The Reliability of GPS Ambiguity Resolution

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ABSTRACT
GPS ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data as integers. It is the key to fast and high-precision relative GPS positioning. Critical in the application of ambiguity resolution is its reliability. Unsuccessful ambiguity resolution, when passed unnoticed, will too often lead to unacceptable errors in the positioning results. High success rates are required, for ambiguity resolution to be reliable. In this contribution we will introduce and evaluate such diagnostic measures. They complement existing methods of ambiguity resolution and allow the user and/or analyst to infer their reliability.

1. INTRODUCTION

Integer carrier phase ambiguity resolution is often a prerequisite for high precision GPS parameter estimation. It applies to a great variety of GPS models currently in use. Ambiguity resolution consists of two distinct parts: the ambiguity estimation problem and the ambiguity validation problem. The estimation part addresses the problem of finding optimal estimates for the unknown integer ambiguities. In this contribution we will use the least-squares principle and assume the data to be normally distributed. Validation is of importance in its own right and quite distinct from the estimation problem. One will namely always be able to compute an integer ambiguity solution, whether it is of good quality or not. The question addressed by validation is therefore whether the quality of the computed solution is such that one is also willing to accept this solution.
In this contribution we will consider the expected performance of validation. The chance of successful ambiguity resolution can be inferred once the probability mass function of the integer ambiguities is known. Of this distribution, the probability of correct integer ambiguity estimation is of particular interest. It describes the reliability of ambiguity resolution in terms of its expected success rate.
The variance matrix of the (real-valued) least-squares ambiguities contains all the information necessary to infer *a priori* whether or not the estimated integer ambiguities have enough chance to coincide with the true, but unknown integer ambiguities. It is shown how this matrix can be used to evaluate the probabilities of correct integer estimation. These success rates are given for the ambiguity estimator that follows from integer bootstrapping.
Although less optimal than integer least-squares, integer bootstrapping provides useful and easy-to-compute approximations to the integer least-squares solution. In a similar manner, the bootstrapped success rates provide bounds for the probability of correct integer least-squares estimation. In fact, when the bootstrapped success rates are close enough to one, the simple bootstrapping ambiguity estimator may be considered a useful alternative to the integer least-squares estimator.
The success rates are evaluated for two types of GPS models, the geometry-free model and the geometry-based model. In both cases we neglect the atmospheric delays and thus assume that the baselines are sufficiently short. The success rates are given for different measurement scenarios.

2. INTEGER AMBIGUITY ESTIMATION

Ambiguity resolution applies to a great variety of GPS models currently in use. They range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. GPS models may have the relative receiver-satellite geometry included (geometry-based) or excluded (geometry-free). The geometry is included through the unit direction vectors in the design matrix. When geometry is excluded, the baseline components are not involved as unknowns in the model, but instead, the receiver-satellite ranges themselves. GPS models may also be discriminated as to whether the slave receiver(s) are in motion (non-stationary) or not (stationary). When in motion, one solves for one or more trajectories, since with the receiver-satellite geometry included, one will have new coordinate unknowns for each new epoch. One may also discriminate as to whether the differential atmospheric delays (ionosphere and/or troposphere) are included as unknowns or not. In case of sufficiently short baselines these delays are often neglected.

An overview of these and other GPS models, together with their applications in surveying, navigation and geodesy, can be found in textbooks such as *Hofmann-Wellenhof et al. (1997), Kleusberg and Teunissen (1996), Leick (1995), Parkinson and Spilker (1996)* and *Strang and Borre (1997)*. Despite the differences in application of the various GPS models, it is important to understand that their ambiguity resolution problems are intrinsically the same. That is, the GPS models on which ambiguity resolution is based, can all be cast in the following conceptual frame of linear(ized) observation equations

$$y = Aa + Bb + e \tag{2.1}$$

where y is the given GPS data vector, a and b are the unknown parameter vectors, and where e is the noise vector. The matrices A and B are the corresponding design matrices. The data vector y will usually consist of the 'observed minus computed' single- or dual-frequency DD phase and/or pseudorange (code) observations, accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*. The entries of vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

2.1 The solution in three steps

Since any GPS model can be cast in the above frame of observation equations, any method of ambiguity resolution that solves Eq.(2.1) is automatically applicable to each of the GPS models currently in use. For the estimation part of ambiguity resolution, solving the above model implies computing the 'best' estimates of the integer vector a and the real vector b. When using the least-squares principle, these estimates can be obtained in three steps. In the *first* step one simply disregards the integer constraints on the ambiguities and performs a standard adjustment. As a result one obtains the (real-valued) least-squares estimates of a and b, together with their variance-covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}; \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2.2}$$

This solution is often referred to as the 'float' solution. In the *second* step the integer constraints on the ambiguities are applied. That is, the 'float' ambiguity estimate $\hat{a}$ is now used to compute the corresponding integer ambiguity estimate $\breve{a}$. This can be written symbolically as

$$\breve{a} = F(\hat{a}) \tag{2.3}$$

where *F(.)* denotes the map from the real-valued ambiguity estimates to the integer estimates. This second step is the most demanding. The first difficulty lies in the fact that the map *F(.)* can often not be given explicitly. It has to be mechanised by means of an integer search. The second difficulty, typical for GPS when short observation time spans are used, has to do with the numerical efficiency with which this search process can be executed. In order to have an efficient search, the ambiguities need to be decorrelated first.

Once the integer ambiguities are computed, they are used in the *third* step to finally correct the 'float' estimate of b. As a result one obtains the 'fixed' solution

$$\breve{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \breve{a}) \tag{2.4}$$

In this final step the difference of the real-valued and integer-valued ambiguity estimates is used to adjust the 'float' solution. The complete solution of the model consists now of $\breve{a}$ and $\breve{b}$.

For more details on these three steps, as well as on the numerical implementation of the least-squares ambiguity decorrelation adjustment (LAMBDA), we refer to e.g. *Teunissen (1993)* or *Jonge de and Tiberius (1996)*. Fortran77 code implementing the LAMBDA method can be obtained upon request. For more information, see the appropriate pages on WWW site: http://www.geo.tudelft.nl/mgp/.

3. INTEGER AMBIGUITY VALIDATION

It is of course not enough to compute the solution of Eq.(2.1) and be done with it. One can always compute an integer ambiguity solution, whether it is of good quality or not. One therefore still needs to address the question whether one is willing to accept the solution. This concerns the second part of ambiguity resolution, the validation.

In the actual practice of GPS, there are various schemes in place for checking how well the ambiguity solution fits the model. Some of them are ad hoc and primarily based on experience, while others make use of concepts from hypothesis testing. Although most of these approaches work quite well in practice, none of them provide the user with a rigorous *reliability* description. The user has therefore no way of knowing how often he can expect the computed ambiguity solution to coincide with the correct, but unknown solution. Is this nine out of ten times, ninety-nine out of a hundred, or a higher percentage? It will surely never equal one hundred percent. After all, the integer ambiguities are computed from the data. They are therefore subject to uncertainty, just like the data are.

In order to obtain a proper reliability description, one needs the probability distribution of the integer ambiguities *(Teunissen, 1997)*. This distribution will be a probability *mass* function, due to the integer nature of the ambiguities. Of this probability mass function, the probability of correct integer estimation is particularly of interest. This probability will be denoted as $P(\check{a}=a)$. It describes the frequency with which one can expect to have a successful ambiguity resolution. It equals the expected success rate.

3.1 The integer ambiguity success rate

In case of GPS, one usually requires a high success rate. Thus

$$P(\check{a}=a)=1-\varepsilon \quad \text{with } \varepsilon \text{ small} \tag{3.1}$$

This probability depends on three contributing factors: the observation equations (the functional model), the precision of the observables (the stochastic model) and the chosen method of integer ambiguity estimation. Changes in any one of these will affect the success rate.

As to the method of integer estimation, one has a variety of options available. For instance, one can choose members from the class of unbiased integer ambiguity estimators *(Teunissen, 1998)*. Members from this class are the ambiguity estimators that follow from ‘integer rounding’, ‘integer bootstrapping’ or ‘integer least-squares’. In this contribution we will restrict our attention to ‘integer bootstrapping’.

The integer bootstrapped ambiguity vector follows from applying a *sequential* rounding scheme to the entries of $\hat{a}$. It goes as follows. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$ and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are taken care of. In essence this ‘bootstrapping’ technique boils down to the use of a sequential conditional least-squares adjustment, with a conditioning on the integer ambiguity values obtained in the previous steps. The integer bootstrapped solution reads therefore

$$\check{a}=([\hat{a}_1],\cdots,[\hat{a}_{n|N}])^T \tag{3.2}$$

where ‘[.]’ denotes rounding to the nearest integer and where the shorthand notation $\hat{a}_{i|I}$ stands for the ith least-squares ambiguity obtained through a conditioning on the previous $I=\{1,...,(i-1)\}$ sequentially rounded ambiguities.

The bootstrapped probability of correct integer ambiguity estimation reads *(Teunissen, 1997)*

$$P(\check{a}=a)=\prod_{i=1}^{n}(2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|I}}})-1) \tag{3.3}$$

where $\Phi(x)$ denotes the integral of the standardized normal distribution from minus infinity to x. As Eq.(3.3) shows, for the computation of the probability one only needs the conditional standard deviations of the ambiguities, $\sigma_{\hat{a}_{i|I}}$. Note however that these standard deviations, and therefore the bootstrapped probability as well, depend on the chosen ambiguity parametrization. These standard deviations will already change in value, when one changes the choice of reference satellite in the definition of the DD ambiguities. Since the bootstrapped probability gets larger for smaller standard deviations, one should use an ambiguity parametrization that provides ambiguities with small standard deviations. The DD ambiguities are therefore out of the question. Their precision is usually very poor, in particular in case of short observation time spans. Instead of the DD ambiguities, the ambiguities as provided by the LAMBDA method should be used. The decorrelating ambiguity transformation of this method returns ambiguities which are usually far more precise than the original DD ambiguities. Thus before commencing with the bootstrapping and the subsequent evaluation of the probability, one should first transform the DD ambiguities by means of the LAMBDA method.

4. SUCCESS RATES FOR SOME GPS MODELS

In this section we will apply the bootstrapped success rate Eq.(3.3) to both the geometry-free and geometry-based GPS model. The geometry-free model is the simplest one can think of. It allows one to use the pseudorange (code) data almost directly in combination with the phase data to determine the integer ambiguities, see e.g. *Hatch (1982)*, *Euler and Goad (1991)*, *Dedes and Goad (1994)*, *Euler and Hatch (1994)*, *Teunissen (1996)*, *Jonkman (1998)*. The DD phase and code observation equations of the geometry-free model are given for a single epoch i as

$$\begin{aligned} \phi_1(i) &= \rho(i)-\mu_1 I(i)+\lambda_1 a_1 \\ \phi_2(i) &= \rho(i)-\mu_2 I(i)+\lambda_2 a_2 \\ p_1(i) &= \rho(i)+\mu_1 I(i) \\ p_2(i) &= \rho(i)+\mu_2 I(i) \end{aligned} \tag{4.1}$$

where $\phi_1(i)$ and $\phi_2(i)$ are the DD phase observables on L1 and L2; $p_1(i)$ and $p_2(i)$ are the DD code observables on L1 and L2; $\rho(i)$ is the DD form of the unknown receiver-satellite range, $I(i)$ is the DD form of the unknown ionospheric delay and a_1 and a_2 are the unknown but time-invariant integer DD ambiguities. The known wavelengths are denoted as λ_1 and λ_2. Since the ionospheric delay is to a first order inversely proportional to the square of the frequency, we have to the same degree of approximation $\mu_1=\lambda_1/\lambda_2$ and $\mu_2=\lambda_2/\lambda_1$.

Note, due to the parametrization in terms of the DD ranges, that no linearization is required for the above observation equations. The absence of the receiver-satellite geometry also implies that the model permits both receivers to be either stationary or moving. Furthermore, the parametrization in terms of the DD ranges implies that the tropospheric delays need not be modelled explicitly. When present, these delays will get lumped with the DD ranges. Hence the estimated ambiguities will always be free from tropospheric biases.

The geometry-based model is obtained when the DD ranges in Eq.(4.1) are further parametrized in terms of the baseline components of the two receivers. In this case a linearization is required due to the nonlinear relation between the ranges and the baseline. The relative receiver-satellite geometry enters in the model because of the coupling of the ranges with this baseline. In the linearized version of the model, this geometry manifests itself through the receiver-satellite unit direction vectors.

In the following it will be assumed that the models are solved in a least-squares sense using k number of epochs. For the geometry-free model two satellites are taken, while for the geometry-based model four satellites or more. The ambiguities are considered to be time-invariant for the duration of the observation period. We also assume that time correlation and cross correlation are absent. In all cases we neglect the presence of the atmospheric delays. The results apply therefore only to sufficiently short baselines.

4.1 The geometry-free model

We consider both the single-frequency and dual-frequency case. The results shown are based on an undifferenced phase variance of $\sigma_\phi^2=(3mm)^2$ and a varying undifferenced pseudorange (code) variance of $\sigma_p^2=(10cm)^2$, $(15cm)^2$ and $(30cm)^2$ respectively.

The single-frequency case

In the single-frequency case, only a single ambiguity is present in the model. In this scalar case, integer bootstrapping, integer rounding and integer least-squares become identical. With $n=1$, Eq.(3.3) reduces to

$$P(\breve{a}=a)=2\Phi(\frac{1}{2\sigma_{\hat{a}_1}})-1 \qquad (4.2)$$

with the L1 variance of the least-squares ambiguity given as

$$\sigma_{\hat{a}_1}^2=\frac{4}{\lambda_1^2 k}(\sigma_\phi^2+\sigma_p^2) \qquad (4.3)$$

Figure 4-1 shows the probability of correct integer estimation (the success rate) as function of k (the number of epochs) for the three different values of the code variance. The overall conclusion that can be drawn from this figure is that successful ambiguity resolution is impossible, unless quite a number of epochs are taken into account. For a code standard deviation of $10cm$ more than 10 epochs are needed to get at the 90% level and about 50 epochs to get at the 99.9% level. For a code standard deviation of $15cm$ even more than 100 epochs are needed to reach this level.

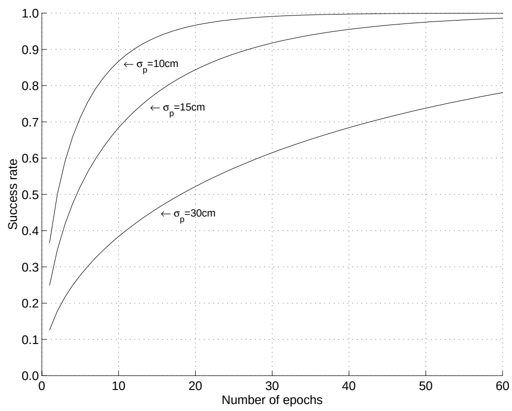


Figure 4-1: The geometry-free, single-frequency success rate as function of the number of epochs used.

The dual-frequency case

In the dual-frequency case, two ambiguities are present in the model. Thus $n=2$ and Eq.(3.3) becomes

$$P(\breve{a}=a)=\left(2\Phi(\frac{1}{2\sigma_{\hat{a}_{2|1}}})-1\right)\left(2\Phi(\frac{1}{2\sigma_{\hat{a}_1}})-1\right) \qquad (4.4)$$

We already observed that this probability depends on the chosen ambiguity parametrization. That is, the bootstrapped probability based on the use of DD ambiguities will differ from the bootstrapped probability based on the use of another set of admissible ambiguities. Here and in what follows the ambiguities obtained through the decorrelation process of the LAMBDA method are used. Figure 4-2 shows the corresponding success rates. Note the different vertical scale used. It now ranges from 0.99 to 1.00. These results show a dramatic improvement when compared with the single-frequency case. The figure shows that instantaneous ambiguity resolution is possible at the 99.5% level when $\sigma_p=10cm$. The 99.9% level is reached for $k=2$ when $\sigma_p=15cm$ and for $k=4$ when $\sigma_p=30cm$.

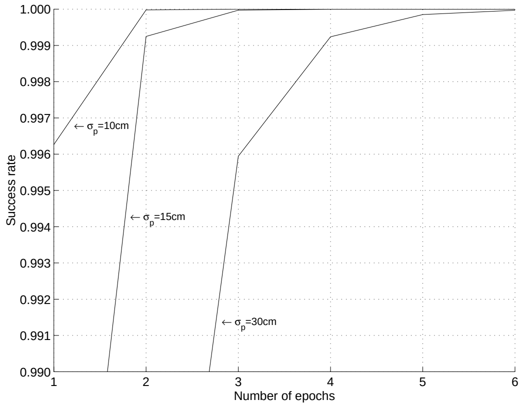


Figure 4-2: The geometry-free, dual-frequency success rates as function of the number of epochs used.

4.2 The geometry-based model

A better performance of ambiguity resolution can be expected when using the geometry-based model instead of the geometry-free model. Additional redundancy enters due to the fact that all ranges are now linked to same baseline. Also the information content of the relative receiver-satellite geometry and its change over time, can now be taken into account.

For the geometry-based model a minimum of four satellites is needed. In the following we will vary the number of satellites, as well as the observation time span. Again both the single-frequency and dual-frequency cases are considered. For the undifferenced pseudorange (code) variance the conservative value of $(30cm)^2$ is used and for the undifferenced phase variance again the value of $(3mm)^2$.

Although a representative satellite configuration was chosen for the examples following, one should bear in mind that changes in relative receiver-satellite geometry do have their effect on the probability of correct integer estimation.

The single-frequency case

Figure 4-3 shows three graphs. Each graph shows the success rate as function of the number of satellites tracked. The three graphs differ in the observation time spans used. For the first (top) graph only a single epoch of data was used. These results show that even with eight satellites only nine out of ten ambiguity resolutions can expected to be successful. For the second (middle) graph two epochs of data were used. The two epochs are separated by 30 seconds. When compared to the first graph, the success rates have improved of course. In case of eight satellites the success rate has now reached the 99.5% level. But this would still not be good enough when aiming at the 99.9% level or better.

For the third (bottom) graph again two epochs of data were used, but now separated by 10 minutes. This graph shows, when compared to the second graph, the impact of the change over time of the relative receiver-satellite geometry. In this case the 99.9% level is already reached when tracking six satellites.

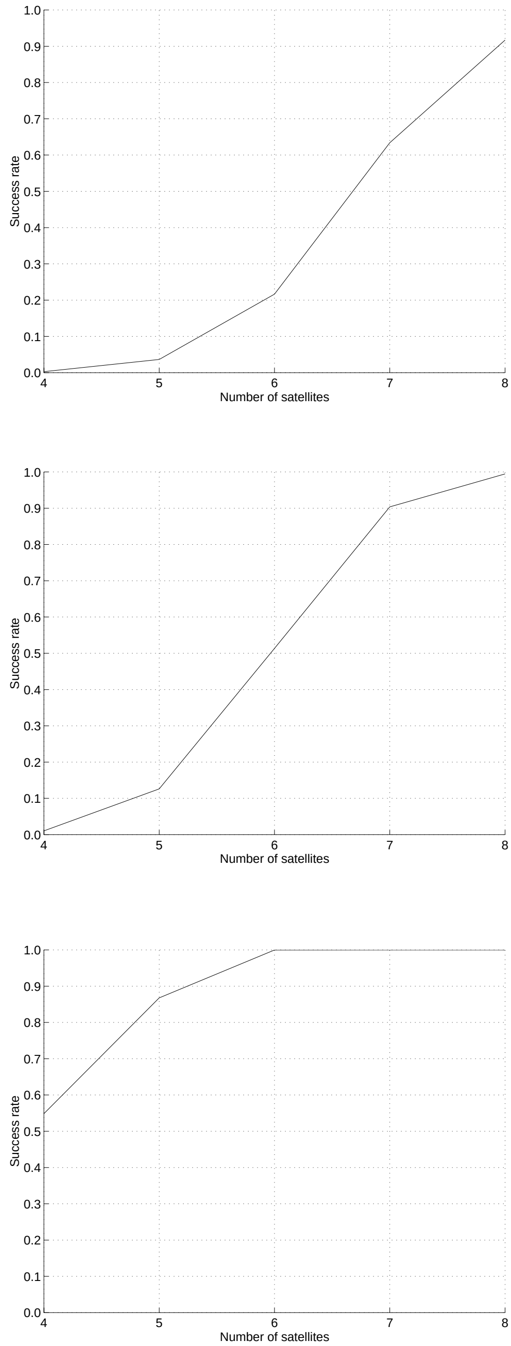
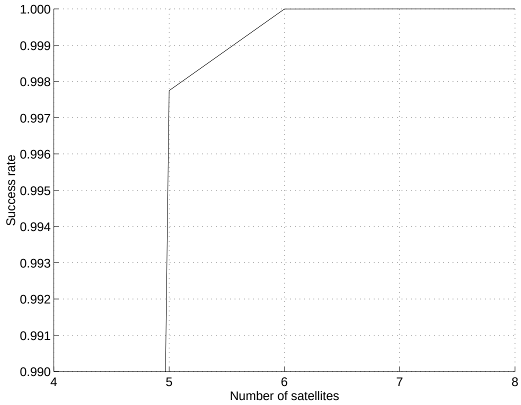


Figure 4-3: The geometry-based, single frequency success rates as function of the number of satellites tracked: (top) one epoch data; (middle) two epochs of data, separated by 30 seconds; (bottom) two epochs of data, separated by 10 minutes.

The dual-frequency case

The previous results showed that instantaneous single frequency ambiguity resolution is not possible at the

99.9% level when σ_ϕ =3*mm* and σ_p=30*cm*. As the results of figure 4-4 show, this becomes possible though in case of dual-frequency data. An instantaneous success rate larger than 99.99% is obtained when using six satellites. The same level is reached for five satellites in case two epochs are used.



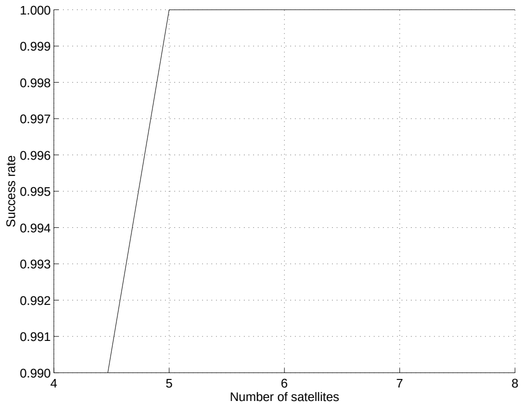


Figure 4-4: The geometry-based, dual-frequency success rates as function of the number of satellites tracked: (top) one epoch of data; (bottom) two epochs of data, separated by 30 seconds.

5. SUMMARY

In this contribution we considered the reliability of ambiguity resolution. We emphasized that the problem of ambiguity resolution is intrinsically the same for all the different GPS models that one may think of. Any rigorous method of ambiguity resolution should therefore be applicable to each of these models and should be able to efficiently provide the integer ambiguity estimates together with a proper description of the quality of the solution so obtained.

The expected performance of ambiguity resolution is measured by its success rate. Without it the user and/or analyst has no way of knowing how often he can expect the computed ambiguity solution to coincide with the true, but unknown solution. For many applications this is not acceptable. We therefore introduced the success rate as reliability measure of ambiguity resolution. For integer bootstrapping this success rate is given as

$$P(\breve{a} = a) = \prod_{i=1}^{n} (2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|I}}}) - 1)$$

It requires the conditional standard deviations of the ambiguities as input. Since this probability depends on the chosen ambiguity parametrization, the decorrelation process should be applied first, before commencing with the integer bootstrapping.

The above success rate was evaluated for two type of GPS models: the geometry-free model and the geometry-based model. It was shown that it is virtually impossible to have a fast and successful ambiguity resolution when using the single-frequency geometry-free model. It becomes possible however when the second frequency is included. Only a few epochs are then needed to reach the 99.9% level. Provided enough satellites are tracked, the same level can be reached within minutes when using the single-frequency geometry-based model. And in the dual-frequency case this level can even be reached instantaneously with six satellites or more.

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