *Let $H \subseteq G$ be an inclusion of discrete groups with H abelian. Then $L(H)$ is a singular masa in $L(G)$ if and only if condition* (C3) *is satisfied.*

Proof. Suppose that $L(H)$ is a singular masa in $L(G)$. From [21], the inclusion $L(H) \subseteq L(G)$ has the weak asymptotic homomorphism property, so it is immediate from the definition that the triple $L(H) \subseteq L(H) \subseteq L(G)$ has the relative form. Theorem 3.1 then gives $q\mathcal{N}_{L(G)}^{(1)}(L(H)) \subseteq L(H)$, and so condition (C3) holds.

Conversely, suppose that condition (C3) is valid. Then $q\mathcal{N}_{L(G)}^{(1)}(L(H)) \subseteq L(H)$ follows from Theorem 5.1, and the weak asymptotic homomorphism property holds for $L(H) \subseteq L(G)$ by Theorem 3.1. As noted before Corollary 5.4, condition (C1) is a consequence of condition (C3), so $L(H)$ is a masa in $L(G)$. Singularity now follows from [21]. $\square$

In the case that $L(H)$ is a masa in $L(G)$, we can now describe $\mathcal{N}_{L(G)}(L(H))''$ in terms of the normalizer $\mathcal{N}_G(H) := \{g \in G : gHg^{-1} = H\}$ at the group level. For this we need a preliminary group theoretic result.

Lemma 5.6. *Let $H \subseteq G$ be an inclusion of discrete groups with H abelian, and suppose that condition* (C1) *holds. Let $g \in G$ be such that there exists a finite set $\{g_1, \ldots, g_n\} \subseteq G$ satisfying*

$$Hg \subseteq \cup_{i=1}^{n} g_i H. \tag{5.12}$$

Then $g \in \mathcal{N}_G(H)$.

Proof. We may assume that the left cosets in (5.12) are a minimal set for which (5.12) holds. Thus they are pairwise distinct, and so disjoint, and minimality implies that

$$Hg \cap g_i H \neq \emptyset, \quad 1 \leq i \leq n. \tag{5.13}$$

Since there exist $h \in H$ and some integer i such that $g = g_i h$, we may replace g_i by $g_i h$ and renumber to further assume that $g = g_1$.

Now (5.12) implies that $HgH \subseteq \cup_{i=1}^{n} g_i H$ while the reverse containment follows from (5.13). Since HgH is invariant under left multiplication by elements $h \in H$, we obtain a representation π of H into the permutation group of $\{1, \ldots, n\}$ by defining $\pi_h(i)$ to be that (unique) integer j so that $hg_i \in g_j H$, $1 \leq i \leq n$. Let $K \subseteq H$ be the kernel of π, a finite index subgroup of H. Since $g = g_1$, we see that $kg \in gH$ for all $k \in K$. Let $\alpha \in \mathrm{Aut}\,(G)$ be defined by $\alpha(r) = g^{-1} r g$, for $r \in G$. Then, by definition of K, we have $\alpha(K) \subseteq H$. Thus

$$\alpha(K) \subseteq H \cap \alpha(H) \subseteq \alpha(H), \tag{5.14}$$

and so $K_1 := H \cap \alpha(H)$ has finite index in $\alpha(H)$ and satisfies

$$K \subseteq \alpha^{-1}(K_1) \subseteq H. \tag{5.15}$$

Then $\alpha^{-1}(K_1)$ has finite index in H, so we may list the cosets as $\alpha^{-1}(K_1)h_1, \ldots, \alpha^{-1}(K_1)h_m$ for some integer m and elements $h_1, \ldots, h_m \in H$. For any $h \in H$ and $k \in K_1$,

$$\alpha^{-1}(k) h_i \alpha^{-1}(h) h_i^{-1} \alpha^{-1}(k^{-1}) = h_i \alpha^{-1}(h) h_i^{-1}, \quad 1 \leq i \leq m, \tag{5.16}$$

since $\alpha^{-1}(k)$ commutes with both h_i and $\alpha^{-1}(h)$. Thus $\alpha^{-1}(h)$ has only a finite number of H-conjugates, showing that $\alpha^{-1}(h) \in H$ from the hypothesis that condition (C1) holds. Thus $gHg^{-1} \subseteq H$. But condition (C1) implies that H (and hence gHg^{-1}) is maximal abelian in G, showing that $gHg^{-1} = H$. It follows that $g \in \mathcal{N}_G(H)$. □

Corollary 5.7. *Let $H \subseteq G$ be an inclusion of discrete groups with H abelian, and satisfying condition* (C1)*, so that $L(H)$ is a masa in $L(G)$. Then*

$$\mathcal{N}_{L(G)}(L(H))'' = L(\mathcal{N}_G(H)). \tag{5.17}$$

In particular, $L(H)$ is a singular masa if and only if $\mathcal{N}_G(H) = H$, and is Cartan precisely when H is a normal subgroup of G.

Proof. By Theorem 5.1, any $u \in \mathcal{N}_{L(G)}(L(H))$ lies in $L(\Gamma)$. By Lemma 5.6, $\Gamma \subseteq \mathcal{N}_G(H)$, so $\mathcal{N}_{L(G)}(L(H))'' \subseteq L(\mathcal{N}_G(H))$. Since the reverse inclusion is true for any subgroup H, the result follows. □

In specific cases this corollary is easy to apply. The properties of the various types of masas presented in [4] or in [20, §2.2] can now be verified trivially by using the equality of (5.17). We also note that Corollary 5.7 solves a question posed in [22, Remark 5.5]. The conclusion of (5.17) was reached by Cameron in a different situation. As a special case of [1, Theorem 5.4], he showed that (5.17) holds when H and G are both I.C.C. satisfying $L(H)' \cap L(G) = \mathbb{C}1$.

Remark 5.8. The results of this section can be extended to the more general setting of inclusions $N \rtimes_\theta H \subseteq N \rtimes_\theta G$ where N is a finite von Neumann algebra with a faithful normal trace τ, $H \subseteq G$ are discrete groups, and θ is an action of G on N by trace preserving automorphisms. We make no assumptions that G acts either freely or ergodically. The analog of Corollary 5.2 is then the relations

$$q\mathcal{N}_{N\rtimes_\theta G}(N \rtimes_\theta H)'' = N \rtimes_\theta H_1, \quad W^*(q\mathcal{N}^{(1)}_{N\rtimes_\theta G}(N \rtimes_\theta H)) = N \rtimes_\theta H_2, \tag{5.18}$$

which are seen to be generalizations by taking $N = \mathbb{C}1$ and θ the trivial action. We omit the details since they are so similar to what has already been presented, and we mention only the one small change that is necessary. The Fourier series $\sum_{g\in G} \alpha_g g$ of Theorem 5.1 is replaced by $\sum_{g\in G} x_g g$ with $x_g \in N$ and $\sum_{g\in G} \|x_g\|_2^2 < \infty$, and $\|x_g\|_2$ is substituted in all calculations involving $|\alpha_g|$. □

6 Concluding remarks

Let B be a von Neumann subalgebra of M. Various notions of "normalizers" have been introduced:

(i) normalizers $\mathcal{N}_M(B)$ ([4]): a unitary operator $u \in M$ is a normalizer of B if $uBu^* = B$;

(ii) one sided normalizers $\mathcal{ON}_M(B)$ ([22]): a unitary operator $u \in M$ is a one sided normalizer of B if $uBu^* \subseteq B$;

(iii) groupoid normalizers $\mathcal{GN}_M(B)$ ([5]): a partial isometry $v \in M$ is a groupoid normalizer of B if $vBv^* \subseteq B$ and $v^*Bv \subseteq B$;

(iv) intertwiners $\mathcal{GN}^{(1)}_M(B)$ ([6]): a partial isometry $v \in M$ is an intertwiner of B if $v^*v \in B$ and $vBv^* \subseteq B$;

(v) quasi-normalizers $q\mathcal{N}_M(B)$ ([14]): an operator $x \in M$ is a quasi-normalizer of B if there exists a finite number of elements $x_1, \ldots, x_n \in M$ such that $Bx \subseteq \sum_{i=1}^n x_i B$ and $xB \subseteq \sum_{i=1}^n Bx_i$;

(vi) one sided quasi-normalizers $q\mathcal{N}_M^{(1)}(B)$: an operator $x \in M$ is a one sided quasi-normalizer of B if there exist a finite number of elements $x_1, \ldots, x_n \in M$ such that $Bx \subseteq \sum_{i=1}^n x_i B$.

The relations between von Neumann algebras generated by the above "normalizers" are the following:

$$\begin{array}{ccccc} \mathcal{N}_M(B)'' & \subseteq & \mathcal{GN}_M(B)'' & \subseteq & q\mathcal{N}_M(B)'' \\ \cap\!\vert & & \cap\!\vert & & \cap\!\vert \\ \mathcal{ON}_M(B)'' & \subseteq & W^*(\mathcal{GN}_M^{(1)}(B)) & \subseteq & W^*(q\mathcal{N}_M^{(1)}(B)) \end{array}. \tag{6.1}$$

By Corollary 4.5, if B is a masa in a type II_1 factor M, then $\mathcal{N}_M(B)'' = W^*(q\mathcal{N}_M^{(1)}(B))$ and therefore all of the above "normalizer algebras" are the same. On the other hand, for each "$X \subseteq Y$" in the above diagram, there are examples of inclusions of finite von Neumann algebras such that $X \neq Y$ (see [6, 22]). Among the above "normalizer algebras", $W^*(q\mathcal{N}_M^{(1)}(B))$ has the following two special properties. The first of these is the formula (6.4) for tensor products. This is an outgrowth of the analogous formulas

$$\mathcal{GN}_{M_1 \bar{\otimes} M_2}(B_1 \bar{\otimes} B_2)'' = \mathcal{GN}_{M_1}(B_1)'' \bar{\otimes} \mathcal{GN}_{M_2}(B_2)'' \tag{6.2}$$

and

$$W^*\left(\mathcal{GN}^{(1)}_{M_1 \bar{\otimes} M_2}(B_1 \bar{\otimes} B_2)\right) = W^*\left(\mathcal{GN}^{(1)}_{M_1}(B_1)\right) \bar{\otimes} W^*\left(\mathcal{GN}^{(1)}_{M_2}(B_2)\right) \tag{6.3}$$

established in [6] under the hypothesis that $B_i' \cap M_i \subseteq B_i$, and which can fail without some such assumption. In contrast, the next proposition requires no restrictions.

Proposition 6.1. *Let $B_i \subseteq M_i$ be inclusions of finite von Neumann algebras, $i = 1, 2$. Then*

$$W^*\left(q\mathcal{N}^{(1)}_{M_1 \bar{\otimes} M_2}(B_1 \bar{\otimes} B_2)\right) = W^*(q\mathcal{N}^{(1)}_{M_1}(B_1)) \bar{\otimes} W^*(q\mathcal{N}^{(1)}_{M_2}(B_2)). \tag{6.4}$$

Proof. Suppose that $x_1 \in M_1$ satisfies $B_1 x_1 \subseteq \sum_{i=1}^{n_1} y_i B_1$ for a finite number of elements $y_1, \ldots, y_{n_1}$ in M_1, and $x_2 \in M_2$ satisfies $B_2 x_2 \subseteq \sum_{i=1}^{n_2} z_i B_2$ for a finite number of elements $z_1, \ldots, z_{n_2}$ in M_2. Then $(B_1 \bar{\otimes} B_2)(x_1 \otimes x_2) \subseteq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (y_i \otimes z_j)(B_1 \bar{\otimes} B_2)$. This proves that

$$W^*\left(q\mathcal{N}^{(1)}_{M_1 \bar{\otimes} M_2}(B_1 \bar{\otimes} B_2)\right) \supseteq W^*(q\mathcal{N}^{(1)}_{M_1}(B_1)) \bar{\otimes} W^*(q\mathcal{N}^{(1)}_{M_2}(B_2)). \tag{6.5}$$

On the other hand, the triple $B_i \subseteq W^*(q\mathcal{N}^{(1)}_{M_i}(B_i)) \subseteq M_i$ has the relative weak asymptotic homomorphism property by Corollary 4.1, $i = 1, 2$, and so

$$B_1 \bar{\otimes} B_2 \subseteq W^*(q\mathcal{N}^{(1)}_{M_1}(B_1)) \bar{\otimes} W^*(q\mathcal{N}^{(1)}_{M_2}(B_2)) \subseteq M_1 \bar{\otimes} M_2 \tag{6.6}$$

also has the relative weak asymptotic homomorphism property. By Theorem 3.1, we have the reverse containment

$$W^*\left(q\mathcal{N}^{(1)}_{M_1\,\bar{\otimes}\,M_2}(B_1\,\bar{\otimes}\,B_2)\right) \subseteq W^*(q\mathcal{N}^{(1)}_{M_1}(B_1))\,\bar{\otimes}\,W^*(q\mathcal{N}^{(1)}_{M_2}(B_2)), \tag{6.7}$$

completing the proof. □

We end with two results that discuss the situation of a cut down of $B \subseteq M$ to an inclusion $eBe \subseteq eMe$ for a projection $e \in B$.

Proposition 6.2. *Let $e \in B$ be a projection. Then $W^*(q\mathcal{N}^{(1)}_{eMe}(eBe)) = eW^*(q\mathcal{N}^{(1)}_M(B))e$.*

Proof. We only prove that $e\left(q\mathcal{N}^{(1)}_M(B)\right)e \subseteq W^*(q\mathcal{N}^{(1)}_{eMe}(eBe))$. The proof of $q\mathcal{N}^{(1)}_{eMe}(eBe) \subseteq eW^*(q\mathcal{N}^{(1)}_M(B))e$ is similar. Suppose that z is a central projection in B such that $z = \sum_{j=1}^n v_jv_j^*$ with the v_j's partial isometries in B and $v_j^*v_j \leq e$. Write $e_0 = ez$. If $x \in M$ satisfies $Bx \subseteq \sum_{i=1}^r x_iB$, then

$$\begin{aligned} eBee_0xe_0 \subseteq eBzxe_0 = ezBxe_0 \subseteq e_0\sum_{i=1}^r x_iBe_0 = e_0\sum_{i=1}^r x_izBe_0 \\ = \sum_{i=1}^r\sum_{j=1}^n (e_0x_iv_j)(v_j^*Be_0) \subseteq \sum_{i=1}^r\sum_{j=1}^n (e_0x_iv_j)(eBe). \end{aligned} \tag{6.8}$$

Therefore, $e_0xe_0 \in q\mathcal{N}^{(1)}_{eMe}(eBe)$. Since the central support of e in B can be approximated arbitrarily well by such special central projections z, e_0 approximates e arbitrarily well, and $exe \in W^*(q\mathcal{N}^{(1)}_{eMe}(eBe))$. □

Combining Proposition 6.2 and Corollary 4.2, we obtain the following consequence.

Corollary 6.3. *Suppose that B has the weak asymptotic homomorphism property in M and $e \in B$ is a projection. Then eBe has the weak asymptotic homomorphism property in eMe.*

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