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THE REPRESENTATION OF LATTICES BY MODULES

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1. **A quasivariety characterization of lattices representable by Λ-modules.** If Λ is a nontrivial ring with 1, a lattice L is "representable by Λ-modules" if it can be embedded in the lattice of submodules of some unitary left Λ-module M. This lattice of submodules is denoted $\Gamma(M;\Lambda)$.

A (lattice) "Horn formula" is an open formula:

$$(e_1 = e_2 \,\&\, e_3 = e_4 \,\&\ldots\&\, e_{n-3} = e_{n-2}) \Rightarrow e_{n-1} = e_n,$$

where $e_1, e_2, \ldots, e_n$ are lattice polynomials.

MAIN THEOREM. *For every commutative ring Λ, there exists a set $J(\Lambda)$ of Horn formulas such that a lattice L is representable by Λ-modules if and only if every formula of $J(\Lambda)$ is satisfied in L. Each member of $J(\Lambda)$ is constructible by a finite sequence of four basic operations.*

That is, the class $\mathscr{L}(\Lambda)$ of lattices representable by Λ-modules is the "quasivariety" of lattices satisfying $J(\Lambda)$, for commutative Λ.

OUTLINE OF PROOF. For Λ commutative, let $\iota: L \to \Gamma(M;\Lambda)$ be an embedding for some M. Without loss of generality, assume that L has a smallest element ω, and $\iota(\omega) = 0$. Motivated by the "abelian" lattice $\Gamma_f(G^N)$ of [**2**, 4.2] with $G = M$, we consider "constraint systems" in variables a_k (corresponding to coordinate positions in $M^{\mathbf{N}}$) and "auxiliary" variables b_k (with existential quantifiers understood) for k in $N = \{1, 2, 3, \ldots\}$. Consider $\boldsymbol{r} = (d_1, d_2, d_3, d_4)$ below.

(d_1) $\quad a_1 \in x_1, \quad a_2 \in x_2, \quad a_k \in \omega \quad \text{for} \quad k \geqq 3\ (x_1, x_2 \in L).$

(d_2) $\quad b_1 \in x_3, \quad b_2 \in x_1, \quad b_k \in \omega \quad \text{for} \quad k \geqq 3\ (x_3 \in L).$

(d_3) $\quad a_1 - a_2 - b_1 = 0.$

(d_4) $\quad a_1 - \lambda_0 b_2 = 0 \qquad (\lambda_0 \in \Lambda).$

A "solution" $f: N \to M$ of $\boldsymbol{r}$ satisfies

(e_1) $\quad f(1) \in \iota(x_1), \quad f(2) \in \iota(x_2), \quad f(k) \in \iota(\omega) = 0 \quad \text{for } k \geqq 3\ (d_1).$

(e_2) $\quad f(1) - f(2) \in \iota(x_3) \qquad (d_3, b_1 \in x_3).$

(e_3) $\quad$ There exists $v \in \iota(x_1)$ such that $\lambda_0 v = f(1)\ (d_4, b_2 \in x_1).$

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Formally, let $N_1 = \{a_k : k \in N\}$, let $N_2 = N_1 \cup \{b_k : k \in N\}$, and let M_1^∞ and M_2^∞ be the Λ-modules of all functions $N_1 \to M$ and $N_2 \to M$, respectively. Let a "Λ-equation" be a function $g : N_2 \to \Lambda$ such that $g(a_k) = g(b_k) = 0$ except for finitely many k in N; g determines the "linear solution set" g^* in $\Gamma(M_2^\infty; \Lambda)$:

$$g^* = \left\{ m \in M_2^\infty : \sum_{k=1}^{\infty} (g(a_k)m(a_k) + g(b_k)m(b_k)) = 0 \right\}.$$

A "constraint function" is a function $\alpha : N_2 \to L$ such that $\alpha(a_k) = \alpha(b_k) = \omega$ except for finitely many k; it determines a "box" $\iota_*(\alpha)$ in $\Gamma(M_2^\infty; \Lambda)$:

$$\iota_*(\alpha) = \{ m \in M_2^\infty : m(c_k) \in \iota\alpha(c_k) \quad \text{for} \quad c_k \in N_2 \}.$$

If α is a constraint function and $G = \{g_1, g_2, \ldots, g_n\}$ is a finite (possibly empty) set of Λ-equations, the pair (G, α) is a "constraint system". An "extended solution" $m : N_2 \to M$ of (G, α) is a member of

$$\mu_0(G, \alpha) = \iota_*(\alpha) \cap g_1^* \cap g_2^* \cap \cdots \cap g_n^* \quad \text{in } \Gamma(M_2^\infty; \Lambda).$$

A "solution" $m' : N_1 \to M$ of (G, α) is a restriction $m' = m|N_1$ of an extended solution m. Let $D(L; \Lambda)$ denote the set of all constraint systems. Given M and ι, define $\mu : D(L; \Lambda) \to \Gamma(M_1^\infty; \Lambda)$ by the "solution set" $\mu(G, \alpha) = \{ m|N_1 : m \in \mu_0(G, \alpha) \}$. Since $\iota(\omega) = 0$, $\mu(G, \alpha)$ has "finite support" as in [**2**, p. 181].

Now, $D(L; \Lambda)$ can be defined for any lattice L, not just those in $\mathscr{L}(\Lambda)$. Meet and join operations, corresponding to solution set intersection and sum, can be defined abstractly in $D(L; \Lambda)$. We can also define "equivalence" of constraint systems, obtaining a congruence $E(L; \Lambda)$ on $D(L; \Lambda)$. If an embedding $\iota : L \to \Gamma(M; \Lambda)$ with $\iota(\omega) = 0$ exists, the corresponding $\mu : D(L; \Lambda) \to \Gamma(M_1^\infty; \Lambda)$ preserves meet and join and takes equivalent constraint systems modulo $E(L; \Lambda)$ into the same solution set. Seven "rules of equivalence" generate $E(L; \Lambda)$; we reconsider $\boldsymbol{r} = (d_1, d_2, d_3, d_4)$ to suggest them:

Constraint decrease: The lattice constraint of a_1 can be changed to $x_1 \wedge (x_2 \vee x_3)$, since d_3 can be solved for $a_1, a_1 = a_2 + b_1$, and $a_2 + b_1$ is in $x_2 \vee x_3$. *Linear combination augmentation*: Any Λ-equation of the form $\lambda d_3 + \lambda' d_4$ can be added to $\boldsymbol{r}$. *Defined variable augmentation*: We can "define" an unused auxiliary variable, say b_4, by adding a Λ-equation, say $\lambda a_2 + \lambda' b_2 + b_4 = 0$, if we change the lattice constraint of b_4 to $x_2 \vee x_1$ ($-\lambda a_2 - \lambda' b_2$ is in $x_2 \vee x_1$). *Union augmentation*: Add the Λ-equation $a_2 - b_7 - b_9 = 0$, for example, expressing a variable a_2 as a sum of two unused auxiliary variables. Then change the lattice constraints of b_7 and b_9 to some x_4 and x_5 in L, respectively, such that $x_2 \subset x_4 \vee x_5$ ($a_2 \in x_2$). *Null variable augmentation*: Terms λa_5 and $\lambda' a_5$

can be added to the Λ-equations d_3 and d_4, respectively, since $a_5 \in \omega$ and $\iota(\omega) = 0$. *Inessential variables augmentation*: We can add finitely many Λ-equations in the variables b_k, $k \geqq 3$, and make finitely many arbitrary changes in the lattice constraints of those variables. *Renumbering*: b_1 and b_2 can be replaced throughout $\boldsymbol{r}$ by any two other auxiliary variables.

The solution set of $\boldsymbol{r}$ is unchanged by any of the above modifications. The primary fact about $M(L;\Lambda) = D(L;\Lambda)/E(L;\Lambda)$ is that it is an abelian lattice under the induced meet and join. Intuitively, $M(L;\Lambda)$ acts like the lattice of submodules with finite support of $M^{\boldsymbol{N}}$, for some hypothetical Λ-module M.

Associated with any abelian lattice X is a small abelian category $\boldsymbol{A}_X$ [**2**, Main Theorem]. We next construct for each object A of $\boldsymbol{A}_{M(L;\Lambda)}$ a ring homomorphism ζ_A preserving 1 from Λ into the ring of endomorphisms of A ($\zeta_A(\lambda)$ is a formal analogue of $\lambda 1_A$). Let **Ab** and Λ-**Mod** be the usual categories of abelian groups and of Λ-modules, respectively. By [**1**. Theorem 7.14], there exists an exact embedding functor $F:\mathbf{A}_{M(L;\Lambda)} \to \mathbf{Ab}$. Defining $\lambda x = (F(\zeta_A(\lambda)))(x)$ makes $F(A)$ into a Λ-module, denoted $G(A)$ ($F\zeta_A(\lambda) = \lambda 1_{G(A)}$). We can prove that $\zeta_B(\lambda)f = f\zeta_A(\lambda)$ for $f:A \to B$ in $\boldsymbol{A}_{M(L;\Lambda)}$, so $Ff:G(A) \to G(B)$ is Λ-linear. But then $G(A)$ and $Gf = Ff$ define an exact embedding functor $G:\boldsymbol{A}_{M(L;\Lambda)} \to \Lambda$-**Mod**. Because of G, the lattice of subobjects of each object of $\boldsymbol{A}_{M(L;\Lambda)}$ is in $\mathscr{L}(\Lambda)$. But then every interval sublattice of $M(L;\Lambda)$ is in $\mathscr{L}(\Lambda)$ by [**2**, 3.24], and $M(L;\Lambda) \in \mathscr{L}(\Lambda)$ follows, using a direct limit of Λ-modules.

We now define a lattice homomorphism $\psi:L \to M(L;\Lambda)$, similar to ψ in [**2**, 4.3]. For x in L, $\psi(x)$ is the equivalence class in $M(L;\Lambda)$ of $(\varnothing, \theta_x)$ in $D(L;\Lambda)$ given by $\theta_x(a_1) = x$, $\theta_x(c_k) = \omega$ for $c_k \in \boldsymbol{N}_2 - \{a_1\}$. If ψ is one-to-one, it embeds L into $M(L;\Lambda)$, and so L is in $\mathscr{L}(\Lambda)$. Suppose L is in $\mathscr{L}(\Lambda)$ with embedding $\iota:L \to \Gamma(M;\Lambda)$, $\iota(\omega) = 0$. Since equivalent constraint systems have equal solution sets, $\mu:D(L;\Lambda) \to \Gamma(M_1^\infty;\Lambda)$ induces a function $\bar{\mu}: M(L;\Lambda) \to \Gamma(M_1^\infty;\Lambda)$. Clearly $\bar{\mu}\psi(x) = \mu(\varnothing, \theta_x) = \bar{\psi}\iota(x)$, where $\bar{\psi}:\Gamma(M;\Lambda) \to \Gamma(M_1^\infty;\Lambda)$ is given by

$$\bar{\psi}(M') = \{m \in M_1^\infty : m(a_1) \in M', m(a_k) = 0 \text{ for } k > 1\}.$$

So, $\bar{\psi}\iota = \bar{\mu}\psi$. Since $\bar{\psi}\iota$ is one-to-one, so is ψ. Therefore, L is in $\mathscr{L}(\Lambda)$ if and only if ψ is one-to-one.

Four of the rules generating $E(L;\Lambda)$ are called "direct reductions", namely constraint decrease, linear combination augmentation, defined variable augmentation and union augmentation. A key argument shows that ψ is one-to-one iff, for each x in L and sequence $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ in $D(L;\Lambda)$ such that $\boldsymbol{r}_1 = (\varnothing, \theta_x)$, $\boldsymbol{r}_n = (G, \alpha)$ and $\boldsymbol{r}_{i+1}$ is obtained by a direct reduction of $\boldsymbol{r}_i$ ($1 \leqq i < n$), we have $\alpha(a_1) = x$. Each of the infinitely many Horn formulas of $J(\Lambda)$ is generated by a finite sequence of four operations. These operations imitate the four rules of direct reduction, with lattice polynomials replacing elements of L. Using the above, we show that ψ is one-to-one iff every formula of $J(\Lambda)$ is satisfied in L, and the main theorem follows.

COROLLARY. *Every abelian lattice is representable by abelian groups.*

2. **Comparison of classes of representable lattices.** Let Λ and Λ' be rings with 1, not necessarily commutative. Then $\mathscr{L}(\Lambda) \subset \mathscr{L}(\Lambda')$ if there exists a ring homomorphism $\Lambda' \to \Lambda$ preserving 1, or if there exists a (Λ', Λ)-bimodule M which is faithfully flat as a right Λ-module. A simple change of rings argument proves the first result. For the other: the exact embedding functor $M \otimes_\Lambda$ from Λ-**Mod** into Λ'-**Mod** induces an embedding from the lattice of subobjects of any M_0 in Λ-**Mod** into the lattice of subobjects of $M \otimes_\Lambda M_0$ in Λ'-**Mod**. Then $\mathscr{L}(\Lambda) = \mathscr{L}(\Lambda')$ if Λ is a regular ring and unitary subring of Λ', by known ring theory. Let $\boldsymbol{Q}$ denote the field of rationals and $\boldsymbol{Z}_n$ the ring of integers modulo n, $n \geqq 2$. So, $\mathscr{L}(\Lambda) = \mathscr{L}(\boldsymbol{Q})$ if Λ has a unitary subring isomorphic to $\boldsymbol{Q}$. Also, $\mathscr{L}(\Lambda) = \mathscr{L}(\boldsymbol{Z}_n)$ if Λ has characteristic n for n a square-free number (prime, or a product of distinct primes). Let P_Λ be the set of primes p such that $1 + 1 + \cdots + 1$ (p times) is invertible in Λ. If P is a set of primes, let $\boldsymbol{Q}(P)$ be the unitary subring of $\boldsymbol{Q}$ generated by $\{p^{-1} : p \in P\}$. If Λ has characteristic zero, $\mathfrak{a}$ is the two-sided ideal of torsion elements of Λ and $P_{\Lambda/\mathfrak{a}} = P_\Lambda$, then $\mathscr{L}(\Lambda) = \mathscr{L}(\boldsymbol{Q}(P_\Lambda))$. So, $\mathscr{L}(\Lambda) = \mathscr{L}(\boldsymbol{Q}(P_\Lambda))$ if Λ is torsion-free.

Some of the above results are the best possible. Under various hypotheses, $\mathscr{L}(\Lambda) - \mathscr{L}(\Lambda') \neq \varnothing$ is proved by constructing a Horn formula satisfied in all lattices in $\mathscr{L}(\Lambda')$ but not in all lattices in $\mathscr{L}(\Lambda)$. These formulas reflect properties of the (additive) multiples $k \cdot 1_M = 1_M + 1_M + \cdots + 1_M$ for M an arbitrary Λ-module. For example, $k \cdot 1_M = 0$ if the characteristic of Λ divides k, and $k \cdot 1_M$ is an automorphism if $k \cdot 1$ is invertible in Λ. So, we can show that $\mathscr{L}(\Lambda) - \mathscr{L}(\Lambda') \neq \varnothing$ if the characteristic of Λ does not divide the (nonzero) characteristic of Λ', and therefore $\mathscr{L}(\Lambda) \neq \mathscr{L}(\Lambda')$ if Λ and Λ' have different characteristics. If p is a prime invertible in Λ' but not in Λ, then $\mathscr{L}(\Lambda) - \mathscr{L}(\Lambda') \neq \varnothing$, and so $\mathscr{L}(\Lambda) \neq \mathscr{L}(\Lambda')$ if $P_\Lambda \neq P_{\Lambda'}$. If n is not square-free, then there exists Λ with characteristic n such that $\mathscr{L}(\Lambda) \neq \mathscr{L}(\boldsymbol{Z}_n)$. Also, if Λ has characteristic zero and torsion ideal $\mathfrak{a}$, then $\mathscr{L}(\Lambda) \neq \mathscr{L}(\boldsymbol{Q}(P_\Lambda))$ if $P_{\Lambda/\mathfrak{a}} \neq P_\Lambda$. If P is a proper subset of the primes or is empty, then Λ with characteristic zero exists such that $P_\Lambda = P$ but $\mathscr{L}(\Lambda) \neq \mathscr{L}(\boldsymbol{Q}(P))$.

The detailed proofs of these results have been submitted for publication.

C. Herrmann and W. Poguntke have recently communicated to the author a theorem which implies that $\mathscr{L}(\Lambda)$ admits ultraproducts, for any ring Λ with 1. It then follows nonconstructively that $\mathscr{L}(\Lambda)$ is always a quasivariety, using the known result that a class of algebras admitting isomorphic images, subalgebras, products and ultraproducts is a quasivariety. Another of their results implies that $\mathscr{L}(\Lambda)$ is not finitely first-order axiomatizable if Λ is a unitary subring of $\boldsymbol{Q}$.

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